# VALUATION AND ASSET PRICING IN INFINITE HORIZON SEQUENTIAL MARKETS WITH PORTFOLIO CONSTRAINTS

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## Abstract

There are three ways of measuring the value of a payoff stream in sequential markets with portfolio constraints: the market price, the replication price, and the fundamental value. In this paper we characterize constraints for which these measures coincide in the absence of arbitrage, and in equilibrium. We show that the replication price functional is linear in finite horizon markets, but only sub-linear in general in infinite horizon unless markets are complete. We provide constraints for which the linearity holds regardless whether markets are complete or incomplete. Applying a duality technique, we determine an optimal replicating strategy through solving a sequence of independent linear programs. These results do not depend on investors' preferences (other than monotonicity), probability beliefs, endowments of goods, or supply of assets.

# 1 Introduction

This paper is concerned with the issue of payoff valuation and asset pricing in sequential markets with portfolio constraints. In the literature on sequential asset trading there are usually three ways of measuring the value of a payoff stream:
(i) The **market price** of an asset representing a claim to the payoff stream, which is typically regarded as arising from equilibrium or no-arbitrage conditions. The determination of the market price is a central issue of asset pricing.
(ii) The **replication price**, which is the minimum cost of replicating the payoff stream. The determination of the replication price and associated optimal replicating strategies is at the heart of option hedging and valuation, hedging with futures, portfolio insurance, and financial innovation.
(iii) The **fundamental value**, which is the sum of the *appropriately* discounted payoffs.
In finite horizon frictionless markets these three measures coincide, as shown by Cox and Ross (1976) and Harrison and Kreps (1979).[1] This is sometimes called "the fundamental theorem of asset pricing".

In recent years a large body of literature focuses on relaxing the assumptions that markets are frictionless and of finite horizon. In the presence of portfolio constraints or transaction costs, Cvitanic and Karatzas (1993), Edirisinghe, Naik and Uppal (1993), and Naik and Uppal (1994), Jouini and Kallal (1995a,b), Luttmer (1996), Broadie, Cvitanic and Soner (1998) characterize arbitrage-free replication prices and optimal replicating strategies. Recent work related to this strand of literature includes Chen (1995) and Charupat and Prisman (1997) in the context of financial innovation. Yet, these studies are all carried out in finite horizon models in which there is a final date at which all assets are liquidated. However, markets are of infinite horizon in nature if assets of no maturity date (such as stocks), or if an infinite sequence of assets of finite maturity, are traded. Much recent research on asset pricing bubbles[2] is carried out in infinite horizon models. In infinite horizon sequential markets a portfolio constraint is necessary to prevent unbounded Ponzi schemes.[3] Kocherlakota (1992) illustrates the role of different types of constraints on the existence of price bubbles in a model without uncertainty. Magill and Quinzii (1994) and Santos and Woodford (1997) provide sufficient conditions for the absence of price bubbles when investors are faced with non-positive lower bounds on their portfolio net worth. Huang and Werner (1998) analyze the nature of price bubbles under two market arrangements: the sequential spot markets studied by the previous authors and the simultaneous forward markets studied

[1] In their continuous time model, Harrison and Kreps (1979) restrict portfolio choice to ***simple trading strategies*** in the sense trade can only occur at finitely many times chosen in advance. Consequently, arbitrage activities via continuous trading, such as doubling strategies, are precluded.

[2] A bubble refers to the wedge between the market price of an asset and the fundamental value of its dividends.

[3] Ponzi scheme is a strategy of borrowing at some date and rolling over the debt forever thereby never paying it back. It has been recognized that for some portfolio constraints equilibrium and bounded Ponzi scheme or other bounded arbitrage may not be inconsistent. See, for example, Huang and Werner (1998).

by Gilles and LeRoy (1992a, 1992b). In contrast to the extensive bubbles literature, the replication price in infinite horizon markets has received much less attention. Some results from this perspective can be inferred from Santos and Woodford (1997) with a constraint that portfolio net worth be nonnegative and Huang and Werner (1998) with an assumption of no uncertainty.

In this paper we characterize portfolio constraints under which the three measures of value coincide and the replication price functional is linear or sub-linear, in the absence of arbitrage, and in equilibrium. We are interested in determining optimal replicating strategies as well. We show that in finite horizon markets the fundamental theorem of asset pricing continues to hold whenever strategies with nonnegative portfolio net worth are admissible. More specifically, there is no arbitrage if and only if the replication price functional is strictly positive and if and only if there is a system of discount factors, and only if the three measures of value coincide and the replication price functional is linear. This result applies to any equilibrium in which investors are faced with non-positive lower bounds on their portfolio net worth.

Our main results pertain to infinite horizon markets. Our first main result is that there is no Ponzi scheme or other arbitrage if and only if the replication price functional is strictly positive, and only if there is a system of discount factors and the replication price coincides with the fundamental value on nonnegative payoff streams, provided that constraints are forward looking (its meaning will be made clear later) and admit strategies with nonnegative portfolio net worth. The replication price functional is sub-linear, but not necessarily linear when markets are incomplete. We provide constraints for which the linearity holds regardless whether markets are complete or incomplete. Applying a duality technique, we determine an optimal replicating strategy through solving a sequence of independent linear programs. Independence here refers to the fact that obtaining an optimal portfolio in one date-event does not require knowledge of that in others.

In terms of the market price we show that the price of an asset of finite maturity (traded in infinite horizon markets) coincides with the fundamental value of its dividends, whenever there is no finite arbitrage and strategies with nonnegative portfolio net worth are admissible.[4] Yet, examples are provided in which there is no finite or infinite arbitrage and constraints are forward looking and admit strategies with nonnegative portfolio net worth, but in which there is a bubble in the market price of an asset of no maturity date. This implies that the market price of the asset exceeds the replication price of its dividends. Therefore, the dividends can be purchased in sequential markets at a cost lower than the asset price. The portfolio constraints prevent one from earning arbitrage profits resulting from this price-cost wedge. Our second main result is that the absence of finite and infinite arbitrage implies the absence of price bubbles, provided that strategies with deflated portfolio net worth above a certain negative position (which can be arbitrarily close to zero) are admissible. We provide

[4] A finite arbitrage involves nonzero asset holdings only at finitely many dates. Any arbitrage that is not finite is an infinite arbitrage, of which Ponzi scheme is an example.

constraints for which our two main results are applicable, and for which the three measures of value coincide in any equilibrium. We present an example of such equilibrium.

This paper studies the three measures of value in finite and infinite horizon stochastic models in one unified setting. What makes the results of this paper distinct is that they do not depend on investors' preferences (other than monotonicity), probability beliefs, endowments of goods, or supply of assets. The duality approach used in constructing the algorithm to determine an optimal replicating strategy can be viewed as a methodological contribution to the literature of derivative hedging and valuation, hedging with futures, and portfolio insurance, especially in the case of infinite horizon markets.[5] In contrast to this algorithm, standard methods in determining optimal replicating strategies in finite horizon multi-period markets involve either solving a set of simultaneous equations or using a backward recursion. Therefore, determining an optimal portfolio in one date-event requires knowledge of that in others. Another contribution of the duality results in this paper lies in the fact that they are readily applicable to optimal hedging problems with general polyhedral cone constraints on asset holdings in both finite and infinite horizon markets.[6]

The paper is organized as follows. In section 2 we introduce basic concepts of sequential markets. In section 3 we characterize constraints in finite horizon markets for which the fundamental theorem of asset pricing continues to hold. In section 4 we characterize constraints in infinite horizon markets for which the replication price coincides with the fundamental value on nonnegative payoff streams and the replication price functional is sub-linear, in the absence of arbitrage. In this course, we determine an optimal replicating strategy using the duality technique of linear programming. Linear replication price functional is obtained in section 5 in the presence of two constraints and in the case of complete markets. In section 6 we characterize constraints in infinite horizon markets for which the three measures of value coincide in the absence of arbitrage, and in equilibrium. Section 7 contains a number of examples. Section 8 concludes. Most proofs can be found in the appendix.

[5] This parallels the standard duality approach in the literature on the existence and characterization of optimal consumption policies in the presence of portfolio constraints, such as Cvitanic and Karatzas (1992), He and Pearson (1989), Cuoco (1997), and Xu and Shreve (1992). Application of a duality technique in determining an optimal replicating strategy can be found in Naik and Uppal (1994), which relies on a backward recursion in finite horizon markets.

[6] Examples of polyhedral cone constraints on asset holdings include, but are not limited to, margin requirements, target debt to equity ratios, zero and bounded short-sales constraints, as well as transversality constraints.

## 2 Sequential Markets

Dynamic uncertainty is modeled by a set $\Omega$ of states of the world and an increasing sequence $\{\mathcal{N}_t\}_{t=0}^{\infty}$ of finite information partitions with $\mathcal{N}_0 = \{\Omega\}$. The information structure can be naturally mapped onto an *event-tree* $\mathcal{D}$, where an information set $s^t \in \mathcal{N}_t$ is referred to as a date-event or a node of the event-tree. For each $s^t$, $s^t_-$ denotes its unique immediate predecessor if $t \neq 0$, $\{s^t_+\}$ a finite set of its immediate successors, and $\mathcal{D}_{s^t}$ a subtree with root $s^t$. With this notation we have $\mathcal{D}_{s^0} = \mathcal{D}$.

In each date-event a finite number of primitive assets is traded on spot markets in exchange for a single consumption goods which is taken as the unit of account. Throughout this paper $(q, d)$ denotes a price-dividend process adapted to $\{\mathcal{N}_t\}_{t=0}^{\infty}$. A holder of one share of an asset $j$ traded for a price $q_j(s^t)$ at $s^t$ is entitled to a payoff $R_j(s^{t+1})$ at each $s^{t+1} \in \{s^t_+\}$, where $R_j(s^{t+1}) = q_j(s^{t+1}) + d_j(s^{t+1})$ if the asset continues to be traded for a price $q_j(s^{t+1})$ at $s^{t+1}$ and $R_j(s^{t+1}) = d_j(s^{t+1})$ if the asset is liquidated at $s^{t+1}$. We denote by $q(s^t)$ a vector of prices for assets traded at $s^t \in \mathcal{D}$ and by $R(s^t)$ a vector of one-period payoffs for assets traded at $s^t_-$ for $s^t \in \mathcal{D}\backslash\{s^0\}$. That is, a holder of one share of each of the assets issued for price $q(s^t)$ at $s^t$ is entitled to payoff $R(s^{t+1})$ at each $s^{t+1} \in \{s^t_+\}$. At each $s^t \in \mathcal{D}\backslash\{s^0\}$ new assets can be issued while existing assets can be liquidated, so the dimensions of $R(s^t)$ and $q(s^t)$ can be different. The difference is equal to the number of existing assets liquidated subtracting the number of new assets issued at $s^t$.

We assume that dividends are nonnegative, so primitive assets are of limited liability. We also assume that there is *free disposal* of assets that are purchased in the sense an investor can give up positive shares of any asset he holds. Throughout this paper, we are only interested in the case in which at each $s^t$ at least one asset is traded for a positive price. This implies that the norm $\| q(s^t) \|$ of spot prices is strictly positive for each $s^t$.[7] We assume that for each $s^t$ there is some $s^{t+1} \in \{s^t_+\}$ such that $R_j(s^{t+1}) > 0$ for some asset $j$ traded at $s^t$. This means that there is always some way of carrying wealth into the future.[8]

### 2.1 Arbitrage and Portfolio Constraints

A *portfolio strategy* $\theta$ specifies the number of shares of primitive assets to be held in each date-event. It generates a payoff stream $z^\theta$:

$$z^\theta(s^t) \equiv R(s^t)'\theta(s^t_-) - q(s^t)'\theta(s^t), \qquad \forall\ s^t \in \mathcal{D}\backslash\{s^0\}.$$

[7] The norm $\| q(s^t) \|$ can be, alternatively, defined by $\sum_j |q_j(s^t)|$, $[q(s^t)'q(s^t)]^{\frac{1}{2}}$, or $\max_j |q_j(s^t)|$. It is immaterial for the purpose of this paper which of the three norms is chosen.

[8] See Santos and Woodford (1997) for a discussion of a similar assumption.

For any payoff stream $z$ adapted to $\{\mathcal{N}_t\}_{t=0}^{\infty}$, we denote by $z|s^t$ its projection on subtree $\mathcal{D}_{s^t}\backslash s^t$ in the sense $z|s^t$ coincides with $z$ on $\mathcal{D}_{s^t}\backslash s^t$ and with zeros elsewhere. An *arbitrage* is a portfolio strategy $\theta$ that generates a positive payoff stream at a nonnegative cost or a nonnegative payoff stream at a negative cost, i.e., such that

$$q(s^0)'\theta(s^0) \leq 0, \;\; z^{\theta}(s^t) \geq 0, \;\; \forall \; s^t \in \mathcal{D}\backslash\{s^0\},$$

with at least one strict inequality. One can distinguish two types of arbitrage: finite and infinite arbitrage. A *finite arbitrage* is an arbitrage $\theta$ that involves nonzero asset holdings only at finitely many dates, i.e., such that $\theta(s^t) = 0$ for $s^t \in \mathcal{D}, \;\; t > \tau$ for some $\tau$, of which *one-period arbitrage* is an example. An one-period arbitrage at a node $s^\tau$ is a finite arbitrage $\theta$ such that $\theta(s^t) = 0$ for $s^t \neq s^\tau$. Thus, it involves zero asset holdings except at $s^\tau$ for which $\theta(s^\tau)$ satisfies

$$q(s^\tau)'\theta(s^\tau) \leq 0, \;\; R(s^{\tau+1})'\theta(s^\tau) \geq 0, \;\; \forall \; s^{\tau+1} \in \{s^\tau_+\},$$

with at least one strict inequality. It should be clear that there is no finite arbitrage if and only if there is no one-period arbitrage. Any arbitrage that is not finite is an *infinite arbitrage*, of which *Ponzi scheme* is an example. Ponzi scheme is an infinite arbitrage $\theta$ that involves borrowing in some date-event $s^\tau$ and rolling over the debt into the infinite future thereby never paying it back, i.e., such that

$$\theta(s^t) = 0, \;\; s^t \notin \mathcal{D}_{s^\tau}; \;\; q(s^\tau)'\theta(\tau) < 0, \;\; z^{\theta}(s^t) = 0, \;\; s^t \in \mathcal{D}_{s^\tau}\backslash\{s^\tau\}.$$

As is now well-known, there is no finite arbitrage if and only if the price of each primitive asset is equal to the sum of its discounted one-period payoffs. More specifically, the absence of one-period arbitrage at a node $s^t$ is necessary and sufficient for the existence of strictly positive discount factors $\{a(s^t), \; a(s^{t+1}), \; s^{t+1} \in \{s^t_+\}\}$ such that

$$q(s^t) = \sum_{s^{t+1} \in \{s^t_+\}} \frac{a(s^{t+1})}{a(s^t)} R(s^{t+1}). \tag{1}$$

Throughout this paper, the discount factors determined by (1) are referred to as *state-prices.* Since only the ratios $\{a(s^{t+1})\backslash a(s^t)\}$ are restricted by (1), the absence of one-period arbitrage allows one to define a state-price system consistent with (1) at each node. We denote by $A_{s^t}$ the set of state-price systems on subtree $\mathcal{D}_{s^t}$. To simplify, we denote $A_{s^0}$ by $A$.

In contrast to finite arbitrage, infinite arbitrage cannot be precluded by prices and dividends alone. Indeed, for arbitrary strictly positive prices and dividends one can generate a positive payoff stream at arbitrary negative cost if one can borrow at some date and roll over the debt forever. In other words, unbounded Ponzi schemes can be feasible. Since unbounded Ponzi schemes and equilibrium are inconsistent, any consistent model of infinite horizon sequential markets has to involve a constraint on portfolio strategies. Constraints that restrict the end- or the beginning-of-trade portfolio net worth have been extensively adopted in the literature. We name a few examples:

- the *borrowing limits*, $q(s^t)'\theta(s^t) \geq -B(s^t)$ for some $B(s^t) \geq 0$ and each $s^t \in \mathcal{D}$;
- the *debt constraint*, $R(s^t)'\theta(s^t_-) \geq -D(s^t)$ for some $D(s^t) \geq 0$ and each $s^t \in \mathcal{D}\backslash\{s^0\}$;
- the *bounded borrowing limits*, $\inf_{s^t \in \mathcal{D}} q(s^t)'\theta(s^t) > -\infty$;
- the *bounded debt constraint*, $\inf_{s^t \in \mathcal{D}\backslash\{s^0\}} R(s^t)'\theta(s^t_-) > -\infty$;
- the *uniform transversality constraint*, $\liminf_{t\to\infty} \sum_{s^t \in \mathcal{D}_s \cap \mathcal{N}_t} a(s^t)q(s^t)'\theta(s^t) \geq 0$, for each $s \in \mathcal{D}$ and for each $a \in A$.

The borrowing limits adopted by Santos and Woodford (1997) bounds from below the end-of-trade portfolio net worth in each date-event, while the bounded borrowing limits bounds from below the overall end-of-trade portfolio net worth. The debt constraint adopted by Levine and Zame (1992) bounds from below the beginning-of-trade portfolio net worth in each date-event, while the bounded debt constraint bounds from below the overall beginning-of-trade portfolio net worth. The bounds $-B(s^t)$ and $-D(s^t)$ may depend on asset prices and other variables. The uniform transversality constraint adopted by Hernandez and Santos (1991, 1994) requires the state-price weighted sum of portfolio net worth over each information partition on every subtree be eventually nonnegative (the existence of a state-price system is implicitly assumed). These constraints can also be found in Florenzano and Gourdel (1996). The wealth constraints of Dybvig and Huang (1988) are continuous-time versions of these discrete-time examples.

## 2.2 Asset Span and Replication Price

From now on, we shall denote by $\Theta$ a set of portfolio strategies, and by $\Theta_{s^t}$ the projection of $\Theta$ on subtree $\mathcal{D}_{s^t}$ in the sense a strategy $\tilde{\theta}$ is in $\Theta_{s^t}$ if and only if there is a strategy $\theta$ in $\Theta$ such that $\tilde{\theta}$ coincides with $\theta$ on $\mathcal{D}_{s^t}$ and with null asset holdings elsewhere. The set of payoff streams generated by portfolio strategies in $\Theta_{s^t}$ is referred to as the *asset span*[9] on subtree $\mathcal{D}_{s^t}$. It is given by

$$\mathcal{M}_{s^t} \equiv \{z \ : \ z^{\tilde{\theta}} = z, \ \textit{for some } \tilde{\theta} \in \Theta_{s^t}\}.$$

We denote by $\mathcal{M}^+_{s^t}$ the subset of nonnegative payoff streams of $\mathcal{M}_{s^t}$ and by $\Theta^+_{s^t}$ the subset of $\Theta_{s^t}$ whose strategies generate the payoff streams of $\mathcal{M}^+_{s^t}$. *Replication price functional* $V_{s^t}$ on an asset span $\mathcal{M}_{s^t}$ is given by

$$V_{s^t}(z) \equiv inf\{q(s^t)'\theta(s^t) \ : \ z^{\theta} = z, \ \ \theta \in \Theta_{s^t}\}$$

[9]As pointed out by Edward Green, this is a little abuse of terminology since such a set may not be a linear subspace.

for each $z \in \mathcal{M}_{s^t}$. Therefore, $V_{s^t}(z)$ is the infimum cost of a strategy in $\Theta_{s^t}$ that generates $z$.[10] To simplify, we omit expressions of the dependence of the asset span and replication price functional on the price-dividend process and portfolio constraint. We denote by $\mathcal{M}$ the asset span $\mathcal{M}_{s^0}$ and by $V$ the replication price functional $V_{s^0}$.

## 2.3 Fundamental Value and Bubbles

The equivalence of the absence of one-period arbitrage and the existence of state-prices continues to hold when strategies with nonnegative portfolio net worth are admissible. This can be inferred from lemma 3.1 and 4.1. A state-price system $a \in A_{s^t}$ then provides weights to be used in calculating the present value of a payoff stream, say, $z \in \mathcal{M}_{s^t}$, which is defined as the state-price weighted sum of the payoffs in subsequent date-events. Assuming that the sum is well-defined at $s^t$, the present value is given by

$$\sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in \mathcal{D}_{s^t} \cap \mathcal{N}_\tau} \frac{a(s^\tau)}{a(s^t)} z(s^\tau). \tag{2}$$

We note that the present value is calculated only with reference to a particular state-price system. Thus it needs not be unique except in the case of complete markets. Markets are complete at $s^t$ if the rank of the one-period payoff matrix $\{R(s^{t+1})\}_{s^{t+1} \in \{s^t_+\}}$ is equal to $card\{s^t_+\}$, the number of immediate successors of $s^t$.[11] If markets are complete at $s^t$, equation (1) uniquely determines the ratios $\{a(s^{t+1}) \backslash a(s^t)\}$ for $s^{t+1} \in \{s^t_+\}$. If markets are complete at every node there exists a unique state-price system, while if markets are incomplete at some node there may exist more than one state-price systems equally consistent with (1).

In the case when markets are incomplete, the present value of a payoff stream may differ for different state-price systems. Nevertheless, the present value of the dividend stream of an asset is always well-defined and finite regardless of the system of state-prices chosen. To see this we can iterate (1) to get, for each asset $j$ traded at $s^t$,

$$a(s^t) q_j(s^t) = \sum_{\tau=t+1}^{T} \sum_{s^\tau \in \mathcal{D}_{s^t} \cap \mathcal{N}_\tau} a(s^\tau) d_j(s^\tau) + \sum_{s^T \in \mathcal{D}_{s^t} \cap \mathcal{N}_T} a(s^T) q_j(s^T)$$

[10]This form of replication price functional is standard in the literature of asset pricing with portfolio constraints or transaction costs. See, for example, Luttmer (1996). In Huang and Werner (1998) it is referred to as ***payoff pricing functional.*** An alternative super-replication price functional, adopted by Jouini and Kallal (1995a) and Santos and Woodford (1997) and often seen in the literature of option hedging and portfolio insurance, assigns each payoff the infimum cost of a feasible portfolio strategy that generates a dominating stream, i.e., one whose payoffs are at least as large as the targeted. In general, the super-replication price functional lies below the replication price functional. However, they coincide, at least for marketed payoffs, in all results presented in this paper.

[11]For markets to be complete at $s^t$, it is necessary that there are at least as many assets traded at $s^t$ as the number of immediate successors of $s^t$. Therefore, whether markets are complete or incomplete depends on both dividends and asset prices.

for any $r > t$. Since the second term of the right-hand side of above equality is nonnegative, the following inequality holds:

$$q_j(s^t) \geq \sum_{\tau=t+1}^{r} \sum_{s^\tau \in \mathcal{D}_{s^t} \cap \mathcal{N}_\tau} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau).$$

Since the right-hand side of above inequality is non-decreasing in $r$ and bounded from above by the asset price, it must converge to a limit that is no greater than the price. This limit defines the present value of the dividend stream with reference to the given state-price system, which is expressed by the right-hand side of the following inequality:

$$q_j(s^t) \geq \sum_{s^\tau \in \mathcal{D}_{s^t} \setminus \{s^t\}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau). \tag{3}$$

The difference between the current price and the present value of the future dividend stream of an asset is sometimes called a price bubble. This definition of price bubbles does not present any conceptual difficulties when markets are complete. Even with incomplete markets, this definition is unambiguous for any asset of finite maturity, since the present value of the asset dividends is the same regardless of the system of state-prices chosen, and is equal to the asset price. This will be shown by theorem 6.1. The true conceptual difficulty emerges in the case of an asset of no maturity date: the present value of the asset dividends may differ for different state-price systems, so it is somewhat ambiguous what is the measure of a bubble with this definition. To avoid this potential ambiguity, we draw attention to inequality (3) which says that there cannot be negative bubbles regardless of the system of state-prices chosen. In other words, the asset price is at least as high as the supremum of the present value of its dividends over the set of state-price systems. We define this supremum as the *fundamental value* of the dividend stream of the asset. This fundamental value forms a lower bound on the asset price. Following the lead of Santos and Woodford (1997) we measure a *price bubble* by the wedge between the market price of an asset and the fundamental value of its dividend stream, denoted by

$$\eta_j(s^t) \equiv q_j(s^t) - \sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t} \setminus \{s^t\}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau) \geq 0 \tag{4}$$

for each asset $j$ traded at $s^t$. A price bubble in the remaining of this paper refers to (4).

## 2.4 Equilibrium

In a sequential markets economy populated by a countable number of investors indexed by $h \in \mathcal{H}$, we denote by $\mathcal{H}(s^t) \subseteq \mathcal{H}$ a subset of investors who present at $s^t$ and by $\mathcal{N}^h$ a subset of date-events in which investor $h$ presents. We assume $\mathcal{H}(s^t) \neq \emptyset$ so that at each $s^t \in \mathcal{D}$ at least one investor presents. Both the infinitely lived agents model and the overlapping generations model are special cases of this general setting. Each investor $h \in \mathcal{H}$ has a strictly monotone, complete preference $\preceq^h$ on his consumption possibility set

$$\mathcal{X}^h \equiv \prod_{s^t \in \mathcal{N}^h} I\!R_+ .$$

This in particular implies that the investor prefers more consumption to less in each date-event he presents. Each investor $h \in \mathcal{H}(s^t)$ is endowed with $y^h(s^t) \in I\!R_+$ units of the consumption goods and a vector $\bar{\theta}^h(s^t)$ of shares of assets traded at $s^t$, where $\bar{\theta}^h_j(s^t) = 0$ if asset $j$ is not issued at $s^t$. The net aggregate supply $\bar{\theta}_j(s^t)$ of asset $j$ traded at $s^t$ is then given by

$$\bar{\theta}_j(s^t) = \sum_{h \in \mathcal{H}(s^t)} \bar{\theta}^h_j(s^t)$$

if asset $j$ is issued at $s^t$, and

$$\bar{\theta}_j(s^t) = \bar{\theta}_j(s^t_-)$$

if asset $j$ is issued at some predecessor of $s^t$ and continues to be traded at $s^t$. We assume that the sum in the first equation is well-defined and finite and denote it by $\bar{\theta}(s^t)$. The economy-wide aggregate supply of the consumption goods is then given by

$$\begin{aligned}
\bar{y}(s^0) &= \sum_{h \in \mathcal{H}(s^0)} y^h(s^0), \\
\bar{y}(s^t) &= \sum_{h \in \mathcal{H}(s^t)} y^h(s^t) + d(s^t)'\bar{\theta}(s^t_-), \qquad s^t \in \mathcal{D}\backslash\{s^0\}.
\end{aligned}$$

A portfolio constraint that an investor $h \in \mathcal{H}$ is faced with induces a set $\Theta^h$ of feasible portfolio strategies that he can choose from. An *equilibrium* is a price-consumption-portfolio process $\{\hat{q}, (\hat{c}^h, \hat{\theta}^h)_{h \in H}\}$ that satisfies:

(i) given $\hat{q}$, $(\hat{c}^h, \hat{\theta}^h)$ maximizes $\preceq^h$ subject to

$$\begin{aligned}
& c^h(s^0) + \hat{q}(s^0)'(\theta^h(s^0) - \bar{\theta}^h(s^0)) \le y^h(s^0), \\
& c^h(s^t) + \hat{q}(s^t)'(\theta^h(s^t) - \bar{\theta}^h(s^t)) \le y^h(s^t) + \hat{R}(s^t)'\theta^h(s^t_-), \\
& s^t \in \mathcal{N}^h\backslash\{s^0\}, \ c^h \in \mathcal{X}^h, \ \theta^h \in \Theta^h;
\end{aligned}$$

(ii) for each $s^t \in \mathcal{D}$,

$$\hat{q}(s^t) \ge 0, \qquad \| \hat{q}(s^t) \| > 0;$$

(iii) for each $s^t \in \mathcal{D}$,

$$\begin{aligned}
\sum_{h \in \mathcal{H}(s^t)} \hat{c}^h(s^t) &\le \bar{y}(s^t), \\
\sum_{h \in \mathcal{H}(s^t)} \hat{\theta}^h(s^t) &\le \bar{\theta}(s^t).
\end{aligned}$$

The non-negativity requirement of equilibrium asset prices in (ii) follows from the assumption of free disposal of assets. Note that the sum on the left-hand side of the first equation in (iii) is well-defined since consumptions are nonnegative. We take it part of the equilibrium definition that the sum on the left-hand side of the second equation in (iii) is well-defined.

# 3 The Fundamental Theorem of Asset Pricing in Finite Horizon Markets with Portfolio Constraints

Markets are of *finite horizon* if all assets are liquidated by some date $T$. The setup in section 2 can be easily modified to suit to the case of finite horizon markets.[12] The objective of this section is to characterize portfolio constraints for which the fundamental theorem of asset pricing continues to hold. We draw attention to the following property of a set $\Theta$ of portfolio strategies:[13]

**(FUL)** if $q(s^t)'\theta(s^t) \geq 0$ for each $s^t \in \mathcal{D}$ and $R(s^t)'\theta(s^t_-) \geq 0$ for each $s^t \in \mathcal{D}\backslash\{s^0\}$ with $t < T$ and $d(s^T)'\theta(s^T_-) \geq 0$ for each $s^T \in \mathcal{D}$, then $\theta \in \Theta$.

Property (FUL) means that strategies with nonnegative end- and beginning-of-trade portfolio net worth are admissible. In the literature, a negative portfolio net worth is often referred to as "borrowing".[14] From this perspective, property (FUL) can be interpreted as *Finite-horizon Unrestricted Lending.* The portfolio constraints introduced in section 2 suited to the case of finite horizon settings satisfy (FUL). It turns out that the presence of a constraint that satisfies (FUL) does not alter the set of arbitrage strategies.

**LEMMA 3.1** *Suppose that a set $\Theta$ of portfolio strategies satisfies property (FUL). There is no arbitrage if and only if there is no arbitrage in $\Theta$.*

Lemma 3.1 can be applied to establish the fundamental theorem of asset pricing for portfolio constraints that satisfy property (FUL). The following result is concerned with the determination of the replication price.

**THEOREM 3.1** *Let $\Theta$ be a set of portfolio strategies that satisfies property (FUL). The*

[12]For instance, the uniform transversality constraint and the concept of infinite arbitrage no longer make sense in finite horizon models.

[13]If $A \neq \emptyset$, the condition "$R(s^t)'\theta(s^t_-) \geq 0$ for each $s^t \in \mathcal{D}\backslash\{s^0\}$ with $t < T$ and $d(s^T)'\theta(s^T_-) \geq 0$ for each $s^T \in \mathcal{D}$" implies "$q(s^t)'\theta(s^t) \geq 0$ for each $s^t \in \mathcal{D}$". That is, a strategy with nonnegative beginning-of-trade portfolio net worth must have nonnegative end-of-trade portfolio net worth as well, whenever there is no arbitrage. This can be inferred from relation (1).

[14]Borrowing and lending do have other meanings in both theory and practice. For instance, under margin requirements on stocks and bonds borrowing and lending usually refer to short and long positions in risk-free bonds.

*following three conditions are equivalent:*

$$\begin{array}{rl} (i) & \text{there is no arbitrage in } \Theta, \\ (ii) & A \neq \emptyset, \\ (iii) & V \text{ is strictly positive.} \end{array}$$

*Further, condition (i) implies that $V_{s^t}$ is linear*[15] *for each non-terminal date-event $s^t$ and satisfies*

$$V_{s^t}(z) = \sum_{s^\tau \in \mathcal{D}_{s^t} \setminus \{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \tag{5}$$

*for each $z \in \mathcal{M}_{s^t}$ and arbitrary $a \in A_{s^t}$.*

According to (5) the replication price coincides with the fundamental value on the asset span and the replication price functional is linear. The next result is concerned with the determination of the market price.

**THEOREM 3.2** *Let $\Theta$ be a set of portfolio strategies that satisfies property (FUL). If there is no arbitrage in $\Theta$, then $A \neq \emptyset$, and*

$$q_j(s^t) = \sum_{s^\tau \in \mathcal{D}_{s^t} \setminus \{s^t\}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau) \tag{6}$$

*for each asset $j$ traded at $s^t$ and arbitrary $a \in A_{s^t}$.*

Theorems 3.1 and 3.2 together establish the fundamental theorem of asset pricing: the three measures of value coincide whenever there is no arbitrage and the portfolio constraint satisfies property (FUL). The fundamental value is given by the present value, which is the same regardless of the system of state-prices chosen. These theorems apply to any equilibrium in which investors are faced with non-positive lower bounds on their portfolio net worth.

**THEOREM 3.3** *The three measures of value coincide and the replication price functional is linear in any equilibrium in which investors are faced with non-positive lower bounds on their portfolio net worth.*

[15] A functional $F$ is *linear* on a set $\mathcal{X}$ if for any $x_1, x_2, x \in \mathcal{X}$, $\alpha \in \mathbb{R}$, and $x_1 + x_2, \alpha x \in \mathcal{X}$,

$$F(x_1 + x_2) = F(x_1) + F(x_2), \;\; F(\alpha x) = \alpha F(x)$$

holds as long as the right-hand sides of the two equalities are well-defined.

# 4 Replication Price and Fundamental Value in Infinite Horizon Markets: Sub-linear Valuation

An asset is of finite maturity if it is to be liquidated. Examples are corporate and government bonds.[16] Certain assets, such as corporate stocks[17] and fiat money, do not have a maturity date. Markets are of *infinite horizon* in nature if assets of no maturity date, or if an infinite sequence of assets of finite maturity, are traded. The objective of this section is to characterize portfolio constraints for which the replication price coincides with the fundamental value on nonnegative payoff streams in the absence of finite and infinite arbitrage. We draw attention to the following two properties of a set $\Theta$ of portfolio strategies:

**(FL)** $\Theta_{s^t}$ is contained in $\Theta$ for each $s^t \in \mathcal{D}$;

**(UL)** if $q(s^t)'\theta(s^t) \geq 0$ for each $s^t \in \mathcal{D}$ and $R(s^t)'\theta(s^t_-) \geq 0$ for each $s^t \in \mathcal{D}\backslash\{s^0\}$, then $\theta \in \Theta$.

Property (FL) means that from the perspective of any date-event whether or not a trading plan is feasible only depends on its current and future characteristics. This is interpreted as *Forwarding Looking.* Forward looking makes the condition of no-arbitrage more robust for it requires that one cannot arbitrage even if one's trading history is ignored. Property (UL) is an infinite horizon version of property (FUL) of section 3 and is interpreted as *Unrestricted Lending.* The portfolio constraints introduced in section 2 satisfy properties (FL) and (UL).[18] It turns out that the presence of a constraint that satisfies property (UL) does not alter the set of finite arbitrage strategies. The following result is similar to lemma 3.1.

**LEMMA 4.1** *Suppose that a set $\Theta$ of portfolio strategies satisfies property (UL). There is no finite arbitrage if and only if there is no finite arbitrage in $\Theta$.*

Lemma 4.1 can be applied to show that there is no finite or infinite arbitrage if and only if the replication price functional is strictly positive, and only if there is a state-price system and the replication price coincides with the fundamental value on nonnegative payoff streams, whenever constraints are forward looking and admit strategies with nonnegative portfolio net worth. The duality technique of linear programming is applied in establishing the following result.

[16]Certain government bonds, such as *consols*, are perpetuities and do not have a maturity date.

[17]Provided that the corporation does not go bankrupt with probability one.

[18]Properties (FL) and (UL) extend properties (D) and (UL) in Huang and Werner (1998) to stochastic settings. Please see footnote 13 as well.

**THEOREM 4.1** *Let $\Theta$ be a set of portfolio strategies that satisfies properties (FL) and (UL). The following two conditions are equivalent:*

$$\begin{array}{ll} (i) & there\ is\ no\ finite\ or\ infinite\ arbitrages\ in\ \Theta, \\ (ii) & V\ is\ strictly\ positive. \end{array}$$

*Further, condition (i) implies $A \neq \emptyset$, and that $V_{s^t}$ is sub-linear*[19] *on $\mathcal{M}^+_{s^t}$ for each $s^t$ and satisfies*

$$V_{s^t}(z) = \sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t} \backslash \{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \tag{7}$$

*for each $z \in \mathcal{M}^+_{s^t}$.*

Proof: Suppose that condition $(i)$ holds. Then there is no finite arbitrage due to lemma 4.1, and consequently $A(s^0) \neq \emptyset$. We first establish two inequalities for any feasible strategy $\theta$ that generates a nonnegative payoff stream $z$. The following two inequalities will be used to establish equation (7),

$$\begin{aligned} q(s^t)'\theta(s^t) &\geq 0, \qquad \forall\, s^t \in \mathcal{D}, & (8)\\ R(s^t)'\theta(s^t_-) &\geq 0, \qquad \forall\, s^t \in \mathcal{D} \backslash \{s^0\}. & (9) \end{aligned}$$

To prove (8) suppose, by contradiction, that there is some $s^t$ at which $q(s^t)'\theta(s^t) < 0$. Then the projection $\tilde{\theta}$ of $\theta$ on subtree $\mathcal{D}_{s^t}$ is an arbitrage in $\Theta_{s^t}$. Since $\Theta$ satisfies property (FL), $\tilde{\theta}$ is an arbitrage in $\Theta$ as well. This contradicts condition $(i)$. Therefore (8) must hold. Inequality (8) together with $z \geq 0$ gives rise to inequality (9). Recall that $\Theta^+$ is the subset of $\Theta$ whose strategies generate nonnegative payoff streams. Inequalities (8) and (9) together with property (UL) imply that $\Theta^+$ is a convex cone. Consequently, $V$ is sub-linear on $\Theta^+$.

To establish (7) for the event-tree, we first use inequality (8) to establish

$$V(z) \geq \sup_{a \in A} \sum_{s^t \in \mathcal{D} \backslash \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t) \tag{10}$$

for each $z \in \mathcal{M}^+$. Let $\theta$ be a portfolio strategy in $\Theta$ that finances $z$. Choose an arbitrary $a \in A$ and use relation (1) and $z^\theta = z \geq 0$ to get, for any $\tau \geq 1$,

$$\begin{aligned} a(s^0)q(s^0)\theta(s^0) &= \sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) + \sum_{s^\tau \in \mathcal{N}_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau) \\ &\geq \sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t), \end{aligned}$$

[19] A functional $F$ is ***sub-linear*** on a set $\mathcal{X}$ if for any $x_1, x_2, x \in \mathcal{X}$, $\alpha > 0$, and $x_1 + x_2, \alpha x \in \mathcal{X}$,

$$F(x_1 + x_2) \leq F(x_1) + F(x_2), \;\; F(\alpha x) = \alpha F(x)$$

holds as long as the right-hand side of the first inequality is well-defined.

where the inequality follows from (8). Taking $\tau \to \infty$ on the right-most side of the above (in)equalities leads to

$$a(s^0)q(s^0)'\theta(s^0) \geq \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) \equiv \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} a(s^t)z(s^t).$$

That $a$ is arbitrarily chosen implies

$$q(s^0)'\theta(s^0) \geq \sup_{a \in A} \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t).$$

That $\theta$ is an arbitrary strategy in $\Theta$ that finances $z$ implies

$$V(z) \geq \sup_{a \in A} \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t).$$

This establishes (10). Strict positivity of $V$ then follows from (10).

We next use the duality technique of linear programming and inequalities (8) and (9) to establish

$$V(z) \leq \sup_{a \in A} \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t) \tag{11}$$

for each $z \in \mathcal{M}^+$. Note that (11) is non-trivial only if the right-hand side is finite, so we assume this is the case. For each $s^t \in \mathcal{D}$, consider the following minimization problem:

$$\min_{\theta(s^t)} \quad q(s^t)'\theta(s^t) \tag{12}$$

$$\text{s.t.} \quad R(s^{t+1})'\theta(s^t) \geq \sup_{a \in A_{s^{t+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau), \;\; s^{t+1} \in \{s^t_+\}; \tag{13}$$

and its dual:

$$\max_{\substack{\alpha(s^{t+1}) \\ s^{t+1} \in \{s^t_+\}}} \quad \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1}) [\sup_{a \in A_{s^{t+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau)] \tag{14}$$

$$\text{s.t.} \quad \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1}) R(s^{t+1}) = q(s^t), \;\; \alpha(s^{t+1}) \geq 0, \;\; s^{t+1} \in \{s^t_+\}. \tag{15}$$

We claim that both (13) and (15) have feasible solutions. That (15) has a feasible solution simply follows from the existence of a state-price system. We now prove that any feasible strategy $\theta$ that generates $z$ induces a portfolio $\theta(s^t)$ at $s^t$ that is a feasible solution to (13). To proceed we use (1), (8) and $z^\theta = z$ to obtain the following (in)equalities for each $s^{t+1} \in \{s^t_+\}$, arbitrary state-price system $a \in A_{s^{t+1}}$, and any $r \geq t+1$,

$$
\begin{aligned}
a(s^{t+1})R(s^{t+1})'\theta(s^t) &= \sum_{\tau=t+1}^{r} \sum_{s^\tau \in \mathcal{D}_{s^{t+1}} \cap \mathcal{N}_\tau} a(s^\tau)z(s^\tau) + \sum_{s^r \in \mathcal{D}_{s^{t+1}} \cap \mathcal{N}_r} a(s^r)q(s^r)'\theta(s^r) \\
&\geq \sum_{\tau=t+1}^{r} \sum_{s^\tau \in \mathcal{D}_{s^{t+1}} \cap \mathcal{N}_\tau} a(s^\tau)z(s^\tau),
\end{aligned}
$$

where the inequality follows from (8). Taking $r \to \infty$ on the right-most side of the above (in)equalities leads to

$$
a(s^{t+1})R(s^{t+1})'\theta(s^t) \geq \sum_{s^\tau \in \mathcal{D}_{s^{t+1}}} a(s^\tau)z(s^\tau).
$$

That $a$ is arbitrarily chosen from $A(s^{t+1})$ implies

$$
R(s^{t+1})'\theta(s^t) \geq \sup_{a \in A_{s^{t+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau).
$$

Thus, $\theta(s^t)$ is a feasible solution to (13).

By the duality theorem of linear programming, both the primal and dual problems have finite optimal solutions and the values of their optimal objectives (12) and (14) are equal. Since the objective (14) in the dual problem is continuous in $\alpha(s^{t+1})$, $s^{t+1} \in \{s^t_+\}$, the problem can be re-written as:

$$
\sup_{\substack{\alpha(s^{t+1}) \\ s^{t+1} \in \{s^t_+\}}} \quad \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1})\Big[\sup_{a \in A_{s^{t+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau)\Big] \tag{16}
$$

$$
\text{s.t} \quad \sum_{s^{t+1} \in \{s^t_+\}} \alpha(s^{t+1})R(s^{t+1}) = q(s^t), \;\; \alpha(s^{t+1}) > 0, \;\; s^{t+1} \in \{s^t_+\}. \tag{17}
$$

The value of the optimal objective of the problem (16)-(17) is equal to

$$
\sup \sum_{s^{t+1} \in \{s^t_+\}} \frac{a(s^{t+1})}{a(s^t)} \Big[\sup_{a \in A_{s^{t+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau)\Big] \tag{18}
$$

where the outer supremum is taken over the state-prices $\{a(s^{t+1})\backslash a(s^t)\}$ given by relation (1). By a dynamic programming argument, (18) is equal to

$$
\sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t} \backslash \{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau). \tag{19}
$$

Repeating the above procedure for every node of the event-tree shows that there is a portfolio strategy $\hat{\theta}$ such that for each $s^t \in \mathcal{D}$, $\hat{\theta}(s^t)$ is an optimal solution to the primal problem (12)-(13). Therefore

$$
\begin{aligned}
q(s^t)'\hat{\theta}(s^t) &= \sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t} \backslash \{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \geq 0, \;\; s^t \in \mathcal{D}, && (20)\\
R(s^t)'\hat{\theta}(s^t_-) &\geq \sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \geq 0, \;\; s^t \in \mathcal{D} \backslash \{s^0\}. && (21)
\end{aligned}
$$

Relations (20) and (21) imply $\hat{\theta} \in \Theta$ since $\Theta$ satisfies property (UL), and that the following holds for each $s^t \in \mathcal{D} \backslash \{s^0\}$:

$$
\begin{aligned}
z^{\hat{\theta}}(s^t) &\equiv R(s^t)'\hat{\theta}(s^t_-) - q(s^t)'\hat{\theta}(s^t)\\
&\geq \sup_{a \in A_{s^t}} [z(s^t) + \sum_{s^\tau \in \mathcal{D}_{s^t} \backslash \{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau)] - \sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t} \backslash \{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau)\\
&= z(s^t).
\end{aligned}
$$

Therefore $\hat{\theta}$ generates a payoff $z^{\hat{\theta}} \geq z$ at a date-0 cost equal to:

$$
q(s^0)'\hat{\theta}(s^0) = \sup_{a \in A} \sum_{s^t \in \mathcal{D} \backslash \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t). \tag{22}
$$

Since at least one asset is traded for a positive price at each node, we can, if necessary, increase the portfolio weights of $\hat{\theta}$ at each non-initial node to obtain a strategy in $\Theta$ that generates $z$ at the date-0 cost (22). This establishes (11) and, combined with (10), gives rise to (7) for the event-tree. We can similarly establish the sub-linearity and the representation (7) for $V_{s^t}$ on $\mathcal{M}^+_{s^t}$ for each subtree $\mathcal{D}_{s^t}$. We finally note that the implication $(ii) \Longrightarrow (i)$ follows from the definition of $V$. □

The replication price coincides with the fundamental value on nonnegative payoff streams whenever there is no finite or infinite arbitrage and the portfolio constraint satisfies properties (FL) and (UL). According to equation (7) the replication price functional is sub-linear, but not necessarily linear since the present value of a payoff stream may differ for different state-price systems in incomplete markets. Theorem 4.1 generalizes the result of Santos and Woodford (1997) establishing the equality between the replication price and the fundamental value on nonnegative payoff streams with a constraint that portfolio net worth be nonnegative.

In the course of proving theorem 4.1 we construct an algorithm to determine an optimal replicating strategy. It is as simple as solving the linear program (12)-(13) for each node independently. Independence here refers to the fact that obtaining a solution at one node does not require knowledge of that at others. Therefore it is easy to implement in both finite and infinite horizon markets. In contrast, existing optimal replicating algorithms in finite horizon multi-period asset markets usually involve either solving a set of simultaneous equations, as is the case of Cox, Ross, and Rubinstein (1979), or using a backward recursion, as is the case of Naik and Uppal (1994). With these algorithms, determining an optimal portfolio at one node requires knowledge of that at others. Moreover, the algorithm developed here is readily applicable to optimal replication problems with general polyhedral cone constraints on asset holdings, thanks to the technique of convex duality.

Before closing this section, we provide a constraint which violates property (FL), but under which the replication price coincides with the fundamental value on nonnegative payoff streams on the event-tree. It is given by

$$\liminf_{t\to\infty} \sum_{s^t\in\mathcal{N}_t} a(s^t)q(s^t)'\theta(s^t) \geq 0, \quad \textit{for each } a \in A. \tag{23}$$

Constraint (23) requires the state-price weighted sum of portfolio net worth over each information partition be eventually nonnegative. Imposed only on the event-tree rather than on every subtree, it gives rises to a set of portfolio strategies that violates property (FL). Nevertheless, the replication price coincides with the fundamental value on nonnegative payoff streams on the event-tree in the presence of (23), with an additional assumption on asset structure.

**THEOREM 4.2** *Suppose that for each $s^t \in \mathcal{D}\backslash\{s^0\}$ there is some asset $j$ traded at $s^t_-$ with $R_j(s^t) > 0$. Let $\Theta$ be a set of portfolio strategies induced by (23). Then there is no finite or infinite arbitrage in $\Theta$, and $V$ is strictly positive and sub-linear on $\mathcal{M}^+$ and satisfies*

$$V(z) = \sup_{a\in A} \sum_{s^t\in\mathcal{D}\backslash\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t) \tag{24}$$

*for each $z \in \mathcal{M}^+$.*

If (23) were imposed on a subtree the replication price would coincide with the fundamental value on nonnegative payoff streams on that subtree.

# 5 Replication Price and Fundamental Value in Infinite Horizon Markets: Linear Valuation

In the absence of arbitrage under the portfolio constraints of section 4 the replication price functional is sub-linear, but not necessarily linear when markets are incomplete. The objective of this section is to derive linear replication price functional in the presence of certain constraints and in the case of complete markets. We proceed by first providing constraints for which the linearity holds regardless whether markets are complete or incomplete. One simple constraint is that portfolio net worth be eventually zero in each date-event. With this constraint the replication price coincides with the fundamental value on the asset span of each subtree and the replication price functional is linear, whenever there is no arbitrage. Another is the following transversality constraint:

$$\lim_{t\to\infty} \sum_{s^t\in\mathcal{N}_t} a(s^t)q(s^t)'\theta(s^t) = 0, \quad \textit{for each} \;\; a \in A. \tag{25}$$

Constraint (25) requires the state-price weighted sum of portfolio net worth over each information partition be eventually zero. In the presence of (25), the replication price coincides with the fundamental value on the asset span of the event-tree and the replication price functional is linear.

**THEOREM 5.1** *Let $\Theta$ be a set of portfolio strategies induced by (25). Then there is no finite or infinite arbitrage in $\Theta$, and $V$ is strictly positive, linear and countably additive on $\mathcal{M}$ and satisfies*

$$V(z) = \sum_{t=1}^{\infty} \sum_{s^t\in\mathcal{N}_t} \frac{a(s^t)}{a(s^0)} z(s^t) \tag{26}$$

*for each $z \in \mathcal{M}$ and each $a \in A$, provided that the right-hand side of (26) is well-defined.*

Equation (26) displays linear and countably additive valuation. If (25) were imposed on a subtree (26) would hold for that subtree as well. We close this section by establishing linear replication price functional for the portfolio constraints of section 4 in the case of complete markets. The following is a direct corollary of theorem 4.1:

**COROLLARY 5.1** *Suppose that markets are complete. Let $\Theta$ be a set of portfolio strategies that satisfies properties (FL) and (UL). If there is no finite or infinite arbitrage in $\Theta$, then there is a unique state-price system $a$, and $V_{s^t}$ is linear and countably additive on $\mathcal{M}^+_{s^t}$ for each $s^t$ and satisfies*

$$V_{s^t}(z) = \sum_{s^\tau\in\mathcal{D}_{s^t}\setminus\{s^t\}} \frac{a(s^\tau)}{a(s^t)} z(s^\tau) \tag{27}$$

*for each* $z \in \mathcal{M}_{st}^{+}$.

Equation (27) displays linear and countably additive valuation for nonnegative payoff streams. Corollary 5.1 extends theorem 5.1 of Huang and Werner (1998) to stochastic settings with complete markets.

# 6 The Three Measures of Value in Infinite Horizon Markets

We have established the equality between the replication price and the fundamental value in the absence of arbitrage under certain portfolio constraints. This in particular implies that the infimum cost to purchase the dividends of an asset in sequential markets is equal to the fundamental value of the dividends. According to (3), the fundamental value must be no greater than the market price of the asset. It turns out that the market price of an asset of finite maturity coincides with the fundamental value of its dividends and (3) holds as an equality regardless of the system of state-prices chosen, provided that there is no finite arbitrage and strategies with nonnegative portfolio net worth are admissible.

**THEOREM 6.1** *Let $\Theta$ be a set of portfolio strategies that satisfies property (UL). If there is no finite arbitrage in $\Theta$, then $A \neq \emptyset$, and*

$$q_j(s^t) = \sum_{s^\tau \in \mathcal{D}_{s^t} \setminus \{s^t\}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau) \tag{28}$$

*for each $s^t$, each finite maturity asset $j$ traded at $s^t$, and arbitrary $a \in A_{s^t}$.*

The objective of this section is to characterize portfolio constraints for which the three measures of value coincide in the absence of arbitrage, and in equilibrium. We first characterize constraints for which there is no price bubbles whenever there is no arbitrage. The following is a relevant property of a set $\Theta$ of portfolio strategies:

**(NB)** there is $\epsilon > 0$ such that, if $\frac{q(s^t)'\theta(s^t)}{\|q(s^t)\|} \geq -\epsilon$ for each $s^t \in \mathcal{D}$ and $\frac{R(s^t)'\theta(s^t_-)}{\|q(s^t)\|} \geq -\epsilon$ for each $s^t \in \mathcal{D} \setminus \{s^0\}$ then $\theta \in \Theta$.

Property (NB) means that strategies with deflated portfolio net worth above a certain negative position are admissible. This is interpreted as *Non-zero Borrowing.* It turns out that the absence of finite and infinite arbitrage implies the absence of price bubbles whenever the constraint satisfies property (NB). The duality technique is applied in establishing the following result.

**THEOREM 6.2** *Let $\Theta$ be a set of portfolio strategies that satisfies property (NB). If there is no finite or infinite arbitrage in $\Theta$, then $A \neq \emptyset$, and*

$$q_j(s^t) = \sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t} \setminus \{s^t\}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau) \tag{29}$$

*for each $s^t$ and each asset $j$ traded at $s^t$.*

Proof: Since $\Theta$ satisfies property (NB) it satisfies property (UL) as well. By lemma 4.1, the absence of finite arbitrage in $\Theta$ implies the absence of finite arbitrage, thus $A \neq \emptyset$ and the right-hand side of (29) is well-defined. The remaining of the proof is equivalent to showing that whenever there is a price bubble there is infinite arbitrage. Note that this is non-trivial only if there is no fiat money with a positive price. Suppose that there is a price bubble for some asset $j$ traded at some $s^t$, i.e.,

$$\eta_j(s^t) \equiv q_j(s^t) - \sup_{a \in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t} \setminus \{s^t\}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau) > 0.$$

For each $s^r \in \mathcal{D}_{s^t}$, consider the following minimization problem:

$$\min_{\theta(s^r)} \quad q(s^r)'\theta(s^r) \tag{30}$$

$$\text{s.t.} \quad R(s^{r+1})'\theta(s^r) \geq \sup_{a \in A_{s^{r+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{r+1}}} \frac{a(s^\tau)}{a(s^{r+1})} d_j(s^\tau), \quad s^{r+1} \in \{s^r_+\}; \tag{31}$$

and its dual:

$$\max_{\substack{\alpha(s^{r+1}) \\ s^{r+1} \in \{s^r_+\}}} \quad \sum_{s^{r+1} \in \{s^r_+\}} \alpha(s^{r+1}) [\sup_{a \in A_{s^{r+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{r+1}}} \frac{a(s^\tau)}{a(s^{r+1})} d_j(s^\tau)] \tag{32}$$

$$\text{s.t} \quad \sum_{s^{r+1} \in \{s^r_+\}} \alpha(s^{r+1}) R(s^{r+1}) = q(s^r), \quad \alpha(s^{r+1}) \geq 0, \quad s^{r+1} \in \{s^r_+\}. \tag{33}$$

We claim that both (31) and (33) have feasible solutions. That (33) has a feasible solution simply follows from the existence of a state-price system. To show that (31) has a feasible solution, let $\theta$ be a strategy of purchasing one share of asset $j$ at $s^r$ without further re-trading. We can use relation (1) and $z^\theta = d_j | s^r$ to obtain the following (in)equalities for each $s^{r+1} \in \{s^r_+\}$, arbitrary $a \in A_{s^{r+1}}$, and any $u \geq r+1$,

$$\begin{aligned} a(s^{r+1}) R(s^{r+1})'\theta(s^r) &= \sum_{\tau=r+1}^{u} \sum_{s^\tau \in \mathcal{D}_{s^{r+1}} \cap \mathcal{N}_\tau} a(s^\tau) d_j(s^\tau) + \sum_{s^u \in \mathcal{D}_{s^{r+1}} \cap \mathcal{N}_u} a(s^u) q_j(s^u) \\ &\geq \sum_{\tau=r+1}^{u} \sum_{s^\tau \in \mathcal{D}_{s^{r+1}} \cap \mathcal{N}_\tau} a(s^\tau) d_j(s^\tau). \end{aligned}$$

Taking $u \to \infty$ on the right-most side of the above (in)equalities leads to

$$a(s^{r+1}) R(s^{r+1})'\theta(s^r) \geq \sum_{s^\tau \in \mathcal{D}_{s^{r+1}}} a(s^\tau) d_j(s^\tau).$$

That $a$ is arbitrarily chosen from $A_{s^{r+1}}$ implies

$$R(s^{r+1})'\theta(s^r) \geq \sup_{a\in A_{s^{r+1}}} \sum_{s^\tau \in \mathcal{D}_{s^{r+1}}} \frac{a(s^\tau)}{a(s^{r+1})} d_j(s^\tau).$$

Thus, $\theta(s^r)$ is a feasible solution to (31). By the duality theorem of linear programming, both the primal and dual problems have finite optimal solutions and the values of their optimal objectives (30) and (32) are equal. By similar continuity and dynamic programming arguments as in the proof of theorem 4.1, the value of the optimal objective of the dual problem is equal to

$$\sup_{a\in A_{s^r}} \sum_{s^\tau \in \mathcal{D}_{s^r}\backslash\{s^r\}} \frac{a(s^\tau)}{a(s^r)} d_j(s^\tau).$$

Repeating the above procedure for every $s^r \in \mathcal{D}_{s^t}$ shows that there is a portfolio strategy $\hat{\theta}$ such that, $\hat{\theta}(s^r)$ is an zero asset holding if $s^r \notin \mathcal{D}_{s^t}$ and an optimal solution to the primal problem (30)-(31) if $s^r \in \mathcal{D}_{s^t}$. It follows that

$$\begin{aligned} q(s^r)'\hat{\theta}(s^r) &= \sup_{a\in A_{s^r}} \sum_{s^\tau \in \mathcal{D}_{s^r}\backslash\{s^r\}} \frac{a(s^\tau)}{a(s^r)} d_j(s^\tau) \geq 0, \;\; s^r \in \mathcal{D}_{s^t}, & (34)\\ R(s^r)'\hat{\theta}(s^r_-) &\geq \sup_{a\in A_{s^r}} \sum_{s^\tau \in \mathcal{D}_{s^r}} \frac{a(s^\tau)}{a(s^r)} d_j(s^\tau) \geq d_j(s^r) \geq 0, \;\; s^r \in \mathcal{D}_{s^t}\backslash\{s^t\}. & (35)\end{aligned}$$

Note that (34) and (35) imply $z^{\hat{\theta}} \geq d_j|s^t$.

Let $\bar{\theta}$ be a portfolio strategy of short-selling $\epsilon$ shares of asset $j$ at $s^t$ without further re-trading. Define a strategy $\tilde{\theta}$ by $\tilde{\theta} \equiv \epsilon\hat{\theta} + \bar{\theta}$. It follows that,

$$\begin{aligned} \frac{q(s^r)'\tilde{\theta}(s^r)}{\| q(s^r) \|} &= \frac{q(s^r)'(\epsilon\hat{\theta}(s^r) + \bar{\theta}(s^r))}{\| q(s^r) \|} \qquad (36)\\ &\geq \frac{q(s^r)'\bar{\theta}(s^r)}{\| q(s^r) \|} = \frac{-\epsilon q_j(s^r)}{\| q(s^r) \|} \geq -\epsilon \end{aligned}$$

holds for each $s^r \in \mathcal{D}_{s^t}$ where the first inequality follows from (34), and

$$\begin{aligned} \frac{R(s^r)'\tilde{\theta}(s^r_-)}{\| q(s^r) \|} &= \frac{R(s^r)'(\epsilon\hat{\theta}(s^r_-) + \bar{\theta}(s^r_-))}{\| q(s^r) \|} \qquad (37)\\ &\geq \frac{\epsilon d_j(s^r) - \epsilon(q_j(s^r) + d_j(s^r))}{\| q(s^r) \|} = \frac{-\epsilon q_j(s^r)}{\| q(s^r) \|} = -\epsilon \end{aligned}$$

holds for each $s^r \in \mathcal{D}_{s^t}\backslash\{s^t\}$ where the first inequality follows from (35). Since $\Theta$ satisfies property (NB), (36) and (37) imply $\tilde{\theta} \in \Theta$. In fact, $\tilde{\theta}$ is an arbitrage since (34) implies

$$q(s^t)'\tilde{\theta}(s^t) = q(s^t)'(\epsilon\hat{\theta}(s^t) + \bar{\theta}(s^t)) = \epsilon \sup_{a\in A_{s^t}} \sum_{s^\tau \in \mathcal{D}_{s^t}\backslash\{s^t\}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau) - \epsilon q_j(s^t) = -\epsilon\eta_j(s^t) < 0,$$

and $z^{\hat{\theta}} \geq d_j|s^t$ and $z^{\bar{\theta}} = -\epsilon d_j|s^t$ imply

$$z^{\tilde{\theta}}(s^r) = z^{\epsilon\hat{\theta}}(s^r) + z^{\bar{\theta}}(s^r) = \epsilon z^{\hat{\theta}}(s^r) + z^{\bar{\theta}}(s^r) \geq \epsilon d_j(s^r) - \epsilon d_j(s^r) = 0$$

for each $s^r \in \mathcal{D}_{s^t} \backslash \{s^t\}$. $\square$

Theorems 4.1 and 6.2 together imply that the three measures of value coincide whenever there is no arbitrage and the constraint satisfies properties (FL) and (NB). Note that both theorems are applicable to the following portfolio constraints:

$$\inf_{s^t \in \mathcal{D}} \frac{q(s^t)'\theta(s^t)}{\| q(s^t) \|} > -\infty; \tag{38}$$

$$\inf_{s^t \in \mathcal{D}\backslash\{s^0\}} \frac{R(s^t)'\theta(s^t_-)}{\| q(s^t) \|} > -\infty; \tag{39}$$

$$\inf_{s^t \in \mathcal{D}} \frac{q(s^t)'\theta(s^t)}{\| q(s^t) \|} > -\infty \textit{ and } \inf_{s^t \in \mathcal{D}\backslash\{s^0\}} \frac{R(s^t)'\theta(s^t_-)}{\| q(s^t) \|} > -\infty. \tag{40}$$

In fact, a set $\Theta$ of portfolio strategies induced by any of the constraints (38)-(40) not only satisfies properties (NB) and (FL) but is a convex cone. Therefore, there cannot be arbitrage in equilibria with such constraints. This leads to the following result.

**THEOREM 6.3** *The three measures of value coincide and the replication price functional is sub-linear (linear, if markets are complete) in any equilibrium with one of the constraints (38)-(40).*

The no-bubble result implied by theorem 6.3 complements theorem 3.1 of Santos and Woodford (1997) establishing sufficient conditions on aggregate asset supply and fundamental value of aggregate endowments for the absence of equilibrium price bubbles when investors are faced with non-positive lower bounds on their portfolio net worth. According to theorem 6.3 there is no equilibrium price bubbles regardless of the supply of assets and the endowment of goods. We provide an example of such equilibrium in next section.

# 7 Examples

Under a number of assumptions the three measures of value coincide: they are equal to the supremum of the present value over the set of state-price systems. In finite horizon markets the present value is the same regardless of the system of state-prices chosen. In this section, we provide examples of infinite horizon incomplete markets in which the present value is state-price dependent, and in which the equivalence of the three measures of value breaks down in one or another way when one or another of these assumptions are violated.

Our first example shows that even when the assumptions in theorem 4.1 or 4.2 are satisfied the present value of some payoff stream can differ for different state-price systems, so the replication price functional can be non-linear. It also shows that when one or another of these assumptions are violated the replication price can be strictly lower than the fundamental value for some payoff stream, so the replication price functional can fail to be sub-linear.

**EXAMPLE 7.1** Consider an event-tree with uncertainty only at date-1. More specifically, the root $s^0$ of the event-tree has two immediate successors $\xi^1$ and $\zeta^1$, where both $\{\xi^t_+\} = \{\xi^{t+1}\}$ and $\{\zeta^t_+\} = \{\zeta^{t+1}\}$ are singleton sets for $t \geq 1$. One asset of no maturity date is traded that has a price-dividend process $q(s^0) = q(\xi^t) = q(\zeta^t) = 1,\ d(\xi^t) = d(\zeta^t) = 0$, for $t \geq 1$. Markets are incomplete since only one asset is traded at date-0 but two possible events can be realized at date-1. This price-dividend process admits no finite arbitrage. Therefore the set $A$ of state-price systems is nonempty and can be indexed by $\alpha \in (0,1)$ as follows:

$$A = \left\{ \left( \frac{a(\xi^t)}{a(s^0)}, \frac{a(\zeta^t)}{a(s^0)} \right)_{t=1}^{\infty} \in I\!R^{\infty}_{++} \times I\!R^{\infty}_{++} : \frac{a(\xi^t)}{a(s^0)} = \alpha,\ \frac{a(\zeta^t)}{a(s^0)} = 1-\alpha,\ t \geq 1,\ 0 < \alpha < 1 \right\}.$$

We denote by $\bar{z}$, $\hat{z}$ and $z$ the following payoff streams on the event-tree:

$$\begin{aligned}
&\bar{z}(\xi^1) = 1,\ \bar{z}(\zeta^1) = 0,\ \bar{z}(\xi^t) = \bar{z}(\zeta t) = 0,\ t \geq 1,\\
&\hat{z}(\xi^1) = 0,\ \hat{z}(\zeta^1) = 1,\ \hat{z}(\xi^t) = \hat{z}(\zeta^t) = 0,\ t \geq 1,\\
&z(\xi^1) = 1,\ z(\zeta^1) = 1,\ z(\xi^t) = z(\zeta^t) = 0,\ t \geq 1,
\end{aligned}$$

and by $\bar{\theta}$, $\hat{\theta}$ and $\theta$ the following portfolio strategies:

$$\begin{aligned}
&\bar{\theta}(s^0) = 1,\ \bar{\theta}(\xi^t) = 0,\ \bar{\theta}(\zeta^t) = 1,\ t \geq 1,\\
&\hat{\theta}(s^0) = 1,\ \hat{\theta}(\xi^t) = 1,\ \hat{\theta}(\zeta^t) = 0,\ t \geq 1,\\
&\theta(s^0) = 1,\ \theta(\xi^t) = 0,\ \theta(\zeta^t) = 0,\ t \geq 1.
\end{aligned}$$

It can be verified that strategies $\bar{\theta}$, $\hat{\theta}$ and $\theta$ finance payoff streams $\bar{z}$, $\hat{z}$ and $z$, respectively. We consider three portfolio constraints in sequence:

*Zero-borrowing limits:* Properties (FL) and (UL) are satisfied and there is no infinite arbitrage. It follows that condition (i) in theorem 4.1 is met. Since strategies $\bar{\theta}$, $\hat{\theta}$ and $\theta$ are feasible, payoff streams $\bar{z}$, $\hat{z}$ and $z$ are in the asset span. We show that $\bar{\theta}$, $\hat{\theta}$ and $\theta$ are in fact optimal replicating strategies for $\bar{z}$, $\hat{z}$ and $z$, respectively. The optimal strategy of replicating $z$ is buying one share at date-0 and selling it at date-1, which is strategy $\theta$. To replicate $\bar{z}$, the best one can do is buying one share at date-0 and selling it at date-1 if event $\xi^1$ is realized, which is strategy $\bar{\theta}$. The best one can do to replicate $\hat{z}$ is buying one share at date-0 and selling it at date-1 if event $\zeta^1$ is realized, which is strategy $\hat{\theta}$. Obviously, these three strategies incur the same date-0 cost equal to one. The above argument can be verified by applying theorem 4.1 to calculate the replication price for $\bar{z}$, $\hat{z}$ and $z$:

$$\begin{aligned} V(\bar{z}) &= \sup_{0<\alpha<1} \{\alpha \cdot 1 + (1-\alpha)\cdot 0\} = 1, \\ V(\hat{z}) &= \sup_{0<\alpha<1} \{\alpha \cdot 0 + (1-\alpha)\cdot 1\} = 1, \\ V(z) &= \sup_{0<\alpha<1} \{\alpha \cdot 1 + (1-\alpha)\cdot 1\} = 1. \end{aligned}$$

For payoff stream $\bar{z}$ or $\hat{z}$, there is a continuum of the present value corresponding to different state-price systems and the replication price is strictly higher each of them. Consequently, the replication price functional $V$ is non-linear since $\bar{z}+\hat{z}=z$ but $1 = V(z) = V(\bar{z}+\hat{z}) < V(\bar{z}) + V(\hat{z}) = 2$.

*Constraint (23):* The assumptions in theorem 4.2 are satisfied and the same argument and conclusion for the above case of zero-borrowing limits remain valid.

*The transversality constraint:*

$$\liminf_{t\to\infty} \sum_{s^t \in \mathcal{N}_t} a(s^t) q(s^t)'\theta(s^t) \geq 0, \;\; for \;\; some \;\; a \in A. \tag{41}$$

Property (FL) is violated and there is no infinite arbitrage. Strategies $\bar{\theta}$, $\hat{\theta}$ and $\theta$ are feasible, so payoff streams $\bar{z}$, $\hat{z}$ and $z$ are in the asset span. While $\theta$ remains an optimal replicating strategy for $z$, $\bar{\theta}$ and $\hat{\theta}$ are no longer optimal replicating strategies for $\bar{z}$ and $\hat{z}$. We claim $V(\bar{z})$, $V(\hat{z}) \leq 0$.[20] To prove $V(\bar{z}) \leq 0$, consider for an arbitrary $\epsilon \in (0,1)$ a strategy of buying $\epsilon$ shares at $s^0$ and selling 1 share at $\xi^1$ without other trading activities. This strategy finances $\bar{z}$. It satisfies constraint (41) for any state-price system indexed by $\alpha \in (0,\epsilon)$, thus is feasible. This means $V(\bar{z}) \leq \epsilon$. Since $\epsilon \in (0,1)$ is arbitrarily chosen, $V(\bar{z}) \leq 0$. We can similarly prove $V(\hat{z}) \leq 0$. Therefore, the following holds for any state-price system indexed by $\alpha^* \in (0,1)$:

$$\begin{aligned} V(\bar{z}) \leq 0 \;\; &< \;\; \alpha^* \cdot 1 + (1-\alpha^*)\cdot 0 = \alpha^* < \sup_{0<\alpha<1} \{\alpha \cdot 1 + (1-\alpha)\cdot 0\} = 1, \\ V(\hat{z}) \leq 0 \;\; &< \;\; \alpha^* \cdot 0 + (1-\alpha^*)\cdot 1 = 1-\alpha^* < \sup_{0<\alpha<1} \{\alpha \cdot 0 + (1-\alpha)\cdot 1\} = 1. \end{aligned}$$

[20] It can be shown $V(\bar{z}) = V(\hat{z}) = 0$.

For payoff stream $\bar{z}$ or $\hat{z}$, the replication price is strictly less than the fundamental value. Consequently, the replication price functional $V$ fails to be sub-linear since $1 = V(z) = V(\bar{z} + \hat{z}) > 0 \geq V(\bar{z}) + V(\hat{z})$. □

The above example involves fiat money with a positive price so a price bubble occurs. In the following example, the assumptions in theorem 6.2 are satisfied so there is no bubble in the price of a traded asset. In each date-event, however, there is a continuum of the present value of the dividend stream of the asset corresponding to different state-price systems.

**EXAMPLE 7.2** Consider the following event-tree with root $s^0$: node $s^t$ has two immediate successors $s^{t+1}$ and $\xi_1^{t+1}$ for each $t \geq 0$ and node $\xi_r^t$ has one immediate successor $\xi_{r+1}^t$ for each $t \geq 1$ and $r \geq 1$. One asset is traded that has the following price-dividend process:

$$\begin{aligned} q(s^t) &= 1, \; d(s^t) = 0, \qquad \forall\, t \geq 0, \\ q(\xi_r^t) &= d(\xi_r^t) = 1/2^r, \qquad \forall\, t \geq 1, \qquad \forall\, r \geq 1. \end{aligned}$$

This price-dividend process admits no finite arbitrage. Therefore the set $A$ of state-price systems is nonempty and can be indexed as follows:

$$A = \left\{ \left( a(s^0) = \alpha_0 \equiv 1; a(s^t) = \prod_{j=0}^{t} \alpha_j; a(\xi_r^t) = \prod_{j=0}^{t-1} \alpha_j (1 - \alpha_t), r \geq 1 \right)_{t=1}^{\infty} : 0 < \alpha_j < 1, \; \forall\, j \geq 1 \right\}.$$

Consider any of the constraints (38)-(40). Clearly, property (NB) is satisfied. We claim that there is no infinite arbitrage. To prove it, note that any infinite arbitrage strategy has to involve short-selling the asset in some date-event. Consider a strategy $\theta$ that involves short-selling $\epsilon$ shares at, say, $s^t$. In order for $\theta$ to be an arbitrage, it is necessary $\theta(\xi_r^{t+1}) \leq -2^r \epsilon$ for each $r \geq 1$. But such $\theta$ is not feasible. We can similarly show that no strategy that involves short-selling at some $\xi_r^t$ can be a feasible arbitrage. This proves our claim. Therefore the assumptions in theorem 6.2 are met. It follows that in each date-event, the asset price is equal to the fundamental value of its future dividend stream. To verify this we first note that the present value of the dividend stream at, say, $s^0$, for a given state-price system is:

$$\lim_{t \to \infty} \sum_{\tau=0}^{t} \left( \prod_{j=0}^{\tau} \alpha_j \right) (1 - \alpha_{\tau+1}) = 1 - \lim_{t \to \infty} \prod_{j=0}^{t} \alpha_j \;\in\; (0, 1). \tag{42}$$

The fundamental value of the dividend stream is obtained by taking supremum of (42) over the set of state-price systems:

$$\sup_{\substack{\alpha_j \in (0,1) \\ j \geq 0}} 1 - \lim_{t \to \infty} \prod_{j=0}^{t} \alpha_j = 1,$$

which is equal to the asset price at $s^0$. We can similarly verify that there is no price bubbles at other nodes. In each date-event, however, there is a continuum of the present value of the dividend stream of the asset corresponding to different state-price systems and the price of the asset is strictly higher than each of them. To verify this, we note that the following indexes a state-price system:

$$\alpha_0 = 0, \ \ \alpha_j = e^{-\frac{b}{j(j+1)}}, \ j \geq 1 \tag{43}$$

for any $b \in (0, \infty)$. Applying the state-price system (43) to (42) yields the present value of the dividend stream at $s^0$ to be $1 - e^{-b}$, which ranges continuously from 0 to 1 as $b$ ranges continuously from 0 to $\infty$. Therefore, there is a continuum of the present value of the dividend stream of the asset at $s^0$. We can similarly verify this for each $s^t$, $t \geq 1$. □

In the following example, we present an equilibrium with constraint (38) as of theorem 6.3. The three measures of value coincide in this equilibrium.

**EXAMPLE 7.3** Consider an infinite horizon sequential markets economy with no uncertainty. Two infinitely lived investors have the same utility function

$$u(c) = \sum_{t=0}^{\infty} \beta^t ln(c_t),$$

where $0 < \beta < 1$. Their consumption endowments are

$$\begin{array}{lll} y_t^1 = b - 1, & y_t^2 = b + \beta, & for\ t\ even, \\ y_t^1 = b + \beta, & y_t^2 = b - 1, & for\ t\ odd, \end{array}$$

where $b \geq 1$. One asset is traded that has a dividend stream $\{d_t\}_{t=0}^{\infty}$, where $d_t = 1 - \beta$ for each $t \geq 0$, and a supply equal to 1. The investors' initial holdings of the asset are $\bar{\theta}_0^1 = 1$ and $\bar{\theta}_0^2 = 0$. Each investor is faced with constraint (38):

$$\inf_{t \geq 0} \frac{q_t \theta_t}{\| q_t \|} = \inf_{t \geq 0} \theta_t > -\infty.$$

There is an equilibrium with price path

$$q_t = \beta, \qquad \forall\, t \geq 0, \tag{44}$$

and allocation of consumption and asset holding

$$\begin{array}{rcll} c_t^1 & = & c_t^2 = b & for\ \forall\, t \geq 0; \\ \theta_t^1 & = & 0, & for\ t\ even, \\ & = & 1, & for\ t\ odd, \\ \theta_t^2 & = & 1, & for\ t\ even, \\ & = & 0, & for\ t\ odd. \end{array}$$

To verify that this is indeed an equilibrium, note that markets clear and the budget and portfolio constraints are all satisfied. It remains to verify that the consumption and asset holding plans are optimal for the investors. One can easily verify that,

$$u_t(c^h)q_t = u_{t+1}(c^h)(q_{t+1} + d_{t+1}) \tag{45}$$

holds for each investor $h = 1, 2$ and each date $t \geq 0$, where $u_t(c^h) = \beta^t / c_t^h$ is the partial derivative of $u$ with respect to date-$t$ consumption. It follows that it is suboptimal for either investor to increase his consumption at one date by reducing his consumption at others. Moreover, confronted with the portfolio constraint, neither investor can increase his consumption at some date without reducing his consumption at others. To verify this, consider a strategy of lowering an investor's asset holding by $\epsilon$ shares to finance an extra consumption at some date $t \geq 0$. In order to avoid reducing his consumption at future dates, the investor has to lower his asset holding by at least $\epsilon\beta^{-\tau}$ shares at date $t + \tau$ for each $\tau \geq 1$, as required by his budget constraint. This violates his portfolio constraint since the assumption $0 < \beta < 1$ implies $-\beta^{-\tau} \to -\infty$ as $\tau \to \infty$. This shows that the given price-allocation constitutes an equilibrium. According to theorem 6.3, the three measures of value coincide in this equilibrium. To verify that there is no price bubbles, we apply relation (1) to obtain the unique state price system $a_t = \beta^t$ for each $t \geq 0$. It follows that

$$\sum_{\tau=t+1}^{\infty} \frac{a_\tau}{a_t} d_\tau = (1 - \beta) \sum_{\tau=1}^{\infty} \beta^\tau = \beta = q_t$$

for each $t \geq 0$. □

# 8 Concluding Remark

We have developed a theory of valuation and asset pricing in finite and infinite horizon sequential markets with portfolio constraints. We characterize constraints for which the market price, the replication price, and the fundamental value coincide and the replication price functional is linear or sub-linear, in the absence of arbitrage, and in equilibrium. To help exposition our results are presented under a simple asset structure, but they hold equally well for general asset markets such as Santos and Woodford (1997) in which there are multiple consumption goods and an asset represents claims to not only consumption dividends but also shares of own and other assets. Therefore our results are readily applicable to markets with a broad range of assets such as convertible bonds, stock splits and derivatives, etc. The duality approach can be extended to account for other types of market frictions such as transaction cost or general polyhedral cone constraints on asset holdings, and for valuation and pricing of other types of resource streams.

# 9 Appendix

Proof of lemma 3.1: Suppose that $\theta$ is an one-period arbitrage at some $s^t$. If $q(s^t)'\theta(s^t) = 0$, then $R(s^{t+1})'\theta(s^t) \geq 0$ for each $s^{t+1} \in \{s_+^t\}$ with at least one strictly inequality. Therefore $\theta$ is an one-period arbitrage in $\Theta$ for $\Theta$ satisfies property (FUL). If $q(s^t)'\theta(s^t) < 0$, then some asset $j$ is traded for a positive price at $s^t$ with $\theta_j(s^t) < 0$. The assumption that there is always some way of carrying wealth into the future implies that there is some asset $k$ traded at $s^t$ with $R_k(s^{t+1}) > 0$ for some $s^{t+1} \in \{s_+^t\}$. Hence, we can modify $\theta(s^t)$ by increasing the shares of asset $k$ and, if $q_k(s^t) = 0$, also increasing the shares of asset $j$, to obtain a portfolio strategy $\hat{\theta}$ such that $q(s^t)'\hat{\theta}(s^t) = 0$ and $R(s^{t+1})'\hat{\theta}(s^t) \geq 0$ for each $s^{t+1} \in \{s_+^t\}$, with strict inequality for those $s^{t+1} \in \{s_+^t\}$ at which $R_k(s^{t+1}) > 0$. Therefore $\hat{\theta}$ is an one-period arbitrage in $\Theta$. The conclusion then follows from the observation that in finite horizon markets there is no arbitrage if and only if there is no one-period arbitrage. □

Proof of theorems 3.1 and 3.2: The implication $(iii) \Longrightarrow (i)$ follows from the definition of $V$. Lemma 3.1 gives rise to the implication $(i) \Longrightarrow (ii)$. Suppose that $(ii)$ holds and choose an arbitrary $a \in A$. We now establish $(iii)$, (5) and (6) for the event-tree. For each $z \in \mathcal{M}$, let $\theta$ be a strategy in $\Theta$ that generates $z$. Since $T$ is the terminal date of the finite horizon markets, we have

$$\begin{aligned} z(s^t) &= R(s^t)'\theta(s_-^t) - q(s^t)'\theta(s^t), \ \ s^t \in \mathcal{N}_t, \ \ 0 < t < T, \\ z(s^T) &= R(s^T)'\theta(s_-^T), \ \ s^T \in \mathcal{N}_T. \end{aligned}$$

The above equations and (1) imply

$$q(s^0)'\theta(s^0) = \sum_{t=1}^{T} \sum_{s^t \in \mathcal{N}_t} \frac{a(s^t)}{a(s^0)} z(s^t) \equiv \sum_{s^\tau \in \mathcal{D} \setminus \{s^0\}} \frac{a(s^\tau)}{a(s^0)} z(s^\tau). \tag{46}$$

That $\theta$ is an arbitrary strategy in $\Theta$ that generates $z$ implies

$$V(z) = \sum_{t=1}^{T} \sum_{s^t \in \mathcal{N}_t} \frac{a(s^t)}{a(s^0)} z(s^t) \equiv \sum_{s^\tau \in \mathcal{D} \setminus \{s^0\}} \frac{a(s^\tau)}{a(s^0)} z(s^\tau). \tag{47}$$

Equation (47) implies that $V$ is linear and strictly positive, i.e., $(iii)$ holds, and establishes (5) for the event-tree. In the case $z = d_j$ and $\theta$ is a strategy of buying one share of asset $j$ at $s^0$ without further re-trading, equation (46) gives rise to (6) for the event-tree. Similarly, we can obtain (5) and (6) for each non-terminal node $s^t$. □

Proof of theorem 3.3: Consider an equilibrium with price-payoff process $\hat{q}$-$\hat{R}$. Suppose, by contradiction, that there is an arbitrage. Then, by the proof of lemma 3.1, there is an one-period arbitrage $\theta$ at some $s^t$, such that $\hat{q}(s^t)'\theta(s^t) = 0$ and $\hat{R}(s^{t+1})'\theta(s^t) \geq 0$ for each $s^{t+1} \in \{s_+^t\}$ with at least one strictly inequality. Consider an investor $h$ who presents in some event $s^{t+1} \in \{s^t_+\}$ in which $\hat{R}(s^{t+1})'\theta(s^t) > 0$, and has an optimal consumption-portfolio plan $(\hat{c}^h, \hat{\theta}^h)$. He can choose strategy $\hat{\theta}^h + \theta$, which is feasible for $\hat{\theta}^h$ is, to increase his consumption by $\hat{R}(s^{t+1})'\theta(s^t)$ at each $s^{t+1} \in \{s^t_+\}$ without reducing his consumption in other date-events or violating his budget constraint. Therefore, $(\hat{c}^h, \hat{\theta}^h)$ cannot be his optimal plan for he prefers more consumption to less in any date-event he presents. The conclusion then follows from theorems 3.1 and 3.2. □

Proof of lemma 4.1: Similar as the proof of lemma 3.1. □

Proof of theorem 4.2: Since $\Theta$ is given by (23), $A \neq \emptyset$. We first establish

$$V(z) \geq \sup_{a \in A} \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t) \tag{48}$$

for each $z \in \mathcal{M}^+$. Let $\theta$ be a portfolio strategy in $\Theta$ that finances $z$. Choose an arbitrary $a \in A$, and use relation (1) and $z^\theta = z \geq 0$ to get, for any $\tau \geq 1$,

$$a(s^0)q(s^0)\theta(s^0) = \sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) + \sum_{s^\tau \in \mathcal{N}_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau).$$

Taking liminf on both sides of the above equation with respect to $\tau$ leads to

$$\begin{aligned} a(s^0)q(s^0)\theta(s^0) &= \liminf_{\tau\to\infty}[\sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) + \sum_{s^\tau \in \mathcal{N}_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau)] \\ &\geq \liminf_{\tau\to\infty}\sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) + \liminf_{\tau\to\infty} \sum_{s^\tau \in \mathcal{N}_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau) \\ &\geq \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} a(s^t)z(s^t), \end{aligned}$$

where the last inequality holds since $\Theta$ is given by (23) and $\theta \in \Theta$, and the first inequality holds since the sum of the two liminfs on the right-hand side is well-defined. That $a$ is arbitrarily chosen implies

$$q(s^0)'\theta(s^0) \geq \sup_{a \in A} \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t).$$

That $\theta$ is an arbitrary strategy in $\Theta$ that finances $z$ implies

$$V(z) \geq \sup_{a \in A} \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t).$$

This establishes (48). Strict positivity of $V$ then follows from (48).

To establish

$$V(z) \leq \sup_{a \in A} \sum_{s^t \in \mathcal{D}\setminus\{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t) \tag{49}$$

for each $z \in \mathcal{M}^+$, we take a similar duality approach as in the proof of theorem 4.1, noting that there is always a feasible solution to the primal problem due to the assumption that for each $s^t \in \mathcal{D}\setminus\{s^0\}$ there is some asset $j$ traded at $s^t_-$ such that $R_j(s^t) > 0$. The remaining proof is similar. This establishes (49) and, combined with (48), gives rise to (24). The sub-linearity of $V$ then follows from (24). □

Proof of theorem 5.1: Let $z \in \mathcal{M}$ and $\theta$ be an arbitrary strategy in $\Theta$ that generates $z$. Choose an arbitrary $a \in A$ for which the right-hand side of (26) is well-defined. Multiplying both sides of (1) by $\theta(s^t)$ and iterating, we obtain, for any $\tau \geq 1$,

$$a(s^0)q(s^0)\theta(s^0) = \sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) + \sum_{s^\tau \in \mathcal{N}_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau).$$

Taking limit on both sides of the above equation with respect to $\tau$ leads to

$$\begin{aligned} a(s^0)q(s^0)\theta(s^0) &= \lim_{\tau\to\infty}[\sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) + \sum_{s^\tau \in \mathcal{N}_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau)] \\ &= \lim_{\tau\to\infty}\sum_{t=1}^{\tau} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t) + \lim_{\tau\to\infty}\sum_{s^\tau \in \mathcal{N}_\tau} a(s^\tau)q(s^\tau)'\theta(s^\tau) \\ &= \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{N}_t} a(s^t)z(s^t), \end{aligned}$$

where the last equality holds since $\Theta$ is given by (25) and $\theta \in \Theta$, and the second equality holds since the two limits and their sum on the right-hand side are well-defined. That $\theta$ is an arbitrary strategy in $\Theta$ that generates $z$ implies

$$V(z) = \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{N}_t} \frac{a(s^0)}{a(s^t)} z(s^t).$$

This gives rise to (26), which implies that $V$ is strictly positive, linear and countably additive, and that there is no arbitrage in $\Theta$. □

Proof of corollary 5.1: Since markets are complete, there is a unique state-price system conditioning on that there is one. The conclusion then follows from theorem 4.1. □

Proof of theorem 6.1: Since $\Theta$ satisfies property (UL) and there is no finite arbitrage in $\Theta$, $A \neq \emptyset$ due to lemma 4.1. Since the asset is to be liquidated, iterating (1) for arbitrary $a \in A$ gives rise to (28). □

Proof of theorem 6.3: Consider an equilibrium in sequential markets with one of the constraints (38)-(40). Suppose, by contradiction, that there is a feasible arbitrage $\theta$. This in particular implies that $\theta$ generates a positive portfolio income at some $s^t \in \mathcal{D}$ without committing to further expenditure in other date-events. Consider an investor $h$ who presents at $s^t$ and has an optimal consumption-portfolio plan $(\hat{c}^h, \hat{\theta}^h)$. Portfolio strategy $\hat{\theta} + \theta$ is feasible for the infimum of the sum of two sequences is no less than the sum of the infimum of each. Therefore the investor can choose this strategy to increase his consumption at $s^t$, without reducing his consumption at other dates or violating his budget constraint. This contradicts the assumption that $(\hat{c}^h, \hat{\theta}^h)$ is his optimal plan for he prefers more consumption to less at $s^t$. The conclusion then follows from theorems 4.1 and 6.2. $\square$

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