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Existence, Uniqueness, and Determinacy of Equilibria in Complete Security Markets with Infinite Dimensional Martingale Generator

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Abstract. There is a strong evidence that most of financial variables are better described by a combination of diffusion and jump processes. Considering such evidence, researchers have studied security market models with jumps, in particular, in the context of option pricing. In most of their models, jump magnitude is specified as a continuously distributed random variable at each jump time. Then, the *dimensionality of martingale generator*, which can be interpreted as the "number of sources of uncertainty" in markets is infinite, and no finite set of securities can complete markets. In security market economy with infinite dimensional martingale generator, no equilibrium analysis has been conducted thus far. We assume *approximately complete markets* (Björk *et al.* [10] [11]) in which a continuum of bonds are traded and any contingent claim can be approximately replicated with an arbitrary precision. We introduce the notion of *approximate security market equilibrium* in which an agent is allowed to choose a consumption plan approximately supported with any prescribed precision. We prove that an approximate security market equilibrium in approximately complete markets can be identified with an Arrow-Debreu equilibrium. Then, we present sufficient conditions for the existence of equilibria in the case of stochastic differential utilities with Inada condition, and for the existence, uniqueness, and determinacy of equilibria in the case of additively separable utilities.
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1. Introduction

Many empirical studies have shown that the dynamics of most financial processes such as equity prices, interest rates, and exchange rates are better described by a combination of diffusion and jump processes (Akgiray and Booth [2], Andersen, Benzoni, and Lund [3], Bakshi, Cao, and Chen [6], Bates [8] [9], Jorion [28], *etc.*). Considering such studies, researchers have studied security market models (Ahn and Thompson [1], Back [5], Bakshi, Cao, and Chen [6], Bates [7] [8], Björk, Kabanov, and Runggaldier [10], Duffie, Pan, and Singleton [20], Naik and Lee [37], *etc.*) with jumps, in particular, in the context of derivative asset pricing. In most of their models, jump magnitude is specified as a random variable with a continuous distribution at each jump time. Then, the *dimensionality of martingale generator*,[1] which can be interpreted as the "number of sources of uncertainty" in markets is infinite, and no finite set of continuously traded securities can complete markets. In security market economy with infinite dimensional martingale generator, no equilibrium analysis has been conducted thus far.[2] The purpose of this paper is to develop equilibrium analysis of security market economy with infinite dimensional martingale generator.

Björk, Kabanov, and Runggaldier [10] introduce *approximately complete security markets* with an infinite dimensional martingale generator consisting of a jump process given by the *marked point process* (see Appendix A.1) and a Wiener process. In approximately complete markets, a continuum of bonds are traded and any contingent claim can be approximately replicated with an arbitrary precision by a suitable portfolio of bonds. In this paper, we introduce the notion of *approximate security market equilibrium* in which an agent is allowed to choose any consumption plan that can be approximately supported with any prescribed precision by a budgetary admissible portfolio. We present sufficient conditions for the existence of approximate security market equilibria in approximately complete markets in the case of stochastic differential utilities (SDUs) with Inada condition, and for the existence, uniqueness, and determinacy of approximate security market equilibria in the case of additively separable utilities (ASUs). In a companion paper (Kusuda [34]), using the framework of this paper, we have derived the Consumption-Based Capital Asset Pricing Model (CCAPM) under jump-Wiener and non-Markovian information in each case of heterogeneous agents with ASUs and of homogeneous agents with a common SDU.[3] We have also derived explicit formulas for market prices of risks, and presented an economic framework of jump-diffusion option pricing

[1] For example, if the *filtration*, which can be interpreted as the "information," in markets is generated by d-dimensional Wiener process, then a martingale generator is the Wiener process and its dimensionality is d.

[2] In security market economy in which the filtration is generated by finite dimensional Wiener process, Duffie [16], Duffie and Zame [22], and Huang [25] show sufficient conditions for the existence of equilibria, and Karatzas, Lakner, Lehoczky, and Shreve [29], and Karatzas, Lehoczky, and Shreve [30] present sufficient conditions for the existence and uniqueness of equilibria. Dana and Pontier [15], and Duffie [16] show sufficient conditions for the existence of equilibria in security markets in which the filtration is not restricted to the one generated by finite dimensional Wiener process. However, in the security markets, martingale generator is assumed to be still finite dimensional.

[3] The CCAPM says that the risk premium between a risky security and the nominal-risk-free security can be decomposed into two groups of terms. One is related to the price fluctuation of the risky security, and the other is related to that of the commodity. Each group can be decomposed into two terms related to consumption volatility and consumption jump in the case of ASU, and

models. Moreover, in subsequent two papers (Kusuda [32] [33]), we have proposed jump-diffusion LIBOR rate models using the framework of Kusuda [34].

We first prove that an approximate security market equilibrium in approximately complete markets can be identified with an Arrow-Debreu equilibrium. To implement Arrow-Debreu equilibria in security markets, Dana and Pontier [15], Duffie [16], Duffie and Zame [22], and Huang [25] assume that every nominal bond price is one at every date, or equivalently the nominal-risk-free rate is always zero. We do not specify nominal bond prices like this. We introduce a class of nominal bond prices at which markets are arbitrage-free and approximately complete. We prove that for every nominal bond prices in the class, an approximate security market equilibrium can be identified with an Arrow-Debreu equilibrium.

Next, we present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of SDUs (Stochastic Differential Utilities). Duffie and Epstein [18] introduce the notion of SDU, and show that any SDU can be *normalized* under Wiener information. Duffie, Geoffard, and Skiadas [19] present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of normalized SDUs. It is not true that any SDU can be normalized under jump-Wiener information. We present a necessary and sufficient condition for an SDU to be normalized under jump-Wiener information. Then, we can apply the results of Duffie, Geoffard, and Skiadas [19] on the existence of Arrow-Debreu equilibria, to our class of SDUs satisfying the necessary and sufficient condition. Our class of SDUs is a subclass of SDUs, but still includes the standard ASU (Additively Separable Utility), the Uzawa utility (Uzawa [40]), the Kreps-Porteus utility (Kreps and Porteus [31]), *etc.* However, sufficient conditions for the existence of Arrow-Debreu equilibria given by Duffie, Geoffard, and Skiadas [19] include Inada condition, so we need to present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of general ASUs.

Finally, we present sufficient conditions for the existence, uniqueness, and determinacy of Arrow-Debreu equilibria in the case of ASUs. We extend results of Dana [13] [14] from static economy to our continuous-time economy. Her proof uses the Negishi approach (Negishi [38]) and consists of the following steps. (1) An Arrow-Debreu equilibrium can be identified with a ***representative agent equilibrium.*** (2) There exists a representative agent equilibrium under a regularity condition. In addition, (3) if every agent's risk tolerance coefficient satisfies an integrability condition and every agent's endowment process is bounded away from zero, then the set of equilibria is generically finite, and (4) if every agent's relative risk aversion coefficient is less than or equal to one, then the representative agent equilibrium is unique.

Our proof of the existence of Arrow-Debreu equilibria in the case of ASUs is similar to the one given in Dana and Pontier [15], but their condition that the aggregate endowment process is bounded away from zero is stronger than ours. Also, in order to show the existence of Arrow-Debreu equilibria, we could assume strictly monotonic, continuous, convex, and uniformly proper preferences instead of assuming SDUs with Inada condition or ASUs. However, uniform properness

into three terms related to consumption volatility, utility volatility, and jumps of consumption and utility in the case of SDU.

of preferences is a restrictive condition in the sense that it does not allow Inada condition on utilities.[4]

Björk, Masi, Kabanov, and Runggaldier [11] and Jarrow and Madan [27] consider approximately complete security markets[5] with a slightly more general information than ours. The equivalence of ASM and Arrow-Debreu equilibria could be extended to economies with their security markets, but it would not be necessary to do so from viewpoints of applications such as deriving the CCAPM and constructing option pricing models.

This paper is organized as follows. In Section 2, we provide a specification of security market economy with infinite dimensional martingale generator. In Section 3, we review arbitrage-free approximately complete security markets following Björk *et al.* [10] [11]. In Section 4, we introduce the notion of approximate security market equilibrium, and prove that an approximate security market equilibrium can be identified with an Arrow-Debreu equilibrium. In Section 5, we introduce a class of SDUs, and present that the results of Duffie, Geoffard, and Skiadas [19] on the existence of Arrow-Debreu equilibria can be applied to the class of SDUs. In Section 6, we show sufficient conditions for the existence, uniqueness, and determinacy of Arrow-Debreu equilibria in the case of ASUs.

2. Security Market Economy with Jump-Diffusion Uncertainty

In this section, we provide a specification of security market economy with jump-diffusion uncertainty.

We consider a continuous-time frictionless security market economy with time span $\mathbf{T} \stackrel{\text{def}}{=} [0, T^\dagger]$ for a fixed horizon time $T^\dagger > 0$. Agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, \mathbb{P})$ where $\mathbb{F}^{W,\nu} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is the natural filtration generated by a d_1-dimensional Wiener process W and a marked point process $\nu(dt \times dz)$ on a Lusin space $(\mathbb{Z}, \mathcal{Z})$ (in usual applications, $\mathbb{Z} = \mathbb{R}^{d_2}$, or $\mathbb{N}^{d_2}$, or a finite set) with the $\mathbb{P}$-intensity kernel $\lambda_t(dz)$ (for marked point process, see Appendix A.1). Note that Martingale Representation Theorem (see Chapter III Corollary 4.31 in Jacod and Shiryaev [26]) shows that martingale generators in this economy is $(W, (\nu(dt \times \{z\}) - \lambda_t(\{z\}))_{z\in\mathbf{Z}})$. Thus, if the mark set $\mathbb{Z}$ is infinite, then the dimensionality of martingale generator is infinite.

There is a single perishable consumption commodity. The commodity space is a Banach space $\mathbf{L}^2 \stackrel{\text{def}}{=} \mathbf{L}^2(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$ where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbf{T}$, μ is the product measure of $\mathbb{P}$ and Lebesgue measure on $\mathbf{T}$.

There are I agents, and each of them is represented by $(U^i, \bar{c}^i)$, where U^i is a strictly increasing and continuous utility on the positive cone $\mathbf{L}^2_+$ of consumption process and $\bar{c}^i \in \mathbf{L}^2_+$ is an endowment process which is assumed to be nonzero, for $i \in \mathbf{I} \stackrel{\text{def}}{=} \{1, 2, \cdots, I\}$.

The economy is described by a collection

$$\mathbf{E} \stackrel{\text{def}}{=} ((\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, \mathbb{P}), (U^i, \bar{c}^i)_{i\in\mathbf{I}}).$$

[4]Araujo and Monteiro [4] and Duffie and Zame [22] show the existence of Arrow-Debreu equilibria in a static economy with non-uniformly proper preferences. However, they assume ASUs to prove the existence of Arrow-Debreu equilibria in a dynamic economy.

[5]To be exact, Jarrow and Madan [27] introduce the notion of ***quasicomplete markets***, but it is similar to the notion of approximately complete markets.

There are markets for the consumption commodity and securities at every date $t \in \mathbf{T}$. The traded securities are nominal-risk-free security (NOT the risk-free security) called the *money market account* and a continuum of zero-coupon bonds whose maturity dates are $(0, T^\dagger]$, each of which has \$1 payoff (NOT one unit payoff of the commodity) at its maturity date. Let p, B, and $(B^T)_{T\in(0,T^\dagger]}$ denote the consumption commodity price process, nominal money market account price process and nominal bond price processes, respectively. We write $\mathbf{B} = (B, (B^T)_{T\in(0,T^\dagger]})$ and call it *bond price family*.

3. Approximately Complete Markets

In this section, we briefly review approximately complete markets given in Björk *et al.* [10] [11] and introduce the notion of implementable approximately complete bond price family.

Let $n \in \mathbb{N}$. Let $\mathcal{L}^n$ denote the set of real-valued $\mathcal{P}$-measurable process X satisfying the integrability condition $\int_0^{T^\dagger} |X_s|^n \, ds < \infty$ $\mathbb{P}$-almost surely. Also let $\mathcal{L}^n(\lambda_t(dz) \times dt)$ denote the set of real-valued $\mathcal{P} \otimes \mathcal{Z}$-measurable process H satisfying the integrability condition $\int_0^{T^\dagger}\int_{\mathbf{Z}} |H_s(z)|^n \, \lambda_s(dz)\, ds < \infty$ $\mathbb{P}$-a.s.

We say that a bond price family $\mathbf{B}$ is *regular* if and only if the following conditions hold:

1. For every $T \in (0, T^\dagger]$, the dynamics of nominal bond price process B^T satisfies the following stochastic differential equation (SDE)

$$\frac{dB_t^T}{B_{t-}^T} = r_t^T dt + v_t^T \cdot dW_t + \int_{\mathbf{Z}} H_t^T(z) \{ \nu(dt \times dz) - \lambda_t(dz)\, dt \} \qquad \forall t \in [0, T)$$

   with $B_T^T = 1$ and $B_t^T = 0$ for every $t \in (T, T^\dagger]$ for some $r^T \in \mathcal{L}^1$, $v^T \in \prod_{j=1}^{d_1} \mathcal{L}^2$, and $H^T \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Moreover, it follows that:
   (a) For every $(\omega, t) \in \Omega \times \mathbf{T}$, $r_t^{\cdot}(\omega), v_t^{\cdot}(\omega) \in \mathbf{C}^1((t, T^\dagger])$ and for every $(\omega, t, z) \in \Omega \times \mathbf{T} \times \mathbb{Z}$, $H_t^{\cdot}(\omega, z) \in \mathbf{C}^1((t, T^\dagger])$.
   (b) For every $T \in (0, T^\dagger]$, $H_t^T(\omega, z)$ is bounded.
   (c) The processes $(B^T)_{T\in\mathbf{T}}$ are regular enough to allow for the differentiation under the integral sign and the interchange of integration order.[6]
2. The dynamics of nominal money market account price process B satisfies the following SDE

$$\frac{dB_t}{B_t} = r_t^B \, dt \qquad \forall t \in [0, T^\dagger)$$

   with $B_0 = 1$ where r_t^B is given by $r_t^B = -\left.\dfrac{\partial \ln B_t^T}{\partial T}\right|_{T=t}$.

When the number of traded securities is finite, we set the portfolio of securities a predictable process of corresponding dimension. In our economy, the continuum of bonds are traded and each agent is allowed to hold a portfolio of the continuum of bonds, so we set the portfolio component of continuum of bonds a signed finite Borel measure on $[t, T^\dagger]$ for every event $\omega \in \Omega$ and time $t \in \mathbf{T}$.

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[6] For the marked point process integrals, we can apply the ordinary Fubini Theorem, and for the interchange of integration with respect to dW and dt, we can apply the Stochastic Fubini Theorem (see Protter [39]).

Definition 1. A *portfolio* is a stochastic process $\vartheta = (\vartheta^0, \vartheta^1(\cdot))$ that satisfies:

1. The component ϑ^0 is a real-valued $\mathcal{P}$-measurable process.
2. The component ϑ^1 is such that:
   (a) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\vartheta^1_t(\omega, \cdot)$ is a signed finite Borel measure on $[t, T^\dagger]$.
   (b) For every Borel set A, the process $\vartheta^1(A)$ is $\mathcal{P}$-measurable.

Let the bond price family $\mathbf{B}$ be regular. We say that a portfolio ϑ is *feasible at* $\mathbf{B}$ if and only if the following integrability conditions are satisfied:

$$\int_t^{T^\dagger} |B_t^T|\,|\vartheta^1_t(dT)| < \infty \qquad \mathbb{P}\text{-a.s.} \qquad \forall t \in \mathbf{T},$$

$$B_t r_t^B \vartheta_t^0, \int_t^{T^\dagger} |B_t^T r_t^T|\,|\vartheta^1_t(dT)| \in \mathcal{L}^1, \qquad \int_t^{T^\dagger} |B_t^T v_t^T|\vartheta^1_t(dT)| \in \mathcal{L}^2,$$

$$\int_t^{T^\dagger} |B_t^T H_t^T(z)|\,|\vartheta^1_t(dT)| \in \mathcal{L}^1(\lambda_t(dz) \times dt).$$

Let $\Theta(\mathbf{B})$ denote the set of all feasible portfolios at $\mathbf{B}$. The *value process* $V^{\mathbf{B}}(\vartheta)$ *of a feasible portfolio* $\vartheta \in \Theta(\mathbf{B})$ *at* $\mathbf{B}$ is given by

$$V_t^{\mathbf{B}}(\vartheta) = B_t\,\vartheta_t^0 + \int_t^{T^\dagger} B_t^T\,\vartheta_t^1(dT) \qquad\qquad \forall t \in \mathbf{T}.$$

We say that a feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is *self-financing at* $\mathbf{B}$ if and only if

$$V_t^{\mathbf{B}}(\vartheta) = V_0^{\mathbf{B}}(\vartheta) + \int_0^t \vartheta_s^0\,dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_s^1(dT)\,dB_s^T \qquad\qquad \forall t \in \mathbf{T}.$$

Also, we say that a self-financing portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is an *arbitrage portfolio at* $\mathbf{B}$ if and only if either of the following condition holds:

1. $V_0^{\mathbf{B}}(\vartheta) \le 0$, and $V_{T^\dagger}^{\mathbf{B}}(\vartheta) > 0$, *i.e.* $V_{T^\dagger}^{\mathbf{B}}(\vartheta) \ge 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(\{V_{T^\dagger}^{\mathbf{B}}(\vartheta) > 0\}) > 0$.
2. $V_0^{\mathbf{B}}(\vartheta) < 0$, and $V_{T^\dagger}^{\mathbf{B}}(\vartheta) \ge 0$ $\mathbb{P}$-a.s.

For a real-valued $\mathcal{P}$-measurable process X, the *discounted process of* X *at* $\mathbf{B}$ is denoted by $\tilde{X}$. Thus, $\tilde{X} = \frac{X}{B}$. We write $\tilde{\mathbf{B}} = (\tilde{B}, (\tilde{B}^T)_{T \in \mathbf{T}})$. To eliminate unrealistic portfolios such as those based on *doubling strategy* (see Chapter 6 in Duffie [17]), we restrict the space of portfolios to the space of credit-constrained portfolios proposed in Dybvig and Huang [23].

Definition 2. Let $\mathbf{B}$ be regular. A feasible portfolio $\vartheta \in \Theta(\mathbf{B})$ at $\mathbf{B}$ is *admissible at* $\mathbf{B}$ if and only if the discounted value process $\tilde{V}^{\mathbf{B}}(\vartheta)$ is bounded below $\mathbb{P}$-a.s.

Let $\underline{\Theta}(\tilde{\mathbf{B}})$ denote the set of all admissible portfolios at $\mathbf{B}$. We give definitions of *arbitrage-free markets* and the *spot martingale measure.*

Definition 3. Let $\mathbf{B}$ be regular.

1. Markets are *arbitrage-free at* $\mathbf{B}$ if and only if there exists no admissible arbitrage portfolio at $\mathbf{B}$.
2. A probability measure $\tilde{\mathbb{P}}^{\mathbf{B}}$ on $(\Omega, \mathcal{F})$ is a *spot martingale measure at* $\mathbf{B}$ if and only if $\tilde{\mathbb{P}}^{\mathbf{B}}$ is equivalent to $\mathbb{P}$ and the discounted bond price family $\tilde{\mathbf{B}}$ is a local martingale under $\tilde{\mathbb{P}}^{\mathbf{B}}$.

One can see that the existence of spot martingale measures implies that markets are arbitrage-free.

Lemma 1. *Let* $\mathbf{B}$ *be regular. If there exists a spot martingale measure at* $\mathbf{B}$*, then markets are arbitrage-free at* $\mathbf{B}$.

Proof. See the proofs of Theorem 6.F and Corollary 6.F in Duffie [17]. □

Suppose that the bond price family $\mathbf{B}$ is regular. Then, the following lemma shows a necessary and sufficient condition on $\mathbf{B}$ for the existence of spot martingale measures.

Lemma 2. *Let* $\mathbf{B}$ *be regular. Then it follows that:*

1. *There exists a spot martingale measure* $\tilde{\mathbb{P}}^{\mathbf{B}}$ *at* $\mathbf{B}$ *if and only if there exists a martingale process* $\Lambda^{\mathbf{B}}$ *such that*

$$\frac{d\Lambda_t^{\mathbf{B}}}{\Lambda_{t-}^{\mathbf{B}}} = -v_t^{\mathbf{B}} \cdot dW_t - \int_{\mathbf{Z}} H_t^{\mathbf{B}}(z) \{ \nu(dt \times dz) - \lambda_t(dz)\, dt \} \qquad \forall t \in [0, T^\dagger)$$

with $\Lambda_0^{\mathbf{B}} = 1$ *where* $(v^{\mathbf{B}}, H^{\mathbf{B}}) \in \left(\prod_{j=1}^{d_1} \mathcal{L}^2\right) \times \mathcal{L}^1(\lambda_t(dz) \times dt)$ *satisfies the following equation*

$$r_t^T = r_t^B + v_t^{\mathbf{B}} \cdot v_t^T + \int_{\mathbf{Z}} H_t^{\mathbf{B}}(z) H_t^T(z)\, \lambda_t(dz) \qquad \forall t \in [0, T^\dagger).$$

2. *If there exists a martingale process* $\Lambda^{\mathbf{B}}$ *satisfying the above conditions, then it follows that:*
   (a) *The probability measure* $\tilde{\mathbb{P}}^{\mathbf{B}}$ *given by the Radon-Nikodym derivative*

$$d\tilde{\mathbb{P}}^{\mathbf{B}} = \Lambda_{T^\dagger}^{\mathbf{B}}\, d\mathbb{P}$$

   is a spot martingale measure at $\mathbf{B}$.
   (b) *The process* $\tilde{W}^{\mathbf{B}}$ *given by*

$$\tilde{W}_t^{\mathbf{B}} = W_t + \int_0^t v_s^{\mathbf{B}}\, ds \qquad \forall t \in \mathbf{T}$$

   is a $\tilde{\mathbb{P}}^{\mathbf{B}}$*-Wiener process.*
   (c) *The marked point process* $\nu(dt \times dz)$ *has the* $\tilde{\mathbb{P}}^{\mathbf{B}}$*-intensity kernel* $\tilde{\lambda}_t^{\mathbf{B}}(dz)$ *such that*

$$\tilde{\lambda}_t^{\mathbf{B}}(dz) = (1 - H_t^{\mathbf{B}}(z))\, \lambda_t(dz) \qquad \forall (t, z) \in \mathbf{T} \times \mathbb{Z}. \tag{3.1}$$

Proof. The result immediately follows from Ito's formula (see Appendix B.1) and Girsanov's Theorem (see Appendix B.2). □

Remark 1. We call $v^{\mathbf{B}}$ and $H^{\mathbf{B}}$ *market price of nominal volatility risk* and *market price of nominal jump risk*, respectively. In incomplete markets, it is difficult to obtain a tractable market price of nominal risks, especially market price of nominal jump risk. The formula (3.1) shows that if the market price of nominal jump risk $H^{\mathbf{B}}$ is intractable then the CCAPM becomes intractable, and it becomes hard to construct a tractable intensity under the spot martingale measure $\tilde{\mathbb{P}}^{\mathbf{B}}$ at $\mathbf{B}$, which makes it difficult to price derivative assets. As shown below, we can construct approximately complete markets in our security markets. In a companion paper (Kusuda [34]), using the framework of this paper, we have derived the tractable CCAPM in each case of heterogeneous agents with ASUs and of homogeneous agents with a common SDU. We have also derived explicit formulas for market prices of risks, and presented an economic framework of jump-diffusion option pricing models.

Let $\mathcal{B}$ denote the set of all regular bond price families satisfying conditions in Lemma 2. A process $\Lambda^{\mathbf{B}}$ is called the *density process of* $\tilde{\mathbb{P}}^{\mathbf{B}}$ *relative to* $\mathbb{P}$. Note that the density process $\Lambda^{\mathbf{B}}$ of $\tilde{\mathbb{P}}^{\mathbf{B}}$ relative to $\mathbb{P}$ is not necessarily unique for every $\mathbf{B} \in \mathcal{B}$, which implies that spot martingale measures at $\mathbf{B}$ are not necessarily unique. In order to make markets complete, we need to impose a condition under which $\Lambda^{\mathbf{B}}$ is unique, since the market completeness implies the uniqueness of spot martingale measures. The converse, *i.e.* the proposition that the uniqueness of spot martingale measures implies the market completeness, is also true for security markets with Wiener information. Unfortunately, this is not true for our bond markets with jump-Wiener information, that is, even if spot martingale measures are unique then markets are not necessarily complete. However, Björk, Kabanov, and Runggaldier [10] show that the uniqueness of spot martingale measures are equivalent to the *market approximate completeness*, which is defined in the following.

Definition 4. Let $\mathbf{B} \in \mathcal{B}$.

1. For every $T \in (0, T^{\dagger}]$, a *contingent* T*-claim* is a $\mathcal{F}_T$-measurable random variable X_T such that $\tilde{X}_T \stackrel{\text{def}}{=} \frac{X_T}{B_t} \in \mathbf{L}^{\infty}_{+}(\Omega, \mathcal{F}_T)$ where $\mathbf{L}^{\infty}(\Omega, \mathcal{F}_T)$ is the space of almost surely bounded $\mathcal{F}_T$-measurable random variables.
2. A contingent T-claim X_T is *replicable at* $\mathbf{B}$ if and only if there exists an admissible self-financing portfolio $\vartheta \in \underline{\Theta}(\tilde{\mathbf{B}})$ such that the value process satisfies $V^{\mathbf{B}}_T(\vartheta) = X_T$.
3. Markets are *complete at* $\mathbf{B}$ if and only if every T-contingent claim X_T is replicable for every $T \in (0, T^{\dagger}]$.
4. Markets are *approximately complete at* $\mathbf{B}$ if and only if for any $T \in (0, T^{\dagger}]$ and any T-contingent claim X_T, there exists a sequence of replicable claims $(X_{Tn})_{n \in \mathbb{N}}$ converging to X_T in $\mathbf{L}^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}^{\mathbf{B}})$ where $\tilde{\mathbb{P}}^{\mathbf{B}}$ is a spot martingale measure at $\mathbf{B}$.

Let $\mathbf{B} \in \mathcal{B}$. Björk, Masi, Kabanov, and Runggaldier [11] prove that if the mark set $\mathbb{Z}$ is infinite, then spot martingale measures are unique at $\mathbf{B}$ if and only if markets are approximately complete at $\mathbf{B}$, and that if $\mathbb{Z}$ is finite, then spot martingale measures at $\mathbf{B}$ are unique if and only if markets are complete at $\mathbf{B}$.

Proposition 1. *Let* $\mathbf{B} \in \mathcal{B}$.

1. *If the mark set* $\mathbb{Z}$ *is infinite, then each of the following conditions*[7] *is necessary and sufficient for* $\mathbf{B}$ *to have a unique spot martingale measure.*
   (a) *Markets are approximately complete at* $\mathbf{B}$.
   (b) *For every* $(\omega, t) \in \Omega \times \mathbf{T}$, *the equation*

$$\tilde{O}^*_t(\omega)\, \vartheta^1_t(\omega) = \begin{pmatrix} v_t(\omega) \\ H_t(\omega, \cdot) \end{pmatrix} \tag{3.2}$$

can be solved on a dense proper subset of $\mathbb{R}^{d_1} \times \mathbf{L}^2(\mathbb{Z}, \mathcal{Z}, \tilde{\lambda}^{\mathbf{B}}_t(\omega, dz))$ *where the operator* $\tilde{O}^*_t(\omega) : \mathbf{C}^*_{\mathbf{T}} \to \mathbb{R}^{d_1} \times \mathbf{L}^2(\mathbb{Z}, \mathcal{Z}, \tilde{\lambda}^{\mathbf{B}}_t(\omega, dz))$ *is defined by*

$$\tilde{O}^*_t(\omega) : \vartheta^1_t(\omega) \mapsto \begin{pmatrix} \int_t^{T^{\dagger}} \tilde{B}^T_{t-}(\omega)\, v^T_t(\omega)\, \vartheta^1_t(\omega, dT) \\ \int_t^{T^{\dagger}} \tilde{B}^T_{t-}(\omega)\, H^T_t(\omega, \cdot)\, \vartheta^1_t(\omega, dT) \end{pmatrix}.$$

[7] The conditions 1.(b) and 2.(b) in Proposition 1 are utilized to implement an Arrow-Debreu equilibrium.

2. *If the mark set $\mathbb{Z}$ is finite, then each of the following conditions is necessary and sufficient for* $\mathbf{B}$ *to have a unique spot martingale measure.*
   (a) *Markets are complete at* $\mathbf{B}$.
   (b) *For every* $(\omega, t) \in \Omega \times \mathbf{T}$, *the equation (3.2) can be solved on* $\mathbb{R}^{d_1} \times \mathbf{L}^2(\mathbb{Z}, \mathcal{Z}, \tilde{\lambda}_t^{\mathbf{B}}(\omega, dz))$.

Proof. See the proof of Proposition 6.10 in Björk, Masi, Kabanov, and Runggaldier [11]. □

We introduce the notion of *implementable bond price family.*

Definition 5. A bond price family $\mathbf{B} \in \mathcal{B}$ is an *implementable bond price family* if and only if the following two conditions hold:

1. Spot martingale measures at $\mathbf{B}$ are unique.
2. The discounted density process $\tilde{\Lambda}^{\mathbf{B}}$ of $\tilde{\mathbb{P}}^{\mathbf{B}}$ relative to $\mathbb{P}$ is bounded above and bounded away from zero μ-a.e.

Let $\bar{\mathcal{B}}$ denote the set of all implementable bond price families.

4. Approximate Security Market Equilibrium and Implementation of Arrow-Debreu Equilibria

In this section, we first introduce the notion of approximate security market equilibrium, and then we show a general implementation method of Arrow-Debreu equilibria. Finally, we prove that for every implementable bond price family, an approximate security market equilibrium can be identified with an Arrow-Debreu equilibrium.

4.1. Approximate Security Market Equilibrium.

Before introducing the approximate security market equilibrium, we consider the *security market equilibrium* defined as follows.

Definition 6. A collection $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B}) \in \prod_{i\in\mathbf{I}} \mathbf{L}^2_+ \times \mathbf{L}^2 \times \mathcal{B}$ constitutes a *security market equilibrium for* $\mathbf{E}$ if and only if:

1. For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \mathcal{C}^i(p,\mathbf{B})} U^i(c^i)$$

   where

$$\mathcal{C}^i(p,\mathbf{B}) = \Big\{ c^i \in \mathbf{L}^2_+ \, : \, \exists \vartheta^i \in \underline{\Theta}(\tilde{\mathbf{B}}) \quad \text{s.t.}$$
$$V_t^{\mathbf{B}}(\vartheta^i) = \int_0^t \vartheta_s^{i0}\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_s^{i1}(dT)\, dB_s^T + \int_0^t p_s(\bar{c}_s^i - c_s^i)\, ds \qquad \forall t \in \mathbf{T},$$
$$V_{T^\dagger}^{\mathbf{B}}(\vartheta^i) = 0 \Big\}.$$

2. The commodity market is cleared as $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} \bar{c}^i$.

Note that if $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ constitutes a security market equilibrium for $\mathbf{E}$, then the security market clearing condition is satisfied in the sense that there exists a

$(\hat{\vartheta}^i)_{i\in\mathbf{I}} \in \prod_{i\in\mathbf{I}} \underline{\Theta}(\tilde{\mathbf{B}})$ with $\sum_{i\in\mathbf{I}} \hat{\vartheta}^i = 0$ such that $\hat{\vartheta}^i$ supports $\hat{c}^i$, *i.e.*,

$$V_t^{\mathbf{B}}(\hat{\vartheta}^i) = \int_0^t \hat{\vartheta}_s^{i0}\, dB_s + \int_0^t \int_s^{T^\dagger} \hat{\vartheta}_s^{i1}(dT)\, dB_s^T + \int_0^t p_s(\bar{c}_s^i - \hat{c}_s^i)\, ds \qquad \forall t \in \mathbf{T},$$
$$V_{T^\dagger}^{\mathbf{B}}(\hat{\vartheta}^i) = 0$$

for every $i \in \mathbf{I}$. This immediately follows from the commodity market clearing condition and linearity of value process. Hence, we have removed the security market clearing condition out of the definition of security market equilibrium.

Suppose that the mark set $\mathbb{Z}$ is infinite. Then, markets may not be complete but approximately complete at any $\mathbf{B} \in \bar{\mathcal{B}}$ as shown in Proposition 1. In this case, for some price system $(p, \mathbf{B}) \in \mathbf{L}_+^2 \times \bar{\mathcal{B}}$, an agent's maximization problem may not be well defined since the consumption plan that should be a maximizer $\hat{c}^i \in \mathbf{L}_+^2$ may not be exactly supported by any portfolio in the budget constraint set $\mathcal{C}^i(p, \mathbf{B})$. However, as shown later, such a consumption plan can always be approximately supported with any prescribed precision by an budgetary admissible portfolio. Hence, we allow an agent to choose any consumption plan that can be approximately supported with any prescribed precision by budgetary admissible portfolio, and introduce the notion of ***approximate security market equilibrium.***

Definition 7. A collection $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B}) \in \prod_{i\in\mathbf{I}} \mathbf{L}_+^2 \times \mathbf{L}^2 \times \mathcal{B}$ constitutes an *approximate security market equilibrium for* $\mathbf{E}$ if and only if:

1. For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})} U^i(c^i)$$

where

$$\bar{\mathcal{C}}^i(p, \mathbf{B}) = \Big\{ c^i \in \mathbf{L}_+^2 \,:\, \exists (\vartheta_n^i)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \underline{\Theta}(\tilde{\mathbf{B}}) \quad \text{s.t.}$$
$$V_t^{\mathbf{B}}(\vartheta_n^i) = \int_0^t \vartheta_{ns}^{i0}\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta_{ns}^{i1}(dT)\, dB_s^T + \int_0^t p_s(\bar{c}_s^i - c_s^i)\, ds \qquad \forall (n,t) \in \mathbb{N} \times \mathbf{T},$$
$$\lim_{n\to\infty} V_{T^\dagger}^{\mathbf{B}}(\vartheta_n^i) = 0 \Big\}.$$

2. The commodity market is cleared as $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} \bar{c}^i$.

We refer to approximate security market equilibrium as ASM equilibrium, hereafter.

4.2. **Implementation of Arrow-Debreu Equilibria.** Let $\mathcal{L}(\mathbf{L}_+^2)$ denote the space of bounded linear functions $\Pi : \mathbf{L}_+^2 \to \mathbb{R}$. We say that a collection $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi) \in \prod_{i\in\mathbf{I}} \mathbf{L}_+^2 \times \mathcal{L}(\mathbf{L}_+^2)$ constitutes an *Arrow-Debreu equilibrium for* $\mathbf{E}$ if and only if:

1. For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \mathcal{C}^i(\Pi)} U^i(c^i)$$

where $\mathcal{C}^i(\Pi) = \{c^i \in \mathbf{L}_+^2 \,:\, \Pi(\bar{c}^i - c^i) = 0\}$.
2. The commodity market is cleared as $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} \bar{c}^i$.

We want to implement an Arrow-Debreu equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ into a unique ASM equilibrium, or a finite number of ASM equilibria, otherwise we would face an indeterminacy problem in which there exist an infinite number of ASM even if the Arrow-Debreu equilibrium is unique. To avoid such an indeterminacy problem, Dana and Pontier [15], Duffie [16], Duffie and Zame [22], and Huang [25] assume that every nominal security price process $S(D)$ with a nominal cumulative dividend process D given by

$$S_t(D) = \mathbb{E}[D_{T^\dagger} - D_t|\mathcal{F}_t] \qquad \forall t \in \mathbf{T}.$$

One can show that this assumption is equivalent to the one that every nominal bond price is one at every date or that the nominal-risk-free rate is always zero.

Instead of specifying nominal bond prices like this, we just pick an implementable bond price family $\mathbf{B} \in \bar{\mathcal{B}}$.[8] Then, we present that for $\mathbf{B}$, an approximate security market equilibrium can be identified with an Arrow-Debreu equilibrium.

4.3. **Equivalence of ASM and Arrow-Debreu Equilibria.** We prove that for every implementable bond price family $\mathbf{B} \in \bar{\mathcal{B}}$, an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$ can be identified with an Arrow-Debreu equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ for $\mathbf{E}$ under the relation $\tilde{\Lambda}^{\mathbf{B}} p = \pi$ where $\tilde{\Lambda}^{\mathbf{B}}$ is the discounted density process of $\tilde{\mathbb{P}}^{\mathbf{B}}$ relative to $\mathbb{P}$ and π is the Riesz kernel of Π. We also show that if the mark set $\mathbb{Z}$ is finite, then any ASM equilibrium is reduced to be an security market equilibrium. To see these, we prove the following proposition using Martingale Representation Theorem and Proposition 1.

Proposition 2. *Let $i \in \mathbf{I}$. It follows that:*

1. *Let $\Pi \in \mathcal{L}(\mathbf{L}^2_+)$ be given by a Riesz kernel $\pi \in \mathbf{L}^2_{++}$ and $c^i \in \mathcal{C}^i(\Pi)$. Let $\mathbf{B} \in \bar{\mathcal{B}}$ and $p = (\tilde{\Lambda}^{\mathbf{B}})^{-1}\pi$. Then, $c^i \in \bar{\mathcal{C}}^i(p, \mathbf{B})$. Moreover, if the mark set $\mathbb{Z}$ is finite then $c^i \in \mathcal{C}^i(p, \mathbf{B})$.*
2. *Conversely, let $c^i \in \bar{\mathcal{C}}^i(p, \mathbf{B})$ where $(p, \mathbf{B}) \in \mathbf{L}^2_{++} \times \mathcal{B}$. Define Π by the Riesz kernel $\pi = \tilde{\Lambda}^{\mathbf{B}} p$. If $\tilde{\Lambda}^{\mathbf{B}}$ is bounded μ-a.e., then $\hat{c}^i \in \mathcal{C}^i(\Pi)$.*

Proof. Let $i \in \mathbf{I}$. First, it follows from Bayes' rule and integration by part that

(4.1)
$$\begin{aligned}
\tilde{\mathbb{E}}^{\mathbf{B}}\left[\int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\,ds\right] &= \frac{1}{\Lambda^{\mathbf{B}}_0}\mathbb{E}\left[\Lambda^{\mathbf{B}}_{T^\dagger}\int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\,ds\right] \\
&= \mathbb{E}\left[\int_0^{T^\dagger} \Lambda^{\mathbf{B}}_s \tilde{p}_s(\bar{c}^i_s - c^i_s)\,ds + \int_0^{T^\dagger}\int_0^s \tilde{p}_{s_1}(\bar{c}^i_{s_1} - c^i_{s_1})\,ds_1\,d\Lambda^{\mathbf{B}}_s \right.\\
&\qquad\qquad \left. + \int_0^{T^\dagger} d\left[\Lambda^{\mathbf{B}}_s, \int_0^s \tilde{p}_{s_1}(\bar{c}^i_{s_1} - c^i_{s_1})\,ds_1\right]\right] \\
&= \mathbb{E}\left[\int_0^{T^\dagger} \pi_s(\bar{c}^i_s - c^i_s)\,ds\right].
\end{aligned}$$

[8]Such an exogenously given nominal bond price family can be interpreted as agents' common subjective probability for term structure process of nominal-interest-rates. Considering that a central bank controls term structures process of nominal-interest-rates in order to control the commodity price process, we might justify such an agents' common subjective probability for term structure process of nominal-interest-rates.

Define $\tilde{C}^i(p,\mathbf{B})$ for every $p \in \mathbf{L}^2_+$ by

$$\tilde{C}^i(p,\mathbf{B}) = \Big\{c^i \in \mathbf{L}^2_+ \mid \exists(\vartheta^i_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \underline{\Theta}(\tilde{\mathbf{B}}) \quad \text{s.t.}$$

$$\tilde{V}^{\mathbf{B}}_t(\vartheta^i_n) = \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s + \int_0^t \tilde{p}_s(\tilde{c}^i_s - c^i_s)\, ds \qquad \forall (n,t) \in \mathbb{N}\times\mathbf{T}$$

$$\lim_{n\to\infty} \tilde{V}^{\mathbf{B}}_{T^\dagger}(\vartheta^i_n) = 0\Big\}.$$

Step 1 - $\tilde{C}^i(p,\mathbf{B}) = \tilde{C}^i(p,\mathbf{B})$ where $p \in \mathbf{L}^2_+$: See Appendix C.1.

Step 2 - Proof of 2: See Appendix C.1.

Step 3 - Proof of 1: Let $\Pi \in \mathcal{L}(\mathbf{L}^2_+)$ be given by a Riesz Kernel $\pi \in \mathbf{L}^2_{++}$ and $c^i \in C^i(\Pi)$. Then, we have

$$\mathbb{E}\left[\int_0^{T^\dagger} \pi_s(\bar{c}^i_s - c^i_s)\, ds\right] = 0. \tag{4.2}$$

Let $\mathbf{B} \in \tilde{\mathcal{B}}$ and define $p = (\tilde{\Lambda}^{\mathbf{B}})^{-1}\pi$. Then, we have $p \in \mathbf{L}^2_{++}$, since $(\tilde{\Lambda}^{\mathbf{B}})^{-1}$ is bounded μ-a.e. It follows from (4.1) and (4.2) that

$$\tilde{\mathbb{E}}^{\mathbf{B}}\left[\int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds\right] = 0. \tag{4.3}$$

On the other hand, $\tilde{\mathbb{E}}^{\mathbf{B}}_t\left[\int_0^{T^\dagger} \tilde{p}_s(c^i_s - \bar{c}^i_s)\, ds\right]$ is a $\tilde{\mathbb{P}}^{\mathbf{B}}$-martingale, so it follows from Martingale Representation Theorem and (4.3) that there exists a unique $(v^i, H^i) \in \left(\prod_{j=1}^{d_1} \mathcal{L}^2\right) \times \mathcal{L}^1(\lambda_t(dz)\times dt)$ such that

$$\tilde{\mathbb{E}}^{\mathbf{B}}\left[\int_0^{T^\dagger} \|v^i_s\|^2\, ds\right] < \infty, \qquad \tilde{\mathbb{E}}^{\mathbf{B}}\left[\int_0^{T^\dagger}\int_{\mathbf{Z}} |H^i_s(z)|^2\, \lambda_t(dz)\, ds\right] < \infty,$$

and such that for every $t \in \mathbf{T}$

(4.4)

$$\begin{aligned}
&\tilde{\mathbb{E}}^{\mathbf{B}}_t\left[\int_0^{T^\dagger} \tilde{p}_s(c^i_s - \bar{c}^i_s)\, ds\right] \\
&= \tilde{\mathbb{E}}^{\mathbf{B}}\left[\int_0^{T^\dagger} \tilde{p}_s(c^i_s - \bar{c}^i_s)\, ds\right] + \int_0^t v^i_s \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^t \int_{\mathbf{Z}} H^i_s(z)\, \{\nu(ds\times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds\} \\
&= \int_0^t v^i_s \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^t \int_{\mathbf{Z}} H^i_s(z)\, \{\nu(ds\times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds\}.
\end{aligned}$$

Since $\mathbf{B} \in \tilde{\mathcal{B}}$, by Proposition 1 there exists a pair of sequences $(v^i_n, H^i_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}\left(\left(\prod_{j=1}^{d_1}\mathcal{L}^2\right)\times\mathcal{L}^1(\lambda_t(dz)\times dt)\right)$ such that $(v^i_{nt}(\omega), H^i_{nt}(\omega))$ converging to $(v^i_t(\omega), H^i_t(\omega))$ in $\mathbb{R}^{d_1} \times \mathbf{L}^2(\mathbb{Z}, \mathcal{Z}, \tilde{\lambda}^{\mathbf{B}}_t(\omega, dz))$ as $n\to\infty$ for every $(\omega,t) \in \Omega\times\mathbf{T}$ and such that for every $n \in \mathbb{N}$ there exists $\vartheta^{i1}_n \in \mathbf{C}^*_{\mathbf{T}}$ satisfying

$$\tilde{O}^*_t(\omega)\, \vartheta^{i1}_{nt}(\omega) = \begin{pmatrix} \int_t^{T^\dagger} \tilde{B}^T_{t-}(\omega)\, v^T_t(\omega)\, \vartheta^{i1}_t(\omega, dT) \\ \int_t^{T^\dagger} \tilde{B}^T_{t-}(\omega)\, H^T_t(\omega,\cdot\,)\, \vartheta^{i1}_t(\omega, dT) \end{pmatrix} = \begin{pmatrix} v^i_{nt}(\omega) \\ H^i_{nt}(\omega,\cdot\,) \end{pmatrix} \tag{4.5}$$

for every $(\omega, t) \in \Omega \times \mathbf{T}$. Thus, it follows from (4.4) that

(4.6)

$$
\begin{aligned}
&\int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s \\
&\quad = \int_0^t \int_s^{T^\dagger} \tilde{B}^T_s v^T_s \vartheta^{i1}_{ns}(dT) \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^t \int_s^{T^\dagger} \int_{\mathbf{Z}} \tilde{B}^T_s H^T_s(z)\, \vartheta^{i1}_{ns}(dT) \left\{ \nu(ds \times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds \right\} \\
&\quad = \int_0^t v^i_{ns} \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^t \int_{\mathbf{Z}} H^i_{ns}(z) \left\{ \nu(ds \times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds \right\} \\
&\qquad \to \qquad \int_0^t v^i_s \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^t \int_{\mathbf{Z}} H^i_s(z) \left\{ \nu(ds \times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds \right\}
\end{aligned}
$$

in $\mathbf{L}^2(\Omega, \mathcal{F}_t, \tilde{\mathbb{P}}^{\mathbf{B}})$ as $n \to \infty$ for every $t \in \mathbf{T}$. Define $(\vartheta^{i0}_n)_{n \in \mathbb{N}}$ by

$$
\vartheta^{i0}_{nt} = -\int_t^{T^\dagger} \tilde{B}^T_t \vartheta^{i1}_{nt}(dT) + \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s + \int_0^t \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds \qquad \forall (n,t) \in \mathbb{N} \times \mathbf{T}.
$$

Substituting this into $\tilde{V}^{\mathbf{B}}_t(\vartheta^i_n) = \vartheta^{i0}_{nt} + \int_t^{T^\dagger} \tilde{B}^T_t \vartheta^{i1}_{nt}(dT)$ yields

(4.7)

$$
\tilde{V}^{\mathbf{B}}_t(\vartheta^i_n) = \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s + \int_0^t \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds \qquad \forall (n,t) \in \mathbb{N} \times \mathbf{T}.
$$

Finally, it follows from (4.4), (4.6), and (4.7) that

(4.8)

$$
\begin{aligned}
&\lim_{n\to\infty} \tilde{V}^{\mathbf{B}}_{T^\dagger}(\vartheta^i_n) = \lim_{n\to\infty} \left\{ \int_0^{T^\dagger} \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s + \int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds \right\} \\
&\quad = \lim_{n\to\infty} \left\{ \int_0^{T^\dagger} v^i_{ns} \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^{T^\dagger} \int_{\mathbf{Z}} H^i_{ns}(z) \left\{ \nu(ds \times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds \right\} \right\} + \int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds \\
&\quad = \int_0^{T^\dagger} v^i_s \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^{T^\dagger} \int_{\mathbf{Z}} H^i_s(z) \left\{ \nu(ds \times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds \right\} + \int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds \\
&\quad = \tilde{\mathbb{E}}^{\mathbf{B}}_{T^\dagger} \left[\int_0^{T^\dagger} \tilde{p}_s(c^i_s - \bar{c}^i_s)\, ds \right] + \int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds = 0.
\end{aligned}
$$

Equations (4.7) and (4.8) show $c^i \in \tilde{\mathcal{C}}^i(p, \mathbf{B})$, and therefore $c^i \in \bar{\mathcal{C}}^i(p, \mathbf{B})$. Next let us consider the case when the mark set $\mathbb{Z}$ is finite. First, we can obtain (4.4) for some unique $(v^i, H^i) \in \left(\prod_{j=1}^{d_1} \mathcal{L}^2\right) \times \mathcal{L}^1(\lambda_t(dz) \times dt)$. Then, since $\mathbf{B} \in \bar{\mathcal{B}}$ and $\mathbb{Z}$ is finite, by Proposition 1 there exists a supporting portfolio $\vartheta^{i1} \in \mathbf{C}^*_{\mathbf{T}}$ such that

$$
\begin{aligned}
\int_0^t \int_s^{T^\dagger} \vartheta^{i1}_s(dT)\, d\tilde{B}^T_s &= \int_0^t \int_s^{T^\dagger} \tilde{B}^T_s v^T_s \vartheta^{i1}_s(dT) \cdot d\tilde{W}^{\mathbf{B}}_s \\
&\quad + \int_s^{T^\dagger} \int_{\mathbf{Z}} \tilde{B}^T_s H^T_s(z)\, \vartheta^{i1}_s(dT) \left\{ \nu(ds \times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds \right\} \\
&= \int_0^t v^i_s \cdot d\tilde{W}^{\mathbf{B}}_s + \int_0^t \int_{\mathbf{Z}} H^i_s(z) \left\{ \nu(ds \times dz) - \tilde{\lambda}^{\mathbf{B}}_s(dz)\, ds \right\}
\end{aligned}
$$

for every $t \in \mathbf{T}$. Define ϑ^{i0} by

$$\vartheta_t^{i0} = -\int_t^{T^\dagger} \tilde{B}_t^T \vartheta_t^{i1}(dT) + \int_0^t \int_s^{T^\dagger} \vartheta_s^{i1}(dT)\, d\tilde{B}_s^T + \int_0^t \tilde{p}_s(\bar{c}_s^i - c_s^i)\, ds \qquad \forall t \in \mathbf{T}.$$

Then, we can show $c^i \in \tilde{C}^i(p, \mathbf{B}) = \bar{C}^i(p, \mathbf{B})$ in the same way as before. □

Using Proposition 2, we can show the following theorem.

Theorem 1. *It follows that:*

1. *Let $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ be an Arrow-Debreu equilibrium for* $\mathbf{E}$. *Let $\mathbf{B} \in \bar{\mathcal{B}}$ and $p = (\tilde{\Lambda}^{\mathbf{B}})^{-1}\pi$ where $\pi \in \mathbf{L}^2_{++}$ is the Riesz kernel of Π. Then, $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is an ASM equilibrium for* $\mathbf{E}$. *Moreover, if the mark set $\mathbb{Z}$ is finite then $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is a security market equilibrium for* $\mathbf{E}$.
2. *Conversely, let $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ be an ASM equilibrium for* $\mathbf{E}$. *Define Π by the Riesz kernel $\pi = \tilde{\Lambda}^{\mathbf{B}}p$. If $\tilde{\Lambda}^{\mathbf{B}}$ is bounded μ-a.e., then $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ is an Arrow-Debreu equilibrium for* $\mathbf{E}$.

Proof. Proof of 1. Let $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ be an Arrow-Debreu equilibrium for $\mathbf{E}$. Then, since agents' utilities are strictly increasing, Π is strictly increasing. Thus, by Riesz Representation Theorem, there uniquely exists a Riesz kernel $\pi \in \mathbf{L}^2_{++}$. Let $\mathbf{B} \in \bar{\mathcal{B}}$ and define $p = (\tilde{\Lambda}^{\mathbf{B}})^{-1}\pi$. First, by definition of Arrow-Debreu equilibrium, $(\hat{c}^i)_{i\in\mathbf{I}}$ satisfies the commodity market clearing condition in ASM equilibrium. Let $i \in \mathbf{I}$. It follows from Proposition 2.1 that $\hat{c}^i \in \bar{C}^i(p, \mathbf{B})$. Suppose that $\hat{c}^i$ is not a utility maximizer in $\bar{C}^i(p, \mathbf{B})$, Then, Proposition 2.2 implies that $\hat{c}^i$ is not a utility maximizer in $C^i(\Pi)$, which contradicts that $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ is an Arrow-Debreu equilibrium for $\mathbf{E}$. Thus, $\hat{c}^i$ is not a utility maximizer in $\bar{C}^i(p, \mathbf{B})$, and hence $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is an ASM equilibrium for $\mathbf{E}$.

Proof of 2. Let $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ be an approximate security market equilibrium for $\mathbf{E}$. Then, since agents' utility functions are strictly increasing, $p \in \mathbf{L}^2_{++}$. Define Π by the Riesz kernel $\pi = \tilde{\Lambda}^{\mathbf{B}}p$. Let $i \in \mathbf{I}$. It suffices to show that $\hat{c}^i$ is a utility maximizer in $C^i(\Pi)$. First, it follows from Proposition 2.2 that $\hat{c}^i \in C^i(\Pi)$. Suppose that $\hat{c}^i$ is not a utility maximizer in $C^i(\Pi)$. Then, Proposition 2.1 implies that $\hat{c}^i$ is not a utility maximizer in $\bar{C}^i(p, \mathbf{B})$. This is a contradiction, and therefore $\hat{c}^i$ is a utility maximizer in $C^i(\Pi)$. □

We obtain the following corollary, which says that for every implementable bond price family $\mathbf{B} \in \bar{\mathcal{B}}$, an ASM equilibrium can be identified with an Arrow-Debreu equilibrium.

Corollary 1.1. *For every $\mathbf{B} \in \bar{\mathcal{B}}$, $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is an ASM equilibrium for* $\mathbf{E}$ *if and only if $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ is an Arrow-Debreu equilibrium for* $\mathbf{E}$ *where Π is given by the Riesz kernel $\pi = \tilde{\Lambda}^{\mathbf{B}}p$.*

5. Existence of Equilibria in Case of SDUs with Inada Condition

In this section, we present sufficient conditions for the existence of ASM equilibria in the case of SDUs (Stochastic Differential Utilities) with Inada condition. It is enough to show sufficient conditions for the existence of Arrow-Debreu equilibria since for every implementable bond price family, an ASM equilibrium can be identified with an Arrow-Debreu equilibrium as shown in Corollary 1.1. Duffie and Epstein [18] introduce the notion of SDU, and show that any SDU can be normalized under Wiener information. Duffie, Geoffard, and Skiadas [19] present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of normalized SDUs. It is not true that any SDU can be normalized under jump-Wiener information. We present a necessary and sufficient condition for an SDU to be normalized under jump-Wiener information. Then, we can apply the results of Duffie, Geoffard, and Skiadas [19] on the existence of Arrow-Debreu equilibria, to our class of SDUs satisfying the necessary and sufficient condition. Our class of SDUs is a subclass of SDUs, but still includes the standard ASU (Additively Separable Utility), the Uzawa utility (Uzawa [40]), the Kreps-Porteus utility (Kreps and Porteus [31]), *etc.* However, sufficient conditions for the existence of Arrow-Debreu equilibria given by Duffie, Geoffard, and Skiadas [19] include Inada condition, so we need to present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of general ASUs.

We consider security market economy in which agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is a filtration satisfying usual conditions.

The notion of an SDU was first shown in Epstein and Zin [24] in discrete-time setting, and then extended to continuous-time setting in Duffie and Epstein [18]. An SDU is a utility with expected recursive utility representation, and an extension of the standard ASU (Additively Separable Utility). It is well known that in the standard ASU, both of risk aversion and intertemporal substitution depend on the curvature of the von-Neumann Morgenstern utility function, for instance, relative risk aversion is reciprocal of elasticity of intertemporal substitution in the CRRA utility. These two properties of utility can be independently given in some SDUs such as the *Kreps-Porteus utility* (Kreps and Porteus [31]).

5.1. **Normalizable SDUs under Jump-Wiener Information.** We first review the notion of *SDU for Wiener information* given in Duffie and Epstein [18] (for definitions of *aggregator*, *certainty equivalent*, and its *local gradient representation*, see Duffie and Epstein [18]).

Definition 8. Let $\mathbb{F} = \mathbb{F}^W$. Then, a utility $\bar{U} : \mathbf{L}^2_+ \to \mathbb{R}$ is an *SDU for Wiener information* if and only if U is characterized by an aggregator $(\bar{f}, \bar{m})$ such that $\bar{U}(c) = \bar{Y}_0$ for every $c \in \mathbf{L}^2_+$ where $\bar{Y}$ is the unique square-integrable process satisfying the following SDE

$$d\bar{Y}_t = \mu^{\bar{Y}}_t\, dt + \sigma^{\bar{Y}}_t \cdot dW_t \qquad \forall t \in \mathbf{T}$$

with $\bar{Y}_{T^\dagger} = 0$ where $\mu^{\bar{Y}} \in \mathcal{L}^1$, $\sigma^{\bar{Y}} \in \prod_{j=1}^d \mathcal{L}^2$, and

$$\mu^{\bar{Y}}_t = -\bar{f}(c_s, \bar{Y}_s) - \frac{1}{2}\bar{M}_{11}(\bar{Y}_s, \bar{Y}_s)\,\|\sigma^{\bar{Y}}_s\|^2$$

where $\bar{M} : \mathbb{R}^2 \to \mathbb{R}$ is the local gradient representation (LGR) of certainty equivalent $\bar{m}$, and satisfies $\bar{M} \in \mathbf{C}^{2,0}$ and $\bar{M}_1(x, x) = 1$ for every $x \in \mathbb{R}$.

We introduce the notion of *SDU for jump-Wiener information*, which is a natural extention of the notion of SDU for Wiener information.

Definition 9. Let $\mathbb{F} = \mathbb{F}^{W,\nu}$. Then, a utility $\bar{U} : \mathbf{L}^2_+ \times \mathbb{R} \to \mathbb{R}$ is an *SDU for jump-Wiener information* if and only if $\bar{U}$ is characterized by an aggregator $(\bar{f}, \bar{m})$

such that $\bar{U}(c) = \bar{Y}_0$ for every $c \in \mathbf{L}^2_+$ where $\bar{Y}$ is the unique square-integrable process satisfying the following SDE

$$d\bar{Y}_t = \mu_t^{\bar{Y}}\,dt + \sigma_t^{\bar{Y}} \cdot dW_t + \bar{Y}_{t-} \int_{\mathbf{Z}} H_t^{\bar{Y}}(z)\,\{\,\nu(dt \times dz) - \lambda_t(dz)\,dt\,\} \qquad \forall t \in \mathbf{T}$$

with $\bar{Y}_{T^\dagger} = 0$ where $\mu^{\bar{Y}} \in \mathcal{L}^1$, $\sigma^{\bar{Y}} \in \prod_{j=1}^{d_1} \mathcal{L}^2$, $H^{\bar{Y}} \in \mathcal{L}^1(\lambda_t(dz) \times dt)$, and

$$\begin{aligned} (5.1) \quad \mu_t^{\bar{Y}} &= -\bar{f}(c_s, \bar{Y}_s) - \frac{1}{2}\bar{M}_{11}(\bar{Y}_s, \bar{Y}_s)\,\|\sigma_s^{\bar{Y}}\|^2 \\ &\quad - \int_{\mathbf{Z}} \Big\{ \bar{M}((1 + H_s^{\bar{Y}}(z))\bar{Y}_s, \bar{Y}_s) - \bar{M}(\bar{Y}_s, \bar{Y}_s) - \bar{Y}_s H_s^{\bar{Y}}(z) \Big\} \lambda_s(dz) \end{aligned}$$

where $\bar{M}$ is the LGR of $\bar{m}$, and satisfies $\bar{M} \in \mathbf{C}^{2,0}$ and $\bar{M}_1(x,x) = 1$ for every $x \in \mathbb{R}$.

Remark 2. Equation (5.1) is derived from definitions of the aggregator $(\bar{f}, \bar{m})$ and LGR $\bar{M}$ of $\bar{m}$:

$$\begin{aligned} \bar{f}(c_t, \bar{Y}_t) &= -\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathbb{E}_t \left[\bar{M}(\bar{Y}_t, \bar{Y}_{t-\Delta}) - \bar{M}(\bar{Y}_{t-\Delta}, \bar{Y}_{t-\Delta}) \right] \\ &= -\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathbb{E}_t \left[\int_{t-\Delta}^{t} \Big\{ \bar{M}_1(\bar{Y}_s, \bar{Y}_s)\,\mu_s^{\bar{Y}} + \frac{1}{2}\bar{M}_{11}(\bar{Y}_s, \bar{Y}_s)\,\|\sigma_s^{\bar{Y}}\|^2 \right. \\ &\quad \left. + \int_{\mathbf{Z}} \Big\{ \bar{M}((1 + H_s^{\bar{Y}}(z))\bar{Y}_s, \bar{Y}_s) - \bar{M}(\bar{Y}_s, \bar{Y}_s) - \bar{M}_1(\bar{Y}_s, \bar{Y}_s)\bar{Y}_s H_s^{\bar{Y}}(z) \Big\} \lambda_s(dz) \Big\}\, ds \right] \\ &= -\mu_t^{\bar{Y}} - \frac{1}{2}\bar{M}_{11}(\bar{Y}_t, \bar{Y}_t)\,\|\sigma_t^{\bar{Y}}\|^2 \\ &\quad - \int_{\mathbf{Z}} \Big\{ \bar{M}((1 + H_t^{\bar{Y}}(z))\bar{Y}_t, \bar{Y}_t) - \bar{M}(\bar{Y}_t, \bar{Y}_t) - \bar{Y}_t H_t^{\bar{Y}}(z) \Big\} \lambda_t(dz). \end{aligned}$$

Here we use the property $\bar{M}_1(\bar{Y}_s, \bar{Y}_s) = 1$.

Remark 3. We have the following expected recursive utility representation of $\bar{U}$:

(5.2)

$$\begin{aligned} \bar{Y}_t &= -\mathbb{E}_t \left[\int_t^{T^\dagger} \mu_s^{\bar{Y}}\, ds \right] \\ &= \mathbb{E}_t \left[\int_t^{T^\dagger} \Big\{ \bar{f}(c_s, \bar{Y}_s) + \frac{1}{2}\bar{M}_{11}(\bar{Y}_s, \bar{Y}_s)\,\|\sigma_s^{\bar{Y}}\|^2\, ds \right. \\ &\quad \left. + \int_{\mathbf{Z}} \Big\{ \bar{M}((1 + H_t^{\bar{Y}}(z))\bar{Y}_t, \bar{Y}_t) - \bar{M}(\bar{Y}_t, \bar{Y}_t) - \bar{Y}_t H_t^{\bar{Y}}(z) \Big\} \lambda_t(dz) \Big\} \right] \qquad \forall t \in \mathbf{T}. \end{aligned}$$

The expected recursive utility representation (5.2) of $\bar{U}$ is intractable. Exploiting the notion of *ordinally equivalent utility*,[9] Duffie and Epstein [18] introduce the notion of an *normalizable SDU*, which is defined for every $\mathbb{F}$.

[9] We say that a utility $U : \mathbf{L}^2_+ \to \mathbb{R}$ is an ***ordinally equivalent utility*** to a utility $\bar{U} : \mathbf{L}^2_+ \to \mathbb{R}$ if and only if there exists a strictly increasing and continuous function $\varphi : \mathbf{R} \to \mathbb{R}$ with $\varphi(0) = 0$ such that $U = \varphi \circ \bar{U}$.

Definition 10. A utility $\bar{U} : \mathbf{L}^2_+ \to \mathbb{R}$ is a *normalizable SDU* if and only if there exists an ordinally equivalent utility U that is characterized by an aggregator (f, m) such that $U(c) = Y_0$ for every $c \in \mathbf{L}^2_+$ where Y is the unique square-integrable process satisfying

$$(5.3) \qquad Y_t = \mathbb{E}_t \left[\int_t^{T^\dagger} f(c_s, Y_s)\, ds \right] \qquad \forall t \in \mathbf{T}.$$

We call (f, m) or f the ***normalized aggregator***. The class of normalizable SDUs depends on $\mathbb{F}$, so let it be denoted by $\mathcal{U}_{\mathrm{SD}}(\mathbb{F})$. Let $\mathbb{F}$ be generated by a Wiener process. Then, by definition, any normalizable SDU is an SDU for Wiener information. Duffie and Epstein [18] show that any SDU for Wiener information can be normalized. Next, let $\mathbb{F} = \mathbb{F}^{W,\nu}$. Then, it is not true that any SDU for jump-Wiener information can be normalized. We present a necessary and sufficient condition for an SDU for jump-Wiener information to be normalized.

Proposition 3. *Let $\mathbb{F} = \mathbb{F}^{W,\nu}$. Let $\bar{U}$ be an SDU for jump-Wiener information characterized by an aggregator $(\bar{f}, \bar{m})$. Then, $\bar{U} \in \mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu})$ if and only if $\bar{U}$ satisfies*

$$(5.4) \qquad \bar{M}_1(x, y) = \exp\left[\int_y^x \bar{B}(x_1)\, dx_1 \right] \qquad \forall (x, y) \in \mathbb{R}^2$$

for some continuous function $\bar{B} : \mathbb{R} \to \mathbb{R}$ where $\bar{M}$ is the LGR of $\bar{m}$.

Remark 4. Let $\mathcal{U}^{W,\nu}_{\mathrm{SD}}$ denote the class of SDUs for jump-Wiener information satisfying the condition (5.4). The class $\mathcal{U}^{W,\nu}_{\mathrm{SD}}$ is a subclass of SDUs for jump-Wiener information, but still includes the class of SDUs, each of which is characterized by an *expected-utility certainty equivalent* (for definition, see Duffie and Epstein [18]). This class of SDUs includes the Kreps-Porteus utility and the Uzawa utility (Uzawa [40]) as well as the standard ASU.

Proof. Suppose that a utility $\bar{U} : \mathbf{L}^2_+ \to \mathbb{R}$ is characterized by an unnormalized aggregator $(\bar{f}, \bar{m})$ such that $\bar{U}(c) = \bar{Y}_0$ for every $c \in \mathbf{L}^2_+$ where $\bar{Y}$ is the unique square-integrable process satisfying the SDE (9) with (5.1).

Step 1 – $\bar{U}$ is normalized if and only if there exists a continuous function $\bar{B}$: $\mathbb{R} \to \mathbb{R}$ satisfying (5.4): Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing and $\mathbf{C}^2$ function with $\varphi(0) = 0$ and let $Y_t = \varphi(\bar{Y}_t)$ for every $t \in \mathbf{T}$ and $f(x, \varphi(y)) = \varphi'(y)\bar{f}(x, y)$. Applying Ito's formula to $Y_t = \varphi(\bar{Y})$ yields

(5.5)

$$dY_t = \mu^Y_t\, dt + \varphi'(\bar{Y}_t)\, \sigma^{\bar{Y}}_t \cdot dW_t + \int_{\mathbf{Z}} \left\{ \varphi((1 + H^{\bar{Y}}_{t-}(z))\bar{Y}_{t-}) - \varphi(\bar{Y}_{t-}) \right\} \{ \nu(dt \times dz) - \lambda_t(dz)\, dt \}$$

for every $t \in \mathbf{T}$ where

$$(5.6) \quad \mu^Y_t = \varphi'(\bar{Y}_t) \left\{ \mu^{\bar{Y}}_t - \bar{Y}_t \int_{\mathbf{Z}} H^{\bar{Y}}_t(z)\, \lambda_t(dz) \right\} + \frac{1}{2} \varphi''(\bar{Y}_t)\, \|\sigma^{\bar{Y}}_t\|^2 + \int_{\mathbf{Z}} \left\{ \varphi((1 + H^{\bar{Y}}_t(z))\bar{Y}_t) - \varphi(\bar{Y}_t) \right\} \lambda_t(dz).$$

Substituting $\varphi'(\bar{Y}_t)\bar{f}(c_t, \bar{Y}_t) = f(c_t, Y_t)$ and (5.1) into (5.6) yields

(5.7)

$$\begin{aligned}
\mu_t^Y =& -\varphi'(\bar{Y}_t)\bar{f}(c_t,\bar{Y}_t) - \frac{1}{2}\varphi'(\bar{Y}_t)\,\bar{M}_{11}(\bar{Y}_t,\bar{Y}_t)\,\|\sigma_t^{\bar{Y}}\|^2 \\
& -\varphi'(\bar{Y}_t)\int_{\mathbf{Z}}\Big\{\bar{M}((1+H_t^{\bar{Y}}(z))\bar{Y}_t,\bar{Y}_t) - \bar{M}(\bar{Y}_t,\bar{Y}_t) - \bar{Y}_t H_t^{\bar{Y}}(z)\Big\}\lambda_t(dz) \\
& + \frac{1}{2}\varphi''(\bar{Y}_t)\,\|\sigma_t^{\bar{Y}}\|^2 + \int_{\mathbf{Z}}\Big\{\varphi((1+H_t^{\bar{Y}}(z))\bar{Y}_t) - \varphi(\bar{Y}_t) - \varphi'(\bar{Y}_t)\bar{Y}_t H_t^{\bar{Y}}(z)\Big\}\,\lambda_t(dz) \\
=& - f(c_t,Y_t) - \frac{1}{2}\Big\{\varphi'(\bar{Y}_t)\,\bar{M}_{11}(\bar{Y}_t,\bar{Y}_t) - \varphi''(\bar{Y}_t)\Big\}\,\|\sigma_t^{\bar{Y}}\|^2 \\
& - \int_{\mathbf{Z}}\Big[\varphi'(\bar{Y}_t)\Big\{\bar{M}((1+H_t^{\bar{Y}}(z))\bar{Y}_t,\bar{Y}_t) - \bar{M}(\bar{Y}_t,\bar{Y}_t)\Big\} - \Big\{\varphi((1+H_t^{\bar{Y}}(z))\bar{Y}_t) - \varphi(\bar{Y}_t)\Big\}\Big]\,\lambda_t(dz).
\end{aligned}$$

Thus, $\bar{U}$ can be normalized if and only if the set of conditions hold:

$$\varphi''(y) = \bar{M}_{11}(y,y)\,\varphi'(y), \tag{5.8}$$

$$\varphi(x) - \varphi(y) = \varphi'(y)\{\bar{M}(x,y) - \bar{M}(y,y)\} \tag{5.9}$$

for every $(x, y) \in \mathbb{R}^2$. However, twice partial differentiating both sides of (5.9) with respect to x and substituting $x = y$ yields (5.8). Hence, $\bar{U}$ is normalized if and only if the condition (5.9) holds. Considering $\bar{M}_1(y,y) = 1$ for every $y \in \mathbb{R}$, the condition (5.9) is equivalent to

$$\tilde{\varphi}'(x) = \tilde{\varphi}'(y)\bar{M}_1(x,y) \qquad \forall (x,y) \in \mathbb{R}^2. \tag{5.10}$$

Taking log and partial differentiating both sides of (5.10) with respect to x, we have

$$\frac{\tilde{\varphi}''(x)}{\tilde{\varphi}'(x)} = \frac{\bar{M}_{11}(x,y)}{\bar{M}_1(x,y)} \qquad \forall (x,y) \in \mathbb{R}^2. \tag{5.11}$$

Conversely, we can obtain (5.10) from (5.11) using $\bar{M}_1(y,y) = 1$. Thus, the condition (5.11) is equivalent to the condition (5.10). It is straightforward to see that the condition (5.11) holds if and only if there exists a continuous function $\bar{B} : \mathbb{R} \to \mathbb{R}$ satisfying the condition (5.4). Then, a function φ satisfying (5.11) is given by

$$\varphi(x) = \int_0^x \exp\left[\int_0^{x_2} \bar{B}(x_1)\,dx_1\right] dx_2 \qquad \forall x \in \mathbb{R}. \tag{5.12}$$

Step 2 – $\mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu}) \supset \mathcal{U}_{\mathrm{SD}}^{W,\nu}$: Let $\bar{U} \in \mathcal{U}_{\mathrm{SD}}^{W,\nu}$ be characterized by an unnormalized aggregator $(\bar{f}, \bar{m})$. Then, it immediately follows from Step1 that $\bar{U} \in \mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu})$.

Step 3 – $\mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu}) \subset \mathcal{U}_{\mathrm{SD}}^{W,\nu}$: Let $\bar{U} \in \mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu})$ be characterized by an aggregator $(\bar{f}, \bar{m})$. We assume w.l.o.g. that $(\bar{f}, \bar{m})$ is an unnormalized aggregator. Then, it follows from Step 1 that there exists a continuous function $\bar{B} : \mathbb{R} \to \mathbb{R}$ satisfying (5.4), and therefore $\bar{U} \in \mathcal{U}_{\mathrm{SD}}^{W,\nu}$. □

5.2. **Existence of Equilibria.** In Definition 10, we postulate the existence and uniqueness of square-integrable process Y satisfying (5.3) for every $c \in \mathbf{L}_+^2$. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a measurable function. Duffie and Epstein [18] show a set of sufficient conditions on the aggregator f for the existence and uniqueness of square-integrable process Y satisfying (5.3) for every $c \in \mathbf{L}_+^2$.

Proposition 4. *Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a measurable function. Suppose that f satisfies*

1. *a growth condition in consumption, in the sense that there exist constants* k_0 *and* k_1 *such that for every* $x \in \mathbb{R}_+$, *we have* $|f(x,0)| \leq k_0 + k_1\|c\|$, *and*
2. *a uniform Lipschitz condition in utility, in the sense that there exists a constant* k *such that for every* $x \in \mathbb{R}_+$ *and every* $(y_1, y_2) \in \mathbb{R}^2$, *we have* $|f(x,y_1) - f(x,y_2)| \leq k\|y_1 - y_2\|$.

Then, for every $c \in \mathbf{L}^2_+$, *there exists a unique square-integrable process* Y *satisfying (5.3).*

Proof. See Duffie and Epstein [18]. □

We assume that every agent's utility is a normalized SDU for jump-Wiener information.

Assumption 1. *For every* $i \in \mathbf{I}$, $U^i \in \mathcal{U}_{\mathrm{SD}}^{W,\nu}$ *is characterized by a normalized aggregator* (f^i, m^i) *where* f^i *satisfies the growth condition in consumption and the uniform Lipschitz condition in utility.*

We introduce the following set of assumptions to ensure the existence of equilibria.

Assumption 2. 1. *For every* $i \in \mathbf{I}$, *it follows that:*
   (a) *For every* $y \in \mathbb{R}$, $f^i(\,\cdot\,, y)$ *is strictly increasing.*
   (b) *The aggregator* f^i *is continuously differentiable on the interior of its domain.*
   (c) *The aggregator* f^i *is concave.*
   (d) *For every* $x > 0$, $\sup_{y\in\mathbb{R}} f^i_c(x,y) < \infty$.
   (e) *The aggregator* f^i *satisfies* $\lim_{x\downarrow 0}\inf_{y\in\mathbb{R}} f^i_c(x,y) = \infty$.

2. *The aggregate endowment is bounded away from zero* μ*-a.e.*

Remark 5. Consider a standard ASU of the form

$$U(c) = \mathbb{E}\left[\int_0^{T^\dagger} e^{-\rho s} u(c_s)\, ds\right].$$

Then, it follows from Ito's formula that U can be interpreted as an SDU of the form

$$Y_t = \mathbb{E}_t\left[\int_t^{T^\dagger} \big(u(c) - \rho y\big)\, ds\right] \qquad \forall t \in \mathbf{T}.$$

It is straightforward to see that the condition 2.1.(e) is equivalent to the Inada condition in the case of ASU. Thus, we need to present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of general ASUs.

Under Assumptions 1 and 2, Duffie, Geoffard, and Skiadas [19] prove the existence of Arrow-Debreu equilibria exploiting the Negishi approach (Negishi [38]) and results given in Duffie and Epstein [18], Duffie and Zame [22], and Mas-Collel and Zame [36].

Proposition 5. *Under Assumptions 1 and 2, there exists an Arrow-Debreu equilibrium* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *for* $\mathbf{E}$. *Moreover,* $(\hat{c}^i)_{i\in\mathbf{I}}$ *is a Pareto optimal allocation.*

Proof. See Duffie, Geoffard, and Skiadas [19]. □

Combining Proposition 5 with Corollary 1.1, we obtain the following proposition.

Proposition 6. *Let $\mathbb{F} = \mathbb{F}^{W,\nu}$. Under Assumptions 1 and 2, for every $\mathbf{B} \in \bar{\mathcal{B}}$, there exists an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$. Moreover, $(\hat{c}^i)_{i\in\mathbf{I}}$ is a Pareto optimal allocation.*

6. Existence, Uniqueness, and Determinacy of Equilibria in Case of ASUs

In this section, we show sufficient conditions for the existence, uniqueness, and determinacy of Arrow-Debreu equilibria in the case of ASUs (Additively Separable Utilities). To do so, we extend results given in Dana [13] [14] for a static economy with ASUs, to the results for our continuous-time economy. Her proof uses the Negishi approach (Negishi [38]) and consists of the following steps. (1) An Arrow-Debreu equilibrium can be identified with a representative agent equilibrium. (2) There exists a representative agent equilibrium under a regularity condition. In addition, (3) if every agent's risk tolerance coefficient satisfies an integrability condition and every agent's endowment process is bounded away from zero, then the set of equilibria is generically finite, and (4) if every agent's relative risk aversion coefficient is less than or equal to one, then the representative agent equilibrium is unique.

We suppose that every agent's utility has an additively separable utility representation.

Assumption 3. *For every agent $i \in \mathbf{I}$, the utility U^i is an additively separable utility of the form*

$$U^i(c) = \mathbb{E}\left[\int_0^{T^\dagger} u^i(t, c_t^i)\, dt\right]$$

where the von Neumann-Morgenstern (VNM) utility function u^i is a real-valued $\mathbf{C}^{1,2}$ function on $\mathbf{T}\times\mathbb{R}_+$ such that $u^i(t, \cdot)$ is strictly increasing and strictly concave on $\mathbb{R}_+$ for every $t \in \mathbf{T}$.

6.1. Equivalence of Arrow-Debreu and Representative Agent Equilibria.

We consider the aggregate utility to use the Negishi approach. Let $\alpha \in \Delta_+^I$ where $\Delta_+^I = \{\alpha \in \mathbb{R}_+^I \mid \sum_{i\in\mathbf{I}} \alpha_i = 1\}$ and define the aggregate utility $U^\alpha : \mathbf{L}_+^2 \to \mathbb{R}$ by

$$U^\alpha(c) = \max_{(c^1,c^2,\cdots,c^I)\in\prod_{i\in\mathbf{I}}\mathbf{L}_+^2} \sum_{i\in\mathbf{I}} \alpha_i U^i(c^i) \qquad \text{s.t.} \qquad \sum_{i\in\mathbf{I}} c^i \le c.$$

We also define a function $c^* : \mathbf{T} \times \mathbb{R}_+ \times \mathbb{R}_+^I \to \mathbb{R}_+^I$ by

$$(c_i^*(t,x,\alpha))_{i\in\mathbf{I}} = \operatorname{argmax}_{\{(x_1,x_2,\cdots,x_I)\in\mathbb{R}_+^I:\sum_{i\in\mathbf{I}} x_i \le x\}} \sum_{i\in\mathbf{I}} \alpha_i u^i(t, x_i).$$

Then, we have

Lemma 3. *Under Assumption 3, the aggregate utility U^α has an additively separable expected utility representation*

$$U^\alpha(c) = \mathbb{E}\left[\int_0^{T^\dagger} u(t, c_t, \alpha)\, dt\right] \quad \textit{where} \quad u(t,x,\alpha) = \sum_{i\in\mathbf{I}} \alpha_i u^i(t, c_i^*(t,x,\alpha)). \tag{6.1}$$

Moreover, u and $(c_i^*)_{i\in\mathbf{I}}$ satisfy the following conditions.

1. (a) *The function u is a real-valued $\mathbf{C}^{1,1,0}$ function on $\mathbf{T} \times \mathbb{R}_+ \times \mathbb{R}_+^I$ such that $u(t, \cdot, \alpha)$ is strictly increasing and strictly concave on $\mathbb{R}_+$ for every $(t, \alpha) \in \mathbf{T} \times \mathbb{R}_+^I$.*
   (b) *Let $i \in \mathbf{I}$. For every $(t, x, \alpha) \in \mathbf{T} \times \mathbb{R}_+ \times \mathbb{R}_+^I$ satisfying $c_i^*(t, x, \alpha) > 0$, the first partial derivative with respect to x denoted by $u_c(t, x, \alpha)$ satisfies*

$$u_c(t,x,\alpha) = \alpha_i u_c^i(t, c_i^*(t,x,\alpha)). \tag{6.2}$$

2. *Let $i \in \mathbf{I}$.*
   (a) *The function c_i^* is continuous on $\mathbf{T} \times \mathbb{R}_+ \times \mathbb{R}_+^I$.*
   (b) *For every $(t, x) \in \mathbf{T} \times \mathbb{R}_{++}$, the function $c_i^*(t, x, \cdot)$ is homogeneous of degree zero.*
   (c) *For every $(t, \alpha) \in \mathbf{T} \times \mathbb{R}_+^I$, $c_i^*(t, 0, \alpha) = 0$.*
3. *Let $t \in \mathbf{T}$ and $i \in \mathbf{I}$. Functions $u_c(t, \cdot, \cdot)$ and $c_i^*(t, \cdot, \cdot)$ are:*
   (a) *differentiable off the set of Lebesgue zero measure:*

$$\mathcal{D} = \{ (t,x,\alpha) \in \mathbf{T} \times \mathbb{R}_{++} \times \mathbb{R}_{++}^I \; : \; u_c(t,x,\alpha) = \alpha_i u_c^i(t, c_i^*(t,0,\alpha)) \textit{ for some } i \in \mathbf{I} \},$$

   and
   (b) *Lipschitz continuous on compact subsets of $\mathbb{R}_+ \times \mathbb{R}_+^I$.*
   (c) *Let $(t, x, \alpha) \in \mathcal{D}^c$. Assume that $c_i^*(t, x, \alpha) > 0$ for every $i \in \mathbf{I}$. Then, it follows that for every $i, j \in \mathbf{I}$*

$$\frac{\partial c_i^*}{\partial \alpha_j}(t,x,\alpha) = \frac{u_c^j(t, c_j^*(t,x,\alpha))}{\alpha_i \alpha_j u_{cc}^i(t, c_i^*(t,x,\alpha)) u_{cc}^j(t, c_j^*(t,x,\alpha)) \eta(t,x,\alpha)} \tag{6.3}$$

   where

$$\eta(t,x,\alpha) = \sum_{i \in \mathbf{I}} \frac{1}{\alpha_i u_{cc}^i(t, c_i^*(t,x,\alpha))}.$$

Proof. Proofs of 1 and 2 are easy. For the proofs of 3.(a)(b), see the proof of Proposition 2.3 in Dana [13]. We can obtain 3.(c) differentiating the first order condition

$$\alpha_1 u_c^i(t, c_i^*(t,x,\alpha)) = \alpha_2 u_c^2(t, c_2^*(t,x,\alpha)) = \cdots = \alpha_I u_c^I(t, c_I^*(t,x,\alpha))$$

and the relation $\sum_{i \in \mathbf{I}} c_i^*(t, x, \alpha) = x$ with respect to α_j. □

We introduce the notion of *representative agent equilibrium for* $\mathbf{E}$.

Definition 11. A utility weight $\hat{\alpha}$ constitutes a *representative agent equilibrium for* $\mathbf{E}$ if and only if $\hat{\alpha} \in \Delta_+^I$ satisfies $\mathcal{T}(\hat{\alpha}) = 0$ where $\mathcal{T} : \mathbb{R}_{++}^I \to \mathbb{R}^I$ is the *weighted transfer payment function*[10] defined by

$$\mathcal{T}_i(\alpha) = \frac{1}{\alpha_i} \mathbb{E}\Big[\int_0^{T^\dagger} u_c(s, \bar{c}_s, \alpha)(c_i^*(s, \bar{c}_s, \alpha) - \bar{c}_s^i)\, ds \Big] \qquad \forall i \in \mathbf{I}.$$

For a representative agent equilibrium $\hat{\alpha} \in \Delta_+^I$, $(c_i^*(s, \bar{c}_s, \hat{\alpha}))$ is a Pareto optimal allocation without transfer payments under the supporting price $u_c(s, \bar{c}_s, \hat{\alpha})$. We can show that a representative agent equilibrium for $\mathbf{E}$ can be identified with an Arrow-Debreu equilibrium for $\mathbf{E}$. To do so, we need the following lemma.

[10]Dana [13] [14] calls $\mathcal{T}$ the "excess utility map." For each $i \in \mathbf{I}$, $\mathcal{T}_i$ is the agent i's transfer payment weighted with α_i^{-1}. Thus, we call $\mathcal{T}$ the weighted transfer payment function.

Lemma 4. *Under Assumption 3, for any Pareto optimal allocation $(c^i)_{i\in\mathbf{I}}$ for $\mathbf{E}$, there exists a utility weight $\hat{\alpha} \in \Delta^I_+$ such that*

$$c^*(t, \bar{c}_t(\omega), \hat{\alpha}) = (c^i_t(\omega))_{i\in\mathbf{I}} \qquad \forall \mu\text{-a.e.}$$

Proof. See Huang [25]. □

Proposition 7. *Under Assumption 3, it follows that:*

1. *Let $\hat{\alpha}$ be a representative agent equilibrium for $\mathbf{E}$. Define $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ by $(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, \bar{c}_t(\omega), \hat{\alpha})$ and the Riesz kernel $\pi_t = u_c(t, \bar{c}_t(\omega), \hat{\alpha})$ for every $(\omega, t) \in \Omega \times \mathbf{T}$. Then, $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ is an Arrow-Debreu equilibrium for $\mathbf{E}$.*
2. *Conversely, let $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ be an Arrow-Debreu equilibrium for $\mathbf{E}$. Then, there exists a representative agent equilibrium $\hat{\alpha}$ for $\mathbf{E}$ such that $c^*(t, \bar{c}_t(\omega), \hat{\alpha}) = \hat{c}_t(\omega)$ μ-a.e.*

Proof. See Appendix C.2. □

6.2. **Existence of Representative Agent Equilibria.** Now our task is reduced to show the existence of representative agent equilibria. To prove the existence of representative agent equilibria, we introduce the following assumption.[11]

Assumption 4.

$$\max_{\alpha\in\Delta^I} u_c(t, \bar{c}_t(\omega), \alpha) \stackrel{\text{def}}{=} \bar{\pi}_t(\omega) \in \mathbf{L}^2_+ \tag{6.4}$$

Then, the weighted transfer payment function has the following desired properties for proving the existence of representative agent equilibria.

Lemma 5. *Under Assumptions 3 and 4, it follows that:*

1. *The weighted transfer payment function $\mathcal{T}$ is homogeneous of degree zero, and satisfies $\alpha \cdot \mathcal{T}(\alpha) = 0$ for every $\alpha \in \mathbb{R}^I_+$, and bounded above on $\mathbb{R}^I_+$.*
2. *The weighted transfer payment function $\mathcal{T}$ is continuous on $\mathbb{R}^I_{++}$, and $\mathcal{T}_i(\alpha) \to -\infty$ whenever $\alpha_i \to 0$ for some $i \in \mathbf{I}$.*

Proof. The proof of 1 is easy, so we prove 2.

Step 1 – Continuity on $\mathbb{R}^I_{++}$: We use the proof given in Dana [13]. Let S be a compact subset of $\mathbb{R}^I_{++}$ bounded away from the boundary. It suffices to show the continuity of $\mathcal{T}$ on S. Since $\mathcal{T}$ and c^* are homogeneous of degree zero on α, it

[11] In static economy, the assumption

$$\int_\Omega |\max_{\alpha\in\Delta^I} u_c(\bar{c}(\omega), \alpha)|^2 \mu(d\omega) < \infty$$

is proven to be the minimal assumption (see Theorem 11.1 in Mas-Collel and Zame [36]). In continuous-time economy, Dana and Pontier [15], Duffie and Zame [22], Karatzas, Lakner, Lehoczky, and Shreve [29], and Karatzas, Lehoczky, and Shreve [30] assume that the aggregate endowment is bounded away from zero, *i.e.* $\bar{c} > b$ μ-a.e. for some $b > 0$. This assumption is stronger than Assumption 4, since

$$\max_{\alpha\in\Delta^I_+} u_c(t, \bar{c}_t(\omega), \alpha) \le \max_{\alpha\in\Delta^I_+} u_c(t, b, \alpha)$$

and $u_c(\,\cdot\,, b, \alpha)$ is continuous on $\mathbf{T}$.

follows that for every $i \in \mathbf{I}$,

$$\left| \frac{1}{\alpha_i} u_c(t, \bar{c}_t(\omega), \alpha) \{ c_i^*(t, \bar{c}_t(\omega), \alpha) - \bar{c}_t^i(\omega) \} \right| \leq \frac{\sqrt{I}\, \|\alpha\|}{\alpha_i} \max_{\alpha \in \Delta_+^I} [\, u_c(t, \bar{c}_t(\omega), \alpha) \,]\, \bar{c}_t(\omega).$$

Thus, the continuity of $\mathcal{T}$ on S from Assumption (4), $\bar{c} \in \mathbf{L}_+^2$, Cauchy-Schwartz Inequality in $\mathbf{L}^2$, and Lebesgue Dominated Convergence Theorem.

Step 2 – Boundary condition: Let $i \in \mathbf{I}$. Recall that every agent's endowment process is not zero, so there exists $\mathbf{A} \in \mathcal{P}$ such that $\mu(\mathbf{A}) > 0$ and $\bar{c}_t^i(\omega) > 0$ for every $(\omega, t) \in \mathbf{A}$. Then, it follows that

$$\begin{aligned} \mathcal{T}_i(\alpha) \leq \frac{1}{\alpha_i} \Big[& \| \max_{\alpha \in \Delta^I} u_c(t, \bar{c}_t(\omega), \alpha) \|_{\mathbf{L}^2} \, \| c_i^*(t, \bar{c}_t(\omega), \alpha) \|_{\mathbf{L}^2} \\ & - \int_{\mathbf{A}} u_c(s, \bar{c}_s(\omega), \alpha)\, \bar{c}_s^i(\omega)\, \nu(d\omega \times ds) \Big] \quad \to \quad -\infty \end{aligned}$$

as $\alpha_i \to 0$. □

6.3. Uniqueness of Representative Agent Equilibria. To prove uniqueness of equilibria, we impose the following two assumptions.

Assumption 5.
1. *For every $i \in \mathbf{I}$, the agent i's relative risk aversion coefficient satisfies*
$$\gamma^i(t,x) \stackrel{\text{def}}{=} -\frac{x\, u_{cc}^i(t,x)}{u_c^i(t,x)} \leq 1 \qquad \forall (t,x) \in \mathbf{T} \times \mathbb{R}_+.$$
2. *Either of the following two conditions is satisfied:*
   (a) *Every agent's endowment is positive μ-a.e., i.e. $\bar{c}^i > 0$ μ-a.e. for every $i \in \mathbf{I}$.*
   (b) *Every agent's utility satisfies the Inada condition, i.e. $\lim_{x \downarrow 0} u_c^i(t,x) = \infty$ for every $i \in \mathbf{I}$.*

Then, the weighted transfer payment function has the strong gross substitution in the following.

Lemma 6. *Under Assumptions 3-5, the weighted transfer payment function $\mathcal{T}$ is strongly gross substitute, i.e.:*
1. *For every (i,j) such that $i \neq j$, $\mathcal{T}_i(\alpha_1, \alpha_2, \cdots, \alpha_{j-1}, \cdot, \alpha_{j+1}, \cdots, \alpha_I)$ is non-increasing and for every i, $\mathcal{T}_i(\alpha_1, \cdots, \alpha_{i-1}, \cdot, \alpha_{i+1}, \cdots, \alpha_I)$ is non-decreasing.*
2. *If $c_i^*(t, \bar{c}_t(\omega), \alpha) > 0$ on some $\mathbf{A} \in \mathcal{P}$ with $\mu(\mathbf{A}) > 0$, then for every $j \neq i$, $\mathcal{T}_i(\alpha_1, \cdots, \alpha_{j-1}, \cdot, \alpha_{j+1}, \cdots, \alpha_I)$ is strictly decreasing on a neighborhood of α.*

Proof. See the proof of Theorem 3.1 in Dana [13]. □

6.4. Determinacy of Representative Agent Equilibria. Unfortunately, we do not have any strong evidence which supports Assumption 5. Thus, we show that under more reasonable assumptions, the determinacy of equilibria is a generic property of our economies using the Negishi approach given in Dana [13] for static economies. First, we fix agents' common subjective probability and information structure as $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We parameterize the space of economies by keeping utilities and the aggregate endowment fixed and varying the distribution of individual endowments. We introduce the following assumptions.

Assumption 6. *For every $i \in \mathbf{I}$, the VNM utility function satisfies*

$$-\frac{u^i_c(t,x)}{u^i_{cc}(t,x)} \leq \beta^i_1 x + \beta^i_2 \qquad\qquad \forall (t,x) \in \mathbf{T} \times \mathbb{R}_+$$

for some $\beta^i \in \mathbb{R}^2_+$.

Assumption 7. *There exists $\delta \in \mathbb{R}^{\mathbf{I}}_{++}$ such that $\bar{c}^i_t > \delta_i$ μ-a.e. on $\mathbf{T} \times \Omega$ for every $i \in \mathbf{I}$.*

We consider the following space of economies.

$$\mathcal{E}_\delta \stackrel{\text{def}}{=} \Big\{ (\bar{c}^i)_{i\in\mathbf{I}} \in \prod_{i\in\mathbf{I}} \left(\mathbb{R}^2_+ \times \mathbf{L}^2_+\right) \Big| \sum_{i\in\mathbf{I}} \bar{c}^i = \bar{c}, \text{ and } (\bar{c}^i)_{i\in\mathbf{I}} \text{ satisfies Assumption 7 for } \delta \Big\}$$

We define a function $\hat{\mathcal{T}} : \Delta^I_+ \times \mathcal{E}_\delta \to \mathbb{R}^I$ by

$$\hat{\mathcal{T}}_i(\alpha, \mathbf{E}) = \frac{1}{\alpha_i}\mathbb{E}\Big[\int_0^{T^\dagger} u_c(s, \bar{c}_s, \alpha)(c^*_i(s, \bar{c}_s, \alpha) - \bar{c}^i_s)\, ds\Big] \qquad\qquad \forall i \in \mathbf{I}.$$

It follows from Dominated Convergence Theorem that $\hat{\mathcal{T}}$ is continuous on $\Delta^I_+ \times \mathcal{E}_\delta$. We have

Lemma 7. *Under Assumptions 3, 4, 6, and 7, $\hat{\mathcal{T}}$ is differentiable with respect to α on $\Delta^{\mathbf{I}}_{++}$ and its derivative is continuous on $\Delta^I_{++} \times \mathcal{E}_\delta$.*

Proof. See Appendix C.3 □

Let us recall that for every $\alpha \in \Delta^I_+$, $\sum_{i\in\mathbf{I}} \hat{\mathcal{T}}_i(\alpha, \mathbf{E}) = 0$, thus $\operatorname{rank} D_\alpha \hat{\mathcal{T}}(\alpha, \mathbf{E}) \leq I - 1$. We say that the economy $\mathbf{E}$ is *regular* if and only if $\hat{\mathcal{T}}(\hat{\alpha}, \mathbf{E}) = 0$ implies $\operatorname{rank} D_\alpha \hat{\mathcal{T}}(\alpha, \mathbf{E}) = I - 1$. We know that any equilibrium in the regular economy is locally unique (see Mas-Collel, Whinston, and Green [35]).

Define the correspondence $\{\hat{\alpha}\}(\mathbf{E}) : \mathcal{E}_\delta \to \Delta^I_+$ by

$$\{\hat{\alpha}\}(\mathbf{E}) = \{\, \alpha \in \Delta^I_+ \,:\, \hat{\mathcal{T}}(\alpha, \mathbf{E}) = 0 \,\}.$$

Lemma 8. *Under Assumptions 3, 4, 6, and 7, it follows that:*

1. *The correspondence $\{\hat{\alpha}\}$ is u.h.c., and for every $\mathbf{E} \in \mathcal{E}_\delta$, $\{\hat{\alpha}\}(\mathbf{E})$ is compact.*
2. *If $\mathbf{E}$ is regular then $\{\hat{\alpha}\}(\mathbf{E})$ is finite.*

Proof. The proof of 1 immediately follows from the continuity of $\hat{\mathcal{T}}$. Let $\mathbf{E}$ be a regular economy. Suppose $\{\hat{\alpha}\}(\mathbf{E})$ is infinite. Then, since $\{\hat{\alpha}\}(\mathbf{E})$ is compact, it has an accumulation point $\hat{\alpha} \in \{\hat{\alpha}\}(\mathbf{E})$. This implies that $\hat{\alpha}$ is not locally unique. This is a contradiction. □

6.5. **Existence, Uniqueness, and Determinacy of Equilibria.** Now we are ready to prove the existence, uniqueness, and determinacy of ASM equilibria.

Proposition 8. *Under Assumptions 3 and 4, it follows that for every $\mathbf{B} \in \bar{\mathcal{B}}$:*

1. *There exists an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$. In particular, if the mark set $\mathbb{Z}$ is finite, then $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is a security market equilibrium for $\mathbf{E}$. The equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is characterized by the corresponding representative agent equilibrium $\hat{\alpha}$ for $\mathbf{E}$, i.e. $(\hat{c}^i)_{i\in\mathbf{I}}$ and p satisfies $(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, \bar{c}_t(\omega), \hat{\alpha})$ and*

$$p_t(\omega) = \frac{B_t(\omega)}{\Lambda^{\mathbf{B}}_t(\omega)} u_c(t, \bar{c}_t(\omega), \hat{\alpha}) \tag{6.5}$$

for every $(\omega, t) \in \Omega \times \mathbf{T}$, respectively. Moreover, $(\hat{c}^i)_{i\in\mathbf{I}}$ is a Pareto optimal allocation.

2. *If Assumption 5 is satisfied, then the ASM equilibrium is unique.*
3. *If Assumptions 6 and 7 are satisfied, then the set of regular economies $\mathcal{R}_\delta$ of $\mathcal{E}_\delta$ is open and dense in $\mathcal{E}_\delta$.*

Proof. Step 1 – Existence: It follows from Lemma 5 and Fixed Point Theorem that there exists an $\hat{\alpha} \in \Delta_+^I$ such that $\mathcal{T}(\hat{\alpha}) = 0$, *i.e.*, there exists a representative agent equilibrium $\hat{\alpha}$ for $\mathbf{E}$ (for the proof, see pp. 585-587 in Mas-Collel, Whinston, and Green [35]). Define $(\hat{c}^i)_{i\in\mathbf{I}}$ and p by $(\hat{c}_t^i(\omega))_{i\in\mathbf{I}} = c^*(t, \bar{c}_t(\omega), \hat{\alpha})$ and $p_t(\omega) = (\tilde{\Lambda}_t^{\mathbf{B}}(\omega))^{-1} u_c(t, \bar{c}_t(\omega), \hat{\alpha})$ for every $(\omega, t) \in \Omega \times \mathbf{T}$, respectively. Then, by Corollary 1.1 and Proposition 7, $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is an ASM equilibrium for $\mathbf{E}$, and $(\hat{c}^i)_{i\in\mathbf{I}}$ is a Pareto optimal allocation. Suppose that the mark set $\mathbb{Z}$ is finite. Then, it follows from Theorem 1, Corollary 1.1, and Proposition 7 that $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ constitutes a security market equilibrium for $\mathbf{E}$.

Step 2 – Uniqueness: By Corollary 1.1 and Proposition 7, it is sufficient to show that the representative agent equilibrium is unique. We use the proof given in Dana [13]. Assume that there exist two non-collinear solutions for $\mathcal{T}(\alpha) = 0$ and let them $\hat{\alpha}$ and $\check{\alpha}$. Since E is homogeneous of degree zero by Lemma 5, let w.l.o.g. $\hat{\alpha} < \check{\alpha}$ with $\hat{\alpha}_i = \check{\alpha}_i$ for some $i \in \mathbf{I}$. As $\check{\alpha}$ is a solution for $\mathcal{T}(\alpha) = 0$, $c_i^*(t, \bar{c}_t(\omega), \check{\alpha}) \neq 0$ for every j. Therefore, $\mathcal{T}_i$ is strictly increasing at $\check{\alpha}$. Let $\hat{\alpha} < \alpha < \check{\alpha}$. Then, $0 = \mathcal{T}_i(\hat{\alpha}) < \mathcal{T}_i(\alpha) < \mathcal{T}_i(\check{\alpha}) = 0$, which is a contradiction.

Step 3 – Determinacy: See Appendix C.4.

□

Appendix A. Marked Point Process and Integration Theorem

A.1. **Marked Point Process.** We consider a double sequence $(s_n, Z_n)_{n\in\mathbf{N}}$ where s_n is the occurrence time of nth jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n. Define the random counting measure $\nu(dt \times dz)$ by

$$\nu([0,t] \times A) = \sum_{n\in\mathbf{N}} 1_{\{s_n \le t,\, Z_n \in A\}} \qquad \forall (t, A) \in [0, T^\dagger] \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the $\mathbb{Z}$*-marked point process.*

Let λ be such that

1. For every $(\omega, t) \in \Omega \times (0, T^\dagger]$, the set function $\lambda_t(\omega, \cdot)$ is a finite Borel measure on $\mathbb{Z}$.
2. For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is $\mathcal{P}$-measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

If the equation

$$\mathbb{E}\left[\int_0^{T^\dagger} Y_s\, \nu(ds \times A)\right] = \mathbb{E}\left[\int_0^{T^\dagger} Y_s \lambda_s(A)\, ds\right] \qquad \forall A \in \mathcal{Z}$$

holds for any nonnegative $\mathcal{P}$-measurable process Y, then we say that the marked point process $\nu(dt \times dz)$ has the $\mathbb{P}$*-intensity kernel* $\lambda_t(dz)$.

A.2. **Integration Theorem.** Let $\nu(dt \times dz)$ be a $\mathbb{Z}$-marked point process with the $\mathbb{P}$-intensity kernel $\lambda_t(dz)$. Let H be a $\mathcal{P} \otimes \mathcal{Z}$-measurable function. It follows that:

1. If we have

$$\mathbb{E}\left[\int_0^{T^\dagger}\int_{\mathbf{Z}} |H_s(z)|\lambda_s(z)\,ds\right] < \infty,$$

then the process $\int_0^t \int_{\mathbf{Z}} H_s(z)\{\,\nu(ds \times dz) - \lambda_s(dz)\,ds\,\}$ is a $\mathbb{P}$-martingale.
2. If $H \in \mathcal{L}(\lambda_t(dz) \times dt)$, then the process $\int_0^t \int_{\mathbf{Z}} H_s(z)\{\,\nu(ds \times dz) - \lambda_s(dz)\,ds\,\}$ is a local $\mathbb{P}$-martingale.

Proof. See p. 235 in Brémaud [12]. □

Appendix B. Ito's Formula and Girsanov's Theorem

B.1. **Ito's Formula.** Let $X = (X^1, ..., X^d)'$ be a d-dimensional semimartingales, and g be a real-valued $\mathbf{C}^2$ function on $\mathbb{R}^d$. Then, $g(X)$ is a semimartingale of the form

$$g(X_t) = g(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} g(X_{s-})\,dX_s^i + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^t \frac{\partial^2}{\partial x_i \partial x_j} g(X_{s-})\,d\langle X^{ic}, X^{jc}\rangle$$
$$+ \sum_{0 \le s \le t}\left\{ g(X_s) - g(X_{s-}) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} g(X_{s-})\,\Delta X_s^i \right\}$$

where X^{ic} is the continuous part of X^{ic} and $\langle X^{ic}, X^{jc}\rangle$ is the quadratic covariation of X^{ic} and X^{jc}.

B.2. **Girsanov's Theorem.**

1. Let $v \in \prod_{j=1}^{d} \mathcal{L}^2$ and $H \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Define a process Λ by

$$\frac{d\Lambda_t}{\Lambda_{t-}} = -v_t \cdot dW_t - \int_{\mathbf{Z}} H_t(z)\,\{\,\nu(dt \times dz) - \lambda_t(dz)\,dt\} \qquad \forall t \in [0, T^\dagger)$$

with $\Lambda_0 = 1$, and suppose $\mathbb{E}[\Lambda_{T^\dagger}] = 1$. Then there exists a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ given by the Radon-Nikodym derivative

$$d\tilde{\mathbb{P}} = \Lambda_{T^\dagger}\,d\mathbb{P}$$

such that:
(a) The measure $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$.
(b) The process given by

$$\tilde{W}_t = W_t + \int_0^t v_s\,ds \qquad \forall t \in \mathbf{T}$$

is a $\tilde{\mathbb{P}}$-Wiener process.
(c) The marked point process $\nu(dt \times dz)$ has the $\tilde{\mathbb{P}}$-intensity kernel such that

$$\tilde{\lambda}_t(dz) = (1 - H_t(z))\lambda_t(dz) \qquad \forall (t, z) \in \mathbf{T} \times \mathbb{Z}.$$

2. Every probability measure equivalent to $\mathbb{P}$ has the structure above.

Appendix C. Proofs

C.1. **Proofs of Steps 1 and 2 in Proof of Proposition 2.** *Step 1 -* $\bar{\mathcal{C}}^i(p,\mathbf{B}) = \tilde{\mathcal{C}}^i(p,\mathbf{B})$ *where* $p \in \mathbf{L}^2$: First, let $c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})$. Then, we have $\lim_{n\to\infty} \tilde{V}^{\mathbf{B}}_{T^\dagger}(\vartheta^i_n) = \frac{1}{B_t}\lim_{n\to\infty} V^{\mathbf{B}}_{T^\dagger}(\vartheta^i_n) = 0$. Also applying integration by part yields for every $(n,t) \in \mathbb{N}\times\mathbf{T}$,

(C.1)

$$\begin{aligned}
\tilde{V}^{\mathbf{B}}_t(\vartheta^i_n) &= \tilde{V}^{\mathbf{B}}_0(\vartheta^i_n) + \int_0^t B_s^{-1}\, dV^{\mathbf{B}}_s(\vartheta^i_n) + \int_0^t V^{\mathbf{B}}_s(\vartheta^i)\, dB_s^{-1} + \int_0^t d[V^{\mathbf{B}}_s(\vartheta_n), B_s^{-1}] \\
&= \int_0^t B_s^{-1}\left\{\vartheta^{i0}_{ns}\, dB_s + \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, dB^T_s + p_s(\bar{c}^i_s - c^i_s)\, ds\right\} \\
&\qquad\qquad + \int_0^t \left\{\vartheta^{i0}_{ns} B_s + \int_s^{T^\dagger} B^T_s \vartheta^{i1}_{ns}(dT)\right\} dB_s^{-1} \\
&= \int_0^t \vartheta^{i0}_{ns}\left\{B_s^{-1}\, dB_s + B_s\, dB_s^{-1}\right\} \\
&\qquad + \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\left\{B_s^{-1}\, dB^T_s + B^T_s\, dB_s^{-1}\right\} + \int_0^t \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds \\
&= \int_0^t \vartheta^{i0}_{ns}\, d\tilde{B}_s + \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s + \int_0^t \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds \\
&= \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s + \int_0^t \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds
\end{aligned}$$

where $[X^1, X^2]$ is the optional quadratic covariation of X^1 and X^2. Thus, we have $c^i \in \tilde{\mathcal{C}}^i(p,\mathbf{B})$. Second, let $c^i \in \tilde{\mathcal{C}}^i(p,\mathbf{B})$. Then, in the same way, we obtain $c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})$.

Step 2 - Proof of 2: Let $c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})$ where $(p,\mathbf{B}) \in \mathbf{L}^2_{++} \times \bar{\mathcal{B}}$. Then, $c^i \in \tilde{\mathcal{C}}^i(p,\mathbf{B})$ follows from Step 1. Define Π by the Riesz kernel $\pi = \tilde{\Lambda}^{\mathbf{B}} p$. Then, $\pi \in \mathbf{L}^2_{++}$ since $\tilde{\Lambda}^{\mathbf{B}}$ is bounded μ-a.e. Also it follows from $c^i \in \tilde{\mathcal{C}}^i(p,\mathbf{B})$ that

(C.2)

$$\mathbb{E}\left[\int_0^{T^\dagger} \tilde{p}_s(\bar{c}^i_s - c^i_s)\, ds\right] = \tilde{\mathbb{E}}^{\mathbf{B}}\left[\lim_{n\to\infty} \tilde{V}^{\mathbf{B}}_{T^\dagger}(\vartheta^i_n) - \lim_{n\to\infty}\int_0^{T^\dagger}\int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, d\tilde{B}^T_s\right] = 0.$$

Combining (C.2) with (4.1), we obtain $\mathbb{E}[\int_0^{T^\dagger} \pi_s(\bar{c}^i_s - c^i_s)\, ds] = 0$, and therefore $c^i \in \mathcal{C}^i(\Pi)$.

C.2. **Proof of Proposition 7.** Proof of 1. Assume that $\hat{\alpha}$ is a representative agent equilibrium for $\mathbf{E}$. Let $\hat{c}^i_t(\omega) = c^*_i(t, \bar{c}_t(\omega), \hat{\alpha})$ and $\pi_t(\omega) = u_c(t, \bar{c}_t(\omega), \hat{\alpha})$. Then, it follows that $\sum_{i\in\mathbf{I}} \hat{c}^i = \bar{c}$ by definition of c^* and that $\hat{c}^i_t$ satisfies the necessary and sufficient condition for every agent's optimality $u^i_c(t, \hat{c}^i_t) = \frac{1}{\hat{\alpha}_i}\pi_t$ for every $i \in \mathbf{I}$.

Proof of 2. Assume that $((\hat{c}^i)_{i\in\mathbf{I}}, \Pi)$ is an Arrow-Debreu equilibrium for $\mathbf{E}$. Since $(u^i)_{i\in\mathbf{I}}$ are strictly increasing by Assumption 3, $(\hat{c}^i)_{i\in\mathbf{I}}$ is Pareto optimal by First Welfare Theorem (see Mas-Collel and Zame [36]). Then, by Lemma 4, there exists $\hat{\alpha} \in \Delta^I_+$ such that

(C.3) $$c^*(t, \bar{c}_t(\omega), \hat{\alpha}) = (\hat{c}^i_t(\omega))_{i\in\mathbf{I}} \qquad \mu\text{-a.e.}$$

by Lemma 4. Combining (6.2) with (C.3), we have for every $i \in \mathbf{I}$,

$$u_c(t, \bar{c}_t(\omega), \hat{\alpha}) = \hat{\alpha}_i u_c^i(t, \hat{c}_t^i(\omega)) \qquad \mu\text{-a.e.} \tag{C.4}$$

On the other hand, the optimality of consumption plans implies that there exists a rescaled Lagrange multiplier $\hat{\alpha}^- \in \{\alpha \in \mathbb{R}_{++} \mid \sum_{i \in \mathbf{I}} \frac{1}{\hat{\alpha}_i^-} = 1\}$ such that for every $i \in \mathbf{I}$ and

$$u_c^i(t, \hat{c}_t^i) = \hat{\alpha}_i^- \pi_t \qquad \mu\text{-a.e.} \tag{C.5}$$

Comparing (C.4) with (C.5) yields $u_c(t, \bar{c}_t(\omega), \hat{\alpha}) = \pi_t(\omega)$, which implies $\mathcal{T}(\hat{\alpha}) = 0$.

C.3. **Proof of Lemma 7.** We exploit the proof given in Dana [13]. Let S be a compact subset of Δ_+^I bounded away from the boundary. It suffices to prove the differentiability of $\hat{\mathcal{T}}$ with respect to α on S. Define a function $\tau : \mathbf{T} \times \mathbb{R}_+ \times S \to \mathbb{R}^I$ by

$$\tau_i(t, \bar{c}_t, \alpha) = \frac{1}{\alpha_i} u_c(t, \bar{c}_t, \alpha)(c_i^*(t, \bar{c}_t, \alpha) - \bar{c}_t^i).$$

Then, we have

$$\frac{\partial \tau_i}{\partial \alpha_j}(t, \bar{c}_t, \alpha) = \frac{\partial c_i^*}{\partial \alpha_j}(t, \bar{c}_t, \alpha)\{u_{cc}^i(t, c_i^*)(c_i^*(t, \bar{c}_t, \alpha) - \bar{c}_t^i) + u_c^i(t, c_i^*)\}. \tag{C.6}$$

It follows from (6.3) that

$$\frac{\partial c_i^*}{\partial \alpha_j}(t, \bar{c}_t, \alpha)\, u_{cc}^i(t, c_i^*) \le \frac{1}{\alpha_i} u_c^j(t, c_j^*) = \frac{1}{\alpha_i \alpha_j} u_c(t, \bar{c}_t, \alpha). \tag{C.7}$$

It follows from (C.6),(C.7), and Assumptions 6 and 7 that

$$\left| \frac{\partial \tau_i}{\partial \alpha_j}(t, \bar{c}_t, \alpha) \right| \le \frac{1}{\alpha_i \alpha_j} \max_{(\alpha') \in \Delta^I} [u_c(t, \bar{c}_t(\omega), \alpha')]\{(\beta_1^i + 2)\bar{c}_t + \beta_2^i\}. \tag{C.8}$$

It follows from Lebesgue Dominated Convergence Theorem that $\hat{\mathcal{T}}$ is differentiable with respect to α on S, and its derivative is

$$\begin{aligned}\frac{\partial \hat{\mathcal{T}}_i}{\partial \alpha_j}(\alpha, \mathbf{E}) = \mathbb{E}\Bigg[\int_0^{T^\dagger} & \frac{\partial c_i^*}{\partial \alpha_j}(s, \bar{c}_s, \alpha) \\ & \times \Big\{u_{cc}^i(s, c_i^*(s, \bar{c}_s, \alpha)(c_i^*(s, \bar{c}_s, \alpha) - \bar{c}_s^i) + u_c^i(t, c_i^*(s, \bar{c}_s, \alpha)\Big\}\, ds\Bigg]\end{aligned}$$

Since $\bar{c}$ is fixed, $\left|\frac{\partial F_i}{\partial \alpha_j}\right|$ are bounded independently of $(\alpha, \mathbf{E})$ on S. Therefore, $\frac{\partial \hat{\mathcal{T}}_i}{\partial \alpha_j}$ is continuous on $\Delta_{++}^I \times \mathcal{E}_\delta$.

C.4. **Proof of Proposition 8.3.** We use the proof given in Dana [13]. We first show the openness of $\mathcal{R}_\delta$. Let $\mathbf{E}_0 \in \mathcal{R}_\delta$. Then, for any $\alpha_0 \in \delta^I$ such that $\hat{\mathcal{T}}(\alpha_0, \mathbf{E}_0) = 0$, and that rank $D_\alpha \hat{\mathcal{T}}(\alpha_0, \mathbf{E}_0) = I - 1$. Since $\{\hat{\alpha}\}(\mathbf{E})$ is compact and $D_\alpha \hat{\mathcal{T}}$ is continuous, there exists neighborhoods $\mathcal{V} \subset \mathcal{E}_\delta$ of $\mathbf{E}_0$ and $V \subset \Delta_+^I$ of α_0 such that $D_\alpha \hat{\mathcal{T}}(\alpha, \mathbf{E}) = I - 1$ for every $(\alpha, \mathbf{E}) \in V \times \mathcal{V}$. Since $\{\hat{\alpha}\}$ is u.h.c., there exists $\mathcal{V}' \subset \mathcal{V}$ such that $\{\hat{\alpha}\}(\mathcal{V}') \subset \mathcal{V}$. Thus, if $\mathbf{E} \in \mathcal{V}'$, then rank $D_\alpha \hat{\mathcal{T}}(\alpha_0, \mathbf{E}_0) = I - 1$ for every $\alpha \in \{\hat{\alpha}\}(\mathbf{E})$. Therefore, $\mathcal{V}' \subset \mathcal{R}_\delta$ and $\mathcal{R}_\delta$ is open in $\mathcal{E}_\delta$. Second, we prove the denseness of $\mathcal{R}_\delta$. Let $\mathbf{E} \in \mathcal{E}_\delta$. Pick $\varepsilon > 0$ such that $\bar{c}^i - \varepsilon > \delta_i$ μ-a.e. for every $i \in \{1, 2, \cdots, I-1\}$. Let $(X_i^\varepsilon)_{i \in \{1,2,\cdots,I-1\}}$ such that $\max\{\|X_i\|_{\mathbf{L}^2}, \|X_i\|_{\mathbf{L}^\infty}\} \le$

$\varepsilon \quad \forall i \in \{1,2,\cdots,I-1\}$. Let $A = \{(a_i)_{i\in\{1,2,\cdots,I-1\}} \in \mathbb{R}^{I-1} : 0 \le a_i \le 1 \quad \forall i \in \{1,2,\cdots,I-1\}\}$. Define a function $h : \Delta^I \times A \to \mathbb{R}^I$ by

$$h_i(\alpha,a) = \mathbb{E}\left[\int_0^{T^\dagger} u_c(s,\bar{c}_s,\alpha)(c_i^*(s,\bar{c}_s,\alpha) - \bar{c}_s^i - a_i X_{is}^\varepsilon)\,ds\right] \qquad \forall i \in \{1,2,\cdots,I-1\},$$

and

$$h_I(\alpha,a) = \mathbb{E}\left[\int_0^{T^\dagger} u_c(s,\bar{c}_s,\alpha)(c_I^*(s,\bar{c}_s,\alpha) - \bar{c}_s^I + \sum_{i=1}^{I-1} a_i X_{is}^\varepsilon)\,ds\right].$$

One easily checks that rank $D_a g(\alpha,a) = I-1$. By Transversality Theorem, there exists $a \in A$ such that 0 is a regular value of $h(\cdot a)$ that is 0 is a regular value of the economy in $\mathcal{E}$, $(\bar{c}^1 + a_1 X_1^\varepsilon, \bar{c}^2 + a_2 X_2^\varepsilon, \cdots, \bar{c}^{I-1} + a_{I-1} X_{I-1}^\varepsilon, \bar{c}^I - \sum_{i=1}^{I-1} a_i X_i^\varepsilon)$, arbitrarily close to $\mathbf{E}$, since ε can be chosen arbitrarily close to zero.

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