Symmetries of Tensors

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Andrew Schaffer Berget

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Abstract

This thesis studies the symmetries of a fixed tensors by looking at certain group representations this tensor generates. We are particularly interested in the case that the tensor can be written as $v_1 \otimes \cdots \otimes v_n$, where the v_i are selected from a complex vector space. The general linear group representation generated by such a tensor contains subtle information about the matroid $M(v)$ of the vector configuration $v_1, \ldots, v_n$. To begin, we prove the basic results about representations of this form. We give two useful ways of describing these representations, one in terms of symmetric group representations, the other in terms of degeneracy loci over Grassmannians. Some of these results are equivalent to results that have appeared in the literature. When this is the case, we have given new, short proofs of the known results.

We will prove that the multiplicities of hook shaped irreducibles in the representation generated by $v_1 \otimes \cdots \otimes v_n$ are determined by the no broken circuit complex of $M(v)$. To do this, we prove a much stronger result about the structure of vector subspace of $\operatorname{Sym} V$ spanned by the products $\prod_{i \in S} v_i$, where S ranges over all subsets of $[n]$. The result states that this vector space has a direct sum decomposition that determines the Tutte polynomial of $M(v)$. We will use a combinatorial basis of the vector space generated the products of the linear forms to completely describe the representation generated by a decomposable tensor when its matroid $M(v)$ has rank two.

Next we consider a representation of the symmetric group associated to every matroid. It is universal in the sense that if $v_1, \ldots, v_n$ is a realization of the matroid then the representation for the matroid provides non-trivial restrictions on the decomposition of the representation generated by the tensor product of the vectors. A complete combinatorial characterization of this representation is proven for parallel extensions of Schubert matroids. We also describe the multiplicity of hook shapes in this representation for all matroids.

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CHAPTER 1

Introduction

1.1. Motivation

This thesis studies the smallest $\mathrm{GL}(V)$ or $\mathfrak{S}_n$ invariant subspace of a tensor product $V^{\otimes n}$ that contains a given decomposable tensor. Associated to this decomposable tensor is a matroid, and the main question addressed here is to what extent this matroid determines the isomorphism type of the representation. It is unknown whether the matroid completely determines the isomorphism type, but it is known that it determines some important aspects of it. Before we describe these aspects specifically, let us attempt to motivate this study. To do begin doing this, we recall that the determinant of a matrix enjoys many beautiful definitions, the most straightforward being that if $A = (a_{ij})_{1\leq i,j\leq n}$ then

$$\det A = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

This definition motivated Schur to consider generalized matrix functions of the form

$$d_\chi(A) = \sum_{\sigma\in\mathfrak{S}_n} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

where $\chi : \mathfrak{S}_n \to \mathbf{C}$ is any function. For example, if $\chi(\sigma) = 1$ for all σ then d_χ is the permanent.

Of particular interest to Schur where those χ that arise as irreducible characters of the symmetric group $\mathfrak{S}_n$. Generalized matrix functions of this form are called irreducible character immanants. Recall that the irreducible representations of $\mathfrak{S}_n$ are indexed by partitions λ of n. If χ^λ denotes the character of the irreducible indexed by λ, then we write d_λ for d_{χ^λ}. The trivial representation is associated to the partition (n) and the sign character is associated to the partition $(1, \ldots, 1) = (1^n)$, hence per $= d_{(n)}$ and det $= d_{(1^n)}$. Schur proved that

$$\det A \leq \frac{1}{\chi^\lambda(1)} d_\lambda(A)$$

when A is a positive semi-definite Hermitian matrix. Since $0 \leq \det A$ for such matrices we immediately see that $0 \leq d_\lambda(A)$ for such matrices. The following question is now natural: When do we have equality? The answer for the determinant is well known: $\det A = 0$ if and only if the columns of A are linearly dependent. To better answer this question, we rephrase it in a suggestive form.

We know that we may write A as the Gram matrix of a collection of vectors, $A = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n}$, where $\langle , \rangle$ is a Hermitian inner product on a complex vector space V. It follows that we can write

$$\begin{aligned} d_\lambda(A) = \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma) \langle v_1 \otimes \cdots \otimes v_n, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \rangle \\ = \langle v_1 \otimes \cdots \otimes v_n, \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma)(v_1 \otimes \cdots \otimes v_n).\sigma \rangle. \end{aligned}$$

Here $\langle , \rangle$ is is the natural extension of the inner product on V to $V^{\otimes n}$ and the symmetric group is acting on the right of $V^{\otimes n}$ via place permutation. The linear operator

$$\pi_\lambda = \frac{\chi^\lambda(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma)\sigma : V^{\otimes n} \to V^{\otimes n},$$

is idempotent and Hermitian. It follows that

$$\langle v_1 \otimes \cdots \otimes v_n, (v_1 \otimes \cdots \otimes v_n)\pi_\lambda \rangle = \langle (v_1 \otimes \cdots \otimes v_n)\pi_\lambda, (v_1 \otimes \cdots \otimes v_n)\pi_\lambda \rangle$$

and hence $d_\lambda(A) = 0$ if and only if

$$(v_1 \otimes \cdots \otimes v_n)\pi_\lambda = 0.$$

Determining when π_λ applied to a decomposable tensor is zero is the starting point of this thesis.

1.2. Statement of the Main Results

Let $v^\otimes = v_1 \otimes \cdots \otimes v_n$ be a decomposable tensor, where the v_i are selected from a complex vector space V. We let v denote the collection of vectors $\{v_1, \ldots, v_n\}$. It will be shown that $v^\otimes \pi_\lambda$ is zero if and only an irreducible representation indexed by the partition λ appears in the smallest $\mathfrak{S}_n$-module or $\mathrm{GL}(V)$-module that contains $v^\otimes \in V^{\otimes n}$. Recall that $\mathrm{GL}(V)$ is acting on the tensor product $V^{\otimes n}$ on the diagonal:

$$g.(v_1 \otimes \cdots \otimes v_n) = (gv_1) \otimes \cdots \otimes (gv_n).$$

Our first main result is a simple proof of the following theorem, originally proved by Gamas [**Gam88**].

Theorem (Gamas's Theorem)**.** *The symmetrized tensor $v^\otimes \pi_\lambda$ is not zero if and only if there is a tableau T of shape λ whose columns index linearly independent subsets of v.*

The approach that we will use to derive this will yield much more. We will derive the known characterization of when $u^\otimes \pi_\lambda = v^\otimes \pi_\lambda$ [**dCDdS05**].

Theorem (Da Cruz–Dias da Silva)**.** *There is an equality of symmetrized tensors $u^\otimes \pi_\lambda = v^\otimes \pi_\lambda$ if and only if for all tableaux T (1) If the columns of T index linearly independent subsets of u then they also do for v. (2) If the columns of T index linearly independent subsets of u then there is a permutation σ such* *that the subspace of u's spanned by column i is equal to the subspace of v's spanned by column $\sigma(i)$. Further, the product of the determinants of the change of bases between all these spaces should be* 1.

The proofs of both these results took long papers, and our derivations are quick and transparent.

Let $\mathfrak{S}(v^{\otimes})$ denote the smallest symmetric group representation in $V^{\otimes n}$ containing $v^{\otimes}$ and let $G(v^{\otimes})$ denote the smallest $\mathrm{GL}(V)$ representation containing $v^{\otimes}$. We will show that $\mathfrak{S}(v^{\otimes})$ and $G(v^{\otimes})$ carry the same information. The goal of this thesis is to see how Gamas's theorem generalizes, in the sense that we want to see to what extent the linear independence properties of v determine the multiplicity of a given irreducible in $G(v^{\otimes})$ or $\mathfrak{S}(v^{\otimes})$. These properties are known as the matroid of v, and collectively denoted $M(v)$.

Our main positive result in this direction is the following.

Theorem. *Let h_j be the multiplicity of the length j hook λ^j in $\mathfrak{S}(v^{\otimes})$, then*

$$\sum_{j\geq 0} h_j t^{r(M(v))-j} = \frac{1}{1+t} T(M(v); 1+t, 0),$$

where $T(M(v); x, y)$ is the Tutte polynomial associated to the matroid of v.

To prove this we will consider the vector space generated by the "square-free" products of the linear forms $v_1, \ldots, v_n \in \operatorname{Sym} V$. That is, we consider the subspace $P(v)$ of the symmetric algebra $\operatorname{Sym} V$ spanned by the products $\prod_{i\in S} v_i$, $S \subset [n]$. The multiplicity the longest hook shape in $\mathfrak{S}(v^{\otimes})$ and $G(v^{\otimes})$ is the dimension of the subspace of $P(v)$ spanned by products over sets $S \subset [n]$ which are compliments of bases of the matroid $M(v)$. We will show that there is a doubly indexed direct sum $P(v) = \bigoplus_{i,j} P(v)_{i,j}$ from which we will derive the following result.

Theorem. *The coefficients of the Tutte polynomial evaluation* $T(M(v); 1 + x.y)$ *are the dimensions of the vector spaces* $P(v)_{i,j}$.

The theorem about hook shapes will follow by setting $y = 0$ in this theorem.

When v is a rank two vector configuration (i.e., the span of the v_i is two-dimensional) we can associate a partition to v by saying that the i-th part of the partition is the size of the i-th parallelism class of v. Using a combinatorial basis constructed for $P(v)$, we will give a complete characterization of the isomorphism type of $\mathfrak{S}(v^{\otimes})$. Recall that the conjugate partition μ' of μ is the partition whose i-th part is $\#\{j : \mu_j \geq i\}$.

Theorem. *Suppose that* v *is a rank two configuration and the partition associated to* v *is* μ. *The multiplicity of* $(n-k, k)$ *in* $\mathfrak{S}(v^{\otimes})$ *is* 1 *if* $k = 0$, *and if* $k > 0$ *it is*

$$\mu'_1 + \cdots + \mu'_k - 2k + 1,$$

or zero, depending on whether the above number is positive or not.

Lastly, we will attempt to generalize $\mathfrak{S}(v^{\otimes})$ by defining a representation of $\mathfrak{S}_n$ that obviously depends only on the matroid of v. If M is any matroid, we define $U(M)$ to be the quotient of $\mathbf{C}\mathfrak{S}_n$ by the right ideal generated by antisymmetrizers of dependent sets of M. This representation will be universal in the sense that if we have any realization v of a matroid M, i.e., $M = M(v)$ then there is a surjective map of $\mathfrak{S}_n$-modules $U(M) \to \mathfrak{S}(v^{\otimes})$. Using this result we will determine which irreducibles appear in $U(M)$ provided that M is realizable over a field of characteristic zero.

Our first main result about $U(M)$ is that the multiplicity of hook shapes is again determined by the Tutte polynomial of M.

Theorem. *Let M be any matroid. If h_j is the multiplicity of the length j hook λ^j in $U(M)$ then*

$$\sum_{j\geq 0} h_j t^{r(M)-j} = \frac{1}{1+t} T(M; 1+t, 0),$$

where $T(M; x, y)$ is the Tutte polynomial of M.

We will also give a complete characterization of $U(M)$ when M is a particular kind of matroid known as a Schubert matroid (also known as shifted matroids and freedom matroids). These matroids come with a canonical ordering on their ground sets, which is useful in our final main result.

Theorem. *Let M be a Schubert matroid. The multiplicity of λ in $U(M)$ is the number of standard Young tableaux T of shape λ such that every decreasing subword of the reading word of T is independent in M.*

CHAPTER 2

Representations and Symmetrizations

In this chapter, we introduce two representations generated by a decomposable tensor. We show that the representations carry the same information, and then describe which irreducible submodules they contain. We use these results to prove facts about symmetrizations of tensors that have appeared in the literature. We then introduce a third way of studying these representations, and indicate how this approach might be useful in our study. Throughout, we consider some examples of our results. The last section indicates future directions of study.

We will assume that the reader is familiar with Schur-Weyl duality throughout this chapter. The details of this result can be found in Appendix A. We will also use some ideas from matroid theory. A quick review of the concepts needed here can be found in Appendix B.

2.1. Representations Generated by Tensors

Let V be a complex vector space of finite dimension k and $V^{\otimes n}$ its n-fold tensor product. This tensor product is a $\mathrm{GL}(V) \times \mathfrak{S}_n$-bimodule, where $\mathrm{GL}(V)$ acts diagonally and $\mathfrak{S}_n$ acts by permutation of tensor factors. Given an element $w \in V^{\otimes n}$ we will define $G_V(w)$ to be the smallest $\mathrm{GL}(V)$ - submodule of $V^{\otimes n}$ that contains w. When no confusion will arise (and we will prove that none will) we will drop the reference to V and write $G(w)$. One sees that $G_V(w)$ is a finite dimensional polynomial representation of $\mathrm{GL}(V)$, since $V^{\otimes n}$ is. Its character can thus be written as an integral sum of Schur polynomials in Λ_k — the ring of symmetric function in k variables. By mapping the Schur polynomial $s_\lambda(x_1, \dots, x_k)$ to the Schur function $s_\lambda(x_1, x_2, \dots)$, we can associate a symmetric function to $G_V(w)$.

It follows from the definition that $G_V(w)$ is equal to the vector space spanned by the $\mathrm{GL}(V)$-orbit of w. Since every element of $\mathrm{End}(V)$ is a limit of elements of $\mathrm{GL}(V)$ we see that $G_V(w)$ is closed under the diagonal action of $\mathrm{End}(V)$ on the tensor product. There is also an action of the Lie algebra $\mathfrak{gl}(V)$ on $G_V(w)$, where $x \in \mathfrak{gl}(V)$ acts by the Leibniz rule

$$x.(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes x v_i \otimes v_{i+1} \otimes \cdots \otimes v_n.$$

From the tensor $w \in V^{\otimes n}$, we may also form a representation $\mathfrak{S}_V(w)$, which is the smallest $\mathfrak{S}_n$-submodule of $V^{\otimes n}$ that contains w. Here the action of $\mathfrak{S}_n$ is on the right of $V^{\otimes n}$, and is permutation of tensor factors. If $V \subset W$ is a subspace and $w \in V^{\otimes n} \subset W^{\otimes n}$ then $\mathfrak{S}_V(w) \approx \mathfrak{S}_W(w)$ and hence we omit any reference to the ambient tensor product and write $\mathfrak{S}(w)$. The Frobenius character of $\mathfrak{S}(w)$ is an integral sum of Schur functions.

By Schur-Weyl duality we have the following result that relates $\mathfrak{S}(w)$ and $G_V(w)$.

Theorem 2.1.1. *The symmetric function obtained from the character of $G_V(w)$ is equal to the Frobenius character of $\mathfrak{S}(w)$.*

We delay the full proof of the theorem.

Corollary 2.1.2. *The symmetric function associated to $G_V(w)$ does not depend on V in the sense that if $V \subset V'$ and $w \in V^{\otimes n} \subset V'^{\otimes n}$ then the symmetric functions associated to $G_V(w)$ and $G_{V'}(w)$ are equal.*

Proof. The symmetric function associated to $G_V(w)$ and $G_{V'}(w)$ are both equal to the Frobenius character of $\mathfrak{S}(w)$. $\square$

Example 2.1.3. Suppose that $n = k(= \dim V)$ and that $w = e_1 \otimes \cdots \otimes e_k$, where $e_1, \ldots, e_k$ is a basis of V. Then $\mathfrak{S}(w)$ is visibly isomorphic to $\mathbf{C}\mathfrak{S}_k$. The Frobenius character of $\mathbf{C}\mathfrak{S}_k$ is $(\sum_i x_i)^k$.

To show that every decomposable tensor is in $G_V(w)$ one uses the diagonal action of $\mathrm{End}(V)$ on $V^{\otimes n}$. Since these tensors span the whole tensor product, $G_V(w) = V^{\otimes n}$. The character of $V^{\otimes n}$ is $(x_1 + x_2 + \cdots + x_k)^k$ and hence the symmetric function associated to $G_V(w)$ is $(\sum_{i=1}^{\infty} x_i)^k$, as Theorem 2.1.1 predicts.

Example 2.1.4. Much more generally, suppose that $w \in V^{\otimes n}$ and $w' \in (V')^{\otimes n'}$. Form the tensor $w \otimes w' \in (V \oplus V')^{\otimes (n+n')}$. If $s(w)$, $s(w')$ and $s(w \otimes w')$ are the symmetric functions associated to w, w' and $w \otimes w'$ then

$$s(w \otimes w') = s(w)s(w').$$

To show two these symmetric functions are equal, we first recall that for two arbitrary symmetric functions f and g, we write $f \leq_S g$ if $g - f$ is a positive integer sum of Schur functions. This is called the Schur partial order on the ring Λ of symmetric functions. To prove that $s(w \otimes w') = s(w)s(w')$ we will show that

$$s(w \otimes w') \leq_S s(w)s(w') \text{ and } s(w)s(w') \leq_S s(w \otimes w').$$

Recall that if X and X' are representations of $\mathfrak{S}_n$ and there is an injection $X \to X'$ then the Frobenius character of X is at most the Frobenius character of X' in the Schur partial order on Λ.

There is an injection of $\mathfrak{S}_n \times \mathfrak{S}_{n'}$-modules

$$\mathfrak{S}(w) \otimes \mathfrak{S}(w') \hookrightarrow V^{\otimes n} \otimes (V')^{\otimes n'}.$$

Inducing to $\mathbf{C}\mathfrak{S}_{n+n'}$ gives us an injection

$$\operatorname{Ind} \mathfrak{S}(w) \otimes \mathfrak{S}(w') \hookrightarrow (V \oplus V')^{\otimes(n+n')}$$

That induction is left-exact follows from the semi-simplicity of the group algebras of finite groups over $\mathbf{C}$. The image of $\operatorname{Ind} \mathfrak{S}(w) \otimes \mathfrak{S}(w')$ in $(V \oplus V')^{n+n'}$ is a representation of $\mathfrak{S}_{n+n'}$ that contains $w \otimes w'$. It follows that the image of the induction in the tensor product $(V \oplus V')^{n+n'}$ contains $\mathfrak{S}(w \otimes w')$. Taking Frobenius characters, we have $s(w \otimes w') \leq_S s(w)s(w')$.

There is a natural injection of $\mathrm{GL}(V \oplus V')$-modules

$$G_{V \oplus V'}(w \otimes w') \to G_{V \oplus V'}(w) \otimes G_{V \oplus V'}(w')$$

induced by the inclusion $V^{\otimes n} \otimes (V')^{\otimes n'} \to (V \oplus V')^{\otimes(n+n')}$. Taking characters and recalling the semisimplicity of polynomial representations of $\mathrm{GL}(V)$, we have proved that $s(w)s(w') \leq_S s(w \otimes w')$.

We conclude that $s(w)s(w') = s(w \otimes w')$. However, we can conclude more from this. Namely, that the injections given above are actually isomorphisms, i.e.,

$$G_{V \oplus V'}(w \otimes w') \approx G_{V \oplus V'}(w) \otimes G_{V \oplus V'}(w'),$$

and

$$\operatorname{Ind} \mathfrak{S}(w) \otimes \mathfrak{S}(w') \approx \mathfrak{S}(w \oplus w').$$

Example 2.1.5. Suppose that $w = v_1 \otimes v_2 \otimes \cdots \otimes v_n$ where the v_i are selected from an ordered basis of V. Suppose that the i-th basis element of V

is selected μ_i times. The previous example shows that, as $\mathrm{GL}(V)$-modules,

$$G_V(w) \approx \mathrm{Sym}^{\mu_1} V \otimes \mathrm{Sym}^{\mu_2} V \otimes \cdots \otimes \mathrm{Sym}^{\mu_k} V.$$

It follows that the symmetric function associated to $G_V(w)$ is h_μ — the complete homogeneous symmetric function associated to the partition μ. The previous example also shows that $\mathfrak{S}(w) \approx \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbf{C}$, where $\mathfrak{S}_\lambda$ is the parabolic subgroup of $\mathfrak{S}_n$ associated to λ and $\mathbf{C}$ denotes the trivial representation of this subgroup.

We will see that this example subsumes a result of Merris (and others) that characterizes the vanishing of symmetrized tensors which have the same form as those considered in this example [**Mer78, Lam78, LV73**].

2.1.1. The Proof of Theorem 2.1.1. Recall that we wish to prove that the symmetric function obtained from the character of $G_V(w)$ is equal to the Frobenius character of $\mathfrak{S}(w)$. This will become more transparent when done an abstract setting.

Lemma 2.1.6. *Suppose that R and S are unital K-algebras, with K a field. Let $U = {}_RU_S$ be a semisimple R-S bimodule and suppose that*

$$\mathrm{End}_R({}_RU) = S \quad \text{and,} \quad \mathrm{End}_S(U_S) = R$$

For every $x \in U$ we have a natural isomorphism of left R-modules

$$\mathrm{Hom}_R(Rx, {}_RU) \approx xS \qquad (\varphi : Rx \to {}_RU) \mapsto \varphi(x),$$

and a natural isomorphism of right S-modules

$$\mathrm{Hom}_S(xS, U_S) \approx Rx, \qquad (\psi : xS \to U_S) \mapsto \psi(x).$$

If $U = \bigoplus_\lambda M^\lambda \otimes N^\lambda$, where M^λ and N^λ are simple R and S modules, respectively, then

$$\dim_K \mathrm{Hom}_R(Rx, M_\lambda) = \dim_K \mathrm{Hom}_S(xS, N_\lambda).$$

Proof. Since U is a semisimple R-module, Rx is a direct summand of U and hence every homomorphism $\varphi : Rx \to U$ can be extended to a map $\tilde{\varphi} : U \to U$. It follows that such a homomorphism is right action by an element of S, since $\mathrm{End}_R(U) = S$, hence the map $\tilde{\varphi} \mapsto \tilde{\varphi}(x) = \varphi(x) \in U$ has image in xS. This is at once seen to be a surjective map of right S-modules. It is injective because if $\varphi(x) = 0$ then φ satisfies

$$\varphi(rx) = r\varphi(x) = 0$$

for all $r \in R$. This proves that $\mathrm{Hom}_R(Rx, U) = xS$ and the proof that $\mathrm{Hom}_S(xS, U) = Rx$ is the same.

To prove the claim about the multiplicities we see that as right S-modules,

$$\begin{aligned}\mathrm{Hom}_R(Rx, U) &= \bigoplus_{\lambda} \mathrm{Hom}_R(Rx, M^{\lambda} \otimes N^{\lambda}) \\ &= \bigoplus_{\lambda} \mathrm{Hom}_R(Rx, M^{\lambda}) \otimes N^{\lambda} \approx \bigoplus_{\lambda} (N^{\lambda})^{\oplus(\dim_K \mathrm{Hom}_R(Rx, M^{\lambda}))}\end{aligned}$$

Since $\mathrm{Hom}_R(Rx, U) \approx xS$ it follows that the multiplicity of N^{λ} in xS is equal to the multiplicity of M^{λ} in Rx. $\square$

We can now prove Theorem 2.1.1, which stated that the symmetric function associated to $G_V(w)$ is equal to the Frobenius character of $\mathfrak{S}(v)$.

Proof of Theorem 2.1.1. The multiplicity of an irreducible representation indexed by λ in $G_V(w)$ is the coefficient of the Schur function indexed by λ in the symmetric function associated to $G_V(w)$. The multiplicity of an irreducible representation indexed by λ in $\mathfrak{S}(w)$ is the coefficient of the Schur function indexed by λ in the Frobenius character of $\mathfrak{S}(w)$.

According to Schur-Weyl duality, we can apply Lemma 2.1.6 with $R = \mathrm{GL}(V)$, $S = \mathbf{C}\mathfrak{S}_n$, and $U = V^{\otimes n}$ to see that these multiplicities are equal, and hence the symmetric functions associated to $G_V(w)$ and $\mathfrak{S}(w)$ are equal. $\square$

2.2. Irreducible Submodules

We will now restrict our attention to the representations generated by decomposable tensors w, i.e., those tensors which may be written as

$$w = v^{\otimes} := v_1 \otimes v_2 \otimes \cdots \otimes v_n.$$

We will denote the family of vectors $\{v_i\}$ simply by v and think of it as a configuration of vectors in V. Given such a configuration v, an element $g \in \mathrm{GL}(V)$ and an element $t = (t_1, \ldots, t_n) \in (\mathbf{C}^{\times})^n$ we say that the configuration

$$g.v.t = (t_1(gv_1), t_2(gv_2), \ldots, t_n(gv_n))$$

is projectively equivalent to v. This defines an equivalence relation on configurations of n vectors in V. From the definitions we have

$$G(v^{\otimes}) = G((g.v.t)^{\otimes}) \quad \text{and} \quad \mathfrak{S}(v^{\otimes}) \approx \mathfrak{S}((g.v.t)^{\otimes})$$

and hence the symmetric function associated to $G(v^{\otimes})$ and $\mathfrak{S}(v^{\otimes})$ only depends on the projective equivalence class of v. The goal of what follows is to determine how other geometric properties of v are reflected in the irreducible decomposition of $G(v^{\otimes})$ and $\mathfrak{S}(v^{\otimes})$. The most basic question one can pose about these representations is "Which irreducible representations have nonzero multiplicity?" In this section we answer this question. Three proofs of this result appear in the literature. The first proof by Gamas [**Gam88**] is long and the second by Pate [**Pat90**] relies on a series of nonstandard results. In [**Ber09**] we gave a short and self contained proof this result using induction and Pieri's rule. In this section we give an even shorter and more revealing proof using some (standard, classical) results from the algebraic geometry of flag manifolds.

Recall that the irreducible representations of $\mathfrak{S}_n$ are indexed by partitions of n, and the polynomial representations of $\mathrm{GL}(V)$ are indexed by partitions with at most $\dim V$ parts. Recall that a partition is a weakly decreasing sequence of non-negative integers whose sum is n (we will commonly extend a partition by adding parts equal to zero to it). We will say that λ appears in a $\mathfrak{S}_n$-module or $\mathrm{GL}(V)$-module if the module contains a submodule that is isomorphic to an irreducible module indexed by λ. The λ-th isotypic component of a module is the sum of the submodules isomorphic to an irreducible indexed by λ. The multiplicity of λ in a representation is the number of summands in a direct sum decomposition of the λ-th isotypic component where every summand is isomorphic to an irreducible indexed by λ.

Theorem 2.2.1. *Let λ be a partition of n. The following are equivalent.*

(1) *The multiplicity of λ is positive in $\mathfrak{S}(v^{\otimes})$*
(2) *$G(v^{\otimes})$ contains a highest weight vector with weight λ.*
(3) *There is a tableau T of shape λ whose columns index linearly independent subsets of v.*

In this thesis, the term "tableau" refers to a bijective filling of the numbers $1, 2, \ldots, n$ into the Young diagram of λ with no other restriction on the filling.

Remark 2.2.2. A quick sanity check is to assume that one of the v_i is zero. In this case $G(v^{\otimes}) = \mathfrak{S}(v^{\otimes}) = 0$ and every tableau has a column indexing a dependent subset of v.

The multiplicity of λ in a $\mathrm{GL}(V)$-module is the number of linearly independent highest weight vectors of weight λ in that module. Hence, the equivalence of the first two items follows from Theorem 2.1.1. To prove the equivalence of items (1) and (3) in Theorem 2.2.1 we will do a bit of translation: We will rephrase the result as a problem in the coordinate ring of a complete flag variety and appeal to the fact that this ring is an integral domain. We must recall some definitions first. Alternately, we could do an induction using Pieri's rule as was done in [**Ber09**].

Let $\mathcal{F}\ell(\mathbf{C}^k)$ be the projective variety of complete flags in $\mathbf{C}^k$:

$$\mathcal{F}\ell(\mathbf{C}^k) = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbf{C}^k : \dim V_i = i\}.$$

We can embed this in the projective variety $\mathbb{P}^* \bigwedge \mathbf{C}^k$ using the Plücker embedding. This is the product of the maps that send a codimension i subspace $W \subset \mathbf{C}^k$ to the hyperplane given by the quotient mapping $\bigwedge^i \mathbf{C}^k \to \bigwedge^i \mathbf{C}^k/W$.

The coordinate ring of a projective variety $X \subset \mathbb{P}^* W$, denoted $O(W)$, is the quotient of $\operatorname{Sym} W$ by the ideal of polynomials which vanish on X. It follows from this that the coordinate ring of $\mathcal{F}\ell(\mathbf{C}^k)$ is a quotient of the polynomial ring

$$\operatorname{Sym}(\bigwedge \mathbf{C}^k) = \mathbf{C}[X_{i_1 \ldots i_p} : \{i_1, \ldots, i_p\} \subset \{1, 2, \ldots, k\}].$$

The variables $X_{i_1 \ldots i_p}$, which form a basis for $\bigwedge \mathbf{C}^k$, are skew symmetric in the subscripts. The following result is classical, and its proof can be found in Fulton's book [**Ful97**, Ch.9].

Theorem 2.2.3. *The kernel of the quotient map* $\operatorname{Sym}(\bigwedge \mathbf{C}^k) \to O(\mathcal{F}\ell(\mathbf{C}^k))$ *is generated by the quadratic equations*

$$X_{i_1 \ldots i_p} X_{j_1 \ldots j_q} = \sum X_{i'_1 \ldots i'_p} X_{j'_1 \ldots j'_q}$$

the sum over all pairs of subscripts obtained by replacing the first ℓ of the i's with any ℓ of the j's, in order. The ideal these polynomials generate is prime and hence $O(\mathcal{F}\ell(\mathbf{C}^k))$ is an integral domain.

Note that

$$\mathrm{Sym}(\bigwedge \mathbf{C}^k) = \bigoplus_{0 \le a_1, \ldots, 0 \le a_k} \mathrm{Sym}^{a_1} \bigwedge^1 \mathbf{C}^k \otimes \cdots \otimes \mathrm{Sym}^{a_k} \bigwedge^k \mathbf{C}^k$$

We can associate the $a_1, \ldots, a_k$-th graded piece of this direct sum with a partition that has a_i columns of length $k - i + 1$.

Since the equations defining $\mathcal{F}\ell(\mathbf{C}^k)$ are homogeneous (by Theorem 2.2.3), $O(\mathcal{F}\ell(\mathbf{C}^k))$ inherits a direct sum decomposition indexed by partitions with at most k-parts. If λ is a partition with a_i columns of length $k - i + 1$ then define $\mathbb{S}^\lambda \mathbf{C}^k$ to be the image of

$$\mathrm{Sym}^{a_1} \bigwedge^1 \mathbf{C}^k \otimes \cdots \otimes \mathrm{Sym}^{a_k} \bigwedge^k \mathbf{C}^k$$

in $O(\mathcal{F}\ell(\mathbf{C}^k))$. It can be shown that $\mathbb{S}^\lambda \mathbf{C}^k$ satisfies a certain universal property which makes it the image of $\mathbf{C}^k$ under the λ-th *Schur functor*. See Section A.3 of Appendix A for the general definition of a Schur functor.

There is a natural action of $\mathrm{GL}(\mathbf{C}^k)$ on $\bigwedge \mathbf{C}^k$ and since the ideal defining $O(\mathcal{F}\ell(\mathbf{C}^k))$ is stable under this action $O(\mathcal{F}\ell(\mathbf{C}^k))$ is a representation of $\mathrm{GL}(\mathbf{C}^k)$. The action of $\mathrm{GL}(\mathbf{C}^k)$ respects the direct sum over partitions, hence $\mathbb{S}^\lambda \mathbf{C}^k$ is also a representation of $\mathrm{GL}(\mathbf{C}^k)$. For a proof of the following result see Fulton [**Ful97**, Ch.9].

Theorem 2.2.4. *For all partitions λ with at most k parts, $\mathbb{S}^\lambda \mathbf{C}^k$ is the irreducible representation of* $\mathrm{GL}(\mathbf{C}^k)$ *with highest weight λ. All irreducible polynomial representations of* $\mathrm{GL}(\mathbf{C}^k)$ *occur in this way.*

We can now prove that items (3) implies item (2) in Theorem 2.2.1. Pick a basis for V so that we may identify it with $\mathbf{C}^k$. Let λ be a partition of n, μ its conjugate partition. Recall that this is the partition μ' obtained from μ whose i-th part is the number of parts of μ with size at least i. Suppose that the sets

$$\{v_1, \dots, v_{\mu_1}\}, \quad \{v_{\mu_1+1}, \dots, v_{\mu_1+\mu_2}\}, \quad \dots$$

are linearly independent. It follows that the linear forms

$$v_1 \wedge \cdots \wedge v_{\mu_1} \in \mathrm{Sym}(\bigwedge \mathbf{C}^k), \quad v_{\mu_1+1} \wedge \cdots \wedge v_{\mu_1+\mu_2} \in \mathrm{Sym}(\bigwedge \mathbf{C}^k), \quad \dots$$

are each nonzero. These linear forms are nonzero in $O(\mathcal{F}\ell(CC^k))$ since the equations defining $\mathcal{F}\ell(\mathbf{C}^k)$ are quadratic. We conclude that the product of these linear forms in $O(\mathcal{F}\ell(V))$ is not zero, since this ring is an integral domain. However, the product of these wedges lives in $\mathbb{S}^\lambda V$, which is the irreducible representation of $\mathrm{GL}(V)$ with highest weight λ. It follows that the space of homomorphisms

$$\mathrm{Hom}_{\mathbf{C}\,\mathrm{GL}(V)}(G(v^\otimes), \mathbb{S}^\lambda V)$$

is non-zero. Since all the $\mathrm{GL}(V)$-representations in question are semisimple this proves that $G(v^\otimes)$ has a irreducible submodule isomorphic to $\mathbb{S}^\lambda V$. We conclude that $G(v^\otimes)$ has a highest weight vector of weight λ.

We delay the (easy) proof that item (2) implies item (3) in Theorem 2.2.1 for a moment to introduce symmetrizations of tensors. We mention that the main result here generalizes to a characteristic free statement.

Corollary 2.2.5. *Let v be a vector configuration in V, defined over any field. Then*

$$\mathrm{Hom}_{\mathrm{GL}(V)}(G(v^\otimes), \mathbb{S}^\lambda V) \neq 0$$

if there is a tableau of shape λ whose columns index linearly independent subsets of v. If the latter condition is satisfied, the canonical quotient map from the tensor product to the image of V under the Schur functor, $\mathbb{S}^\lambda V$, is non-zero.

The definition of $\mathbb{S}^\lambda V$ goes through as it did above, by taking the appropriate graded piece of the coordinate ring of $\mathcal{F}\ell(V)$.

Proof. The proof of this statement is the same as the one above, noting that the coordinate ring of the flag variety is still an integral domain. □

2.3. Symmetrizations of Tensors

The purpose of this section is give simple proofs of some results from the literature on symmetrizations of tensors.

For a partition λ of n, denote by χ^λ the irreducible character of $\mathfrak{S}_n$ indexed by λ. Let $\pi_\lambda \in \mathbf{C}\mathfrak{S}_n$ be the projector of $\mathbf{C}\mathfrak{S}_n$ to its χ^λ-isotypic submodule. We see at once that

$$\pi_\lambda = \frac{\chi^\lambda(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma)\sigma^{-1}.$$

It is known that π_λ can be written as a scalar multiple of

$$\sum_T c_T$$

where the sum is over all tableaux T (i.e., no row or column filling restrictions) of shape λ and c_T is the Young symmetrizer of T. Recall that the Young symmetrizer of T is the product $b_T a_T$ where a_T is the row symmetrizer of T and b_T is the column antisymmetrizer of T. By Schur-Weyl duality, if the length of T is at most $\dim V$ then π_λ is the projector of $V^{\otimes n}$ onto its $\mathbb{S}^\lambda V$-th isotypic component.

Let T^{ss} be the column super standard tableaux of shape λ filled with the numbers $1, 2, \ldots, n$ in order when read top-to-bottom left-to-right. The final result we will need from Fulton's book [**Ful97**] is the following.

Theorem 2.3.1. *The image of $c_{T^{ss}}$ on $V^{\otimes n}$ is isomorphic to $\mathbb{S}^\lambda V$, as $\mathrm{GL}(V)$-modules. The isomorphism sends the tensor $v^{\otimes} c_{T^{ss}}$ to the product of wedges*

$$v_1 \wedge \cdots \wedge v_{\mu_1}, \quad v_{\mu_1+1} \wedge \cdots \wedge v_{\mu_1+\mu_2}, \quad \ldots$$

when $c_{T^{ss}}$ is the tableaux above and μ is the partition conjugate to λ.

Sketch of Proof. It follows from Schur-Weyl duality that the image of $c_{T^{ss}}$ on $V^{\otimes n}$ is irreducible, hence the spaces in question are isomorphic. Further, any $\mathrm{GL}(V)$-module isomorphism must map the highest weight vector of $\mathbb{S}^\lambda V$ to the highest weight vector of $V^{\otimes n} c_{T^{ss}}$. There is map from $V^{\otimes n} c_{T^{ss}} \to \mathbb{S}^\lambda V$ (induced by a scalar multiple of the quotient map) that has the desired form and so it must be an isomorphism. $\square$

The only difference between the two representations in the theorem arises from the distinction between submodule and quotient module: The images of Young symmetrizers gives the irreducible representations of $\mathrm{GL}(V)$ as submodules, while the coordinate ring of the flag variety gives the representations as quotients.

As was discussed in the Introduction, the problem of determining when a symmetrized decomposable tensor $v^{\otimes}\pi_\lambda$ is zero is classical one. By items (1) and (2) in Theorem 2.2.1 the follows proposition is evident.

Proposition 2.3.2. *The symmetrized tensor $v^{\otimes}\pi_\lambda$ is not zero if and only if $G(v^{\otimes})$ has a highest weight vector of weight λ.*

We can now prove that item (3) implies items (1) and (2) in Theorem 2.2.1. From the description of π_λ as the sum over all Young symmetrizers of tableaux with shape λ it is clear that $v^{\otimes}\pi_\lambda$ is zero if $v^{\otimes}b_T$ is zero for all tableaux T of shape λ. As an endomorphism of $V^{\otimes n}$, the image of b_T is a tensor product of exterior products. It follows at once that $v^{\otimes}\pi_\lambda = 0$ if there is no partition of v into linearly independent sets whose sizes are the parts of the partition conjugate to λ. This completes the proof of Theorem 2.2.1. We have also characterized when a symmetrized tensor is zero, proving a theorem of Gamas [**Gam88**] (see also [**Ber09**]).

Theorem 2.3.3 (Gamas's Theorem)**.** *The symmetrized tensor $v^{\otimes}\pi_\lambda$ is not zero if and only if there is a tableau T of shape λ whose columns index linearly independent subsets of v.*

Example 2.3.4. Suppose that the vectors in v are selected from an ordered basis of V and that the i-th basis element of V is selected μ_i times. In Example 2.1.5 we computed that

$$G(v^{\otimes}) \approx \mathrm{Sym}^{\mu_1} V \otimes \dots \mathrm{Sym}^{\mu_k} V.$$

The multiplicity of λ in $G(v^{\otimes})$ is equal to the Kostka number $K_{\lambda\mu}$ — the number of semistandard Young tableaux of shape λ and content μ. We conclude that $v^{\otimes}\pi_\lambda$ is not zero if and only if $K_{\lambda\mu} \neq 0$ is not zero. This is a theorem of Merris [**Mer78, Lam78, LV73**].

The following result is a slight strengthening of Gamas's theorem.

Corollary 2.3.5 (Dias da Silva–Fonseca [**DdSF**])**.** *The symmetrized tensor $v^{\otimes}\pi_\lambda$ is not zero if and only if there is a standard Young tableau T of shape λ whose columns index linearly independent subsets of v.*

Proof. This follows from Gamas's theorem by noting that $v^{\otimes}\pi_{\lambda}$ is not zero if and only if there is some tableau T such that $v^{\otimes}c_T \neq 0$. Straightening the Young symmetrizer c_T in $\mathbf{C}\mathfrak{S}_n$ allows us to see that we may assume that T is a standard Young tableau. $\square$

Returning to the coordinate ring $O(\mathcal{F}\ell(\mathbf{C}^k))$, Fulton [**Ful97**, Ch.9] proves this ring is a unique factorization domain. This implies that a Young symmetrizer c_T applied to two decomposable tensors $u^{\otimes}$ and $v^{\otimes}$ yields the same result if and only if

(i) the columns of T index linearly independent subsets of v if and only if they index linearly independent subsets of u, and

(ii) when the columns of T index linearly independent subsets of v there is some permutation σ such that the span of the v's indexed by the i-th columns is equal to span of u's indexed by the $\sigma(i)$-th column. Further, the product of the determinants of the change of bases between these spaces is equal to one.

This follows at once by translating the statement $u^{\otimes}c_T = v^{\otimes}c_T$ into $O(\mathcal{F}\ell(V))$. Summing over all tableaux T and dividing by $\chi^{\lambda}(1)/n!$ proves the "if" direction of the following theorem.

Theorem 2.3.6 (da Cruz–Dias da Silva [**dCDdS05**])**.** *Two symmetrized decomposable tensors $u^{\otimes}\pi_{\lambda}$ and $v^{\otimes}\pi_{\lambda}$ are equal if and only if conditions (i) and (ii) above hold for all tableaux T of shape λ.*

Proof. To prove the "only if" direction we assume that $u^{\otimes}\pi_{\lambda} = v^{\otimes}\pi_{\lambda}$. Applying a Young symmetrizer c_T to this equation and noting that $\pi_{\lambda}c_T = c_T\pi_{\lambda} = c_T$ proves that $u^{\otimes}c_T = v^{\otimes}c_T$. Translating this statement into $O(\mathcal{F}\ell(\mathbf{C}^k))$ and using the fact that this ring is a unique factorization domain proves that items (i) and (ii) hold. ☐

Example 2.3.7. Consider the vector configurations $u = (x, x, y, y)$ and $v = (x, x, y, x + y)$, where x and y are linearly independent vectors in $\mathbf{C}^2$. For $\lambda = (2, 2)$ we have $u^{\otimes}\pi_{\lambda} = v^{\otimes}\pi_{\lambda}$. If $\lambda = (3, 1)$ then $u^{\otimes}\pi_{\lambda} \neq v^{\otimes}\pi_{\lambda}$, since item (i) fails when $T = \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}$.

2.4. Matroids and the Rank Partition

Theorem 2.2.1 shows us that the appearance of a particular irreducible representation in $G(v^{\otimes})$ and $\mathfrak{S}(v^{\otimes})$ only depends on the linear independence properties of the configuration v. The simplicial complex on $[n]$ whose facets are those sets I such that $\{v_i : i \in I\}$ is linearly independent is known as the matroid of v and denoted $M(v)$. More generally, we have the following definition.

Definition 2.4.1. A matroid is a finite simplicial complex whose maximal faces satisfy the *exchange property:* If B and B' are bases and $e \in B - B'$ then there is an element $e' \in B'$ such that $B - e \cup e'$ is a base.

The following list of notation is meant to serve as reminder of concepts from matroid theory.

Definition 2.4.2. Let M be a matroid on defined on a set E. The *ground set* of M is E. The faces of M are called *independent sets* and the maximal independent sets are called *bases*.

The *rank* of an independent set of M is its size. For a general subset of the ground set of M, its *rank* is the size of the largest independent set it contains. We denote the rank of a set S by $r(S)$. The rank of M, denoted $r(M)$, is the rank of any base of M. Equivalently it is the rank of the ground set of M.

A subset of the ground set is called a *flat* if it is maximal with a given rank. The smallest flat containing a given set is called its closure.

Definition 2.4.3. Let M be a matroid on E. Let λ be a partition of $|E|$. We say that M *conforms to* λ if there is a set partition of $E(M)$ of type λ whose blocks are independent sets of M.

We have already shown that an irreducible representation indexed by λ has positive multiplicity in $G(v^{\otimes})$ or $\mathfrak{S}(v^{\otimes})$ if and only if the matroid $M(v)$ conforms to λ' — the partition conjugate to λ.

Dias da Silva gave an elegant characterization of when a loopless matroid M conforms to a partition μ. (Loopless means that every element of the ground set belongs to some independent set.) He began by defining a sequence of numbers $\rho(M) = (\rho_1, \rho_2, \dots)$ by the condition that

$$\rho_1 + \rho_2 + \cdots + \rho_i$$

is the size of the largest union of i independent subsets of M. The sequence $\rho(M)$ is called the rank partition of M. This definition makes sense for an arbitrary finite simplicial complex. However, in general the definition does not ensure that that the rank partition is, in fact, a partition. The following example is due to Seth Sullivant.

Example 2.4.4. Let Δ be the simplicial complex with maximal faces $\{1, 2, 3\}$, $\{1, 4\}$, $\{2, 5\}$ and $\{3, 6\}$. Then $\rho(\Delta) = (3, 1, 2)$, which is not decreasing. It is clear that Δ is not a matroid, since it is not a pure complex.

The dominance order on partitions of n is defined by saying that $\lambda \geq \mu$ if

$$\lambda_1 + \cdots + \lambda_j \geq \mu_1 + \cdots + \mu_j$$

for all j (where we have added on parts of size 0 to λ and μ. Dominance order is a partial order and, in fact, turns the set of partitions of n into a lattice. In [**DdS90**] Dias da Silva proved the following.

Theorem 2.4.5 (Dias da Silva)**.** *The rank partition of a matroid is a partition of the ground set of M. The matroid M conforms to μ if and only if μ is less than or equal to $\rho(M)$ in dominance order.*

Example 2.4.6. Two elements e and f of a matroid are said to be parallel if they are not loops and the rank of $\{x, y\}$ is one. A parallel extension of a matroid M on a set E is matroid M' on a set $E' \supset E$ such that every element of $E' - E$ is parallel to an element of E.

Suppose that the rank partition of M has first part equal to k and length m. Then there is a parallel extension M' of M that has rank partition equal to $(k, \ldots, k) = (k^m)$. Indeed, if the ground set of M can be into partitioned into independent sets of size $\rho_1, \ldots, \rho_\ell$ then each of these independent sets can be extended to a base of M. To obtain M', add parallel elements to M so that the ground set of M' is a union of these bases. Since the ground set of M' is a disjoint union of bases m bases, the rank partition of M' is (k^m).

The following example will give an interesting result later (see Example 3.3.7).

Example 2.4.7. Suppose that M is a matroid whose rank partition is the rectangular shape (k^m) (in particular, the rank of M is k). We claim that every cocircuit of M has size at least m. Recall that a cocircuit of M is the complement of a flat of rank $r(M) - 1$. Indeed, if there is a flat X of M with rank $k - 1$ and size larger than $mk - m = m(k - 1)$ then there is no way to fill the elements of the ground set of M into the boxes of a Young diagram of shape (k^m) and have every row independent.

Corollary 2.4.8. *The multiplicity of an irreducible representation indexed by λ in $G(v^{\otimes})$ or $\mathfrak{S}(v^{\otimes})$ is non-zero if and only if λ is larger than $\rho(M(v))'$ in dominance order.*

Corollary 2.4.9. *The* $\mathrm{GL}(V) \times \mathfrak{S}_n$*-bimodule generated by $v^{\otimes}$ in $V^{\otimes n}$ is isomorphic to*

$$\bigoplus_{\lambda \geq \rho(M)'} \mathbb{S}^{\lambda} V \otimes \mathcal{S}(\lambda)$$

where $\mathcal{S}(\lambda)$ denotes the irreducible representation of $\mathfrak{S}_n$ with character χ^{λ}.

Proof. Since the decomposition of $V^{\otimes n}$ as a $\mathrm{GL}(V) \times \mathfrak{S}_n$-bimodule is multiplicity free and we know which irreducible submodule must occur, the result follows from Corollary 2.4.8. $\square$

The rank partition of a matroid is an extremely difficult invariant to work with: It behaves badly with respect to all the standard matroid constructions, except direct sum where $\rho(M \oplus N)_i = \rho(M)_i + \rho(N)_i$. In particular, note that if M has a coloop then $\rho(M^*)$ is not even defined, since M^* has a loop

2.5. Approaching $G(v^{\otimes})$ via Vector Bundles

The work in this section was done with the help of David Treumann. Let X be the Grassmannian of codimension k-planes in $\mathbf{C}^n$. This is a projective variety, just as the flag variety was. For a point $x \in X$, the images of the standard basis vectors in the quotient $\mathbf{C}^n/x$ give rise to a rank k vector configuration whose matroid we denote $M(x)$. For a given matroid M, let $X(M)$ be the set of points $x \in X$ such that $M(x) = M$. It is possible that $X(M)$ is empty. In fact, it is a theorem of Mnëv [**Mnë88**] that $X(M)$ can have arbitrarily bad singularities.

It was noted before that $G(v^{\otimes})$ only depends on the projective equivalence class of the configuration v. In this section we will make this notion more precise by beginning to investigate how the irreducible decomposition of $G(v^{\otimes})$ changes with v. To begin, recall that every rank k vector configuration of n vectors can be thought of the columns of a k-by-n matrix. The nullspace of this matrix is a codimension k subspace of $\mathbf{C}^k$, which is a point x of the Grassmannian X. Thus, there is a quotient map

$$q : \{v \in \mathbf{C}^{k\times n} : v \text{ has rank } k\} \to X$$

that takes v to its nullspace. The matroid associated to $q(v)$ is the same as the matroid associated to v, i.e., $M(q(v)) = M(v)$.

Let $\mathcal{U}$ be a the tautological rank k-bundle over X: The fiber over x consists of $\mathbf{C}^n/x$. Now, associated to $\mathcal{U}$ is its bundle of automorphisms $\mathrm{Aut}(\mathcal{U})$ which is a principal $\mathrm{GL}(\mathbf{C}^k)$-bundle. For each i between 1 and n, we may define a map $f_i : \mathrm{Aut}(\mathcal{U}) \to \mathcal{U}$ that sends an automorphism g over a point x to the map g applied to the image of the i-th standard basis vector in the quotient $\mathbf{C}^n/x$. Taking the tensor product of these maps gives a map of bundles

$$f = f_1 \otimes \cdots \otimes f_n : \mathrm{Aut}(\mathcal{U}) \to \mathcal{U}^{\otimes n}$$

The image of f over x consists of the $\mathrm{GL}(\mathbf{C}^k)$-orbit of the tensor

$$(e_1 + x) \otimes \cdots \otimes (e_n + x) \in (\mathbf{C}^n/x)^{\otimes n}.$$

To take the span of this set of tensors, we form the vector bundle $\mathbf{C}\,\mathrm{Aut}(\mathcal{U})$, which has each fiber isomorphic to the group algebra of $\mathrm{GL}(\mathbf{C}^k)$. The map f extends to a map on this bundle and the image of f we have the following result.

Proposition 2.5.1. *The fiber of* $f : \mathbf{C}\,\mathrm{Aut}(\mathcal{U}) \to \mathcal{U}^{\otimes n}$ *over a point* $x \in X$ *is exactly*

$$G_{\mathbf{C}^n/x}((e_1 + x) \otimes \cdots \otimes (e_n + x)).$$

The degeneracy locus of f is our main object of interest. If $Y \subset X$ is a subvariety then the r-th degeneracy locus of f over Y is the set of points $y \in Y$ where the rank of f over y is at most r.

Proposition 2.5.2. *Let* Y *be an irreducible subvariety of* X*. There is a well defined generic isomorphism type for* $G(v^{\otimes})$ *when* $v \in q^{-1}Y$*.*

Suppose further that every degeneracy locus of f *over* Y *is either all of* Y *or empty. Then for all* $v \in q^{-1}Y$*, the isomorphism type of* $G(v^{\otimes})$ *only depends on* Y*.*

Proof. We claim that it is sufficient to prove that the set of points in $x \in Y$ where the λ-th isotypic component of

$$G(x) = G_{\mathbf{C}^n/x}((e_1 + x) \otimes \cdots \otimes (e_n + x))$$

has dimension at most some fixed number is a subvariety of Y. From this it follows that the set of points where the multiplicity of λ is as large as possible is an open and hence dense in Y. It follows that there is a well defined generic multiplicity of λ and it is the largest of all possible multiplicities of λ.

To see the claim we appeal to the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(V)$, $U(\mathfrak{gl}(V))$. If $x_1, \ldots, x_k$ is a basis of $\mathfrak{gl}(V)$ then the Poincaré-Birkhoff-Witt theorem implies that there is a fixed finite set of monomials in the x_i, $\{x^a = x_1^{a_1} \ldots x_k^{a_k} : a \in A\} \subset U(\mathfrak{gl}(V))$, such that for all configurations v,

$$G(v^{\otimes}) = \mathrm{span}_{\mathbf{C}}\{x^a.v^{\otimes} : a \in A\}.$$

To see this note that each $x_i \in U(\mathfrak{gl}(V))$, $x_i \in \operatorname{End}_{\mathbf{C}} V^{\otimes n}$ has a minimal polynomial and hence we see that any monomial in the x_i's can be assumed to have a bounded degree. If π_λ is the projector of $V^{\otimes n}$ to its λ-th isotypic component then

$$G(v^{\otimes})\pi_\lambda = \operatorname{span}_{\mathbf{C}}\{x^a.(v^{\otimes}\pi_\lambda) : a \in A\}.$$

The assertion that the multiplicity of λ in $G(v^{\otimes})$ is at most r, is the assertion that

$$\dim \operatorname{span}_{\mathbf{C}}\{x^a.(v^{\otimes}\pi_\lambda) : a \in A\} \leq r.$$

It follows that the set of points Y where the multiplicity of λ in $G(v^{\otimes})$ is at most r is a subvariety of Y. This completes the proof of the first part of the theorem.

The second hypothesis is equivalent to the assertion that $\dim G(v^{\otimes})$ is constant over Y. Indeed the rank of f over a point is exactly the dimension of the representation this point gives rise to. Since the generic multiplicity of an irreducible is as large as possible, if $G(v^{\otimes})$ does not have the generic isomorphism type, then $\dim G(v^{\otimes})$ cannot be the generic dimension, which proves the second claim. $\square$

Example 2.5.3. A matroid is said to be uniform of rank k on n elements if its bases are all the k-element subsets of its n-element ground set. The Grassmannian X is irreducible and the rank k uniform matroid strata is an open subset of X. It follows that the strata of uniform matroids is dense in X and there is a well defined generic multiplicity of λ in $G(v^{\otimes})$ when v has a uniform matroid. Unfortunately, it is not known what the generic multiplicity of a general irreducible is.

However, we can show that if λ is larger than or equal to (in dominance order) the length k hook shape (i.e., $\lambda \geq (n-k+1, 1^{k-1})$) then the generic multiplicity of λ is the number of standard Young tableaux of shape λ. To see this, note that in the $\mathrm{GL}(V)$-representation generated by all permutations of the tensor

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{n-k+1\ e_1\text{'s}} \otimes e_2 \otimes \cdots \otimes e_k$$

the multiplicity of $\lambda \geq (n-k+1, 1^{k-1})$ is equal to the multiplicity of λ in $V^{\otimes n}$. This follows at once from Corollary 2.4.9 sine the rank partition of the matroid

$$M(\underbrace{e_1, \ldots, e_1,}_{n-k+1 e_1\text{'s}} e_2, \ldots, e_n)$$

is $(n-k+1, 1^{k-1})$. By Schur-Weyl duality, the multiplicity of λ in the $\mathrm{GL}(V)$-module $V^{\otimes n}$ is the number of standard Young tableaux of shape λ. To see that each of these tensors appears in $G(v^{\otimes})$ when v is generic, note that we may assume that $v_1 = e_1$, $v_{n-k+2} = e_2$, ..., $v_n = e_k$ and all the other v_i are supported on all the e_j. Apply the element of $\mathrm{GL}(V)$ that multiplies all coordinates other than e_1 by ε. If g is this element then

$$gv^{\otimes} = \varepsilon^{k-1}(v_1 \otimes gv_2 \otimes \cdots \otimes gv_{n-k+1} \otimes v_{n-k+2} \otimes \cdots \otimes v_n)$$

Factoring out ε^{k-1} and then setting ε equal to zero gives

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{n-k+1\ e_1\text{'s}} \otimes e_2 \otimes \cdots \otimes e_k \in G(v^{\otimes}).$$

Since the statement did not depend on which v_i we scaled by ε, we get all permutations of the above tensor are in $G(v^{\otimes})$.

It is not true that the multiplicity of $\lambda \geq (n-k+1, 1^{k-1})$ are counted by the standard Young tableaux. For example, we will see that if v has four elements and a uniform matroid then

$$G(v^{\otimes}) = \ydiagram{4} + 3\,\ydiagram{3,1} + \ydiagram{2,2}$$

Here, a shape on the right indicates the irreducible $\mathrm{GL}(V)$-module indexed by that shape, and the number next to indicates the multiplicity of that irreducible in $G(v^{\otimes})$. The number of standard Young tableaux of shape $\ydiagram{2,2}$ is 2.

Generalizing the concept of a uniform matroid, we have the Schubert matroids.

Definition 2.5.4. Let $s = (0 < s_0 < s_1 < \cdots < s_{k-1} < s_k = n+1)$ be a sequence of integers. We define a rank k matroid on $\{1, 2, \ldots, n\}$ by saying that a set is independent if and only if for $0 \leq i \leq k$ the set meets each interval $[1, s_i)$ in at most i elements. Denote this matroid by $M(s)$. Matroids of the form $M(s)$ are called Schubert matroids.

The rank k uniform matroid on n elements is $M((1, 2, \ldots, k, n+1))$. Schubert matroids are realizable over sufficiently large fields and the matroid strata $X(M(s)) \subset X$ is equal to the Schubert cell associated to the sequence s.

Example 2.5.5. Suppose that v is a configuration such that $M(v)$ is the Schubert matroid $M(s)$. Suppose that $s_0 = 1$, which implies that v does not contain the zero vector. By the same process as above we see that

$$\underbrace{e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(1)}}_{s_1 - s_0} \otimes \underbrace{e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(2)}}_{s_2 - s_1} \otimes \cdots \otimes \underbrace{e_{\sigma(k)} \otimes \cdots \otimes e_{\sigma(k)}}_{s_k - s_{k-1}} \in G(v^{\otimes})$$

where $\sigma \in \mathfrak{S}_k$ is any permutation. Denote the sequence $(s_1 - s_0, s_2 - s_1, \ldots, s_k - s_{k-1})$ by μ, so that μ is a composition of n. By Example 2.1.5 the multiplicity of λ in the representation generated by this tensor is the Kostka number $K_{\lambda\mu}$. Thus, $K_{\lambda\mu}$ is a lower bound on the multiplicity of λ in $G(v^{\otimes})$.

2.6. Future Directions

The following conjecture is a vast strengthening of Corollary 2.4.8.

Conjecture 2.6.1. *The multiplicity of λ in $G(v^{\otimes})$ is at least the Kostka number $K_{\lambda,\rho(M)'}$. That is, the multiplicity of λ is at least the number of semi-standard Young tableaux of shape λ and content $\rho(M)'$. Further, there is a surjection*

$$G(v^{\otimes}) \to \mathrm{Sym}^{\rho(M)'_1} V \otimes \cdots \otimes \mathrm{Sym}^{\rho(M)'_\ell} V$$

This is a strengthening of the Corollary since it is known that $K_{\lambda,\mu} \neq 0$ if and only if $\lambda \geq \mu$ in dominance order. Weaker versions of the conjecture (where $\rho(M)'$ is replaced by partition $\mu \geq \rho(M)'$) are seen to hold in Examples 2.5.3 and 2.5.5.

An interesting generalization of the construction of $G(v^{\otimes})$ is to take the smallest $\mathrm{GL}(V)$-module containing a tensor product of decomposable wedges $w_1, \ldots, w_n \in \bigwedge V$. Determining which irreducible representations appear in such a representation should be straightforward.

Conjecture 2.6.2. *Let $v_1 \wedge \cdots \wedge v_{i_1}, \ldots, v_{i_m} \wedge \cdots \wedge v_n$ be decomposable wedges. The multiplicity of $\lambda \vdash n$ in the smallest $\mathrm{GL}(V)$-module containing $v_1 \wedge \cdots \wedge v_{i_1} \otimes \cdots \otimes v_{i_m} \wedge \cdots \wedge v_n$ is not zero if and only if there is a standard Young tableaux of shape λ such that every column indexes an independent set of $M(v)$ and the numbers from each of the sets $\{1, \ldots, i_1\}$, $\ldots$, $\{i_m, \ldots, n\}$ are all in different rows.*

It is straightforward to check that if λ appears in the above representation then the condition described is satisfied.

Let $\operatorname{Sym} V$ be the symmetric algebra of V. There is a problem dual to the one above that involves the representation generated by a tensor product of products of linear forms instead of decomposable wedges.

Problem 2.6.3. *Let $v_1 \cdots v_{i_1}, \ldots, v_{i_m} \cdots v_n \in \operatorname{Sym} V$ be products of linear forms. Determine when the multiplicity of λ is not zero in the smallest $\mathrm{GL}(V)$-module containing $v_1 \cdots v_{i_1} \otimes \cdots \otimes v_{i_m} \cdots v_n$.*

This is expected to be a difficult problem since a precise solution would involve proving Rota's basis conjecture.

Conjecture 2.6.4 (Rota)**.** *Let $\{v_{11}, v_{12}, \ldots, v_{1k}\}$, $\ldots$, $\{v_{k1}, v_{k2} \ldots, v_{kk}\}$ be bases of $V \approx \mathbf{C}^k$. There is a reindexing of the elements of each of the bases such that the sets $\{v_{11}, v_{21}, \ldots, v_{k1}\}$, $\ldots$, $\{v_{1k}, v_{2k}, \ldots, v_{kk}\}$ are bases of V too.*

Indeed it is possible to prove that if the shape $(k, \ldots, k) = (k^k)$ appears in the smallest $\mathrm{GL}(V)$-module containing

$$(*) \qquad v_{11}v_{12} \cdots v_{1k} \otimes \cdots \otimes v_{k1}v_{k2} \cdots v_{kk}$$

then Rota's conjecture is true. Along with Conjecture 2.6.1 and Rota's conjecture, we offer our own.

Conjecture 2.6.5. *The smallest $\mathrm{GL}(V)$-module containing the tensor $(*)$ contains $(\operatorname{Sym}^n V)^{\otimes n}$.*

An unrelated but interesting problem is to generalize the construction of $G(v^{\otimes})$ to other Lie groups.

Problem 2.6.6. *Let $G \subset \mathrm{GL}(V)$ be one of the classical Lie groups. Which irreducible representations of G are submodules of the smallest G-module containing a decomposable tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_n$?*

The coordinate ring of the generalized flag varieties G/B are still integral domains and have a description similar to that described in Section 2.2, so the approach of this section should prove useful.

We will discuss more about the following problem in the next section.

Problem 2.6.7. *How complicated can the degeneracy locus of the map of vector bundles* $f : \mathbf{C}\,\mathrm{Aut}(\mathcal{U}) \to \mathcal{U}^{\otimes n}$ *be when restricted to one of the matroid strata* $X(M)$*? Can it have arbitrarily bad singularities?*

CHAPTER 3

Multiplicities of Irreducibles

Computing the exact multiplicity of an irreducible indexed by a partition λ in $G(v^{\otimes})$ and $\mathfrak{S}(v^{\otimes})$ is a difficult problem. It would be desirable to have a solution that says this multiplicity is counted by the number of tableaux of shape λ subject to some condition on the filling. Unfortunately, no such condition is known and, further, it is not known if there exist two configurations u and v such that their matroids are the same but the irreducible decomposition of $G(u^{\otimes})$ and $G(v^{\otimes})$ are different. Hence we do not know if the desirable solution is even possible!

We currently have three different approaches at our disposal to compute these multiplicities. We can compute linearly independent highest vectors for $G(v^{\otimes})$. This is hard since producing highest weight vectors from $v^{\otimes}$ via the action of $\mathrm{GL}(V)$ (or its Lie algebra) seems difficult. We could also try to use general facts about vector bundles and degeneracy loci to give information about the map f constructed in Section 2.5. This will most likely give global information and not information local to a particular vector configuration. The most fruitful approach is to use explicit decompositions of the symmetric group algebra $\mathbf{C}\mathfrak{S}_n$ to decompose $\mathfrak{S}(v^{\otimes})$.

In this chapter we determine the multiplicity of a hook shape in $\mathfrak{S}(v^{\otimes})$. To do this we will consider the vector space generated by products of linear forms. We will prove that the structure of this vector space is determined by the Tutte polynomial of $M(v)$, and obtain a combinatorial description of the hook multiplicities. We will also show how $\mathfrak{S}(v^{\otimes})$ decomposes when $M(v)$ is a rank two matroid.

3.1. Computing Multiplicities

We will now set up a general mechanism to compute the multiplicities of irreducibles in $\mathfrak{S}(v^{\otimes})$. For a moment, though, let A be a finite-dimensional semisimple K-algebra defined over $\mathbf{C}$. Let $e \in A$ be a primitive idempotent, so that eA is a simple right A-module. Let M be another right A-module. The multiplicity of the irreducible eA in M is the K-vector space dimension of the space of homomorphisms $\mathrm{Hom}_A(eA, M)$. We claim that this is isomorphic to the vector space Me. Indeed, the map giving the isomorphism is $\varphi \mapsto \varphi(e)$, which lands in Me since $\varphi(e) = \varphi(e^2) = \varphi(e)e$. Further, φ is uniquely determined by its value at e so the map is injective. Since the map is visibly surjective, we have the isomorphism.

We will use this result in the following form.

Proposition 3.1.1. *Let $x \in \mathbf{C}\mathfrak{S}_n$ be essentially idempotent (e.g., a row symmetrizer, column antisymmetrizer or a Young symmetrizer of a possibly skew tableau). The dimension of $\mathfrak{S}(v^{\otimes})x$ is equal to the dimension of*

$$\mathrm{Hom}_{\mathbf{C}\mathfrak{S}_n}(x\mathbf{C}\mathfrak{S}_n, \mathfrak{S}(v^{\otimes})).$$

If $x\mathbf{C}\mathfrak{S}_n \approx \mathcal{S}(\lambda^1) \oplus \cdots \oplus \mathcal{S}(\lambda^\ell)$ has a multiplicity free decomposition decomposition then $\dim \mathfrak{S}(v^{\otimes})x$ is the sum of the multiplicities of $\lambda^1, \ldots, \lambda^\ell$ in $\mathfrak{S}(v^{\otimes})$.

3.2. Multiplicities of Hooks

In this section we will use Proposition 3.1.1 to compute the multiplicity of hook shaped irreducibles in $\mathfrak{S}(v^{\otimes})$. Recall that a partition λ is called a hook of length j if its Young diagram is of the form

$$j\left\{\begin{array}{|c|c|c|c|c|c|c|c|}\hline &&&&&&&\\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}\right.$$

We will denote the length j hook partition of n by λ^j, n being fixed throughout our discussion.

Having fixed $\lambda = \lambda^j$, let us form the skew tableau T with shape $(n-j, 1^j)/(1)$ whose long column consists of the numbers $\{1, 2, \ldots, j\} = [j]$, in order, and whose long row consists of the numbers $\{j+1, j+2, \ldots, n\} = [n] - [j]$, also in order. For example, if $n = 9$ and $j = 3$ then

$$T = \begin{array}{c|c|c|c|c|c|c|} \cline{2-7} & 4&5&6&7&8&9 \\ \cline{1-7} \multicolumn{1}{|c|}{1} \\ \cline{1-1} \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}$$

We form the Young symmetrizer of T, which is the sum

$$c_T = \sum_{\sigma \in \mathfrak{S}_{[j]}} \operatorname{sign}(\sigma)\sigma \sum_{\tau \in \mathfrak{S}_{[n]-[j]}} \tau.$$

As usual, it is just the product of the column antisymmetrizer of T and the row symmetrizer of T. It is a basic fact that the right ideal generated by c_T consists of two irreducible representations of $\mathfrak{S}_n$, one indexed by λ^j, the other indexed by λ^{j+1}. The fact follows from a simple application of Pieri's rule. We conclude from Proposition 3.1.1 that the multiplicity of λ^j *plus* the multiplicity of λ^{j+1} is equal to

$$\dim \mathbf{C}\{v^{\otimes}\mathfrak{S}_n c_T\} = \dim \mathbf{C}\left\{\bigwedge_{i\in S} v_i \otimes \prod_{i\notin S} v_i : S \subset \tbinom{[n]}{j}\right\}$$
$$\subset \bigwedge^j V \otimes \operatorname{Sym}^{n-j} V \subset V^{\otimes n}$$

Here $\mathbf{C}\{-\}$ means take the $\mathbf{C}$-linear span of the elements in $\{-\}$.

In order to properly state the following lemma, we must digress and remind the reader of a few definitions. The first is a matter of matroid terminology: If M is a matroid then recall that a subset of its ground set which is maximal with a given rank is called a flat, usually denoted X. The collection of flats of M is a geometric lattice, denote $L(M)$. A base of a flat X is a maximal independent set contained in X and the rank of X, $r(X)$, is the size of any of its bases.

Definition 3.2.1. Let $P(v) \subset \operatorname{Sym} V$ be the vector space spanned by products of the form $v_S := \prod_{i \in S} v_i$, where S is any subset of $[n]$.

Define $P(v)_X$ to be the subspace of $P(v)$ spanned by products v_S where the smallest flat of $M(v)$ containing $[n] - S$ is X.

One sees that $P(v)$ is the sum of all the subspace $P(v)_X$. The following is a result of Orlik and Terao (see [**OT94, Ter02**]).

Lemma 3.2.2 (Orlik-Terao)**.** *There is a direct sum decomposition*

$$P(v) = \bigoplus_{X \in L(M(v))} P(v)_X.$$

We will return to this direct sum decomposition is the next section. Returning to our previous discussion, we were considering the vector space

$$\mathbf{C}\left\{\bigwedge_{i\in S} v_i \otimes \prod_{i \notin S} v_i : S \subset \binom{[n]}{j}\right\} \subset \bigwedge^j V \otimes \operatorname{Sym}^{n-j} V.$$

Since we have the decomposition of $P(v)$ over the flats of $M(v)$ and tensor products commute with direct sums we obtain,

Corollary 3.2.3. *There is a direct sum decomposition*

$$\mathbf{C}\left\{\bigwedge_{i\in S} v_i \otimes \prod_{i \notin S} v_i\right\} = \bigoplus_{\substack{X \in L(M(v)) \\ r(X) = j}} \mathbf{C}\left\{\bigwedge_{i\in S} v_i \otimes \prod_{i \notin S} v_i : S \text{ is a base of } X\right\}.$$

Finally, if we investigate the X-th summand of the direct sum above, we notice that the wedges are non-zero scalar multiples of each other. This means that we may omit the wedges on the left and translate the problems of determining $\dim \mathfrak{S}(v)c_T$ to the following problem in commutative algebra.

Corollary 3.2.4. *The sum of the multiplicities of the hooks of length j and $j+1$ in $\mathfrak{S}(v^{\otimes})$ is the dimension of the vector space spanned by the products v_S where $[n]-S$ is an independent set of size j in $M(v)$.*

In particular, the multiplicity of the longest hook shape appearing in $\mathfrak{S}(v^{\otimes})$ is the dimension of the vector space spanned by products v_S where S is the complement of a base of $M(v)$.

The following generalization of the second assertion of this corollary will allow us to decompose $\mathfrak{S}(v^{\otimes})$ when $M(v)$ has a rank two matroid.

Corollary 3.2.5. *Let $v \subset V$ be a rank k vector configuration spanning V and suppose that $(\ell+j, \ell^{k-1})$ is a partition of n. The multiplicity of the shape $(\ell+j, \ell^{k-1})$ in $\mathfrak{S}(v^{\otimes})$ is the dimension of the vector space spanned by those v_S where the S is the complement of a disjoint union of ℓ bases of $M(v)$.*

Proof. By Proposition 3.1.1 the stated multiplicity is the dimension of the vector space spanned by $v^{\otimes}\mathfrak{S}_n c_T$ where c_T is a skew tableaux of shape $(\ell+j, \ell^k)/(\ell)$. By Pieri's Rule the right ideal $c_T\mathbf{C}\mathfrak{S}_n \subset \mathbf{C}\mathfrak{S}_n$ decomposes as a sum of irreducibles, one of shape $(\ell+j, \ell^{k-1})$ and the others with length longer than k.[1] Since the multiplicity of shapes with length longer than k is zero in $V^{\otimes n}$,

$$\dim \operatorname{span} v^{\otimes}\mathfrak{S}_n c_T$$

[1]Recall that Pieri's rule states the following: If λ is a partition and $\mu = (j)$ is a horizontal strip of length j then product of the Schur functions s_λ and s_μ is the sum $\sum_\nu s_\nu$, the sum over shapes ν obtained from λ by adding j boxes, no two in the same column.

is the multiplicity of $(\ell + j, \ell^{k-1})$ in $\mathfrak{S}(v^{\otimes})$. These tensors live in

$$(\bigwedge^k V)^{\otimes \ell} \otimes \operatorname{Sym}^j V \approx \operatorname{Sym}^j V,$$

since V has dimension k. The image of the symmetrized tensors $v^{\otimes}\mathfrak{S}_n c_T$ in $\operatorname{Sym}^j V$ are the products v_S where $[n] - S$ is a disjoint union of ℓ bases of $M(v)$. $\square$

Example 3.2.6. If $M(v)$ is uniform of rank k the multiplicity of this shape in $\mathfrak{S}(v^{\otimes})$ is the dimension of the vector space spanned by the products v_S where S has size j. We will see in the next section that the dimension of this vector space does not depend on the particular coordinates of v, only the fact that $M(v)$ is uniform.

As a particular case of Corollary 3.2.5, we have

Corollary 3.2.7. *Suppose that $(\ell + 1, \ell^{k-1})$ is a partition of n. The multiplicity of $(\ell + 1, \ell^{k-1})$ in $\mathfrak{S}(v^{\otimes})$ is the dimension of the subspace of V spanned by the set*

$$\{v_i : v - v_i \text{ is a disjoint union of } \ell \text{ bases}\}.$$

In particular, the multiplicity of shapes of this form are matroid invariants.

Note that we do not need to assume that $(\ell + 1, \ell^{k-1})$ is a partition of n, since the set in question will be empty if this is a partition of another number.

Example 3.2.8. Let v be the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 3 & 0 \end{bmatrix}.$$

By changing the base, this collection is at once seen to 5 generic points laying in a plane, and 2 points of $V = \mathbf{C}^3$ in general position (thus the matroid of v is a Schubert matroid). It follows from this description that if we want $v - v_i$ to be a disjoint union of two bases then we cannot delete either of the latter points. In this case

$$\{v_i : v - v_i \text{ is a disjoint union of 2 bases}\} = \{v_3, v_4, v_5, v_6, v_7\}.$$

which is at once seen to span a two dimensional space. We conclude that the multiplicity of $(3, 2, 2)$ in $\mathfrak{S}(v^{\otimes})$ is 2.

Example 3.2.9. Suppose that $(\ell + 1, \ell^{k-1})$ is a partition of n and v is a collection of generic vectors in the sense that the matroid $M(v)$ is uniform. Then

$$v = \{v_i : v - v_i \text{ is a disjoint union of } \ell \text{ bases}\}$$

hence the multiplicity of $(\ell + 1, \ell^{k-1})$ is the rank of $M(v)$.

3.3. Symmetric Tensors and Tutte Polynomials

In this section we study the vector space $P(v)$, which is linearly generated by the 2^n products $\prod_{i \in S} v_i$, $S \subset [n]$. The main result of this section states that a slight refinement of the decomposition of Lemma 3.2.2 completely determines the Tutte polynomial of $M(v)$. This work is also done in the article [**Ber08**]. The Tutte polynomial of a matroid is a two variable polynomial that captures many of the properties of the matroid that behave well with respect to the operations of deletion and contraction. This result, and all results in this section not dealing with representation theory directly, generalize to vector configurations over arbitrary fields.

Definition 3.3.1. Given a flat $X \in L(M(v))$ and a non-negative integer d, let $P(v)_{X,d}$ be the homogeneous degree $n-d$ subspace of $P(v)_X$.

Since $P(v)_X$ is generated by homogeneous linear forms $P(v)_X = \bigoplus_{d\geq 0} P(v)_{X,d}$. From the direct sum decomposition given in Lemma 3.2.2 we can conclude that

$$P(v) = \bigoplus_{\substack{X\in L(M(v))\\ d\geq 0}} P(v)_{X,d}.$$

Recall that the Tutte polynomial of a matroid M is the unique polynomial $T(M;x,y) \in \mathbf{Z}[x,y]$ satisfying the conditions

(T1) Let L denote the rank zero matroid on one element and I denote the rank one matroid on one element. Then

$$T(L;x,y) = x \quad \text{and} \quad T(I;x,y) = y.$$

(T2) If M and N are matroids and $M \oplus N$ is their direct sum then

$$T(M \oplus N;x,y) = T(M;x,y)T(N;x,y).$$

(T3) If M is matroid and e is an element of its ground set that is neither a loop or an isthmus then

$$T(M;x,y) = T(M-e;x,y) + T(M/e;x,y).$$

Here $M-e$ and M/e are the deletion and contraction of M by e.

Recall that the independent sets of the direct sum of two matroids M and N are unions of independent sets of M and N. If the ground set of M is E, then the independent sets of the deletion of e from M are of the form $I-e$ where I is independent in M. The contraction of M by e has ground set $E-e$ and independent sets I such that $I \cup e$ is independent in M.

With all this terminology out of the way, we can state our main result on $P(v)$:

Theorem 3.3.2. *The Tutte polynomial of $M(v)$ can be written as*

$$T(M; 1+x, y) = \sum_{\substack{X \in L(M(v)) \\ d \geq 0}} x^{r(M)-r(X)} y^{d-r(X)} \dim P(v)_{X,d}.$$

See [**Ber08**] for more results related to this as well as [**Ard03, AP09, Wag99, Ter02, PS06, OT94, BV99, Cor02**] for similar constructions and results. We will delay the proof of the theorem.

Corollary 3.3.3. *The subspace of $P(v)$ spanned by products v_S where $[n]-S$ is independent of size j in $M(v)$ has dimension equal to the coefficient of $t^{r(M)-j}$ in $T(M(v); 1+t, 0)$.*

Let h_j be the multiplicity of the length j hook λ^j in $\mathfrak{S}(v^{\otimes})$, then

$$\sum_{j \geq 0} h_j t^{r(M)-j} = \frac{1}{1+t} T(M(v); 1+t, 0).$$

In particular, the multiplicity of hook shapes does not depend on the coordinates of v, only the matroid $M(v)$.

Proof. Setting $y = 0$ in Theorem 3.3.2 proves that the subspace in question has the stated dimension. By Corollary 3.2.4 we know that the dimension of this subspace is the sum of the multiplicities of the length j and length $j+1$ hooks in $\mathfrak{S}(v^{\otimes})$, hence

$$T(M(v); 1+t, 0) = \sum_{j \geq 0} (h_j + h_{j+1}) t^{r(M)-j} = \sum_{j \geq 0} (1+t) h_j t^{r(M)-j}.$$

Factoring out $1+t$ we obtain the generating function for multiplicities of hooks in $\mathfrak{S}(v^{\otimes})$. $\square$

When v is defined over $\mathbf{C}$, the vector space $P(v)$ and some of its subspaces have been studied by many people. Most recently, Ardila and Postnikov [**AP09**] proved that $P(v)$ is the Macaulay inverse system (see Eisenbud [**Eis95**, Ch. 21.5]) of a certain ideal generated by powers of linear forms. Given a linear functional λ we let $\mu_v(\lambda)$ be the number of v_i such that $\lambda(v_i) \neq 0$. Then define $I(v)$ to be the ideal in $\operatorname{Sym} V^*$ generated by $\lambda^{\mu_v(\lambda)+1}$ where λ ranges over all of V^*.

Proposition 3.3.4 (Postnikov)**.** *The quotient* $\operatorname{Sym} V^*/I(v)$ *is isomorphic as a vector space to* $P(v)$.

Proof. Ardila and Postnikov prove that the Macaulay inverse system of $I(v)$ is equal to $P(v)$. It is a basic fact [**Eis95**, Thm 21.6] that $P(v)$ is $\operatorname{Sym} V^*/I(v)$-isomorphic to the canonical module of the local zero-dimensional quotient $\operatorname{Sym} V^*/I(v)$. It is know that the canonical module of $\operatorname{Sym} V^*/I(v)$ is $\operatorname{Hom}_{\mathbf{C}}(\operatorname{Sym} V^*/I(v), \mathbf{C})$, and the dimension of this space is the same as the dimension of $\operatorname{Sym} V^*/I(v)$. □

3.3.1. Two Examples. In this subsection we give two illustrative examples of Theorem 3.3.2. Since Theorem 3.3.2 holds for vector configurations defined over any field we can use it to compute the Tutte polynomial of the Fano matroid.

Example 3.3.5. Let $v = (v_1, \ldots, v_7)$ be the seven nonzero elements of the dual of $V = \mathbb{F}_2^3$. The matroid $M(v)$ is known as the Fano matroid and it can be realized as the columns of the matrix below, with entries in $\mathbb{F}_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

We will compute $T(M(v); 1+x, y)$ by finding $\dim P(v)_{X,k}$ for all flats X and all k such that $r(X) \leq k \leq |X|$. As is usual in matroid theory, we will denote the ground set of $M(v)$ by E instead of [7].

The space $P(v)_{\emptyset,0}$ is spanned by one nonzero element $v_E = \prod_{i=1}^{7} v_i$ and hence has dimension one. The rank one flats of v are in bijection with the elements of v. Hence $P(v)_{\{i\},1}$ is spanned by the single element v_E/v_i and $\dim P(v)_{\{i\},1} = 1$.

There are seven rank two flats of v. If X is such a flat then it corresponds to a set of the form $\{v_i, v_j, v_i + v_j\}$. It follows that $P(v)_{X,2}$ is spanned by the three elements

$$v_i v_{E-X},\ v_j v_{E-X},\ (v_i + v_j)v_{E-X}.$$

Adding these three terms up gives 0 and hence $\dim P(v)_{X,2} \leq 2$. Since none of the v_i are parallel, $\dim P(v)_{X,2} = 2$. Because $P(v)_{X,3}$ is spanned by a single nonzero element, $\dim P(v)_{X,3} = 1$.

The only rank 3 flat of v is the whole set E. The empty product spans $P(v)_{E,7}$ and so it has dimension one. One finds that $P(v)_{E,6}$ is the span of $v_1, \ldots, v_7$, so this space has dimension equal to the dimension of V, which is three. To compute $P(v)_{E,5} \subset \operatorname{Sym}^2 V$, assume that v_1, v_2 and v_3 are a basis for V. By considering leading terms (under any term order) we see that

$$v_1 v_2,\ v_1 v_3,\ v_2 v_3,\ v_1(v_1 + v_2),\ v_2(v_2 + v_3),\ v_3(v_1 + v_3)$$

forms a basis for $\operatorname{Sym}^2 V$. In a similar fashion we see that $P(v)_{E,4}$ is equal to $\operatorname{Sym}^3 V$, which has dimension 10. We resort to a computer to find the dimension of $P(v)_{E,3}$, which is spanned by 28 products. This space is contained in $\operatorname{Sym}^4 V$ which has dimension $\binom{3+4-1}{4} = 15$. One computes that $\dim P(v)_{E,3} = 8$.

Adding all the terms up with the appropriate powers of $x-1$ and y, Theorem 3.3.2 says that

$$(x-1)^3 + 7(x-1)^2 + 14(x-1) + 7(x-1)y + y^4 + 3y^3 + 6y^2 + 10y + 8.$$

is the Tutte polynomial of the Fano matroid.

Recall that a cocircuit of a matroid is the complement of a flat of rank $r(M)-1$. Using Theorem 3.3.2 the following result is proved in [**Ber08**].

Theorem 3.3.6. *There is a containment* $\operatorname{Sym}^d V \subset P(v)$ *if and only if* d *is less than or equal to the size of the smallest cocircuit of* $M(v)$.

Example 3.3.7. Suppose that v is a configuration of k^m vectors whose rank partition is (k^m). By Example 2.4.7 every cocircuit of $M(v)$ has size at least m. By Theorem 3.3.6 we conclude that $\operatorname{Sym}^d V \subset P(v)$ if $d \leq m$.

Remark 3.3.8. Suppose that we are in the context of Rota's Conjecture 2.6.4. This means that $v = (v_{ij})$ is a disjoint union of k bases of $V \approx \mathbf{C}^k$. We conclude that $\operatorname{Sym}^k V \subset P(v)$. An interesting question is how this fact relates Conjecture 2.6.4 and Conjecture 2.6.5. Does it allow us to say anything interesting about $\mathfrak{S}(v^{\otimes})$ or the $\mathrm{GL}(V)$-module generated by

$$v_{11}v_{12}\ldots v_{1k} \otimes v_{21}v_{22}\ldots v_{2k} \otimes \cdots \otimes v_{k1}v_{k2}\ldots v_{kk}.$$

In particular, is $(\operatorname{Sym} V)^{\otimes k}$ a subspace of this representation?

3.3.2. A Basis for $P(v)$ and Certain Multiplicities in $\mathfrak{S}(v^{\otimes})$. In order to define a basis for $P(v)$, and give an even yet more precise description of the hook shaped isotypic components of $\mathfrak{S}(v^{\otimes})$, we will need to talk about the activity of an independent set of a matroid.

Definition 3.3.9. Let M be a matroid on an ordered ground set E. If I is an independent set of M with closure X we say that $e \in X - I$ is externally active in I if e is the minimum element of a circuit of $I \cup e$. The set of elements externally active in I is denoted $ex(I)$.

An independent set whose external activity is empty is called a no broken circuit set, and the collection of these sets forms a simplicial complex called the no broken circuit complex of M. A broken circuit of M is a circuit with its smallest element deleted.

It is easy to see that a no broken circuit sets is one that contains no broken circuits. If I is independent in a matroid M then it is a basic fact of matroid theory that $I \cup e$ is independent or is dependent and contains a unique circuit,

Lemma 3.3.10. *The elements $v_{E-(I \cup ex(I))}$ where I is independent in $M(v)$ span $P(v)$.*

In the proof we use the following fact: Every set $S \subset E$ can be uniquely written as $S = I \cup (S - I)$ where I is independent in M and $(S - I) \subset ex(I)$. To see this, let $I \subset S$ be the lexicographically largest base of S. One verifies at once that the complement of I in S is contained in $ex(I)$.

Proof. Let $P(v)'$ be the span of the stated elements:

$$P(v)' = \mathbf{C}\{v_{E-(J \cup ex(J))} : J \text{ is independent in } M\}$$

Suppose that we have chosen S which is lexicographically least such that $v_{E-S} \notin P(v)'$. As noted above, we may write S uniquely as $I \cup (S - I)$ where I is independent and $(S - I) \subset ex(I)$.

Suppose that $S - I \neq ex(I)$ and pick $f \in ex(I) - (S - I)$. Then v_f divides v_{E-S} since $f \notin S$. We may write $v_f = \sum_{e \in I} c_e v_e$ and using this relation we have

$$v_{E-S} = v_{E-(S \cup f)} v_f = \sum_{e \in I} c_e v_{E-(S-e \cup f)}$$

Now, for all $e \in I \subset S$, $S - e \cup f$ is lexicographically smaller than S. Since $v_{E-S} \notin P(v)'$ there is a $e \in I$ such that $v_{E-(S \cup f - e)} \notin P(v)'$. However, this is a contradiction since S was chosen so that it is the lexicographically smallest set such that $v_{E-S} \notin P(v)'$. It follows that v_{E-S} is in $P(v)'$ for all S, hence $P(v) = P(v)'$. □

To prove the linear independence of the elements from the lemma, note that for any linear form $v_i \in V$ there is a complex of graded vector spaces

$$0 \to \mathrm{Sym}(V) \xrightarrow{\cdot v_i} \mathrm{Sym}(V) \to \mathrm{Sym}(V/\mathbf{C}v_i) \to 0. \tag{1}$$

The second map is induced by the quotient map $V \to V/\mathbf{C}v_i$. When $v_i \neq 0$ this complex is exact by definition. If v_i is neither a loop nor an isthmus of v (i.e., $v_i \neq 0$ and the rank of the configuration $v - v_i$ is equal to the rank of v) let $v - v_i$ be obtained from v by deleting the element in the i-th position and v/v_i be the image of the forms $v - v_i$ in $V/\mathbf{C}v_i$.

Lemma 3.3.11. *The elements described in Lemma 3.3.10 are linearly independent and thus form a basis for $P(v)$. The dimension of $P(v)$ is the number of independent sets of $M(v)$.*

Proof. If v_i is neither a loop nor an isthmus of v there is an exact sequence

$$0 \to P(v - v_i) \xrightarrow{\cdot v_i} P(v) \to P(v/v_i) \to 0$$

where the second map is induced by the quotient. This is a complex since it is the restriction of the respective maps on symmetric algebras. A moment's thought proves that it is exact on the left and right. It is exact in the middle by Lemma 3.3.10 since we see that, by induction,

$$\begin{aligned}\#\mathcal{I}(M(v-v_i)) + \#\mathcal{I}(M(v/v_i)) &= \dim P(v-v_i) + \dim P(v/v_i) \\ &\leq \dim P(v) \leq \#\mathcal{I}(M(v))\end{aligned}$$

and it is a standard result that $\#\mathcal{I}(M(v-v_i))+\#\mathcal{I}(M(v/v_i)) = \#\mathcal{I}(M(v))$. $\square$

For each set $B \subset [n]$ of size k, let c_B be the Young symmetrizer of the tableau of skew shape $(n-k+1, 1^k)/(1)$ where the elements of B are in the long column of the tableau.

Corollary 3.3.12. *Let $B_1, \ldots, B_\chi$ be the no broken circuit bases of $M(v)$. The symmetrized tensors*

$$v^{\otimes} c_{B_1}, \ldots, v^{\otimes} c_{B_\chi}$$

are a minimal set of generators of the λ^k-th isotypic component of $\mathfrak{S}(v^{\otimes})$.

In particular, the multiplicity of the longest hook shape in $\mathfrak{S}(v^{\otimes})$ is the number of no broken circuit bases of $M(v)$.

Corollary 3.3.13. *Suppose that $(\ell + j, \ell^{k-1})$ is a partition of n. For any choice of an ordering of a ground set of $M(v)$, let $T_1, \ldots, T_r$ be the skew tableaux of shape $(\ell+j, \ell^k)/(\ell)$ such that the elements in first ℓ columns of T_i are a disjoint union of bases and of the form $ex(B_i)$ for some base B_i. Then there is a direct sum decomposition*

$$\{v^{\otimes} c_{T_1}, \ldots, v^{\otimes} c_{T_r}\}\mathbf{C}\mathfrak{S}_n = \mathcal{S}((\ell + j, \ell^{k-1}))^{\oplus r}$$

3.4. The Proof of Theorem 3.3.2

This section proves Theorem 3.3.2, which says that the Tutte polynomial of $M(v)$ can be written as

$$T(M; 1+x, y) = \sum_{\substack{X \in L(M(v)) \\ d \geq 0}} x^{r(M)-r(X)} y^{d-r(X)} \dim P(v)_{X,d}.$$

Proof of Theorem 3.3.2. Let $H(v; x, y)$ denote the right side of the equation above. We will check that $H(v; x, y)$ satisfies the defining properties (T1), (T2) and (T3) of the Tutte polynomial. One immediately checks that (T1) holds since $H(\{v_1\}; x, y)$ is y if $v_1 = 0$ and $1 + x$ if otherwise.

Next one checks the multiplicative property (T2). Suppose that

$$v = (v_1, \ldots, v_n), \quad u = (u_1, \ldots, u_m)$$

are sequences of vectors from two vector spaces V and W over K. Let $v \oplus u$ denote the concatenation of the sequences, $(v_1, \ldots, v_n, u_1, \ldots, u_m)$, viewed as as vectors in $V \oplus W$. This agrees with the direct sum of matroids since the matroid of $v \oplus u$ is the direct sum of the matroids of v and u. Recall that there is a natural isomorphism of graded K-algebras,

$$\mathrm{Sym}(V \oplus W) \approx \mathrm{Sym}(V) \otimes \mathrm{Sym}(W). \tag{2}$$

The flats of $M(v \oplus u)$ are in bijection with pairs of flats from $M(v)$ and $M(u)$. If X is a flat of $M(v)$ and Y is a flat of $M(u)$ then, as graded vector spaces,

$$P(v \oplus u)_{(X,Y)} \approx P(v)_X \otimes P(u)_Y$$

Indeed, the isomorphism (2) maps $v_S \otimes u_T$ to $v_S u_T$ which is in $P(v \oplus u)_{(X,Y)}$. Since every monomial defining $P(v \oplus u)_{(X,Y)}$ is of this form, we have the needed isomorphism. Lastly, since the rank of the flat corresponding to (X, Y) is sum of the ranks of X and Y and the isomorphism (2) is of *graded algebras*, and we have

$$H(v \oplus u; x, y) = H(v; x, y)H(u; x, y).$$

To prove the deletion-contraction recurrence (T3), recall the exact sequence

$$0 \to P(v - v_i) \stackrel{\cdot v_i}{\to} P(v) \to P(v/v_i) \to 0. \tag{3}$$

of Lemma 3.3.11. Assume that $i \in E$ is neither a loop nor an isthmus of $M(v)$, that is, $r(i) = 1$ and $r(E - i) = r(E)$. Recall that the flats of the deletion $M(v - v_i)$ are in bijection with the flats of X of $M(v)$ such that $r(X - i) = r(X)$. Also, the flats of $M(v/v_i)$ are bijection with the flats of $M(v)$ containing i. We wish to refine the exact sequence (3) to consider a flat X of v and a degree. To do this, take three cases depending on whether or not $i \in X$, and if $i \in X$, then whether i is an isthmus of X.

First suppose that $i \notin X$. It follows that X is a flat of $v - v_i$. The exact sequence (3) restricts to the exact sequence

$$0 \to P(v - v_i)_{X,k} \stackrel{\cdot v_i}{\to} P(v)_{X,k} \to 0. \tag{4}$$

Every product $v_S \in P(v)_{X,k}$ is of the form v_S where $i \in S$, hence v_{S-i} lies in $P(v - v_i)_{X,k}$ and the map is surjective. Since the map is the restriction of an injection it is an isomorphism.

Suppose that $i \in X$ and i is not an isthmus of X. In this case $X - i$ is a flat of both $v - v_i$ and v/v_i and we claim that (3) restricts to the exact sequence

$$0 \to P(v - v_i)_{X-i,k} \stackrel{\cdot v_i}{\to} P(v)_{X,k} \to P(v/v_i)_{X-i,k-1} \to 0. \tag{5}$$

To see this we pick some $v_S \in P(v - v_i)_{X-i,k}$ and see that $v_{S \cup i}$ is in $P(v)_{X,k}$. This is because $(E - i) - S = E - (S \cup i)$ has closure X in $M(v)$. If the closure were the smaller set $X - i$, then i would be have been an isthmus of the flat

X. The map on the left in (5) is the restriction of an injection, hence we have exactness on the left.

If $v_S \in P(v)_{X,k}$ and $i \notin S$ then the closure in $M(v/v_i)$ of $(E-i)-S$ is $X-i$. The degree of v_S is unchanged under $v_i \mapsto 0$ hence $P(v)_{X,k}$ has image in $P(v/v_i)_{X-i,k-1}$. That every monomial spanning $P(v/v_i)_{X-i,k-1}$ is in the image of $P(v)_{X,k}$ follows from the definition of v/v_i as the restriction of the elements of $v-v_i$ to $\ker(v_i)$. The exactness on the right of (5) follows.

We now prove exactness in the middle of (5). If an element of $P(v)_{X,k}$ is zero upon projecting to $\mathrm{Sym}(V/v_i)$ then it can be written as v_i times some linear combination $\sum c_S v_S$ where $v_S \in P(v-v_i)$. For these S we have $v_i v_S \in P(v)_{X,k}$ and, since i was not an isthmus of X, we have v_S is in $P(v-v_i)_{X-i,k}$.

In the case that $i \in X$ and i is an isthmus of X it follows that (3) restricts to the exact sequence

$$0 \to P(v)_{X,k} \to P(v/v_i)_{X-i,k-1} \to 0. \tag{6}$$

The surjectivity here is clear. If an element is in the kernel of this map we may, as before, write it as a linear combination of terms $v_i v_S \in P(v)_{X,k}$. It follows that $E-(S\cup i)$ has closure X in $M(v)$, which cannot be since $i \notin E-(S\cup i)$ and i is in every base of X. We conclude that the kernel is zero and (6) is exact.

Finally, we can verify the deletion-contraction recurrence. To do so, break up the defining sum for $H(v;x,y)$ according to the three cases we just considered. Let $L_1 \subset L(v)$ be of the set of flats of v not containing i, $L_2 \subset L(v)$ be the set of flats containing i as an isthmus, and let $L_3 \subset L(v)$ be the remaining flats. If $cr(X)$ denotes $r(M(v))-r(X)$, the corank of X, we see that the exact sequences

(4), (5) and (6) imply that we can write

$$H(v;x,y) = \sum_{X \in L_1 \cup L_3, k} x^{cr(X)} y^{k-r(X)} \dim P(v - v_i)_{X-i,k} + \sum_{X \in L_2 \cup L_3, k} x^{cr(X)} y^{k-r(X)} \dim P(v/v_i)_{X-i,k-1}.$$

The first sum is (4) and the left of (6), while the second sum is (5) and the right of (6). We claim that the first sum here is $H(v - v_i)$ and the second is $H(v/v_i)$. Since the flats of $v - v_i$ are in bijection with $\{X - i : X \in L_1 \cup L_3\}$ and the flats of v/v_i are in bijection with $\{X - i : X \in L_2 \cup L_3\}$, we only need to check that the exponents of x and y in each sum are correct. If $X \in L_1 \cup L_3$, then the rank of $X - i$ in the matroid $M(v - v_i)$ is equal to the rank of X in $M(v)$. If $X \in L_2 \cup L_3$ then the rank of $X - i$ in $M(v/v_i)$ is one less than its rank in $M(v)$. It follows that the exponents of x and y are correct and so

$$H(v;x,y) = H(v - v_i;x,y) + H(v/v_i;x,y),$$

which is what we wanted to show. □

3.5. Rank Two Vector Configurations

In this section we show how to compute the isomorphism type of $\mathfrak{S}(v^{\otimes})$ when the matroid $M(v)$ has rank two. The reason this is possible is because rank two matroids without loops are particularly simple to describe: They are in bijection with partitions. The size of the i-th largest part of the partition describes the size of the i-th parallelism class of M. All such matroids are realizable over any sufficiently large field.

Theorem 3.5.1. *Let v be a rank two vector configuration corresponding to the partition $\mu \vdash n$. If $k > 0$ then the multiplicity of $(n-k, k) \vdash n$ in $\mathfrak{S}(v^{\otimes})$ is*

$$\mu_1' + \mu_2' + \cdots + \mu_k' - 2k + 1,$$

provided that this number is at least zero. The multiplicity is zero otherwise.

The rest of this section will devoted to proving this result. First, though, we will consider a few examples of the result.

Example 3.5.2. Let $v = (v_1, v_1, v_1, v_2, v_2, v_3, v_3, v_3, v_4, v_5)$ where $v_i \in \mathbf{C}^2 - 0$ and $v_i \notin \mathbf{C}v_j$ if $i \neq j$. The partition corresponding to $M(v)$ is $(3, 3, 2, 1, 1)$ and the conjugate partition is $(3, 3, 2, 1, 1)' = (5, 3, 2)$. Then

$$\begin{aligned} \mathfrak{S}(v^{\otimes}) \approx \ydiagram{10} &+ (5-1)\ \ydiagram{9,1} \\ &+ (5+3-3)\ \ydiagram{8,2} + (5+3+2-5)\ \ydiagram{7,3} \\ &+ (5+3+2+0-7)\ \ydiagram{6,4} + (5+3+2+0+0-9)\ \ydiagram{5,5}\ . \end{aligned}$$

Example 3.5.3. If v is a rank two configuration of n non-parallel vectors (so that its matroid is uniform) then the partition associated to $M(v)$ is (1^n), whose transpose is (n). The multiplicity of $(n-k, k)$ in $\mathfrak{S}(v^{\otimes})$ is thus $n-2k+1$.

For the rest of the section, let v be a rank two vector configuration of n vectors whose matroid is associated to the partition μ and let $\lambda = (n-k, k)$ be a partition of n. We may assume that the vectors v_i are selected from a two-dimensional vector space V.

Proposition 3.5.4. *The sum $\sum_{i=1}^{k} \mu_i' - 2k + 1$ is positive if and only if Gamas's condition is satisfied for $(n-k, k)$; that is, if and only if it is possible to fill the vectors v_i in the Young diagram of $(n-k, k)$ and have linearly independent columns.*

Proof. Gamas's condition is satisfied if and only if there is a semi-standard Young tableaux of shape λ and content μ. This is true if and only if the Kostka number $K_{\lambda\mu}$ is positive. This is known to be true if and only if $\lambda \geq \mu$ in dominance order, and hence if and only if $\lambda' \leq \mu'$. Now, since $\lambda' = (2^k, 1^{n-2k})$ this is seen to be equivalent to

$$\mu'_1 + \cdots + \mu'_k \geq 2k \iff \mu'_1 + \cdots + \mu'_k - 2k + 1 > 0. \qquad \square$$

It is known by Corollary 3.2.5 that the multiplicity of $\lambda = (n-k, k)$ in $\mathfrak{S}(v^{\otimes})$ is the dimension of the vector space spanned by the products $v_S = \prod_{i \in S} v_i$ where S is the complement of a disjoint union of k bases of $M(v)$. Further, for all choices of an ordering of the ground set E of $M(v)$, the dimension of this space of products is at least the number of bases B of $M(v)$ such that $ex(B)$ is a disjoint union of $k-1$ other bases. We show that the number $\mu'_1+\cdots+\mu'_k-2k+1$ is achieved for a particular ordering of the ground set.

Let T be the column superstandard tableau of shape μ: The numbers $1, 2, \ldots, n$ are filled in to T top-to-bottom, left-to-right. Label the vectors v_i (and hence the ground set of its matroid) so that the elements in a given row are all parallel. It follows from the definition of μ that the elements in each column are all non-parallel. For each number i such that

$$2k \leq i \leq \mu'_1 + \cdots + \mu'_k$$

we claim that there is some $j < i$ (and not in the same row in the filling T of μ) such that $ex(\{ij\})$ is a disjoint union of $k-1$ bases of $M(v)$. As an example take $k = 3$,

$$T = \begin{array}{|c|c|c|} \hline 1 & 5 & 8 \\ \hline 2 & 6 & 9 \\ \hline 3 & 7 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array}$$

and choose $i = 9$. Consider the possible elements j to add to $\{9\}$: $ex\{19\} = \{26\}$, 2 cannot be added to $\{9\}$ to get a base, $ex\{39\} = \{126\}$, $ex\{49\} = \{1236\}$ which is a union of two bases. If we started with $i = 6$ then $ex\{16\} = \{2\}$, $ex\{36\} = \{12\}$, $ex\{46\} = \{123\}$, $ex\{56\} = \{1234\}$ which is a union of two bases.

In general we proceed as follows. Having chosen i from column ℓ (where $\ell \leq k$) we know that the elements directly to the left will be externally active in any base containing i. This gives $\ell - 1$ externally active elements to start with. We know that whatever element j we choose to extend i to a base by, the numbers less than j will be externally active in that base. As we start with the possibilities of which element to add, $1, 2, \ldots,$ the possible size of the external activity increases by one at each step, unless we cannot add that element to form a base. It follows that there is a *unique* element j to add to i so that $ex\{ij\}$ has size $2(k-1)$. It is plain that the set of externally active elements constructed in this way will be a union of bases since the number of elements selected from a given parallelism class is always less at most k. Since there were $\mu'_1 + \cdots + \mu'_k - 2k + 1$ choices for i we have proved the lower bound on the multiplicity of $(n-k,k)$ in $\mathfrak{S}(v^{\otimes})$.

Now, if $\mu'_1 + \cdots + \mu'_k = n$ then we have shown that the multiplicity of $(n-k,k)$ in $\mathfrak{S}(v^{\otimes})$ is at least $n - 2k + 1$. We show that this lower bound is also an upper bound. Since we are assuming that $\dim V = 2$, the multiplicity $(n-k,k)$ in $V^{\otimes n}$ is the number of semi-standard Young tableaux of shape $(n-k,k)$ and entries in $\{1,2\}$. This follows from Schur-Weyl duality, since it is the multiplicity of $\mathcal{S}((n-k,k))$ in the right $\mathfrak{S}_n$-module $V^{\otimes n}$. The number of such tableaux is easily seen to be $n - 2k + 1$, hence our lower bound is an upper bound in this case.

Next we claim that we may always assume that $\mu'_1 + \cdots + \mu'_k = n$. Indeed, if this were not the case then we could factor any v_S, where the complement of S was a disjoint union of k bases of $M(v)$, as $v_{S'}v_{S''}$ where $v_{S''}$ is the product of the all vectors indexed by columns to the right of the k-th column of the tableau T. The reason for this is that since the complement of S is a disjoint union of k bases, we know that the complement contains at most k elements from each row of T. Let v' denote the configuration of n' vectors obtained by deleting the vectors indexed by entries to the right of column k in T. Then the multiplicity of $(n-k,k)$ in $\mathfrak{S}(v^{\otimes})$ is a equal to the multiplicity of $(n'-k,k)$ in $\mathfrak{S}((v')^{\otimes})$, whereby reduce to the case considered in the previous paragraph.

3.6. Future Directions

As we said in the introduction to this chapter, computing the exact multiplicity of an irreducible in $\mathfrak{S}(v^{\otimes})$ or $G(v^{\otimes})$ is a difficult problem. It is a sad fact that there is currently no example of two configurations u and v such that $M(u) = M(v)$ and $G(u^{\otimes}) \not\approx G(v^{\otimes})$, and no good evidence that says that the isomorphism type of $\mathfrak{S}(v^{\otimes})$ depends only on $M(v)$. Perhaps the most pressing open problem in this thesis is to resolve this question.

Problem 3.6.1. *Determine if the isomorphism type of $G(v^{\otimes})$ depends only $M(v)$.*

Phrased in terms of the vector bundle terminology of Section 2.5 this question asks: When restricted to a matroid strata $X(M)$ of the Grassmannian, are the degeneracy loci map of vector bundles $f : \mathbf{C}\,\mathrm{Aut}(\mathcal{U}) \to \mathcal{U}^{\otimes n}$ either empty or all of $X(M)$?

Murphy's law says that since there is no reason that the degeneracy loci should be simple they should be as complicated as one likes.

Perhaps, asking if the isomorphism type of $G(v^{\otimes})$ depends only on $M(v)$ is too much, and we should be looking for results more akin to our result for hook shapes.

Problem 3.6.2. *Determine which partitions λ have the property that their multiplicity in $G(v^{\otimes})$ are matroid invariants.*

For example, is there always an ordering of the ground set of $M(v)$ so that the multiplicity of $(\ell + j, \ell^{r(M)-1}) \vdash n$ is the number of bases of external activity with size $(\ell - 1)r(M)$?

Switching gears, we have a problem related to $P(v)$.

Problem 3.6.3. *What is the structure of $P(v)$ as a $\operatorname{Sym} V^*$-module (equivalently, a $\operatorname{Sym} V^*/I(v)$-module)? How many generators does it need?*

This project would involve measuring how far away $\operatorname{Sym} V^*/I(v)$ is away from being Gorenstein. For, if the quotient $\operatorname{Sym} V^*/I(v)$ was a zero-dimensional Gorenstein ring then is isomorphic to $P(v)$ as $\operatorname{Sym} V^*$-modules. The condition of being Gorenstein means that $P(v)$ is generated by one element as a $\operatorname{Sym} V^*$-module. In general this will not happen, since if it did the product $v_1 \cdots v_n \in P(v)$ would need to be a generator. One can check in the simplest examples that this does not hold.

Lastly, there is the problem of generalizing the construction of $P(v)$ to non realizable matroids.

Problem 3.6.4. *Given an arbitrary matroid M can one naturally define a free abelian group that interprets the Tutte polynomial of M as in Theorem 3.3.2?*

If one is simply interested in the single variable characteristic polynomial of M, then such algebraic objects do exist. Any manifestation of the χ-algebras of Forge and Las Vergnas [**FLV01**] gives an example. Can these χ-algebras be generalized to interpret the Tutte polynomial?

CHAPTER 4

The Universal Representation

The canonical surjection $\mathbf{C}\mathfrak{S}_n \to \mathfrak{S}(v^{\otimes})$ that sends 1 to $v^{\otimes}$ has some obvious elements in its kernel: If D indexes any dependent subset of $M(v)$ then the antisymmetrizer of the set D applied to $v^{\otimes}$ is zero. It follows that the right ideal generated by these antisymmetrizers acts by zero on $v^{\otimes}$. In this chapter we study the quotient of $\mathbf{C}\mathfrak{S}_n$ by ideals of this form.

4.1. The Universal Representation

Given a set of integers D, the antisymmetrizer of D, b_D, is the signed sum of the permutations D: $b_D = \sum_{\sigma \in \mathfrak{S}_D} \operatorname{sign}(\sigma)\sigma \in \mathbf{Z}\mathfrak{S}_D$. It is easy to see that

$$b_{D \cup j} = b_D \prod_{i \in D} (1 - (ij)),$$

which is a fact we will use repeatedly.

We will let $\langle - \rangle$ denote taking the right ideal generated by the elements $-$ in $\mathbf{Z}\mathfrak{S}_n$.

Definition 4.1.1. Let M be an arbitrary matroid with ground set $E = [n]$. Define

$$U(M)_{\mathbf{Z}} = \mathbf{Z}\mathfrak{S}_n / \langle b_D : D \text{ is dependent in } M \rangle.$$

This is a right $\mathbf{Z}\mathfrak{S}_n$-module that we will call the universal representation for M with integer coefficients. We will denote its complexification $U(M)_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$ by $U(M)$ and call this the universal representation for M.

Proposition 4.1.2. *The representation $U(M)$ is universal in the sense that for all complex realizations v of M there is a unique map of $\mathbf{C}\mathfrak{S}_n$-modules $U(M) \to \mathfrak{S}(v^{\otimes})$ extending the map $1 \mapsto v^{\otimes}$.*

Proposition 4.1.3. *Suppose that M is realizable over a field of characteristic zero. The multiplicity of λ is positive if and only if $\lambda \geq \rho(M)'$. The "only if" direction of this statement holds for every matroid.*

If M is not realizable over a field of characteristic zero then it may happen that $\lambda \geq \rho(M)'$ but the multiplicity of λ in $U(M)$ is zero.

Proof. By Dias da Silva's theorem characterizing the rank partition, it is clear from the definition that an irreducible $\mathfrak{S}_n$-module indexed by $\lambda \vdash n$ can appear in $U(M)$ if and only λ is larger in dominance order than the transposed rank partition of M. It is not clear that this condition is sufficient to ensure the appearance of an irreducible. When M is representable over $\mathbf{C}$ we may choose a realization v of M and from the surjection of $\mathfrak{S}_n$-modules

$$U(M) \to \mathfrak{S}(v^{\otimes}), \quad 1 \mapsto v^{\otimes},$$

we see that λ appears in $U(M)$ if and only if it appears in $\mathfrak{S}(v^{\otimes})$.

To finish the proof of the proposition, it is sufficient to give an example of a matroid M that is not realizable over $\mathbf{C}$ and a partition $\lambda \geq \rho(M)'$ such that the multiplicity of λ is zero in $U(M)$. For this, let M be the Fano matroid, consisting of the nonzero elements of $\mathbb{F}_2^3$ (see also Example 3.3.5). It can be shown using a result of Crapo and Schmitt [**CS00**, Example 6.6] that the column antisymmetrizer of any tableaux of shape $(3, 2, 2)$ is 2-torsion in $U(M)_{\mathbf{Z}}$, and hence zero in $U(M)$. Since $(3, 2, 2)$ is the transposed rank partition of M, the second statement of the proposition follows. $\square$

There should be a characterization of when a Specht module appears as a quotient of $K \otimes_{\mathbf{Z}} U(M)_{\mathbf{Z}}$ when M is realizable over K. Let $\mathcal{S}(\lambda)_K$ denote the submodule of $K\mathfrak{S}_n$ generated by a Young symmetrizer of a tableau of shape λ.

Conjecture 4.1.4. *For any matroid M realizable over a field K there is a surjection of $K\mathfrak{S}_n$-modules $K \otimes_{\mathbf{Z}} U(M)_{\mathbf{Z}} \to \mathcal{S}(\lambda)$ if and only if $\lambda \geq \rho(M)'$.*

Some of the basic properties of $U(M)$ are that it behaves nicely with respect to direct sum, truncation and weak maps. Recall that the truncation of a matroid M, written TM, has bases equal to all corank 1 independent sets of M. A weak map (for us) will be a pair of matroids on the same ground set M and N such that every independent set of N is an independent set of M. We indicate that N is a weak map image of N by writing $M \rightsquigarrow N$.

Proposition 4.1.5. *For all matroids M and N,*

(1) *If $M \rightsquigarrow N$ is a weak map of matroids then there is a surjection of $\mathfrak{S}_n$-modules $U(M) \to U(N)$*
(2) *U takes direct sums to products in that if M has m elements and N has n elements then*

$$U(M \oplus N) = \operatorname{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} U(M) \otimes U(N).$$

(3) *The multiplicity of λ in $U(TM)$ is equal to the multiplicity of λ in $U(M)$ if $\ell(\lambda) < r(M)$ and is zero otherwise.*

The first two results hold over the integers.

Proof. If we have a weak map $M \rightsquigarrow N$ then the ideal of antisymmetrizers of M is contained in that of N. It follows that there is a surjective $\mathfrak{S}_n$-module homomorphism $U(M) \to U(N)$.

Let M and N be two matroids on m and n elements, respectively. We first observe that if $U(M) = J_1\backslash\mathbf{Z}\mathfrak{S}_m$ and $U(N) = J_2\backslash\mathbf{Z}\mathfrak{S}_n$ then

$$\begin{aligned} U(M) \otimes U(N) &= \frac{\mathbf{Z}\mathfrak{S}_m \otimes \mathbf{Z}\mathfrak{S}_n}{J_1 \otimes \mathbf{Z}\mathfrak{S}_n + J_2 \otimes \mathbf{Z}\mathfrak{S}_m} \\ &= \frac{\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n)}{J_1\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n) + J_2\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n)} \end{aligned}$$

Inducing to $\mathfrak{S}_{m+n}$ is equivalent to tensoring with $\mathbf{Z}\mathfrak{S}_{m+n}$ over $\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n)$. We have

$$\frac{\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n)}{J_1\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n) + J_2\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n)} \otimes_{\mathbf{Z}(\mathfrak{S}_m \times \mathfrak{S}_n)} \mathbf{Z}\mathfrak{S}_{m+n} = \frac{\mathbf{Z}\mathfrak{S}_{m+n}}{J_1\mathbf{Z}\mathfrak{S}_{m+n} + J_2\mathbf{Z}\mathfrak{S}_{m+n}}$$

It remains to be seen that $J_1\mathbf{Z}\mathfrak{S}_{n+m} + J_2\mathbf{Z}\mathfrak{S}_{n+m}$ is the ideal generated by the antisymmetrizers of $M \oplus N$. This follows since the set of circuits of $M \oplus N$ is the disjoint union of the set of circuits of M and that of N.

The claim about truncation will follow from Young's description of the ideal generated by an antisymmetrizer: If D is a set of size k then the ideal generated by b_D is a direct sum of Specht modules $\mathcal{S}(\mu)$ where the length of μ is at least k. In particular, adding an antisymmetrizer of a set of size k to an ideal does not change the multiplicity of an irreducible indexed by λ provided that the length of λ is less than k. $\square$

4.2. Examples of the Universal Representation

When we considered $\mathfrak{S}(v^{\otimes})$, we hoped that there was a description of the multiplicity of an irreducible $\mathfrak{S}_n$-module in $\mathfrak{S}(v^{\otimes})$ that counts tableaux with some constraint on their filling. It is unknown if a simple solution to this problem exists, since we do not know if the irreducible decomposition of $\mathfrak{S}(v^{\otimes})$ depends on the matroid $M(v)$, or higher order geometric data. The situation for $U(M)$ is more hopeful. Consider the following examples as evidence.

Example 4.2.1. Let $M = U_{k,n}$ be the uniform matroid of rank k on n elements. It is easy to see that M is the repeated truncation of $U_{n,n}$ to a rank k matroid. By item 3 in Proposition 4.1.5, the multiplicity of λ in $U(M)$ is the number of standard Young tableaux of shape λ if $\ell(\lambda) \leq k$ and is zero otherwise.

Example 4.2.2. Given a matroid M, the principal extension of M along the improper flat, denoted $M\Box 0$, is the matroid that adds a new element 0 to the ground set of M generically without increasing the rank of M. More precisely, the bases sets of $M\Box 0$ are:

$$\{B : B \text{ is a base of } M\} \cup \{I \cup 0 : r(I) = |I| = r(M) - 1\}$$

Let $M\Box 1$ denote the direct sum of M with the one element rank one matroid. It is easy to see that $M\Box 0 = T(M\Box 1)$, that is, $M\Box 0$ is the truncation of $M\Box 1$. It follows from Proposition 4.1.5 that we know how to compute $U(M\Box 0)$ in terms of $U(M)$, since we know how to compute $U(T(M\Box 1))$ in terms of $U(M)$.

Let $(b_1, \ldots, b_n)$ be a binary sequence such that $b_1 = 1$. We form a matroid

$$M(b) := 1\Box b_2\Box \ldots \Box b_n = (((1\Box b_2)\Box \ldots \Box b_{n-1})\Box b_n)$$

where the notation $\Box$-notation was introduced in the previous example. If the binary sequence has k bits equal to 1 then $M(b)$ is a rank k matroid. We have met matroids of the form $M(b)$ before as Schubert matroids in Definition 2.5.4 (freedom matroid and shifted matroid also appear in the literature). It is important to note that Schubert matroids come with a canonical ordering of their ground sets. In particular, they satisfy the following property: If I is independent in a Schubert matroid and $e \in I$ then for all $f > e$, $I - e \cup f$ is independent. Klivans proves that this property characterizes Schubert matroids are those matroids whose underlying simplicial complex is *shifted* [**Kli**].

Recall that a matroid is simple if every subset of its ground set with size less than three is independent. Two elements e and f of a matroid are said to be parallel if $r(\{e, f\}) = r(\{e\}) = r(\{f\}) = 1$. Given two matroids M and N on sets $F \supset E$, respectively, we say that M is a parallel extension of N if M contains N as a simplicial complex and every element of M is a parallel to an element of N.

To state our characterization of $U(M)$ for parallel extensions of simple Schubert matroids we recall that the reading word (or row word) of a tableau T is obtained by reading the entries of the tableaux from left-to-right, bottom-to-top. For example, if $T = \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 4 & 5 & 2 & 2 & 1 & 3 \\ \hline 1 & 3 & 2 & 3 \\ \cline{1-4} \end{array}$ then the reading word of T is 13232452213.

Theorem 4.2.3. *Let M be a parallel extension of a Schubert matroid N. Let the composition associated to M be μ. The multiplicity of λ in $U(M)$ is the number of semi-standard Young tableaux T with shape λ and content μ such that every strictly decreasing subword of the reading word of T is independent in N.*

Remark 4.2.4. This theorem recovers Proposition 4.1.3 for Schubert matroids. Indeed, the Robinson-Schensted-Knuth insertion tableau of the reading word of a tableau is just the original tableau. However, Greene's theorem says that if the shape of this tableau is ρ', then $\rho_1 + \cdots + \rho_k$ is the size of the largest union of k-decreasing subwords of the row word. It follows that the smallest shape that appears in $U(M)$ is $\rho(M)'$.

Example 4.2.5. Suppose that M is represented by the columns of the matrix on the left and N by the columns of the matrix on the right

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & 1 & 1 & 2 & 1 & 1 & 2 \\ & & & & & & 1 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 1 & 2 \\ & & & 1 & 4 \end{bmatrix}$$

Then M is a parallel extension of the Schubert matroid N. The composition associated to this parallel extension is $(3, 2, 1, 2, 1)$ and the binary sequence associated to the Schubert matroid N is $(1, 1, 0, 1, 0)$.

The multiplicity of $(3, 3, 3)$ is at most two, as there are only two semi-standard Young tableaux of shape $(3, 3, 3)$ and content $(3, 2, 1, 2, 1)$:

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline 4 & 4 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array}$$

The reading words of these tableaux are 445223111 and 345224111, respectively. The maximal strictly decreasing subwords of the first word are 421, 431, 521, all of which are independent in N. The maximal strictly decreasing subwords of the second word include 321, which is a circuit of N. It follows that the multiplicity of $(3, 3, 3)$ in $U(M)$ is exactly one.

Proof of Proposition 4.2.3. Since Schubert matroids are constructed from other Schubert matroids by taking direct sums with single element rank one matroids or by truncating, we will show that the rule for counting the multiplicities of irreducibles is preserved under these operations.

If T is a semi-standard Young tableau then say that it is an N*-independent tableau* if every strictly decreasing subword of T is independent in N.

Suppose that M is a parallel extension of a simple matroid N on the ordered set $[n]$ and that the composition associated to the extension is μ. Suppose further that the multiplicity of $\lambda \vdash n$ in $U(M)$ is the number of N-independent tableau of shape λ and content μ.

Then TM is a parallel extension of TN and the composition associated to the extension is still μ (unless TN is rank one, but this case is trivial). It is clear from item (3) in Proposition 4.1.5 that the multiplicity of λ in $U(TM)$ is the number of N-independent tableaux with shape λ and content μ.

Suppose that we wish to take the direct sum of N with the one element rank one matroid to get the matroid N'. Label the new element of N' $n+1$. Suppose that M' is a parallel extension of N' and that the composition associated to this is extension is (μ, j), i.e., μ with a j added to the end. It is known how to decompose $U(M')$:

$$U(M') = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_j}^{\mathfrak{S}_{n+j}} U(M) \otimes \mathbf{C}$$

where $\mathbf{C}$ denotes the trivial representation of $\mathfrak{S}_j$. By Pieri's rule, the multiplicity of λ in $U(M')$ is the number of semi-standard Young tableaux of shape λ and content (μ, j) such that removing the $n+1$'s results in a N-independent tableaux with content μ. Since all $n+1$'s in this tableau are on the rim, one immediately notes that this is the number of N'-independent tableaux of shape λ with content (μ, j). $\square$

Corollary 4.2.6. *Let M be a parallel extension of a Schubert matroid N. Let the composition associated to M be μ. The dimension of $U(M)$ is the number of permutations of the multiset*

$$\{\underbrace{1, \ldots, 1}_{\mu_1}, \underbrace{2, \ldots, 2}_{\mu_2}, \ldots\}$$

such that every strictly decreasing subword of the permutation is independent in N.

Let π and π' be words with entries in $[n]$. If $x < y < z$ are integers, then an elementary Knuth relation is performed to π by exchanging the string yxz in π with yzx (or vice versa) or by exchanging the string xzy in π with zxy (or vice versa). The words π and π' are said to be Knuth equivalent if π' is obtained from π by applying a series of elementary Knuth relations.

Proof. The dimension of $U(M)$ is the number of permutations of the multiset above that are Knuth equivalent to the row word of a tableau where every strictly decreasing subword of the row word of the tableaux is independent in N. Say that a permutation π of the above set is *N-independent* if every strictly decreasing of it is independent in N.

It suffices to show that if π is N-independent and π' is obtained from π by applying a basic Knuth relation then π' is N-independent. It is sufficient to consider the cases

$$\pi = u(yxz)v, \quad \text{and} \quad \pi' = u(yzx)v$$

as well as

$$\pi = u(xzy)v, \quad \text{and} \quad \pi' = u(zxy)v$$

since if the roles of π and π' are reversed the statement is trivial. Here u and v are words and $x < y < z$ are integers.

Suppose that $\pi = uyxzv$ and that $u'zxv'$ is a strictly decreasing subword of π'. Since $u'yxv'$ is a strictly decreasing subword of π it is independent in N. Since N is a Schubert matroid, its independence complex is shifted and hence $u'zxv'$ is independent in N. Hence π' is N-independent.

Suppose that $\pi = uxzyv$ and that $u'zxv'$ is a strictly decreasing subword of π'. It follows that $u'zyv'$ is strictly decreasing too and is independent since it is a subword of π. The word $u'xv'$ is independent for the same reason. However,

$u'zyv'$ is longer than $u'xv'$, hence by the matroid exchange axiom one of the sets $u'zxv'$ or $u'yxv'$ is independent. However, if $u'yxv'$ is independent then so is $u'zxv'$ since N is Schubert matroid and hence a shifted complex.

We conclude that if π is N-independent and π' is Knuth equivalent to π then π' is N-independent. □

4.3. Multiplicities of Hooks

The multiplicity of hook shaped irreducibles in $U(M)$ enjoys the same beautiful description as we had in Theorem 3.3.3 for $\mathfrak{S}(v^{\otimes})$.

Theorem 4.3.1. *Let M be an arbitrary matroid and h_j denote the multiplicity of the length j hook shape in $U(M)$. Then*

$$\sum_{j\geq 0} h_j t^{r(M)-j} = \frac{1}{1+t} T(M; 1+t, 0).$$

In particular, the multiplicity of the longest hook shape in $U(M)$ is the number of no-broken-circuit bases of M.

We will abbreviate nbc for "no broken circuit" throughout the rest of this section. It is sufficient to prove the second statement of the theorem, since by Proposition 4.1.5(3) the multiplicity of hook shapes with length less than $r(M)$ can be obtained by truncating. Indeed,

$$\begin{aligned}\{I \subset [n] : I \text{ is a nbc base of } T^j M\} \cup \{I - 1 \subset [n] : I \text{ is a nbc base of } T^{j-1} M\} \\ = \{I \subset [n] : I \text{ is a rank } r(M) - j \text{ nbc set of } M\}\end{aligned}$$

We will show that the Young symmetrizers of the skew tableaux of shape $(n - r(M) + 1, 1^{r(M)})/(1)$ where the entries in their long column are nbc bases of M minimally generate the $(n - r(M) + 1, 1^{r(M)-1})$-isotypic component of $U(M)$. The proof of this result will take up the remainder of this section.

Recall that b_D denotes the antisymmetrizer of the set D. That is to say, $b_D = \sum_{\sigma \in \mathfrak{S}_D} \operatorname{sign}(\sigma)\sigma$. Let $\langle - \rangle$ denote taking the right ideal in $\mathbf{C}\mathfrak{S}_n$ generated by the elements $-$.

Lemma 4.3.2. *The antisymmetrizers of nbc bases of a matroid M generate the submodule*

$$\langle b_B : B \text{ is a base of } M \rangle \subset U(M).$$

Proof. Let B be the lexicographically smallest subset of the ground set of M such that b_B is not in the submodule generated by the antisymmetrizers of nbc bases. Suppose that e is the smallest element of a circuit C and that $C \subset B \cup e$. Then

$$b_{B\cup e} = b_B - \sum_{f\in B} (ef) b_B = b_B - \sum_{f\in B} b_{B-f\cup e}(ef)$$

It follows that

$$b_B = \sum_{f\in C-e} b_{B-f\cup e}(ef) \in U(M)$$

since $b_{B\cup e}$ is the antisymmetrizer of a dependent set, which is zero in $U(M)$. However, every set appearing in the sum of the form $B-f\cup e$ is lexicographically smaller than B, since e was the smallest element of C. This is a contradiction, since there must be some set $B-f\cup e$ such that $b_{B-f\cup e}$ is not in the submodule generated by the antisymmetrizers of the nbc bases. ☐

Given a set of size S of size $r(M)$, let c_S denote the Young symmetrizer of standard Young tableaux of skew shape $(n - r(M) + 1, 1^{r(M)})/(1)$, with the entries of S in their long column. For example, if $r(M) = 3$, $n = 7$ and $S = \{4, 5, 6\}$ then

$$c_S = \sum_{\sigma\in\mathfrak{S}_{4,5,6}} \operatorname{sign}(\sigma)\sigma \sum_{\tau\in\mathfrak{S}_{1,2,3,7}} \tau$$

The right ideal in $\mathbf{C}\mathfrak{S}_n$ generated by such a Young symmetrizer has an image in $U(M)$ that is either zero, or irreducible. To prove the theorem, it is sufficient to prove the following lemma.

Lemma 4.3.3. *The right ideal in* $\mathbf{C}\mathfrak{S}_n$ *generated by*

$$\{c_B : B \text{ is a nbc base of } M\}$$

does not meet the right ideal generated by

$$\{c_D : |D| = r \text{ and } D \text{ is dependent in } M\}.$$

The proof of this lemma is technical, so we break this it up into more manageable pieces. For the rest of this section let D denote a dependent set of size $r(M)$.

Remark 4.3.4. Since every base of a matroid is a nbc base in some ordering of its ground set, $c_B \neq 0 \in U(M)$ for all bases B of M. This is non-obvious and proves that the antisymmetrizer of a base is not zero in $U(M)_{\mathbf{Z}}$.

The following equivalence is easy to prove using the circuit axioms: A dependent set D contains a unique circuit if and only if $D - e$ is independent for some $e \in D$. We call dependent sets that contain a unique circuit unicyclic.

Claim 4.3.5. *We have*

$$\langle c_D : D \text{ contains two circuits}\rangle \subset \langle c_D : 1 \in D \text{ dependent}\rangle.$$

Proof. If $1 \notin D$ and D contains more than one circuit then each of the sets $D - e \cup 1$ is dependent since $D - e$ is. We have $c_D = \sum_{e \in D} c_{D-e\cup 1}(1e)$ which proves the result. ∎

Claim 4.3.6. *Let $cl(D)$ denote the closure of D in M. We have the inclusion,*

$$\langle c_D : D \text{ unicyclic}, 1 \in cl(D)\rangle \subset \langle c_D : 1 \in D \text{ dependent}\rangle.$$

Proof. If $1 \notin D$ then for all $e \in D$, $D - e \cup 1$ is dependent. Hence we have the inclusion by writing $c_D = \sum_{e \in D} c_{D-e\cup 1}(1e)$. $\square$

For the unicyclic sets where $1 \notin cl(D)$ recall that we can write $D = I \cup e$, where $e \in ex(I)$. To see this let I be the lexicographically largest independent set of rank $r(M) - 1$ contained in D. Let I denote an independent set of rank $r(M) - 1$.

Claim 4.3.7. *We have the inclusion,*

$$\langle c_D : 1 \notin cl(D), D \text{ unicyclic}\rangle \subset \langle c_D : 1 \in D \text{ dependent}\rangle + \langle c_{I\cup ex(I)} : |ex(I)| = 1, 1 \notin ex(I)\rangle$$

Proof. Let D be a dependent set such that $D = I \cup e$, $e \in ex(I)$, as above but $\{e\} \neq ex(I)$. Then there is an element $f \in ex(I) - e$. We may write,

$$\left(1 - \sum_{g \in I}(gf)\right) c_{I\cup e} = 0 \in U(M)$$

since this element is in the right ideal generated by $b_{I\cup f}$, by the definition of the Young symmetrizer. We conclude that

$$c_{I\cup e} = \sum_{g \in I} c_{(I-g\cup f)\cup e}(fg)$$

We see that for all $g \in I$, $(I - g \cup f) \cup e$ is unicyclic, does not contain 1 in its closure and is lexicographically smaller than $I \cup e$. Assuming inductively that $c_{(I-g\cup f)\cup e}$ is in the ideal

$$\langle c_D : 1 \in D \text{ dependent}\rangle + \langle c_{I\cup ex(I)} : |ex(I)| = 1, 1 \notin ex(I)\rangle$$

we see that $c_{I\cup e}$ is in this ideal too. $\square$

We now straighten the generators of the ideal

$$\langle c_{I\cup ex(I)} : |ex(I)| = 1, 1 \notin ex(I)\rangle.$$

If I is independent of rank $r(M)-1$, has external activity equal to one but does not contain 1 in its closure then $I \cup 1$ is a broken circuit base, but for all elements $g \in I$, $(I - g \cup 1) \cup ex(I)$ is a no broken circuit base of M. Since a Young symmetrizer c_S, $|S| = r(M)$ is that of a standard Young tableau if and only if $1 \in S$, we have proved

$$\begin{aligned}\langle c_B &: B \text{ an nbc base}\rangle \cap \langle c_D : D \text{ dependent}\rangle \\ &= \langle c_B : B \text{ an nbc base}\rangle \cap \langle c_{I\cup ex(I)} : |ex(I)| = 1, 1 \notin ex(I)\rangle\end{aligned}$$

However, since every Young symmetrizer on the last ideal has support on a unique broken circuit base containing 1, this intersection must be empty. This completes the proof of lemma, and thus the theorem.

4.4. Relationship to The Whitney Algebra

In this section we will define the Whitney algebra of a matroid M and identify $U(M)_{\mathbf{Z}}$ as a $\mathbf{Z}$-submodule of it. The definition of the Whitney algebra is due to Crapo, Rota and Schmitt [**CS00**].

If E is a finite set let $\langle E\rangle$ be the free monoid on E and $\mathbf{Z}\langle E\rangle$ be the free module generated by $\langle E\rangle$. This has an algebra structure induced by the product structure on $\langle E\rangle$. Let $\bigwedge E$ be the free exterior algebra on E, which is the quotient of $\mathbf{Z}\langle E\rangle$ by the two-sided ideal generated by the elements ee, $e \in E$. We endow $\bigwedge E$ with a Hopf algebra structure by defining

$$\delta(e) = 1 \otimes e + e \otimes 1.$$

Given a word $w \in \langle E \rangle$, we denote its image in $\bigwedge E$ by w too. The coproduct of w is given by

$$\delta(w) = \sum_{uv = w \in \langle E \rangle} \sigma(uv) u \otimes v$$

where $\sigma(uv)$ is the sign of the permutation that sorts the monomial $uv \in \langle E \rangle$ into increasing order. Let $\bigotimes \bigwedge E$ be the tensor algebra of $\bigwedge E$,

$$\bigotimes \bigwedge E = \mathbf{Z} \oplus \bigwedge E \oplus \bigotimes^2 \bigwedge E \oplus \bigotimes^3 \bigwedge E \oplus \dots$$

This algebra comes with an internal and external multiplication. The internal multiplication is induced by the product on $\bigwedge E$ and we have

$$(u_1 \otimes \cdots \otimes u_n) \times (v_1 \otimes \cdots \otimes v_n) = \pm(u_1 v_1 \otimes \cdots \otimes u_n v_n),$$

the sign is the parity of $\sum_{i<j} |v_i||u_j|$, where $|u| = k$ if $u \in \bigwedge^k E$. For example,

$$(u_1 \otimes u_2)(v_1 \otimes v_2) = (-1)^{|v_1||u_2|} u_1 v_1 \otimes u_2 v_2.$$

The external multiplication is the multiplication comes from the tensor product. The coproduct on $\bigwedge E$ induces a map of algebras

$$\delta : \bigwedge E \to \bigotimes \bigwedge E.$$

where the latter has the internal multiplication.

Note that the tensor algebra of $\bigwedge E$ has several gradings. The first is the grading by the number of tensor factors (called the degree), the other induced by the grading of the exterior algebra (this induces the grading by shape), and the last is the grading of $\langle E \rangle$ given by content. We illustrate these concepts with the following example. The element

$$abc \otimes 1 \otimes d \otimes bc \otimes 1 \otimes bd$$

has degree 6, shape $(3,0,1,2,0,2)$ and content $ab^3c^2d^2$. Suppose that α is a composition of $k+1$ (i.e., a list of non-negative integers summing to k). The iterated coproduct δ^{k+1} (which is a map $\bigotimes^k \bigwedge E \to \bigotimes^{k+1} \bigwedge E$) gives rise to linear maps δ_α obtained by taking the graded piece of δ^k with shape α. This is called a coproduct slice. For example

$$\begin{aligned}\delta(ab) &= (1 \otimes a + a \otimes 1)(1 \otimes b + b \otimes 1) \\ &= 1 \otimes ab + a \otimes b - b \otimes a + ab \otimes 1 \\ \delta_{(1,1)}(ab) &= a \otimes b - b \otimes a\end{aligned}$$

Let M be any matroid defined on the ordered set E. Let $D(M)$ be the ideal in the $\bigotimes \bigwedge E$ generated by $\delta_\alpha(e_{i_1} \dots e_{i_m})$ where α is any composition and $\{e_{i_1}, \dots, e_{i_n}\}$ is a dependent set in M. Here the ideal is taken with respect to the internal multiplication of the tensor algebra of $\bigwedge E$ (which implies that it also contains the ideal taken with respect to the external product). It is easy to see that $D(M)$ is generated by coproduct slices of wedges of circuits of M. The quotient of $\bigotimes \bigwedge E$ by $D(M)$ is called *the Whitney algebra* of M, and denoted $W(M)$. Note that $W(M)$ has gradings by degree, shape and content since $D(M)$ is generated by homogeneous elements. The Whitney algebra of a matroid was defined by Crapo, Rota and Schmitt and was studied in their paper [**CS00**].

Definition 4.4.1. The $\mathbf{Z}$-submodule of $W(M)$ with shape $(1,\dots,1) = (1^n)$ and content $e_1 \cdots e_n$ is called the *universal representation* of M and denoted $U(M)_{\mathbf{Z}}$. Alternatively, $\mathbf{Z}\mathfrak{S}_n$ maybe embedded in $\bigotimes \bigwedge^1 E$ where 1 maps to $e_1 \otimes \cdots \otimes e_n$. From this we see that $U(M)_{\mathbf{Z}}$ is the image of this copy of $\mathbf{Z}\mathfrak{S}_n$ in the quotient $W(M)$.

Now, $U(M)_{\mathbf{Z}}$ visibly carries an action of the symmetric group. We now give a description of the relations of shape $(1, \dots, 1)$ and content $e_1 \cdots e_n$ in $W(M)$ in order to realize $U(M)_{\mathbf{Z}}$ as a quotient of $\mathbf{Z}\mathfrak{S}_n$.

Proposition 4.4.2. *The submodule of relations of $W(M)$ with shape $(1, \dots, 1)$ and content $e_1 \cdots e_n$ is generated by the elements of the form,*

$$\sum_{\sigma \in \mathfrak{S}_C} (-1)^\sigma (e_1 \otimes \cdots \otimes e_n)\sigma$$

where C is a circuit of M. It follows that $U(M)_{\mathbf{Z}}$ is isomorphic to the quotient of $\mathbf{Z}\mathfrak{S}_n$ by the right ideal generated by antisymmetrizers of circuits of M.

Proof. The only way to obtain an element of shape $(1, \dots, 1)$ and content $e_1 \dots e_n$ from the generators of $D(M)$ by taking coproducts of wedges of a circuit is as follows: Let C be a circuit of M with size k, and take the α-th coproduct slice of the wedge of elements of C, where α is a binary sequence of length n with exactly k entries equal to 1. From this we multiply by an appropriate element of the tensor algebra to bring the content to $e_1 \cdots e_n$ and shape to $(1, \dots, 1)$. Actually doing this for a given circuit proves that such elements are of the described form. □

Example 4.4.3. Suppose that $\{a, b, c\}$ is a circuit of the matroid M defined on the set $\{a, b, c, d, e\}$. Then the $(1, 0, 0, 1, 1)$-th coproduct slice of abc is

$$a \otimes 1 \otimes 1 \otimes b \otimes c - b \otimes 1 \otimes 1 \otimes a \otimes c - c \otimes 1 \otimes 1 \otimes b \otimes a - a \otimes 1 \otimes 1 \otimes c \otimes b$$
$$+ b \otimes 1 \otimes 1 \otimes c \otimes a + c \otimes 1 \otimes 1 \otimes a \otimes b.$$

Multiplying this by $1 \otimes d \otimes e \otimes 1 \otimes 1$ gives

$$(a \otimes b \otimes c \otimes d \otimes e)b_{123}(24)(35) = (a \otimes d \otimes e \otimes b \otimes c)b_{145}.$$

4.5. Future Directions

The obvious generalization of Proposition 4.2.3 to all matroids does not hold. This is to say, it is not true if M is realizable over $\mathbf{C}$ then there is an ordering of the ground set of M so that the multiplicity of λ in $U(M)$ is the number of standard Young tableaux T of shape λ whose reading word has every decreasing subword independent in M. Interestingly, Theorem 4.3.1 says that this description is correct when λ is a hook for arbitrary matroids and orderings of their ground set!

Problem 4.5.1. *Determine a combinatorial description of the multiplicity of λ in $U(M)$ when M is realizable over $\mathbf{C}$.*

Which of these multiplicities are Tutte invariants?

If a solution to this problem exists, it will be interesting to see how it relates and generalizes to the whole Whitney algebra. As a first step towards understanding the Whitney algebra of an arbitrary matroid we propose solving the following problem.

Problem 4.5.2. *Let M be a parallel extension of a simple Schubert matroid. Determine a basis for $W(M)$ in terms of pairs of suitably generalized standard Young tableaux.*

To say something interesting about $W(M)$ when M is realizable over $\mathbf{C}$ it might be interesting to use the following fact: If v is a realization of M in V it is known that there is a unique map of lax-Hopf algebras $W(M) \to \bigotimes \bigwedge V$ that sends the elements of the ground set of M to the vector that they represent. The image of this map generalizes $\mathfrak{S}(v^{\otimes})$ should hold interesting information about M and v.

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APPENDIX A

Schur-Weyl Duality

In this appendix we review Schur-Weyl duality, since it is so necessary to the work in Chapers 1 and 2. Nothing here is remotely new, however, it is convenient to have all the things commonly referred to as "Schur-Weyl duality" located in one place.

A.1. Symmetrizations in The Group Algebra of $\mathfrak{S}_n$

To begin our discussion of Schur-Weyl duality we review some basic facts from the representation theory of the symmetric group $\mathfrak{S}_n$ over the complex numbers. Let $\mathbf{C}\mathfrak{S}_n$ be the group algebra of $\mathfrak{S}_n$: $\mathbf{C}\mathfrak{S}_n$ has the elements of $\mathfrak{S}_n$ as a $\mathbf{C}$-vector space basis and the multiplication of two basis elements σ and τ is the element described by their product $\sigma\tau$ in $\mathfrak{S}_n$.

There is an action of $\mathfrak{S}_n$ on the set of all tableaux of a fixed shape. A permutation $\sigma \in \mathfrak{S}_n$ acts on T by replacing the entry i in T with $\sigma(i)$. The set of permutations that map the set of numbers in each row of T to themselves is called the row group of T, denoted $\mathrm{Row}(T)$. Likewise we define the column group $\mathrm{Col}(T)$. From these subgroups we form two elements of the group algebra $\mathbf{C}\mathfrak{S}_n$,

$$a_T = \sum_{\sigma \in \mathrm{Row}(T)} \sigma, \qquad b_T = \sum_{\tau \in \mathrm{Col}(T)} \mathrm{sign}(\tau)\tau,$$

the row symmetrizer and column antisymmetrizer of T, respectively.

Suppose that T is a tableau of shape λ. It is easy to see that the right ideal in $\mathbf{C}\mathfrak{S}_n$ generated by a_T is equal to

$$\mathrm{Ind}_{\mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_\ell}}^{\mathfrak{S}_n} 1$$

where 1 denotes the trivial representation of the subgroup $\mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_\ell}$. Likewise if λ' denotes the partition conjugate to λ then the right ideal in $\mathbf{C}\mathfrak{S}_n$ generated by b_T is equal to

$$\mathrm{Ind}_{\mathfrak{S}_{\lambda'_1}\times\cdots\times\mathfrak{S}_{\lambda'_m}}^{\mathfrak{S}_n}(-1)$$

where -1 denotes the sign representation of $\mathfrak{S}_{\lambda'_1}\times\cdots\times\mathfrak{S}_{\lambda'_\ell}$.

The product $c_T = b_T a_T \in \mathbf{C}\mathfrak{S}_n$ is called a Young symmetrizer.

Theorem A.1.1. *The right ideal in $\mathbf{C}\mathfrak{S}_n$ generated by a Young symmetrizer is irreducible. If T and T' are tableaux of the same shape then $c_T\mathbf{C}\mathfrak{S}_n \approx c_{T'}\mathbf{C}\mathfrak{S}_n$ as right $\mathfrak{S}_n$-modules.*

If c_λ denotes a Young symmetrizer of a tableau of shape λ then as an $\mathfrak{S}_n\times\mathfrak{S}_n$-bimodule

$$\mathbf{C}\mathfrak{S}_n = \bigoplus_\lambda \mathbf{C}\mathfrak{S}_n c_\lambda \otimes c_\lambda \mathbf{C}\mathfrak{S}_n.$$

The Young symmetrizers of the standard Young tableaux are a minimal set of generators of the right $\mathfrak{S}_n$-module $\mathbf{C}\mathfrak{S}_n$.

If T is a tableau of shape λ let χ^λ denote the character of $c_T\mathbf{C}\mathfrak{S}_n$. The projector of $\mathbf{C}\mathfrak{S}_n$ to its $c_T\mathbf{C}\mathfrak{S}_n$-isotypic component is at once seen to be

$$\pi_\lambda = \frac{\chi^\lambda(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\chi^\lambda(\sigma)\sigma^{-1}.$$

Pate attributes the following result to Littlewood.

Theorem A.1.2. *The sum $\sum_T c_T$ over tableaux T of shape λ is a scalar multiple of π_λ.*

Proof. One checks that the sum $\sum_T c_T \in \mathbf{C}\mathfrak{S}_n$ is not zero and central. It follows from Schur's Lemma that $\sum_T c_T$ acts by a scalar on any irreducible representation, and this scalar only depends on the isomorphism type of the irreducible representation (i.e., not the realization of the irreducible). It is straightforward to check that if S is a tableau of shape $\mu \neq \lambda$ then $c_S \sum_T c_T = 0$. Further, $\sum_T c_T$ is essentially idempotent (see Fulton and Harris [**FH91**, Chapter 4]). This means that some scalar multiple of $\sum_T c_T$ is a $\mathbf{C}\mathfrak{S}_n$-commuting projector to the λ-isotypic subspace of $\mathbf{C}\mathfrak{S}_n$. This projector is unique, and since π_λ is this projector by definition, we must have that $\sum_T c_T$ is a scalar multiple of π_λ. □

A.2. Double Centralizer Theorem

From a k-dimensional vector space V we form the n-fold tensor product $V^{\otimes n}$. This carries a left action of $\mathrm{GL}(V)$ by

$$g.(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$$

and a right action of $\mathfrak{S}_n$ via place permutation:

$$(v_1 \otimes \cdots \otimes v_n).\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Theorem A.2.1 (Schur-Weyl Duality)**.**

$$\mathrm{End}_{\mathfrak{S}_n} V^{\otimes n} = \mathbf{C}\,\mathrm{GL}(V), \qquad \mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n} = \mathbf{C}\mathfrak{S}_n/I,$$

where I is the right ideal in $\mathbf{C}\mathfrak{S}_n$-generated by antisymmetrizers of sets of size larger than $\dim V$*.*

In the case that U is a R-S bimodule and $\operatorname{End}_R(U) = S$ and $\operatorname{End}_S(U) = R$ we say that U is a balanced R-S module. Thus the theorem states that $V^{\otimes n}$ is a balanced $\mathbf{C}\,\mathrm{GL}(V)$-$\mathbf{C}\mathfrak{S}_n$ bimodule.

Proof. To see the first equality, we note that

$$\operatorname{End}_{\mathbf{C}} V^{\otimes n} = (\operatorname{End}_{\mathbf{C}} V)^{\otimes n}$$

and if $\varphi_1 \otimes \dots \varphi_n \neq 0$ is $\mathfrak{S}_n$-equivariant then we know that $\varphi_{\sigma(i)} = \varphi_i$ for all i. The second statement follows from the fact that two decomposable tensors are equal if and only if their respective tensor factors are equal, up to a non-zero scalars all of which together multiply to 1. This means that $\operatorname{End}_{\mathfrak{S}_n} V^{\otimes n} = \operatorname{Sym}^n(\operatorname{End} V)$, which is known to be generated by tensors of the form $\varphi \otimes \cdots \otimes \varphi$. Since $\mathrm{GL}(V)$ is dense in $\operatorname{End} V$ we have proved the first part of the theorem.

The second part of the theorem follows from a general fact: If A is a semi-simple algebra, M is a faithful A-module and $\operatorname{End}_A(M) = B$, then it is a classical result known as Burnside's Theorem that $\operatorname{End}_B(M) = A$ (see Lang [**Lan02**, Chapter XVII]). Note that M is a B-module via $\varphi.m = \varphi(m)$. This proves the result when $V^{\otimes n}$ is a faithful $\mathbf{C}\mathfrak{S}_n/I$-module, which it is. $\square$

In the notation of the proof of the theorem, it is easy to see that $M \otimes_A Ae \approx Me$ as left B-modules, when e is near idempotent. We may write $A = \bigoplus_{i \in I} Ae_i \otimes e_i A$, where $\{e_i : i \in I\}$ is a set of orthogonal idempotents for A, we have

$$\begin{aligned} M \approx M \otimes_A A &\approx M \otimes_A \bigoplus_{i \in I} Ae_i \otimes e_i A \\ &\approx \bigoplus_{i \in I} (M \otimes_A Ae_i) \otimes e_i A \\ &\approx \bigoplus_{i \in I} Me_i \otimes e_i A. \end{aligned}$$

Here all the isomorphisms are as $B \times A$-modules. It is a fact, that we will not prove here, that if eA is irreducible as an A-module then Me is irreducible as a B-module. It follows that if we select a tableau $T(\lambda)$ for every partition λ of n with length at most $\dim V$, then $V^{\otimes n}c_{T(\lambda)}$ is an irreducible $\mathrm{GL}(V)$-module and

$$V^{\otimes n} = \bigoplus_{\lambda} V^{\otimes n} c_{T(\lambda)} \otimes c_{T(\lambda)} \mathbf{C}\mathfrak{S}_n$$

where c_T denotes the Young symmetrizer of the tableau T, which is the product of the column antisymmetrizer b_T of T and the row symmetrizer a_T of T.

One checks at once that $V^{\otimes n}c_T$ has a highest weight vector of weight λ. For, after having chosen a Borel subgroup of $\mathrm{GL}(V)$, and hence an ordered basis $e_1, \dots, e_k$ of V, the highest weight vector is at once seen to be the image of some permutation of the tensor

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{\lambda_1} \otimes \cdots \otimes \underbrace{e_k \otimes \cdots \otimes e_k}_{\lambda_\ell}$$

under the Young symmetrizer c_T (which is easily seen to be non-zero).

When $\dim V \geq n$, a theorem of Morita (see [**AF92**]) combined with the Double Centralizer Theorem allow us to conclude a functorial correspondence between right $\mathfrak{S}_n$-modules and left $\mathrm{GL}(V)$-modules.

Theorem A.2.2. *The functors* $\mathrm{Hom}_{\mathrm{GL}(V)}(-, V^{\otimes n})$ *and* $\mathrm{Hom}_{\mathbf{C}\mathfrak{S}_n}(-, V^{\otimes n})$ *give rise to an equivalence of categories between left* $\mathrm{GL}(V)$*-modules and right* $\mathfrak{S}_n$*-modules.*

A.3. The Schur Functor

We now define a functor from the category of modules over a fixed commutative ring to itself that abstracts the construction of $V^{\otimes n}c_T$ given above. Let E be a module over a commutative ring R and place the n factors of $E \times \cdots \times E$

in bijection with boxes of the Young diagram of λ. Consider those maps

$$\varphi : E \times \cdots \times E \to F$$

where F is an R-module, that are

(1) multlinear,

(2) alternating in the entries of a column of λ,

(3) satisfy the exchange condition, which states that if the p's below representation the positions of one column of the Young diagram of λ, and q's represent the position of a different column, then

$$\varphi(\ldots, p_1, \ldots, p_r, q_1, \ldots, q_s, \ldots) = \sum \varphi(\ldots, p'_1, \ldots, p'_r, q'_1, \ldots, q'_s, \ldots)$$

the sum over all exchanges of some fixed set of the p's with an arbitrary set of the q's, in order.

Let $\mathbb{S}^\lambda E$ be the universal R-module that extends $\varphi : E \times \cdots \times E \to E$ to a map of R-module $\tilde{\varphi} : \mathbb{S}^\lambda E \to F$. Clearly $\mathbb{S}^\lambda E$ is unique up to unique isomorphism, provided that it exists.

Suppose that E is a free R-module, $E \approx R^k$. We can obtain all the modules $\mathbb{S}^\lambda E$ at once with the following construction. Consider symmetric algebra of the exterior algebra of E,

$$\mathrm{Sym} \bigwedge E = \bigoplus_{0 \le a_1, \ldots, 0 \le a_k} \mathrm{Sym}^{a_1} \bigwedge^1 E \otimes \cdots \otimes \mathrm{Sym}^{a_k} \bigwedge^k E$$

We form the quotient by the two-sided ideal Q generated by all quadratic equations of the form,

$$(p_1 \wedge \cdots \wedge p_r)(q_1 \wedge \cdots \wedge q_s) = \sum (p'_1 \wedge \cdots \wedge p'_r)(q'_1 \wedge \cdots \wedge q'_s)$$

the sum over all exchanges of a fixed set of the p's with any set of the q's, This yields an R-algebra which retains the direct sum decomposition as above. We can reindex this sum to be over partitions λ with at most k parts by associating the $a_1, \ldots, a_k$-th summand to the partition with a_i columns of length i. It is clear that the λ-indexed piece of this sum is exactly $\mathbb{S}^\lambda E$, hence,

$$(\mathrm{Sym} \bigwedge E)/Q = \bigoplus_\lambda \mathbb{S}^\lambda E,$$

the sum over partitions with at most k parts. It is clear that $\mathbb{S}^\lambda E$ is now a representation of $\mathrm{GL}(E)$, since $\mathrm{Sym} \bigwedge E$ is one and Q is stable under the natural action of $\mathrm{GL}(E)$.

Returning to the case when $E = V$, a complex vector space with an ordered basis $e_1, \ldots, e_k$ for E and a Borel subgroup of $\mathrm{GL}(V)$, one checks that the image of

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{\lambda_1} \otimes \cdots \otimes \underbrace{e_k \otimes \cdots \otimes e_k}_{\lambda_\ell}$$

in the quotient $\mathbb{S}^\lambda E$ is a highest weight vector of weight λ. Further, the image in the quotient is seen to be non-zero since one checks that the is a unique map of $\mathrm{GL}(V)$-modules $\mathbb{S}^\lambda V \to V^{\otimes n} c_T$ that maps these highest weight vectors to each other. It follows that

$$\mathbb{S}^\lambda V \approx V^{\otimes n} c_T,$$

as $\mathrm{GL}(V)$-modules, for any tableaux T of shape λ.

A.4. A Combinatorial Basis for $\mathbb{S}^\lambda V$

The character of $\mathbb{S}^\lambda V$ is given by Weyl's character formula: Suppose that $g \in \mathrm{GL}(V)$ has eigenvalues $x_1, \ldots, x_k$, then

$$s_\lambda(x_1, \ldots, x_k) := \mathrm{tr}(g : \mathbb{S}^\lambda V \to \mathbb{S}^\lambda V) = \frac{\det(x_i^{\lambda_j - j + 1})}{\det(x_i^{k-j+1})}.$$

The quotient on the right is known as a Schur polynomial, and it is at once observed to be a symmetric polynomial. It can be shown that $s_\lambda(x_1, \ldots, x_k)$

is the generating function for semi-standard Young tableaux of shape T and entries in $1, 2, \ldots, k$, i.e.,

$$s_\lambda(x_1, \ldots, x_k) = \sum_T \prod_i x_i^{\#\ i\text{'s in } T}$$

It follows that $\dim \mathbb{S}^\lambda V$ is counted by semi-standard Young tableaux of shape T and entries in $1, 2, \ldots, k$. There is a simple basis of $\mathbb{S}^\lambda$ indexed by these tableaux. To see this label the boxes of the Young diagram in the order used in the construction of the Schur functor, and then take the tensor product of the basis elements indexed by the numbers in the boxes of the tableaux. For example, if the ordering of the boxes is given by the tableau on the left and the semi-standard tableau is shown on the right

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 9 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$$

then we obtain the tensor $e_1 \otimes e_1 \otimes e_3 \otimes e_1 \otimes e_3 \otimes e_4 \otimes e_2 \otimes e_2 \otimes e_3$. Given two tableaux (P, Q), with P standard, let $e_{P,Q}$ denote the tensor constructed in this way. One checks that the image in $\mathbb{S}^\lambda$ of the tensors constructed in this way span $\mathbb{S}^\lambda V$. Since there are exactly enough of them, they form a basis for $\mathbb{S}^\lambda V$.

Transferring this result to the tensor product $V^{\otimes n}$ we have the following result.

Theorem A.4.1. *The set of tensors $e_{P,Q}c_P$, where P and Q are tableau of the same shape, P is standard, Q is semi-standard and c_P is the Young symmetrizer of P, form a basis of $V^{\otimes n}$.*

APPENDIX B

Matroid Theory

This appendix consolidates all the matroid terminology used in this thesis. Its primary function is that of a glorified glossary.

B.1. Definitions

Matroids are an abstraction of finite collections of vectors.

Definition B.1.1. A matroid is a pair $(E, \mathcal{I})$ where E is finite set and $\mathcal{I}$ is a simplicial complex on E that satisfies the *exchange axiom*: For every $I, I' \in \mathcal{I}$ such that $|I'| > |I|$ there is an $e \in I' - I$ such that $I \cup e \in \mathcal{I}$. Here we denote the singleton set $\{e\}$ by e.

The faces of $\mathcal{I}$ are called independent sets. The non-faces are called dependent sets. The minimal dependent sets are called circuits.

Example B.1.2. Let V be a vector space. Every finite collection of vectors $v = \{v_1, \ldots, v_n\}$ determines a matroid $M(v)$. The independent subsets of $M(v)$ are those sets $I \subset [n]$ that index linearly independent subsets of v.

Definition B.1.3. Let M be a matroid on E. The rank of $S \subset E$, denoted $r(S)$ or $r_M(S)$, is the size of the largest independent set of M contained in S. If $e \in E$ has the property that $r(\{e\}) = 0$ then e is called a loop. If $e \in E$ has the property that $r(S \cup e) = r(S) + 1$ for every subset $S \subset E - e$ then e is called an isthmus (or a coloop).

A base of a matroid is a maximal independent set. The rank of a matroid is the size of a base. Equivalently, it is the rank of its ground set E.

A subset $X \subset E$ is a said to be a flat of M if X is maximal with a given rank, i.e., $X \subset Y \subset E$ implies that $r(X) < r(Y)$. The collection of flats of M forms a geometric lattice under the partial order of containment. We denote the lattice of flats of M by $L(M)$.

Example B.1.4 (Constructions)**.** Let M and N be matroids on disjoint sets E and F. The direct sum of M and N, $M \oplus N$, is the matroid on $E \sqcup F$ whose independent sets are unions of independent subsets of M and N.

The deletion of $e \in E$ from M is the matroid on $E - e$ whose independent subsets are those of M not containing e. We denote this matroid by $M - e$. The deletion of $\{e_1, \ldots, e_k\} \subset E$ from M is just $M - \{e_1, \ldots, e_k\} := ((M - e_1) - \cdots - e_k)$.

The dual matroid M^* of M is the matroid on E whose bases are complements in E of bases of M. The contraction of M by E', denoted M/E', is $(M^* - E')^*$. If v is a vector configuration in V then $M(v)/v_i$ is the matroid realized by the collection $v - v_i$ in the quotient space $V/\mathbf{C}v_i$.

A cocircuit of a matroid is the complement of a circuit of its dual matroid.

The truncation of M, denoted TM, is the matroid whose independent sets are those of M with size less than $r(M)$.

Example B.1.5 (Matroid Union)**.** Let $M_1, \ldots, M_k$ be matroids defined on a common ground set E. Define a new matroid $M_1 \vee \cdots \vee M_k$ on E by saying that $I \subset E$ is independent if

$$I = I_1 \cup \cdots \cup I_k$$

where I_i is independent in M_i; the union can be assumed disjoint. It can be shown that this actually defines a matroid (see Oxley [**Oxl92**]).

B.2. The Rank Partition

Definition B.2.1. Let M be a matroid on E. Let λ be a partition of $|E|$. We say that M *conforms to* λ if there is a set partition of $E(M)$ of type λ whose blocks are independent sets of M.

Definition B.2.2. The *rank partition* of a matroid M is the sequence $(\rho_1, \rho_2, \dots)$ such that

$$\rho_1 + \cdots + \rho_k$$

is size of the largest union of k independent sets of M. This is to say, $\rho_1 + \cdots + \rho_k$ is the rank of the k-fold union of M with itself.

Theorem B.2.3 (Dias da Silva)**.** *The rank partition of a matroid is a partition.*

Proof. This proof is a slight rewording of Dias da Silva's original proof. We prove, by induction on t, the following statement: *Given a base B of $M^{(t)}$ there is a sequence of t disjoint independent (in M) sets B_i such that*

$$B_1 \cup \cdots \cup B_r$$

is a basis for $B^{(r)}$, $1 \leq r \leq t$, and $|B_1| \geq |B_2| \geq \cdots$. For $t = 1$ the claim is obvious. Supposing that $t > 1$ we write $B = I_1 \cup \cdots \cup I_t$ where the I_i are independent in M and we may assume the disjointedness of the union. Choose from I_t a minimal subset I'_t such that $B - I'_t$ is independent in $M^{(t-1)}$. That such an I'_t exists follows since $I_1 \cup \cdots \cup I_{t-1}$ is independent in $M^{(t-1)}$.

We claim that $B - I_t'$ is a basis for $M^{(t-1)}$. If $x \in I_t'$ then $x \in \overline{B - I_t'}$ by the minimality of I_t'. It now follows that $\overline{B - I_t'} = \overline{B - I_t' \cup I_t'} = \overline{B}$ which is all of the ground set.

It follows that $B - I_t'$ can be written as a disjoint union

$$B_1 \cup \cdots \cup B_{t-1}$$

of independent sets which decrease in cardinality: $|B_1| \geq |B_2| \geq \cdots$. Suppose that $|I_t'| > |B_{t-1}|$. It follows that we can extend B_{t-1} by a single element $x \in I_t'$ and remain independent. Since the sets I_i above were chosen to be disjoint it follows that $B_1 \cup \cdots \cup B_{t-2} \cup (B_{t-1} \cup x)$ is an independent set in $M^{(t-1)}$ of strictly larger cardinality than $B_1 \cup B_{t-1}$, which is a contradiction. $\square$

Recall the dominance (partial) order on partitions of n: We say that λ dominates μ and write $\lambda \geq \mu$ if for k less than the length of both λ and μ

$$\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k.$$

Theorem B.2.4 (Dias da Silva)**.** *The rank partition of M is the maximum dominance ordered partition of those partitions λ such that M conforms to λ.*

The following result can be used to give an inductive proof of Gamas's theorem that avoids the use of flag varieties.

Corollary B.2.5. *Let M be a matroid with rank partition $\rho(M)$. There is a basis B of M such that $\rho(M - B) = (\rho_2, \rho_3, \ldots)$.*

B.3. The Tutte Polynomial

The Tutte polynomial of a matroid was introduced by Crapo, generalizing Tutte's previous definition for graphs. There are many ways to define this polynomial.

Definition B.3.1. Let M be a matroid on E. The Tutte polynomial of M is

$$T(M;x,y) = \sum_{S \subset E} (1-x)^{r(M)-r(S)}(1-y)^{|S|-r(S)}.$$

From the definition of T it is evident that

$$T(0,0) = 2^{|E|}, \quad T(1,1) = \text{number of bases of } M, \quad T(2,1) = |\mathcal{I}(M)|$$

These facts can all be proved by showing that these quantities satisfy the appropriate recurrence relation and applying the theorem below.

Theorem B.3.2. *Let $\mathcal{M}$ be the set of isomorphism classes of matroids. There is a unique function $T : \mathcal{M} \to \mathbb{Z}[x,y]$ satisfying*

(1) *for all matroids M and N, $T(M \oplus N; x,y) = T(M;x,y)T(N;x,y)$,*

(2) *for all matroids M and $e \in M$ which is neither a loop nor an isthmus in M*

$$T(M;x,y) = T(M-e;x,y) + T(M/e;x,y),$$

(3) *if e is a loop of M then $T(M;x,y) = yT(M-e;x,y)$ and if e is an isthmus of M then $T(M;x,y) = xT(M-e;x,y)$.*

The definition of the Tutte polynomial given above satisfies these properties.

Fix a total order on E, the ground set of M. Let X be a flat of M. For an independent set $I \subset X$ whose closure is X define $e \in X - I$ to be externally active if e is the minimum element of a circuit in $I \cup e$. If $M|X = M - (E - X)$ then define $e \in I$ to be internally active in I if e is externally active the independent set $X - I$ of $(M|X)^*$. For an independent set I let $ex(I)$ and $in(I)$ to be the set of externally and internally active elements of I in its closure.

Theorem B.3.3. *Fix an ordering of the ground set of M, hence define the notion of internally and externally active elements of an independent set of M. The Tutte polynomial of M can be expressed as*

$$T(M; x, y) = \sum_{\text{bases } B} x^{i(B)} y^{e(B)}$$

A set $I \subset E$ is a no-broken-circuit (nbc) set if it does not contain any circuit with its smallest element deleted. The theorem above says that $T(M; 1, 0)$ enumerates nbc bases of M. It can be shown that $T(M; 1+x, 0)$ is the generating function for nbc sets by their corank (the corank of $S \subset E$ is $r(M) - r(S)$).

APPENDIX C

Examples

Here we collect a list of examples of rank three vector configurations v and the irreducible decomposition of $G(v^{\otimes})$, as well as matroids M and the decomposition of $U(M)$.

We do not list the multiplicity of (n) and $(n-1,1)$ since these are easily read off from the vector configuration. The multiplicity of (n) is 1 (since none of the configurations or matroids contain the zero vector or loops) provided v does not contain the zero vector or M does not contain a loop. The multiplicity of $(n-1,1)$ is the number of parallelism classes of the configuration or matroid, minus one.

In the first row of the Tables 1 and 3 we list the partitions of 6 with at most 3 parts, except (6) and $(5,1)$. In the first row of Table 2 we list the partitions of 7 with at most 3 parts, except (7) and $(6,1)$. In the subsequence rows of the tables, we list a vector configuration (or matroid) followed by the multiplicity of a λ in the $\mathfrak{S}(v^{\otimes})$ (or $U(M)$).

The nontrivial circuits of a rank k matroid are the circuits of size at most k. The nontrivial circuits and rank of matroid completely determine it. In Table 3 we list the loopless rank 3 matroids on 6 elements by their nontrivial circuits as well as the irreducible decomposition of their universal representation. These matroids were compiled using Gordon Royle's online matroid database [**Roy09**].

	$\ydiagram{4,2}$	$\ydiagram{4,1,1}$	$\ydiagram{3,3}$	$\ydiagram{3,2,1}$	$\ydiagram{2,2,2}$
$\begin{bmatrix}1&0&0&1&1&1\\0&1&0&0&-2&3\\0&0&1&1&4&9\end{bmatrix}$	9	9	4	8	1
$\begin{bmatrix}1&0&0&1&0&1\\0&1&0&1&1&4\\0&0&1&0&1&16\end{bmatrix}$	9	8	4	8	1
$\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&2&1\\0&0&1&0&4&4\end{bmatrix}$	9	7	4	8	1
$\begin{bmatrix}1&0&0&1&1&1\\0&1&0&1&2&4\\0&0&1&0&0&16\end{bmatrix}$	8	7	4	6	1
$\begin{bmatrix}1&0&0&5&0&1\\0&1&0&4&3&1\\0&0&1&0&-1&0\end{bmatrix}$	8	6	4	6	1
$\begin{bmatrix}1&0&0&1&1&1\\0&1&0&0&2&-3\\0&0&1&0&4&9\end{bmatrix}$	6	6	3	6	1
$\begin{bmatrix}1&0&0&1&1&0\\0&1&0&0&2&3\\0&0&1&0&4&9\end{bmatrix}$	6	5	3	6	1
$\begin{bmatrix}1&0&0&1&0&1\\0&1&0&1&1&4\\0&0&1&0&0&16\end{bmatrix}$	6	5	3	5	1
$\begin{bmatrix}1&0&0&1&0&1\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix}$	6	4	3	5	1

TABLE 1. The irreducible decomposition of $G(v^{\otimes})$ for some rank 3 configuration v on 6 elements.

	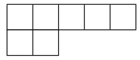	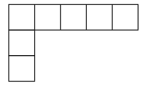	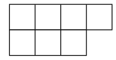	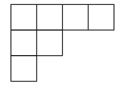	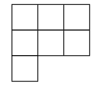	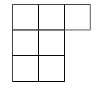
$\begin{bmatrix}1&0&0&1&1&1&1\\0&1&0&2&1&1&2\\0&0&1&4&0&1&0\end{bmatrix}$	13	10	10	13	6	3
$\begin{bmatrix}1&0&0&1&1&1&1\\0&1&0&1&2&1&0\\0&0&1&0&0&1&0\end{bmatrix}$	9	6	7	8	3	2
$\begin{bmatrix}1&0&0&1&1&0&0\\0&1&0&1&2&1&1\\0&0&1&0&0&1&2\end{bmatrix}$	12	9	10	11	6	3
$\begin{bmatrix}1&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&0&1&1&2&3&4\end{bmatrix}$	11	9	8	9	3	2
$\begin{bmatrix}1&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&0&1&1&2&3&0\end{bmatrix}$	11	8	8	9	3	2
$\begin{bmatrix}1&0&0&1&1&1&0\\0&1&0&1&1&0&1\\0&0&1&1&0&1&1\end{bmatrix}$	14	9	10	15	6	3

TABLE 2. The irreducible decomposition of $G(v^{\otimes})$ for some rank 3 configurations v on 7 elements.

	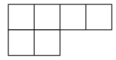	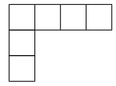	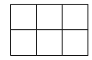	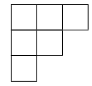	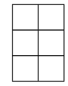
$12, 35, 46$	3	1	1	2	1
$12, 35, 16, 26$	2	1	1	1	0
$12, 15, 25, 16, 26, 56$	1	1	0	0	0
$12, 35, 146, 246$	4	2	2	3	1
$12, 35, 136, 236, 156, 256$	4	2	2	2	0
$12, 15, 25, 346$	3	2	1	2	0
$12, 15, 25, 136, 236, 356$	3	2	1	1	0
$12, 35$	4	3	2	4	1
$12, 15, 25$	3	3	1	2	0
$12, 345, 346, 356, 456$	6	3	3	5	2
$12, 145, 245, 146,$ $246, 156, 256, 456$	6	3	3	3	0
$12, 345, 146, 246$	6	4	3	5	1
$12, 145, 245, 136, 236$	6	4	3	4	1
$12, 345$	6	5	3	7	2
$12, 145, 245$	6	5	3	6	1
12	6	6	3	8	2
$124, 125, 145, 245, 126,$ $146, 246, 156, 256, 456$	9	4	5	5	0
$124, 235, 346, 156$	9	6	5	8	1
$124, 235, 236, 256, 356$	9	7	5	10	2
$124, 125, 145, 245$	9	7	5	10	2
$124, 235$	9	8	5	12	4
124	9	9	5	14	4
none	9	10	5	16	5

TABLE 3. The irreducible decomposition of $U(M)$ for all rank 3 matroids M on 6 elements.