THE OPTIMAL DESIGN OF FIRST-PRICE AUCTIONS WITH FINANCIAL CONSTRAINTS AND DEFAULT RISKS

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Abstract

If the bidders in an auction have financial constraints, how should the seller design the auction to maximize his profit? An observed practice is that the seller offers a loan, or interest subsidy, to the highest bidder. The work by Che and Gale [3] has given a partial answer for second-price auctions, with default risk assumed away. This paper provides a complete solution for first-price auctions, with default risk included. For each level of the interest subsidy, we solve the auction game and give a closed form solution for its symmetric equilibrium.

From this we determine each bidder's behavior as a function of the interest charged by the seller. This behavior exhibits the following novel bifurcation: for low interest rates, poor firms bid high and rich firms bid low; for high interest rates, the reverse is true. At the "critical" rate, bids from poor and rich firms are identical.

From the seller's point of view, we obtain a formula showing expected profit as a function of the interest subsidy. This allows computation of a best subsidy. We show that the best rate is always larger than the critical rate.

These results are especially applicable to auctions of large projects, where bidders' financial constraints are significant. (The FCC C-block auction is a recent example.)

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1 Introduction

If the bidders in an auction have financial constraints, how should the seller design the auction to maximize his profit? An observed practice is that the seller offers a loan, or interest subsidy, to the highest bidder. This paper provides a complete solution for first-price auctions, with default risk included. For each level of the interest subsidy, we solve the auction game and give a closed form solution for its symmetric equilibrium. From this we determine each bidder's behavior as a function of the interest charged by the seller. This behavior exhibits the following surprising bifurcation: for low interest rates, poor firms bid high and rich firms bid low; for high interest rates, the reverse is true. At the "critical" rate, bids from poor and rich firms are identical. From the seller's point of view, we obtain a formula showing expected profit as a function of the interest subsidy. This allows computation of a best subsidy. We show that the best rate is always larger than the critical rate. These results are especially applicable to auctions of large projects, where bidders' financial constraints are significant.

This study was motivated by the recent development in the spectrum auctions conducted by Federal Communications Commission (FCC), where the bidders' financial constraint has dramatically affected bidding behaviors. The most famous example is the C-block auctions conducted in the spring of 1996. Unlike the other FCC auctions, the winning bidders were allowed to delay payments for their licenses for six years in a generous rate pegged to the 30-year Treasury bond. The winning bids for the C-block auctions then totaled $10.2 billion. This figure means that the bidders pledged to pay $40 for each potential customer, almost three times as high as the A-block and B-block spectrum prices. All the winners of the C-block auctions are having trouble financing their payments. One of them, Pocket Communications, has filed for bankruptcy protection. Others are lobbying for lighter payment terms. The FCC has collected only $1.2 billion of the money pledged by the C-block bidders [1].

[1] See, for example, Riva Atlas [1] for a report of recent developments on the C-block.

Another example showing the strong impact of financial constraints is a recent series of spectrum auctions, the WCS (Wireless Communications Services) auctions. Due to the short period for preparation [2] and the tough financial environment for wireless technologies, [3] the potential bidders did not have time to line up financing. [4] This resulted in astonishingly low bids. The Congress expected to fetch \$1.8 billion, but the bidders offered only \$13.6 million. Indeed, the licenses for several metropolitan areas were sold for just \$1 each [5].

Most literature on auctions, however, assumes away the effect of financial constraints. There have been few published papers that considers the impacts of financial constraints on auctions. Among them are the works by Che and Gale [2, 3] and that by Laffont and Robert [5]. The models in Che and Gale [2] and Laffont and Robert [5] do not allow the bidders to bid above their budgets and rule out the possibility of seller-provided financing. The work by Che and Gale [3] is the first one that outlines the framework developed in this paper. In their work, Che and Gale give a rationale for the seller to provide financing services to the winning bidders in a second-price sealed-bid auction, but the seller in their setup is choosing between only two options: either to provide interest-free financing or not to provide any financing at all. Moreover, default risk is assumed away in their rationale.

In this paper, we extend the setup in Che and Gale [3] to address the following issue: Given that the bidders are constrained by their budgets and protected by their limited liabilities, how should the seller design the auction in order to maximize his profit? As a first step, this paper focuses on first-price auctions. The major drive of the model is the assumption that each bidder is financially constrained by a budget, which is the amount of funds already available to the bidder. If a bidder pays within his budget, then the cost is

[2]There was only two months from the time when the auction rule was announced (February 1997) to the time when the auction was conducted (April 1997).

[3]According to *Communications Today*, April 30, 1997, there had been almost no new issues of equity financing in the wireless industry since June 1996, and "junk bonds" were almost the only way for wireless entrepreneurs to raise funds in the days for the WCS auctions.

[4]For example, DigiVox Telecom Inc. argued that it did not have time to obtain financing. See "FCC Briefs" in *Communications Today*, April 15, 1997.

[5]See Bryan Gruley [4].

just the opportunity cost of the payment. If a bidder pledges to pay an amount higher than his budget, however, the cost he has to bear is higher than the opportunity cost of the bid, because the bidder needs to finance the extra amount. In order to stimulate the competition among the bidders, the seller may choose to provide financing services for the extra amount and charge an interest rate that may range from 0 to the cost of financing. The optimal auction design in this simple setup amounts to choosing an interest rate that maximizes the seller's expected profit.

The seller thus faces a trade-off between encouraging high bids through lowering interest rates and reducing financing cost through raising interest rates. In order to choose the best interest rate, the seller needs to know how the interest rate affects bidding behaviors. From a bidder's viewpoint, a bidding strategy is exactly what he is after. It turns out that, for each level of interest rate, there is a closed-form solution for an equilibrium of the auction game. Even better is that the equilibrium is the only symmetric equilibrium of the auction game. A surprising discovery is that the equilibrium bidding strategy "flips" around a critical value of interest rate when default risk exists. While a bidder's bidding strategy slopes upward as a function of his financial capability for those interest rates above this critical value, the strategy is downward sloping for those below the critical value. Thus, with overly generous loans, the highest bidder is the poorest bidder! Moreover, the winning bid pledged by such a "poor" bidder is higher than the expected value of the object being auctioned.

The precise solution of the auction game gives the seller a guide to choose the interest rate to maximize his expected payoff. We are able to compute the exact functional form of the seller's expected profit as a function of the interest rate he charges. This allows computation of a best interest subsidy. The paper also provides several general guidelines for choosing interest rates. One guideline is that the seller "should" (in the sense of strict dominance) always charge an interest rate above the critical value. The intuition is that an interest rate above the critical value selects the most financially capable bidder as the winner, while an interest rate below it "adversely" selects the poorest bidder as the winner. Careful calculations show that the gain from the overly high bid from the poor winner cannot make up the burden of financing for him. Another guideline is that, when default risk can be excluded and there are sufficiently many bidders, the seller "should" charge the lowest possible non-zero interest rate (given discrete options). The reason is that a positive interest rate selects the richest bidder as the winner, as long as default risk is not present. With a large number of bidders, the chance that the richest bidder requires no extra financing is high.

The rest of the paper is organized as follows: Part I focuses on a model where default risk is assumed away. Section 2 spells out the model. Section 3 derives the solution for the symmetric equilibrium of the auction game and proves that it is indeed an equilibrium. Although the core of the derivation is solving a system of differential equations, standard methods in ordinary differential equations do not apply here, due to the budget constraint. This section therefore meticulously spells out the solution to the problem. Section 4 uses the results in Section 3 to compute the profit function for the seller and its dependency on the interest rate. Section 5 gives an example.

Part II extends the previous analysis to a model that incorporates default risks. Section 6 spells out this extended model. It departs from the model in Part I in only two aspects: One is that the object being auctioned may turn out to be valueless. The second aspect is that the winner of the auction has an option to declare bankruptcy after realizing the value of the object. The auction game is solved for each level of interest rates in Section 7: In Section 7.1, we first find out the critical-value interest rate at which the equilibrium bidding strategy "flips". Sections 7.2 and 7.3 then solve the game in each case. While Section 7.2 is a generalization of Section 3, Section 7.3 gives the downward sloping bidding strategy. Using these results, Section 8 computes the seller's expected payoff and compares the payoff among different levels of interest rates. Section 9, the appendix, contains the formal proofs of two lemmas in Section 7.1.

A quick tour through the paper may start with Section 2 for the simple model. Then look at the statements of the theorem and proposition in Section 3.1 for a picture of the equilibrium. Browse through the statements of the lemmas in Section 3.2 for the main steps of the derivation. Then go to Section 6.1 for the model that incorporates default risks.

Browse through Section 7.1 (especially the demonstration of Lemma 7.1) to see how the bidding strategy "flips". Look at the statements of the theorem and proposition in Section 7.2, keeping in mind that they are merely a generalization of the results in Section 3.1. Then take a look at the theorem and proposition in Section 7.3 for an overview of the downward sloping equilibrium bidding strategy. Finally, browse through the statements of the lemmas in Section 7.3.1 for the main steps of the derivation.

Part I

Auctions without Default Risk

2 The Model

We consider a first-price sealed bid auction of an indivisible object. There are n bidders, $n = 2, 3, \ldots$. Each bidder i is endowed with a certain amount of funds, called the budget for bidder i. If a bidder has to make a payment higher than his budget, then he must finance the extra funds from the capital market, where the prevailing net interest rate is $q > 0$, or from the seller, who may choose to offer a loan to the winning bidder at a net interest rate $r \in [0, q]$.

The object being auctioned is valuable only to the bidders. This value is common among the bidders and is publicly known by the seller and bidders as some nonnegative real number v.

For each $i = 1, \ldots, n$, until bidder i wins the object (if he does), the amount of i's budget is known only to bidder i, and this amount is regarded by the seller and other bidders than i to be independently drawn from a publicly known distribution. Denote F for the cumulative distribution function of this distribution. The support of F is $[m_0, \bar{m}]$, with $\bar{m}$ being a nonnegative real number or infinity (When $\bar{m}$ equals infinity, we abuse the notation $[m_0, \bar{m}]$ to mean $[m_0, \infty)$).

The auction proceeds as follows:

1. The seller chooses a (net) interest rate $r \in [0, q]$ for the loan provided to the winning bidder, if needed.
2. Each bidder submits a bid independently and simultaneously. The highest bidder becomes the winner (if more than one bidders submit the highest bid, then the winner

is chosen by a random pick with equal probability). Those who are not the winner each get 0 payoff.

3. The winner pays his bid b to the seller for the object. In doing so, the winner bears a cost

$$C(b, m, r) = \begin{cases} b & \text{if } b \leq m \\ b + r(b - m) & \text{otherwise,} \end{cases} \tag{1}$$

where m is his realized budget (We assume that a bidder's budget can only be used to pay for the object or for consumption). His payoff is then $v - C(b, m, r)$. If the winner needs a loan from the seller, then the latter bears a cost $q - r$ per unit of the loan. Thus, the seller's payoff is

$$b_{(1)} - (q - r) \max\{0, b_{(1)} - m_{(1)}\},$$

where $b_{(1)}$ is the winning bid and $m_{(1)}$ the budget of the winner. (We assume that, in providing the loan, the seller is able to monitor the winner so that none of the loan can be used in any other way than purchasing the object being auctioned. Thus, $\max\{0, b_{(1)} - m_{(1)}\}$ is the amount of the loan.) The game is then over. [6] [7]

We assume that the distribution F of budgets has the following properties:

Assumption 1 $m_0 < v < \bar{m}$.

Assumption 2 *On its support, F is strictly increasing and has a continuous probability density function f; $F(m_0) = 0$ and $F(m_0)/f(m_0) = 0$.*

Assumption 3 *The function φ given by $x \mapsto x + \frac{F(x)}{(n-1)f(x)}$ is strictly increasing on $[m_0, \bar{m}]$.*

[6] This game does not preclude *a priori* the possibility that the seller sells the object to a highest bidder who needs more loan than a lower bidder. Nevertheless, this event does not occur in equilibrium, whose bidding strategy is strict increasing in budgets.

[7] The possibility that a winning bidder may default is assumed away in this model. Section 6 extends the model to study a version of default risk.

Assumption 4 *The function* ψ *given by* $x \mapsto x + (1+r)\frac{F(x)}{(n-1)f(x)}$ *is strictly increasing on* $[m_0, \bar{m}]$, *for every* $r \in (0, q]$.

Remark: Assumptions 2 to 4 are satisfied by any uniform or exponential distributions. Assumptions 3, 4, and 5 (in Section 6) can be replaced by a single assumption that

$$\frac{d}{dx}\left[\frac{F(x)}{f(x)}\right] > -\frac{n-1}{1+q}, \; x \in \text{interior support } F.$$

If this assumption (a version of monotone hazard rate) is strengthened into $\frac{d}{dx}\left[\frac{F(x)}{f(x)}\right] > 0$ for each x in the interior support of F, then Assumption (18) (used in Remarks 4.2, 8.1, and 8.2) can also be replaced.

3 The Strategy for Bidders

Once the seller has chosen an interest rate $r \geq 0$, an auction game among the bidders is given. Both the bidders and the seller want to know the bidding strategies of the auction game. A bidder needs this information to guide his bids, and the seller needs it to choose an interest rate. We have discovered a bidding strategy of the auction game. The nice thing about the strategy is that it comprises the only symmetric equilibrium of the auction game. After an overview of the bidding strategy (Section 3.1), we derive the bidding strategy in Section 3.2, which is also a proof for the uniqueness of the symmetric equilibrium of the auction game. Then in Section 3.3 we will verify that the bidding strategy derived exists and does comprise an equilibrium. Section 3.4 solves the special (yet simple) case when the interest rate is 0.

3.1 An Overview of the Bidding Strategy

With budgets m privately known to the bidders, a bidder's strategy is a mapping from budgets to bids. For positive interest rates, the solution concept we shall use is a symmetric equilibrium whose bidding strategy is continuous, strictly increasing, and piecewise differentiable in budgets. The assumption of strict monotonicity, which is used for expository simplicity, will be partially relaxed in Section 7.1. The following theorem asserts that the symmetric equilibrium exists and is uniquely determined by the interest rate r. The theorem further gives the closed form solution of the strategy of the equilibrium. To save space, we denote, from now on, $F_Y(x) := F(x)^{n-1}$ for each $x \in [m_0, \bar{m}]$. For each bidder i, $F_Y(x)$ is the probability for the highest budget of the other bidders than i to be at least as small as x. Let f_Y denote the derivative of F_Y. Note that $F_Y(x)/f_Y(x) = F(x)/((n-1)f(x))$, and that F_Y is strictly increasing by Assumption 2.

Theorem 3.1 *Let $r \in (0, q]$ be the interest rate chosen by the seller.*

1. *The auction game has a unique symmetric equilibrium whose bidding strategy is strictly increasing, continuous, and piecewise differentiable.*

2. *This bidding strategy $\beta : [m_0, \bar{m}] \to R$ is:*

$$\beta(m) := \begin{cases} \frac{1}{1+r}\left[v + rm - r\int_{m_0}^{m} \frac{F_Y(t)}{F_Y(m)} dt\right] & \text{if } m_0 \le m < m_*(r) \\ m & \text{if } m_*(r) \le m \le m^*(r) \\ v - (v - m^*(r))\frac{F_Y(m^*(r))}{F_Y(m)} & \text{otherwise,} \end{cases} \tag{2}$$

where $m_*(r)$ is the unique non-m_0 root for the equation

$$(v - m_*)F_Y(m_*) = r\int_{m_0}^{m_*} F_Y, \tag{3}$$

and $m^*(r) := \max\{m_*(r), \hat{m}\}$, with $\hat{m}$ a constant given by

$$v = \hat{m} + \frac{F_Y(\hat{m})}{f_Y(\hat{m})}. \tag{4}$$

The theorem will be proved in Sections 3.2 (for uniqueness and the derivation of (2)) and 3.3 (for existence). We here look at the properties of the bidding strategy. Its main feature is the following: those bidders with budgets under a $m_*(r)$ bid over their budgets, those with budgets between $m_*(r)$ and $m^*(r)$ bid exactly at their budgets, and those above $m^*(r)$ bid below their budgets.

Proposition 3.1 *Given any $r > 0$, the function β defined in (2) has the following properties:*
a) It is well-defined, strictly increasing, continuous and bounded from above by v.
b) $\beta(m) > m$ if $m < m_*(r)$, $\beta(m) = m$ if $m_*(r) \le m \le m^*(r)$, and $\beta(m) < m$ otherwise.
c) $m_*(\cdot)$ is a one-to-one function over the domain $(0, \infty)$, and the derivative $m_*' < 0$ over $(0, \infty)$.
d) β is piecewise differentiable and

$$\beta'(m) = \begin{cases} \frac{r}{1+r}\frac{f_Y(m)}{F_Y(m)^2}\int_{m_0}^{m} F_Y(t)dt & \text{if } m_0 < m < m_*(r) \\ (v - m^*(r))F_Y(m^*(r))\frac{f_Y(m)}{F_Y(m)^2} & \text{if } m^*(r) < m < \bar{m} \\ 1 & \text{if } m_*(r) < m < m^*(r). \end{cases} \tag{5}$$

Proof: Property (d) follows directly from Equation (2). The same equation immediately implies the part for continuity and boundedness in Property (a). For the rest of (a): β will be well-defined if $m_*(r)$ and $\hat{m}$ exist and are each unique; by (d), β will be strictly increasing if $v > m^*(r)$. The existence and uniqueness of $m_*(r)$ and $\hat{m}$, as well as $v > m^*(r)$, are the consequences of Lemmas 3.1 and 3.2, which we will establish immediately. For (b), it trivially follows from (2) that $\beta(m) = m$ if $m_*(r) \leq m \leq m^*(r)$. The rest of (b) are also the consequences of Lemmas 3.1 and 3.2. For (c): Lemma 3.1 also implies that the function $m_*(\cdot)$ is well defined. This function is one-to-one because $m_*(r)$ is defined to be the non-m_0 root for (3). To show that $m'_* < 0$, differentiate both sides of Equation (3) and then collect the terms for dm_* and dr. This gives

$$m'_*(r) = \frac{\int_{m_0}^{m_*(r)} F_Y}{g'(m_*(r))}. \tag{6}$$

With $g'(m_*(r)) < 0$ ((c) and (d) of Lemma 3.1), this equation implies that $m'_* < 0$. Thus, (c) will be true if Lemma 3.1 is proved.

We now establish Lemmas 3.1 and 3.2. They study the roots for Equations (3) and (4), respectively. This amounts to studying the following two real-value functions g and h on $[m_0, \bar{m}]$:

$$g(x) := (v - x)F_Y(x) - r\int_{m_0}^{x} F_Y(t)dt, \ x \in [m_0, \bar{m}];$$

$$h(x) := (v - x)F_Y(x), \ x \in [m_0, \bar{m}].$$

Note that both functions are differentiable and

$$g'(x) = f_Y(x)\left[v - x - (1 + r)\frac{F_Y(x)}{f_Y(x)}\right], \ x \in (m_0, \bar{m}); \tag{7}$$

$$h'(x) = f_Y(x)\left[v - x - \frac{F_Y(x)}{f_Y(x)}\right], \ x \in (m_0, \bar{m}). \tag{8}$$

Thus, a root for Equations (3) is a root for $g(x) = 0$, and a root for (4) is a critical point of h.

Lemma 3.1 *Given each $r \geq 0$, the function g satisfies the following properties:*

a) $g(m_0) = 0$;

b) $g(v) \leq 0$ ("$<$" if $r > 0$);

c) g strictly increases up to a certain point $m' \in (m_0, v)$, and g strictly decreases after m';

d) there is a unique $m_* \in (m', v]$ such that $g(x) > 0$ if $x < m_*$, $g(x) = 0$ if $x = m_*$, and $g(x) < 0$ otherwise;

e) this m_* equals v iff $r = 0$.

Proof: Property (a) follows from $F_Y(m_0) = F(m_0)^{n-1} = 0$ by Assumption 2. Property (b) follows from $r \geq 0$, $v > m_0$, and the assumption that F strictly increasing (Assumptions 1 and 2). To prove (c), we need only to show that the derivative g' is positive up to some $m' \in (m_0, v)$ and negative afterwards. But this follows from (7) and Assumption 4. With m' being the maximum for g and $m' \neq m_0$, we have $g(m_0) > 0$. Then the intermediate-value theorem implies the existence part of Property (d). The uniqueness of m_* simply follows from the fact that g strictly decreases for all $x > m'$. Property (e) is trivial. □

Lemma 3.2 *There is exactly one $\hat{m} \in (m_0, v)$ such that $h'(x) > 0$ if $x < \hat{m}$, $h'(x) < 0$ if $x > \hat{m}$, and $h'(\hat{m}) = 0$.*

Proof: By Assumption 3, the function φ is strictly increasing. Moreover, $\varphi(m_0) = 0$ (Assumption 2) and $\varphi(v) > v$. Thus, there is exactly one $\hat{m} \in (m_0, v)$ such that $v > \varphi(x)$ if $x < \hat{m}$, $v < \varphi(x)$ if $x > \hat{m}$, and $v = \varphi(\hat{m})$. As $h'(x) > (\geq)\ 0$ iff $v > (\geq)\ \varphi(x)$ (from the expression of $h'(x)$), the lemma is proved. □

Lemmas 3.1 and 3.2 directly imply that β is well defined and strictly increasing. Thus, Properties (a) and (d) in the Proposition 3.1 are proved. Property (b) in the proposition from Lemmas 3.1 and 3.2 because "$\beta(m) > m$ if $m < m_*(r)$" is equivalent to "$g(m) > 0$ for $m < m_*(r)$", and "$\beta(m) < m$ for $m > m_*(r)$" equivalent to "$h'(m) < 0$ for $m > m_*(r)$". Property (c) follows from Lemma 3.1, as shown before. Thus, the proposition is proved. **Q.E.D.**

3.2 Deriving the Solution (Uniqueness Proof)

This section is a proof for the uniqueness part in Theorem 3.1. Although the core of the derivation is solving a system of differential equations, standard methods in ordinary differential equations do not apply here, because bidding below or above one's budget changes the objective function and hence the first-order-necessary condition (which would give the differential equation). For the simplicity of notations, we will write m_* for $m_*(r)$ and m^* for $m^*(r)$.

Let $r \geq 0$. Let β be a strictly increasing, continuous and piecewise differentiable bidding strategy that constitutes a symmetric equilibrium (the proposition is vacuously true if no such a strategy exists). With β strictly increasing, we can define

$$V(b, m) := (v - C(b, m, r))F_Y(\beta^{-1}(b)), \; b \geq 0, \; m \in [m_0, \bar{m}]. \tag{9}$$

Thus, $V(b, m)$ is the expected payoff for a bidder with budget m to bid b, provided that others play the strategy β and the seller charges an interest rate r (Note that the probability for a tie is 0, since F_Y is continuous by Assumption 2). We shall solve for β by proving the following lemmas.

Lemma 3.3 *For any $r \geq 0$,*

$$\beta(m_0) = \frac{v + rm_0}{1 + r} > m_0.$$

Proof: We first prove that

$$v = C(\beta(m_0), m_0, r). \tag{10}$$

Since $F_Y(m) > 0$ for all $m > m_0$, $v \geq C(\beta(m), m, r)$ for any such m (otherwise, the expected payoff of bidding $\beta(m)$ would be negative, while bidding 0 guarantees a nonnegative payoff). With β and $C(\beta(\cdot), \cdot, r)$ being continuous in m, we have $v \geq C(\beta(m_0), m_0, r)$. To prove that "$\leq$" holds, suppose not so. Then, at budget m_0, bidding slightly above $\beta(m_0)$ would yield a positive payoff (since F_Y is strictly increasing by Assumption 2 and β is continuous and strictly increasing), while bidding $\beta(m_0)$ gives only a payoff 0 ($F_Y(m_0) = 0$). This violates the assumption that β is an equilibrium strategy. Thus, (10) must hold.

We next prove that $\beta(m_0) > m_0$. Suppose not, then the definition of the cost function C would imply that

$$C(\beta(m_0), m_0, r) = \beta(m_0) \leq m_0;$$

but then $v - C(\beta(m_0), m_0, r) \geq v - m_0 > 0$ (Assumption 1), violating the proved equality (10). Thus, $\beta(m_0) > m_0$.

It follows from this inequality that $C(\beta(m_0), m_0, r) = (1+r)\beta(m_0) - rm_0$. Equation (10) then implies that $\beta(m_0) = (v + rm_0)/(1+r)$ (which is indeed greater than m_0 since $v > m_0$), as claimed. Thus, the lemma is proved. □

Lemma 3.4

$$\beta(m) = \frac{1}{1+r}\left[v + rm - r\int_{m_0}^{m} \frac{F_Y(t)}{F_Y(m)} dt\right], \ m_0 \leq m \leq m_*.$$

Proof: With $\beta(m_0) > m_0$ proved in the previous lemma, the continuity of β implies that there is some interval $[m_0, m)$ such that $\beta(x) > x$ if $x \in [m_0, m)$. Choose m' to be the largest among all such m. Such an m' exists by the first two sentences of the proof for Lemma 3.5. The choice of m' implies that $\beta(m') \leq m'$. By the continuity of β, $\beta(m') = m'$.

Denote $\beta_1 := \beta|_{[m_0, m']}$. With β being an equilibrium strategy, $\beta_1(m)$ maximizes $V(\cdot, m)$ over the open set $(\max\{\beta(m_0), m\}, m')$ for any $m \in (m_0, m')$. Further, $\beta_1(m)$ satisfies the first-order-necessary-condition of maximization for all $m \in (m_0, m')$ but finitely many possible exceptions. [8] Since $V(b, m) = (v - (1+r)b + rm)F_Y \circ \beta_1^{-1}(b)$ for each $b \in (\max\{\beta(m_0), m\}, m')$ (The subscript 1 for β comes from the fact that $(\max\{\beta(m_0), m\}, m')$ is contained in the range of β_1), this necessary condition states that (with both f_Y and β_1'

[8] *Proof*: Since β is piecewise differentiable, β_1 is differentiable at all $m \in (m_0, m')$ but finitely many possible exceptions. Pick any $m \in (m_0, m')$ where β is differentiable. Then there is a neighborhood N of m over which β is differentiable. As β is strictly monotone, the inverse function theorem implies that β^{-1} is differentiable over the open set $(\max\{\beta(m_0), m\}, m') \cap \beta(N)$. Thus, the objective function $V(\cdot, m)$ is differentiable over this open set. With β being an equilibrium strategy, $\beta_1(m)$ maximizes $V(\cdot, m)$ over this open set ($\beta_1(m)$ clearly belongs to this set). It thus follows that $\beta_1(m)$ satisfies the first-order-necessary-condition of maximization. □

non-zero everywhere in their domains)

$$\beta_1'(m) + \beta_1(m)\frac{f_Y(m)}{F_Y(m)} = \frac{v+rm}{1+r}\frac{f_Y(m)}{F_Y(m)}.$$

This being true for all $m \in (m_0, m')$ but finitely many possible exceptions, with β continuous, we can solve this linear differential equation to have

$$\beta_1(m) = \frac{1}{1+r}\left[v + rm - r\int_{m_0}^{m}\frac{F_Y(t)}{F_Y(m)}dt\right] + \frac{c}{F_Y(m)}, \; m \in (m_0, m'), \tag{11}$$

where c is some constant. The continuity of β implies that $\beta_1(m_0) = \beta(m_0)$; with $F_Y(m)$ goes to 0 as m approaches m_0, this implies that $c = 0$. Thus,

$$\beta(m) = \frac{1}{1+r}\left[v + rm - r\int_{m_0}^{m}\frac{F_Y(t)}{F_Y(m)}dt\right], \; m \in (m_0, m').$$

With $\beta(m') = m'$ (proved earlier in this proof), this equation implies that $g(m') = 0$, where g has been defined in Section 3.1. Since $m' \in (m_0, v]$ ($\beta(m') = m'$ and $\beta \leq v$), Lemma 3.1 and $g(v) < 0$ imply that $m' = m_*$. Thus, we have completed the proof for the lemma. □

Lemma 3.5 *There is exactly one $m'' \in [m_*, \bar{m})$ such that*

$$\beta(m) = v - (v - m'')\frac{F_Y(m'')}{F_Y(m)}, \; m \in [m'', \bar{m}].$$

Proof: Being an equilibrium strategy, $\beta \leq v$ (since $C(b, m, r) \geq b$). Since $v < \bar{m}$ (Assumption 1) and β continuous, there must be an $m \in (m_*, \bar{m})$ for which $\beta(x) < x$ if $x \in (m, \bar{m}]$. Let m'' be the smallest among these m. Then m'' exists and is unique. The choice of m'' and the continuity of β implies that $\beta(m'') = m''$. By the previous lemma, $m'' \geq m_*$.

Since $\beta \leq v$, $\sup_{m\in[m_0,\bar{m}]}\beta(m)$ exists. Denote it by $\bar{b}$. Note that $\bar{b} \leq v$. Denote $\beta_2 := \beta|_{[m'',\bar{m}]}$. Then $(m'', \bar{b})$ is exactly the interior of the range of β_2. For any $m \in (m'', \bar{m}]$, with β being an equilibrium strategy and $\beta_2(m) < m$, $\beta_2(m)$ maximizes $V(\cdot, m)$ over the open set $(m'', \min\{m, \bar{b}\})$. Further, $\beta_2(m)$ satisfies the first-order-necessary-condition of maximization for all $m \in (m'', \bar{m}]$ but finitely many possible exceptions. [9] Since $V(b, m) = (v-b)F_Y \circ \beta_2^{-1}(b)$ for each $b \in (m'', \min\{m, \bar{b}\})$ (The subscript 2 for β comes from the fact that $(m'', \min\{m, \bar{b}\})$

[9] Prove by the same manner as the previous footnote.

is contained in the range of β_2), this necessary condition states (with both f_Y and β_1' being non-zero everywhere in their domains) that

$$\beta_2'(m) + \beta_2(m)\frac{f_Y(m)}{F_Y(m)} = v\frac{f_Y(m)}{F_Y(m)}.$$

This being true for all $m \in (m'', \bar{m}]$ but finitely many possible exceptions, with β continuous, we solve this linear differential equation to have

$$\beta_2(m) = v - \frac{c'}{F_Y(m)}, \quad m \in (m'', \bar{m}],$$

where c' is some constant. Since $\beta(m'') = m''$ (the first paragraph of the proof), we have $c' = (v - m'')F_Y(m'')$. Thus, the lemma is proved. □

Lemma 3.6 *If $x \geq \hat{m}$ and $\beta(x) = x$, then there is no $x' > x$ for which $\beta(x') = x'$.*

Proof: Suppose not so, i.e., suppose that $x' > x \geq \hat{m}$, $\beta(x) = x$, and $\beta(x') = x'$. Then

$$V(\beta(x'), x') = (v - x')F_Y(x') = h(x') < h(x) = (v - x)F_Y(x) = V(x, x'),$$

where the inequality holds because $x' > x \geq \hat{m}$ and h is strictly decreasing on $[\hat{m}, \bar{m}]$ (Lemma 3.2). But then β would not have been an equilibrium strategy. This proof by contradiction establishes the lemma. □

Lemma 3.7 *For each $m \in [m_*, m'']$, $\beta(m) = m$.*

Proof: The lemma is vacuously true when $[m_*, m'']$ is degenerate. We hence consider the case when the interval is nondegenerate. Suppose that the lemma does not hold. Then, by the continuity of β and Lemmas 3.4 and 3.5, there would be a nondegenerate interval $[m_1, m_2] \subset [m_*, m'']$ such that either

(i) $\beta(m) > m$ for all $m \in (m_1, m_2)$ and $\beta(m) = m$ for $m = m_1, m_2$, or

(ii) $\beta(m) < m$ for all $m \in (m_1, m_2)$ and $\beta(m) = m$ for $m = m_1, m_2$.

We shall prove that each case leads to a contradiction.

Case (i): For any $m \in (m_1, m_2)$, this case implies that $\beta(m)$ maximizes $V(\cdot, m)$ over the open interval (m, m_2). Apply the same reasoning in the derivation of β_1 in the proof of Lemma 3.4. We then have

$$\beta_1(m) = \frac{1}{1+r}\left[v + rm - r\int_{m_0}^{m} \frac{F_Y(t)}{F_Y(m)}dt\right] + \frac{c_1}{F_Y(m)}, \; m \in (m_1, m_2)$$

for some constant c_1. With β continuous, the right-hand-side of the above equation equals m when $m = m_1, m_2$. Consequently, $g(m_1) = -c_1 = g(m_2)$, which is impossible because g is strictly decreasing on $[m_*, \bar{m}]$ (Lemma 3.1) and $m_2 > m_1 \geq m_*$. Thus, Case (i) leads to a contradiction.

Case (ii): Apply the same reasoning as in Case (i), replacing "$\beta(m) > m$" with "$\beta(m) < m$". We then have

$$\beta(m) = v - (v - m_1)\frac{F_Y(m_1)}{F_Y(m)}, \; m \in [m_1, m_2]$$

and hence $h(m_1) = h(m_2)$. This equality, coupled with the fact that the function h has a unique maximum at $\hat{m}$ (Lemma 3.2), implies that

$$m_1 < \hat{m} < m_2. \tag{12}$$

With $m'' \geq m_2$; $\beta(m_2) = m_2$, and $\beta(m'') = m''$, Lemma 3.6 implies that $m'' = m_2$.

It then follows that β is differentiable at m'': By $m'' = m_2$, $h(m_1) = h(m_2)$, and (5) in Proposition 3.1, we have

$$\begin{aligned} \beta'_-(m'') &= \lim_{\Delta m \to 0-} \frac{h(m_1) f_Y(m'' + \Delta m)}{F_Y(m'' + \Delta m)} \\ &= \frac{h(m_1) f_Y(m'')}{F_Y(m'')} \\ &= \frac{h(m_2) f_Y(m'')}{F_Y(m'')} \\ &= \lim_{\Delta m \to 0+} \frac{h(m_2) f_Y(m'' + \Delta m)}{F_Y(m'' + \Delta m)} \\ &= \beta'_+(m''), \end{aligned}$$

where the second and fourth equalities use the continuity of f_Y (Assumption 2). With $m_2 = m''$, $\beta(m) \leq m$ for every m sufficiently closed to m''. Since $\beta(m'') = m''$, we then have

$\beta'_-(m'') \geq 1$ and $\beta'_+(m'') \leq 1$. Coupled with the differentiability of β at m'', this gives

$$\beta'_+(m'') \leq 1 \leq \beta'_-(m'') = \beta'_+(m'');$$

thus, $\beta'(m'') = 1$. In other words,

$$\beta'(m'') = (v - m'')\frac{F_Y(m'')f_Y(m'')}{F_Y(m'')^2} = 1.$$

This equality implies that $h'(m'') = 0$. Lemma 3.2 then implies that $m'' = \hat{m}$, but this contradicts Inequality (12), which requires that $m'' > \hat{m}$. Thus, Case (ii) leads to a contradiction.

As Cases (i) and (ii) are the only consequences from the supposition that the lemma does not hold, we have proved the lemma. □

Lemma 3.8 $m'' = m^*$.

Proof: If $\hat{m} \leq m_*$, then Lemma 3.6 requires that $m'' = m_*$ (otherwise, $\beta(m_*) = m_*$ and $\beta(m'') = m''$ would contradict the proved lemma). If $\hat{m} > m_*$, then the same lemma, together with Lemma 3.7, implies that $m'' \leq \hat{m}$. To finish the proof of the lemma, we need only to prove that $m'' \geq \hat{m}$ when $\hat{m} > m_*$. Recall that $\beta(m) < m$ for all $m > m''$. By Lemma 3.5, this means

$$v - (v - m'')\frac{F_Y(m'')}{F_Y(m)} < m \text{ for all } m > m'',$$

i.e., $h(m) < h(m'')$ for all $m > m''$. It then follows from Lemma 3.2 that $m'' \geq \hat{m}$. Thus, we have proved that $m'' = \max\{m_*, \hat{m}\}$, which, by definition, is m^*. The lemma is hence proved. □

Summing up Lemmas 3.4, 3.5, 3.7, and 3.8, we have solved the bidding strategy β. Thus, the symmetric equilibrium of the auction is unique up to the restriction that the bidding strategy be strictly increasing, continuous, and piecewise differentiable. This finishes the proof for uniqueness. **Q.E.D.**

3.3 Verifying the Solution (Existence Proof)

This section verifies that it is an equilibrium of the auction game for each bidder to bid according to the function β defined in (2). With β well-defined (Property (a) in Proposition 3.1), this proves the existence of a symmetric equilibrium for the auction game.

Since β is not differentiable at m_* and m^* (not differentiable even when the two points collapse) and $C(\cdot, m, r)$ not differentiable at m (recall (1)), the function $V(\cdot, m)$ need not be differentiable at m_*, m^*, and m. Recall that $V(b, m)$ is the expected payoff for a bidder with budget m to bid b, provided that others play the strategy β.

Denote $\beta_1 := \beta|_{[m_0,m_*]}$, $\beta_2 := \beta|_{[m^*,\bar{m}]}$, and $\beta_3 := \beta|_{[m_*,m^*]}$. We shall verify that β comprises an equilibrium by proving the following lemmas.

Lemma 3.9 *For each $m < m_*$,*

(a) $\beta(m)$ maximizes $V(\cdot, m)$ over $[m, m_*]$,

(b) $\beta(m)$ strictly dominates any bid $b \in (m_*, m^*]$,

(c) $\beta(m)$ strictly dominates any bid $b > m^*$, and

(d) $\beta(m)$ strictly dominates any bid $b < m$.

Proof: Part (a): Take any $b \in (m, m_*)$. Then b belongs to the range β_1, and $V(b, m) = (v - (1 + r)b + rm)F_Y \circ \beta_1^{-1}(b)$. Thus, the derivative

$$\begin{aligned} D_1V(b,m) &= \frac{f_Y \circ \beta_1^{-1}(b)}{\beta_1' \circ \beta_1^{-1}(b)} \left\{ v + rm - (1+r)b - (1+r)\beta_1' \circ \beta_1^{-1}(b) \frac{F_Y \circ \beta_1^{-1}(b)}{f_Y \circ \beta_1^{-1}(b)} \right\} \\ &= \frac{f_Y(x)}{\beta_1'(x)} \left\{ v + rm - (1+r)\beta_1(x) - (1+r)\beta_1'(x) \frac{F_Y(x)}{f_Y(x)} \right\}, \end{aligned}$$

where we define $x := \beta_1^{-1}(b)$. Plugging (2) and (5) into the above equation, we obtain

$$D_1V(b,m) = (\text{a positive term}) \times r(m-x) \begin{cases} > 0 & \text{if } m > x, \text{ i.e., } b < \beta_1(m) \\ = 0 & \text{if } m = x, \text{ i.e., } b = \beta_1(m) \\ < 0 & \text{if } m < x, \text{ i.e., } b > \beta_1(m), \end{cases}$$

since $r > 0$. It follows that $\beta_1(m)$ is the maximum of $V(\cdot, m)$ over (m, m_*). By the continuity of V, this result extends to the end points m and m_*. Thus, we have proved (a) of the lemma.

Part (b): Take any $b \in (m_*, m^*]$. Then b belongs to the range of β_3 and $\beta^{-1}(b) = b$. Thus,

$$\begin{aligned} V(b,m) &= [v - b - r(b-m)]F_Y(b) \\ &= (v-b)F_Y(b) - r(b-m)F_Y(b) \\ &< r\int_{m_0}^{b} F_Y(t)dt - r(b-m)F_Y(b) \\ &< r\int_{m_0}^{m} F_Y(t)dt \\ &= V(\beta(m), m), \end{aligned}$$

where the first inequality comes from the fact that $g(b) < 0$ (Lemma 3.1 and $b > m_*$), and the second inequality from the fact that $\int_m^b F_Y < (b-m)F_Y(b)$, which is true since F_Y is strictly increasing (Assumption 2). We have hence shown that $V(b,m) < V(\beta(m), m)$. Thus, Part (b) of the lemma is proved.

Part (c): It suffices to show that $V(\cdot, m)$ is strictly decreasing on $[m^*, v]$ (a bid above v is obviously strictly dominated). To do that, we need only to prove that the derivative $D_1V(\cdot, m) < 0$ over (m^*, v). Thus, take any $b \in (m^*, v)$. Then b belongs to the range of β_2, and $V(b,m) = [v - b - r(b-m)]F_Y \circ \beta_2^{-1}(b)$. Let $x := \beta_2^{-1}(b)$. We have

$$D_1V(b,m) = (\text{a positive term}) \times \left\{ v + rm - (1+r)\beta_2(x) - (1+r)\beta_2'(x)\frac{F_Y(x)}{f_Y(x)} \right\}.$$

Plugging (2) and (5) into the equation, we get

$$D_1V(b,m) = (\text{a positive term}) \times r(m - v) < 0$$

since $m < m_* < v$ ($\beta < v$ in (a) of Proposition 3.1) and $r > 0$. This proves Part (c) of the lemma.

Part (d): It suffices to show that $V(\cdot, m)$ is strictly increasing over $[\beta(m_0), m)$ (note that a bid below $\beta(m_0)$ yields the same payoff as $\beta(m_0)$). To do that, we need only to prove that $D_1V(\cdot, m) > 0$ over $(\beta(m_0), m)$. Hence pick any $b \in (\beta(m_0), m)$, we have $V(b,m) =$

$(v-b)F_Y \circ \beta_1^{-1}(b)$. With $x := \beta_1^{-1}(b)$,

$$\begin{aligned} D_1V(b,m) &= (\text{a positive term}) \times \left\{ v - \beta_1(x) - \beta_1'(x)\frac{F_Y(x)}{f_Y(x)} \right\} \\ &= \frac{r}{1+r}(v-x) \\ &> 0, \end{aligned}$$

as desired. Thus, Part (d) is proved.

We therefore have completed the proof for the lemma. □

Lemma 3.10 *For each $m > m^*$,*

(a) $V(\cdot, m)$ is constant on $[m^*, m]$, and hence $\beta(m)$ maximizes $V(\cdot, m)$ over $[m^*, m]$,

(b) any bid $b \in [m_*, m^*)$ is strictly dominated,

(c) any bid $b < m_*$ is strictly dominated, and

(d) any bid $b > m$ is strictly dominated.

Proof: Part (a): It suffices to show that $D_1V(\cdot, m) = 0$ over (m^*, m). Hence pick any $b \in (m^*, m)$. We have $V(b,m) = (v-b)F_Y \circ \beta_2^{-1}(b)$ and, with $x := \beta_2^{-1}(b)$,

$$D_1V(b.m) = (\text{a positive term}) \times \left\{ v - \beta_2(x) - \beta_2'(x)\frac{F_Y(x)}{f_Y(x)} \right\} = 0$$

by (2) and (5), as desired. Thus, Part (a) of the lemma is proved.

Part (b): This claim is vacuously true if $m_* = m^*$. Thus, suppose that $m_* \neq m^*$. Then by the definition of m^* we have $m^* = \hat{m} > m_*$. The function h is hence strictly increasing on $[m_0, m^*]$ (Lemma 3.2). Thus, for any $b \in [m_*, m^*)$, with b lying in the range of β_3,

$$V(b,m) = (v-b)F_Y(b) = h(b) < h(m^*) = (v-m^*)F_Y(m^*) = V(m^*, m),$$

as claimed. We have thus proved Part (b) of the lemma.

Part (c): Apply the proof for Part (d) of Lemma 3.9, replacing the interval $[\beta(m_0), m)$ in that proof with $[\beta(m_0), m_*)$. This proves Part (c) of the current lemma.

Part (d): Apply the proof for Part (c) of Lemma 3.9, replacing the intervals $[m^*, v]$ in that proof with $[m, v]$, and (m^*, v) with (m, v). Part (d) of this lemma is hence proved.

We have therefore completed the proof for Lemma 3.10. □

Lemma 3.11 *For each $m \in [m_*, m^*]$,*

(a) any bid $b \in [m_*, m)$ is strictly dominated by m,

(b) any bid $b \in (m, m^*]$ is strictly dominated by m,

(c) any bid $b < m_*$ is strictly dominated, and

(d) any bid $b > m^*$ is strictly dominated.

Proof: Part (a): Apply the proof for Part (b) of Lemma 3.10, replacing the interval $[m_*, m^*)$ in that proof with $[m_*, m)$. To see that the same proof works here, notice that $V(b, m) = h(b)$ for each $b \in [m_*, m)$, since $m \leq m^*$. Part (a) of the lemma is hence proved.

Part (b): It suffices to show that $V(\cdot, m)$ is strictly decreasing over $(m, m^*]$. To do that, we need only to prove that $D_1V(\cdot, m) < 0$ over (m, m^*). Take any $b \in (m, m^*)$. Then b belongs to the range of β_3 and $V(b, m) = [v - (1 + r)b + rm]F_Y(b)$. Thus,

$$\begin{aligned}
\frac{D_1V(b, m)}{\text{a positive term}} &= v + rm - (1 + r)b - (1 + r)\frac{F_Y(b)}{f_Y(b)} \\
&< v + rm - (1 + r)m - (1 + r)\frac{F_Y(m)}{f_Y(m)} \\
&= v - \psi(m) \\
&\leq 0,
\end{aligned}$$

where the first inequality comes from the fact that $b > m$ (and hence $\varphi(b) > \varphi(m)$; see Assumption 3), and the second inequality comes from the facts that $g'(x) > (\geq)\ 0$ iff $v > (\geq)\ \psi(x)$ and that $g'(x) \leq 0$ for all $x \geq m_*$ (Lemma 3.1). We have therefore proved Part (b) of the lemma.

Part (c): Use exactly the same proof for Part (c) of Lemma 3.10. This proves Part (c) of the current lemma.

Part (d): Use exactly the same proof for Part (c) of Lemma 3.9. This proves Part (d) of the current lemma.

Summing up Lemmas 3.9, 3.10, and 3.11, with $V(\cdot, m)$ a continuous function for each m, we have therefore proved that, for each budget $m \in [m_0, \bar{m}]$, $\beta(m)$ maximizes the expected payoff $V(\cdot, m)$ of any bidder with budget m, provided that others play the strategy β. In other words, we have proved that each bidder playing β is an equilibrium of the auction game. This completes the existence proof.

3.4 The Bidding Strategy for Zero Interest Rate

Theorem 3.1 has covered all the cases when the the seller charges a positive interest rate. In this section we shall deal with the case when the interest rate is 0 (i.e., the seller bears the entire cost of financing the amount by which the winning bid exceeds the winner's budget). The solution for this case, as stated in the following remark, is quite simple.

Remark 3.1 *If the seller chooses 0 interest rate, then (i) each bidder bidding v is an equilibrium of the auction game and, further, (ii) the winning bid equals v under any equilibrium of the auction game.*

Proof: Given 0 interest rate, a bidder has no longer any financial constraint. Thus, Claim (i) is trivial. To show (ii), let b denote the winning bid under an equilibrium of the auction game with 0 interest rate. Since any bid above v is strictly dominated, $b \leq v$. Suppose that $b < v$ and let $\epsilon := v - b$. Then any bidder can get better-off by bidding slightly higher than b: For example, a bidder can bid $b + \epsilon/3$, which gives him a payoff $2\epsilon/3$, while abiding to the "equilibrium" gives him at most $\epsilon/n \leq \epsilon/2$ (recall that ties are broken by random draws). Thus, $b = v$. This proves the remark. **Q.E.D.**

4 The Seller's Strategy

The precise solution of the auction game discovered in Section 3 gives the seller a guide to choose the interest rate to maximize his expected payoff. Using Theorem 3.1 and Remark 3.1, the first proposition in this section derives a general formula of the seller's expected payoff as a function of the interest rate he charges. Thus, the seller can choose his interest rate according to the following procedure:

1. Partition the domain $[0, q]$ of choice by a finite set $r_1, \ldots, r_k$. The more powerful the computer, the finer the partition can be.

2. For each node r_j, $j = 1, \ldots, k$, compute $m_*(r_j)$ and $m^*(r_j)$ according to (3) and (4) in Theorem 3.1; then compute the seller's expected payoff at r_j according to the formula in Proposition 4.1.

3. Choose the r_j that has the highest payoff according to the above computation.

Besides the formula for the expected payoff, this section provides two general guidelines for choosing interest rates. One guideline is that the seller "should" (in the sense of strict dominance) always charge a positive interest rate (Remark 4.1). The intuition is that a positive interest selects the most financially capable bidder as the winner, while an interest-free loan selects the winner at random. Another guideline is that the seller "should" (same meaning as before) charge the lowest possible non-zero interest rate (given finite options) if the number of bidders is sufficiently large (4.2). The intuition is that the lower the interest rate, the more aggressively the bidders will bid; as long as the interest rate is positive, the winner is the richest. The second guideline should be used with caution, however, since the risk of default is assumed away in the current model.

Define $F_{(1)}(x) := F(x)^n$ for each $x \in [m_0, \bar{m}]$. Then $F_{(1)}(x)$ is the probability for the highest budget among all the bidders to be at least as small as x. Note that $F'_{(1)}(x) = nF_Y(x)f(x)$. We will write m_* for $m_*(r)$ and m^* for m^* for $m^*(r)$.

Proposition 4.1 *Suppose that, for each $r > 0$, the bidders bid according to the symmetric equilibrium strategy β specified by (2). Then the seller's expected payoff $\pi(r)$ (before the selection of the winner) from charging $r \in [0, q]$ is the following:*

$$\pi(0) = v - q \int_{m_0}^{v} F(m)dm \tag{13}$$

and, if $r > 0$,

$$\begin{aligned} \pi(r) &= v - (n-1)\left(1 - \frac{q}{1+r}\right)(v - m_*)F_{(1)}(m_*) - (v - m^*)F_Y(m^*)[n - (n-1)F(m^*)] \\ &+ \left[nr - \frac{q(nr+1)}{1+r}\right] \int_{m_0}^{m_*} F_{(1)}(t)dt - \int_{m_*}^{m^*} F_{(1)}(t)dt, \end{aligned} \tag{14}$$

where m_* and m^* are specified according to (3) and (4) in Theorem 3.1.

Proof: As defined in Section 2, the seller's expected payoff π after choosing an interest rate $r \geq 0$ and before the selection of the winner is

$$\pi(r) = E[\text{ winning bid } - (q - r)\max\{0, \text{ winning bid } - \text{ winner's budget }\}]. \tag{15}$$

We first derive $\pi(0)$. As $r = 0$, a bidder's bid no longer depends on his budget. Thus, the winner's budget is a random draw from F (from the viewpoint of the seller). By Remark 3.1, the winning bid is v. Then (15) implies that

$$\pi(0) = v - q \int_{m_0}^{v} (v - m)dF(m),$$

which is the same as (13) due to $F(m_0) = 0$ (Assumption 2). Thus, (13) is proved.

We now derive $\pi(r)$ when $r > 0$. Hence pick any such r. With the symmetric equilibrium strategy β strictly increases in budgets, the winner's budget $m_{(1)}$ is the highest budget among the bidders, and the winning bid is $\beta(m_{(1)})$. Since the probability for "$m_{(1)} \leq x$" is $F_{(1)}(x)$, we compute $\pi(r)$ according (15) and plug (2) into the integrand. Equation (14) then follows after several integrations by parts, combining terms, and a switch of integrals

$$\begin{aligned} \int_{m_0}^{m_*} \int_{m_0}^{m} \frac{F_Y(t)}{F_Y(m)} dt dF_{(1)}(m) &= n \int_{m_0}^{m_*} \int_{t}^{m_*} F(t)^{n-1} f(m) dm dt \\ &= \frac{n}{r}(v - m_*)F_{(1)}(m_*) - n \int_{m_0}^{m_*} F_{(1)}(t)dt, \end{aligned}$$

where we also use the fact that $g(m_*) = 0$. Thus, Equation (14) is proved. The proposition is hence proved. **Q.E.D.**

The following remark says that providing interest-free financing is dominated by charging an interest rate sufficiently low.

Remark 4.1 *For all $r > 0$ sufficiently small, $\pi(r) > \pi(0)$.*

Proof: We need only to show that, for each $r > 0$, $\pi(r) - \pi(0) > X + O(r)$ for some positive term X independent of r. [10] Thus, let $r > 0$. From 4.1, we have $\pi(r) - \pi(0) = A + B + C$, where

$$\begin{aligned} A &:= q\int_{m_0}^{v} F(m)dm - (q-r)\int_{m_0}^{m_*}(\beta_1(m) - m)dF_{(1)}(m), \\ B &:= \int_{m_0}^{v}(\beta(m) - v)dF_{(1)}(m), \text{ and} \\ C &:= \int_{v}^{\bar{m}}(\beta(m) - v)dF_{(1)}(m). \end{aligned}$$

We first compute that $A > O(r)$: Using the facts that F is uniformly bounded, that $m_* < v$ ((a) of Proposition 3.1), that $g(m_*) = 0$, and that $n \geq 2$, as well as the switch of integrals as done in the proof of Proposition 4.1, we have

$$A > q\int_{m_0}^{v}(F - F_{(1)}) + O(r).$$

Since $F > F_{(1)}$ over $(m_0, v]$, $q\int_{m_0}^{v}(F - F_{(1)})$ is a positive term independent of r. Thus, we will be done if $B \geq O(r)$ and $C \geq O(r)$.

To compute B, we use the following fact: If $m_* < m \leq v$ then $\beta(m) > \beta_1(m)$. This is a straightforward consequence of the definition of β and the fact that $g(m) \leq \ (<)\ 0$ for $m \geq \ (>) m_*$ (Lemma 3.1). This fact, together with the uniform boundedness of F, implies that

$$B > \int_{m_0}^{v}(\beta_1(m) - v)dF_{(1)}(m) = O(r).$$

[10] An expression $\phi(x)$ of x is said to be $O(x)$, written as $\phi(x) = O(x)$, if there is a finite real number k such that $\phi(x) \to kx$ as $x \to 0$. An expression $\zeta(x)$ of x is said to be $o(x)$, written as $\zeta(x) = o(x)$, if $\frac{\zeta(x)}{x} \to 0$ as $x \to 0$. Notice that $\phi(x) = O(x)$ implies $x\phi(x) = o(x)$, and that $O(x) \pm O(x) = O(x)$ and $\alpha O(x) = O(x)$ for any real number α.

We next prove that $C > O(r)$: Since $\beta(m) > \beta(v)$ for every $m > v$, we have

$$C > \int_v^{\bar{m}} (\beta(v) - v) dF_{(1)}(m) \geq -\frac{r}{F_Y(v)}(1 - F_{(1)}(v)) \int_{m_0}^{m^*} F_Y,$$

where the weak inequality follows from the fact that $g(m^*) \leq 0$ (since $m^* \geq m_*$; recall Lemma 3.1). The right-hand-side of the above inequality, by the uniform boundedness of F_Y and the fact that $m^* \leq v$, is $O(r)$.

Therefore, we have $\pi(r) - \pi(0) > q \int_{m_0}^{v} (F - F_{(1)}) + O(r)$. Since $q \int_{m_0}^{v} (F - F_{(1)})$ is positive and independent of r, $\pi(r) - \pi(0) > 0$ for r sufficiently small. The proposition is hence proved. **Q.E.D.**

In the following, we compute the derivative of the seller's expected payoff with respect to the interest rate he charges. This is useful for Remark 4.2.

Lemma 4.1 *(i) The seller's expected payoff function $\pi(\cdot)$ is differentiable at all $r \in (0, q]$ except when $r = m_*^{-1}(\hat{m})$ (if this value exists). (ii) If $m_*(r) < \hat{m}$ then*

$$\pi'(r) = n \left[1 - \frac{q}{(1+r)^2}\right] \int_{m_0}^{m_*} (F_{(1)} - F(m_*)F_Y) + \frac{q}{(1+r)^2} \left[rF(m_*) \int_{m_0}^{m_*} F_Y + \int_{m_0}^{m_*} F_{(1)}\right]. \tag{16}$$

(iii) If $m_*(r) > \hat{m}$ then

$$\pi'(r) = D(r) - n(1 - F(m_*)) \frac{h'(m_*)}{g'(m_*)} \int_{m_0}^{m_*} F_Y, \tag{17}$$

where $D(r)$ stands for the expression on the right-hand-side of (16), and g' and h' are computed in (7) and (8).

Proof: We will first prove statements (ii) and (iii). Statement (i) will then follow from (ii) and (iii) (the expression $m_*^{-1}(\hat{m})$ is meaningful because $\hat{m}$ is constant with respect to r; recall Lemma 3.2). To prove (ii), let $m_*(r) < \hat{m}$. Then $m_* < m^* = \hat{m}$ (recall the definition of m^*). Differentiate $\pi(r)$ with respect to r according to (14), keeping in mind that $m^* = \hat{m}$ is independent of r. We then have (16) after a lengthy and elementary arithmetics. Thus, (ii) is proved.

To prove (iii), let $m_*(r) > \hat{m}$. Then $m^* = m_*(r)$ (recall the definition of m^*). Plug this into (14) and Differentiate $\pi(r)$ with respect to r according to the resulting equation. Equation (17) then results from a lengthy combination of terms and (6). Thus, (iii) is proved. We have therefore proved the Lemma. **Q.E.D.**

The following remark says that the less the seller charges for his financing services, the higher his expected payoff will be, provided that the number of bidders is large relative to the cost q of financing and the value v of the object. For this proposition, we need an additional assumption about the distribution F, which any uniform or exponential distribution satisfies. Coupled with Remark 4.1, the conclusion of this proposition implies that the seller should charge the lowest possible non-zero interest rate (assuming that there are only finitely many levels of interest rate to choose).

Remark 4.2 *Assume that*

$$\frac{f(t)}{f(x)} > \frac{F(t)}{F(x)}, \; x \in (m_0, v), t < x. \tag{18}$$

If

$$n \geq \frac{q}{1-q} \frac{F(v)}{1-F(v)} \tag{19}$$

and $q < 1$, *then* $\pi'(r) < 0$ *for all* $r > 0$.

Proof: First, we prove that $m_*(r) > \hat{m}$ for each $r \in (0, q]$. Recall that $m_*(r)$ is determined by (3) and $\hat{m}$ determined by $h'(\hat{m}) = 0$ (Lemma 3.2). Then

$$\begin{aligned} m_*(r) - \hat{m} &= \int_{m_0}^{\hat{m}} \frac{f_Y(t)}{f_Y(\hat{m})} dt - r \int_{m_0}^{m_*(r)} \frac{F_Y(t)}{F_Y(m_*(r))} dt \\ &> r \left\{ \int_{m_0}^{\hat{m}} \frac{F_Y(t)}{F_Y(\hat{m})} dt - \int_{m_0}^{m_*(r)} \frac{F_Y(t)}{F_Y(m_*(r))} dt \right\}, \end{aligned}$$

where the inequality follows from the condition $q < 1$ (and hence $r < 1$), the assumption (18), and the obvious fact that

$$\frac{f(t)}{f(x)} > \frac{F(t)}{F(x)} \Longrightarrow \frac{f_Y(t)}{f_Y(x)} > \frac{F_Y(t)}{F_Y(x)}. \tag{20}$$

With the same assumption, $\zeta(x) := \int_{m_0}^{x} \frac{F_Y(t)}{F_Y(x)} dt$ is strictly increasing for $x < v$. [11] Now suppose that $m_*(r) \leq \hat{m}$. We would then have

$$0 \geq m_*(r) - \hat{m} > r \left\{ \int_{m_0}^{\hat{m}} \frac{F_Y(t)}{F_Y(\hat{m})} dt - \int_{m_0}^{m_*(r)} \frac{F_Y(t)}{F_Y(m_*(r))} dt \right\} \geq 0,$$

where the last inequality comes from the fact that $\zeta(x)$ is strictly increasing and the supposition that $m_*(r) \leq \hat{m}$. This contradiction implies that the supposition $m_*(r) \leq \hat{m}$ is false. Thus, we have proved that $m_*(r) > \hat{m}$ for each $r \in (0, q]$.

With $m_*(r) > \hat{m}$, Lemma 4.1 gives $\pi'(r)$ according to Equation (17). Recall the notation $D(r)$ in that equation. Since $q < 1$ and $F_{(1)}(t) = F(t)F_Y(t) < F(m_*(r))F_Y(t)$ for each $t < m_*(r)$, we have

$$D(r) < \frac{q}{1+r} F(m_*) \int_{m_0}^{m_*} F_Y.$$

Thus, by (17),

$$\pi'(r) < \left[\frac{q}{1+r} F(m_*) - n(1 - F(m_*)) \frac{h'(m_*)}{g'(m_*)} \right] \int_{m_0}^{m_*} F_Y.$$

It follows that $\pi'(r) < 0$ if

$$n \geq \frac{q}{1+r} \frac{F(m_*)}{1 - F(m_*)} \frac{g'(m_*)}{h'(m_*)}. \tag{21}$$

We compute that

$$\frac{g'(m_*)}{h'(m_*)} = 1 + \frac{rF_Y(m_*)}{F_Y(m_*) - (v - m_*) f_Y(m_*)} = 1 + \frac{r}{1 - \frac{r \int^{m_*} F_Y(\cdot)/F_Y(m_*)}{\int^{m_*} f_Y(\cdot)/f_Y(m_*)}} < 1 + \frac{r}{1-r},$$

where the second equality follows from (3) and the inequality from Assumption (18) and the fact (20). Thus, (21) will hold if

$$n \geq \frac{q}{1+r} \frac{F(m_*)}{1 - F(m_*)} \left[1 + \frac{r}{1-r} \right].$$

Since the right-hand-side of the inequality is less than $(q/(1-q))F(v)/(1-F(v))$, (21) will hold if (19) is satisfied. In other words, $\pi'(r) < 0$ if Condition (19) is met, as claimed by the proposition. Therefore, we have proved the remark. **Q.E.D.**

[11] To see this, we compute that

$$\zeta'(x) = 1 - \frac{f_Y(x)}{F_Y(x)} \int^{x} \frac{F_Y(t)}{F_Y(x)} dt = 1 - \frac{\int^{x} (F_Y(t)/F_Y(x)) dt}{\int^{x} (f_Y(t)/f_Y(x)) dt} > 0,$$

where the inequality follows from Assumption (18) and the fact (20).

5 An Example

Let us compute the solution of the auction game and the seller's profit when the distribution F of budgets is the uniform distribution on $[0,1]$. In this case, for each $x \in [0,1]$, $F(x) = x$, $F_Y(x) = x^{n-1}$, and $F_{(1)}(x) = x^n$; also $m_0 = 0$ and $\bar{m} = 1$. Notice that Assumptions 2, 3, and 4 are all satisfied. Assume that $v \in (0,1)$ so that Assumption 1 is met.

Take any $r > 0$. We first find m_* through solving for the nonzero root of the equation $g(m) = 0$, i.e.,

$$(v-m)m^{n-1} = r\int_0^m x^{n-1}dx.$$

As the only nonzero root of this equation is $nv/(n+r)$, we have

$$m_* = \frac{nv}{n+r}.$$

We next find $\hat{m}$ through solving the equation $v = \varphi(m)$, i.e., $v = m + \frac{m}{n-1}$. Thus, we have

$$\hat{m} = \frac{n-1}{n}v.$$

By the definition of m^*, we know that

$$m^* = \begin{cases} \frac{nv}{n+r} & \text{if } nr \leq n+r \\ \frac{n-1}{n}v & \text{otherwise.} \end{cases}$$

Thus, $m^* = nv/(n+r)$ if $q \leq 1$ (since $r \leq q$).

Once the values of m_* and m^* are determined, Equation (2) gives us the bidding strategy of the symmetric equilibrium for the auction game given $r > 0$:

$$\beta(m) = \begin{cases} \frac{1}{1+r}\left[v + \frac{n-1}{n}rm\right] & \text{if } 0 \leq m \leq \frac{nv}{n+r} \\ v\left[1 - \frac{r}{n+r}\left(\frac{nv}{n+r}\right)^{n-1}\right] & \text{otherwise} \end{cases}$$

when $nr \leq n+r$, and the bidding strategy when $nr > n+r$ can be computed likewise.

The profit function for the seller can be computed by applying Equation (14). When $nr \leq n+r$, we derive from (14) that

$$\pi(r) = v + \left[\frac{n-1}{1+r}qv - r\right]\left(\frac{nv}{n+r}\right)^n + \left[\frac{nr}{n+1} - \frac{qn(n+r)}{(n+1)(1+r)}\right]\left(\frac{nv}{n+r}\right)^{n+1}. \qquad (22)$$

The computation of $\pi(r)$ when $nr > n + r$ is likewise.

The following remark says that providing interest-free financing is strictly dominated, as Remark 4.1 states.

Remark 5.1 *If the distribution F is as specified in this section, and if $q \leq 1$, then $\pi(r) > \pi(0)$ for $r > 0$ sufficiently small.*

Proof: We can directly apply Remark 4.1. Alternatively, we can independently compute as follows. With the assumption $q \leq 1$, we have $nr \leq n + r$ and thus Equation (22) applies for all $r > 0$. As $\pi(0) = v - qv^2/2$ by (13), we compute as follows:

$$\pi(r) - \pi(0) = \frac{1}{2}qv^2 + \left(\frac{nv}{n+v}\right)^n \left[\frac{n-1}{1+r}qv - r + \left(\frac{nr}{n+1} - \frac{qn(n+r)}{(n+1)(1+r)}\right)\frac{nv}{n+r}\right].$$

Denoting the term inside the square bracket $[\cdots]$ by X, we have

$$\begin{aligned} X &= \frac{(n-1)qv}{1+r} - r + \frac{n^2 v}{n+1}\left(\frac{r}{n+r} - \frac{q}{1+r}\right) \\ &= (n-1)qv - (n-1)qvr + o(r) - r + \frac{n^2 v}{n+1}\left(\frac{r}{n} + o(r) - q - qr - o(r)\right) \\ &= (n-1)qv - \frac{n^2 v}{n+1}q + O(r), \end{aligned}$$

where the second equality follows from Taylor's formula. Thus,

$$\begin{aligned} \pi(r) - \pi(0) &= qv\left[\frac{1}{2}v - \frac{1}{n+1}\left(\frac{nv}{n+r}\right)^n\right] + O(r) \\ &> qv\left[\frac{1}{2}v - \frac{1}{n+1}v^n\right] + O(r), \end{aligned}$$

where the inequality follows from the fact that $n \geq 2$ and $nv/(n+r) < v$. Since $qv[\frac{1}{2}v - \frac{1}{n+1}v^n]$ is positive ($n \geq 2$) and independent of r, this proves the remark. **Q.E.D.**

The following remark gives a sufficient condition for the seller's expected payoff to increase with the decrease of interest rate.

Remark 5.2 *If the distribution F is as specified in this section, and if $q < 1$, then $\pi'(r) < 0$ for all $r > 0$ if*

$$n \geq \frac{v}{1-v}\frac{q}{1-q}.$$

Proof: Since Assumption (18) is satisfied by the uniform distribution F, and $q < 1$ by hypothesis, Remark 4.2 applies. Thus, $\pi'(r) < 0$ for all $r > 0$ if Inequality (19) holds. The right-hand-side of the inequality equals $\frac{v}{1-v}\frac{q}{1-q}$ for the uniform distribution F. Thus, the remark is proved. **Q.E.D.**

Part II

Auctions with Default Risk

6 A Formulation of Default Risk

From now on, we extend our analysis to incorporate default risk. Needless to say, the risk that a winning bidder may default should be seriously considered in the design of an auction where bidders are financially constrained. This is dramatically exemplified by the troubles in the C-block FCC auctions, where most of the winners could not pay what they have pledged. In the following, we formulate default risks as being generated by a shock of the value of the object being auctioned. [12] Coupled with this formulation is the limited liability of the bidders. In this model, we have discovered the closed form solution for the equilibrium of the auction game and the seller's expected payoff. One surprising feature of the results is that the equilibrium bidding strategy is strictly increasing in budgets for those interest rates above a critical value, and decreasing for those below it. Consequently, the poorest bidder would end with being the highest bidder when the interest rate is too low. From the seller's viewpoint, charging an interest rate lower than the critical value is strictly dominated.

The analysis of such a "flip" of the bidding strategy forces us to derive the monotonicity of bidding strategies rather than assume this property. The reasoning for the case with upward-sloping strategy turns out to be the general version of our analysis in Part I. The reasoning for the other case is different. The organization of the materials resembles that in Part I.

[12]This formulation resulted from a conversation with Professor Jim Jordan.

6.1 The Model

The model specified in Section 2 remains unchanged except for the following aspects.

There is an uncertainty of the value of the object being auctioned, which equals v with probability $1-\theta$, and equals 0 with probability θ, where $\theta \in [0,1)$ is publicly known among the seller and the bidders. The actual value of the object is not known to any of them until a winner of the auction is selected.

The auction proceeds as follows:

1. Same as Step 1 in Section 2.
2. Same as Step 2 in Section 2.
3. The value of the object is revealed to the bidders and the seller.
4. The winner chooses whether to pay his bid b or declare bankruptcy. If he declares bankruptcy, then the winner loses his entire budget m (i.e., limited liability up to the budget) and gets 0 in return, while the seller gets nothing. Otherwise, the winner pays his bid b to the seller for the object. In doing so, the winner bears a cost $C(b, m, r)$ defined in (1), and the seller's payoff is $b - (q - r)\max\{0, b - m\}$. The game is then over.

We assume that the distribution F of budgets has all the properties specified in Section 2, except for the following changes:

Assumption 1' (replacing Assumption 1) $m_0 < (1-\theta)v < \bar{m}$.

Assumption 5 *(Additional) The function ϕ given by $x \mapsto x + (1-\theta)(1+r)\frac{F_Y(x)}{f_Y(x)}$ is strictly increasing on $[m_0, \bar{m}]$, for every $r \in (0, q]$.*

6.2 When to Default and the Players' Payoffs

Remark 6.1 *Assume that the bidders do not play strictly dominated strategies.*

a) A winning bidder would not declare bankruptcy unless the value of the object is 0.

b) Given each $r \geq 0$, *for each bidder with budget* $m \in [m_0, \bar{m}]$ *who bids* b, *the bidder's expected payoff conditional to winning the object, before the revelation of its value, is* $(1-\theta)u(b,m,r)$, *where*

$$u(b,m,r) = \begin{cases} v - \frac{1}{1-\theta}b & \text{if } b \leq m \\ v - (1+r)b + \left(r - \frac{\theta}{1-\theta}\right)m & \text{otherwise.} \end{cases} \tag{23}$$

c) The seller's expected payoff given the winning bid $b_{(1)}$ *and the winner's budget* $m_{(1)}$, *before the realization of the value of the object, is*

$$(1-\theta)\left[b_{(1)} - (q-r)\max\{0, b_{(1)} - m_{(1)}\}\right] + \theta\chi_{[b_{(1)}<m_{(1)}]}b_{(1)}, \tag{24}$$

where $\chi_{[b_{(1)}<m_{(1)}]}$ *equals 1 if* $b_{(1)} < m_{(1)}$, *equals 0 if the inequality is reversed, and indeterminate (between 0 and 1) if* $b_{(1)} = m_{(1)}$.

Proof: Notice that any bid b such that $C(b,m,r) > v$ is strictly dominated by bidding 0. This implies (a), because the bidder loses m if bankrupt, while making the payment yields a nonnegative $v - C(b,m,r)$ if $v \neq 0$. It follows that the winner's expected payoff (before the revelation of the value) is

$$(1-\theta)[v - C(b,m,r)] + \theta\max\{-m, -C(b,m,r)\},$$

which is equivalent to (23) by the definition (1). Hence we have proved (b). We now prove (c). By (a), the payoff to the seller if the objective has positive value is the term in the [...] in (24). Look at the case when the object is valueless. Note that $b_{(1)} < m_{(1)}$ is equivalent to $C(b_{(1)}, m_{(1)}, r) < m_{(1)}$ (i.e., the winner strictly prefers paying $b_{(1)}$ to bankruptcy). Similarly, $b_{(1)} = m_{(1)}$ is equivalent to that the winner is indifferent between paying $b_{(1)}$ and bankruptcy. Thus, the seller's payoff when the object is valueless is $\chi_{[b_{(1)}<m_{(1)}]}$ (recall the assumption that the seller gets 0 if the winner declares bankruptcy). It follows that (24) is the seller's expected payoff after the selection of the winner and before the revelation of the value. This proves the remark. **Q.E.D.**

7 The Strategy for Bidders with Default Risk

As in Part I, once the seller has chosen an interest rate $r \geq 0$, an auction game among the bidders is given. Both the bidders and the seller want to know the bidding strategies of the auction game. A bidder needs this information to guide his bids, and the seller needs it to choose an interest rate. With default risk, however, the equilibrium bidding strategy is surprisingly different from that in Part I. Protected by their limited liability, the poor bidders bear little penalty when skyrocketing their bids. Thus, it may no longer be taken for granted that an equilibrium bidding strategy increases with budgets. We will first find out the critical value of interest rates at which the bidding strategy "flips" from an upward sloped curve to a downward sloped one (Section 7.1). We will then derive and verify the solution for the bidding strategy in Sections 7.2 and 7.3, the former dealing with the case with interest rates above the critical value, and the latter for those below it. The special (and yet simple) case when the interest rate is exactly the critical value is solved in Section 7.4.

7.1 How the Bidding Strategy "Flips"

As in Part I, with budgets m privately known to the bidders, a bidder's strategy is a mapping from budgets to bids. We shall still use symmetric equilibrium as the solution concept. The following proposition says that, for those bidders who bid over their budgets according to a bidding strategy of such an equilibrium, the dependency of bids on budgets changes from positive to negative as the interest rate falls below $\frac{\theta}{1-\theta}$. That $\frac{\theta}{1-\theta}$ is the critical value would not surprise us if we look back to Equation (23), where the coefficient $r - \frac{\theta}{1-\theta}$, if negative, becomes a penalty for wealthiness.

Proposition 7.1 (Strict Monotonicity) *Let $r \geq 0$ and let $\beta : [m_0, \bar{m}] \to R$ be a bidding strategy that comprises a symmetric equilibrium of the auction game under r. Suppose that β is continuous over some open interval $N \subset [m_0, \bar{m}]$ and $\beta(m) > m$ for each $m \in N$. Then $\beta|_N$ is strictly decreasing if $r < \frac{\theta}{1-\theta}$, and strictly increasing if $r > \frac{\theta}{1-\theta}$.*

Proof: Let $r < \frac{\theta}{1-\theta}$ and let β and N be as specified by the hypotheses. We shall prove that $\beta|_N$ is strictly decreasing. The proof for the case $r > \frac{\theta}{1-\theta}$ is analogous. Let us temporarily assume the following:

1. that $\beta|_N$ is weakly decreasing, as Lemma 7.1 asserts, and
2. that the bids according to β are atomless as long as the payoff conditional to winning the auction is positive, as Lemma 7.2 claims.

Suppose that $\beta|_N$ is not strictly decreasing. Then by the first temporary assumption there is a nondegenerate interval $[c, d] \subset N$ on which β is constant. This, by the second temporary assumption and the fact that $F([c, d]) > 0$ (F is strictly increasing), would be impossible unless $u(\beta(t), t, r) \leq 0$ for all $t \in [c, d]$. With β being an equilibrium strategy, $u(\beta(t), t, r) < 0$ is impossible. Neither can $u(\beta(t), t, r) \equiv 0$, because that would imply $\beta(t) = (v + r't)/(1 + r)$ for all $t \in [c, d]$, which is strictly decreasing in t, while β is supposed to be constant on $[c, d]$ if it were not strictly decreasing. Thus, the supposition that β is not strictly decreasing over N has led to a contradiction. Therefore, the proposition will be proved if Lemmas 7.1 and 7.2 are proved. We hence demonstrate them in the following.

Lemma 7.1 (Weak Monotonicity) *Assume all the hypotheses of Proposition 7.1. Then $\beta|_N$ is weakly decreasing if $r < \frac{\theta}{1-\theta}$, and weakly increasing if $r > \frac{\theta}{1-\theta}$.*

Reason (see Appendix for Formal Proof): Pick any $x, x' \in N$. Since β is an equilibrium strategy, the expected payoff for a bidder with budget x to bid $\beta(x)$ cannot be lower than his expected payoff if he bids $\beta(x')$ instead. A similar relation holds for a bidder with budget x'. If x and x' are sufficiently close to each other so that $\beta(x) > x'$ and $\beta(x') > x$, then the two previous relations imply that

$$r'(x - x')(\text{Prob}[\text{win}|\beta(x)] - \text{Prob}[\text{win}|\beta(x')]) \geq 0, \tag{25}$$

where Prob[win$|b$] denotes the probability that a bid b is the winning bid and $r' := r - \frac{\theta}{1-\theta}$. This inequality is exactly the object showing why the equilibrium bidding strategy "flips":

If $r' > 0$, then $x > x'$ implies that Prob[win$|\beta(x)$] $\geq$ Prob[win$|\beta(x')$]; with Prob[win$|\cdot$] strictly increasing on the range of β (because F is strictly increasing and β continuous), we then have $\beta(x) \geq \beta(x')$. In contrast, if $r' < 0$, then $x > x'$ implies that Prob[win$|\beta(x)$] $\leq$ Prob[win$|\beta(x')$] and hence $\beta(x) \leq \beta(x')$! A standard compactness argument extends this local result to cover those x and x' that are not sufficiently near to each other. □

Lemma 7.2 (Atomless Bids) *Let $r \geq 0$ and let $\beta : [m_0, \bar{m}] \to R$ be a bidding strategy that comprises a symmetric equilibrium of the auction game under r. Then there is no subset $E \subset [m_0, \bar{m}]$ such that $F(E) > 0$, $\beta|_E \equiv b$ for some b, and $u(b, m, r) > 0$ for some $m \in E$.*

Reason (see Appendix for Formal Proof): Recall that $u(b, m, r)$ is equivalent to the payoff conditional to winning the auction (up to a positive constant). This payoff is continuous in the bids b. Thus, if $u(b, m, r)$ is positive, then bidding slightly higher than b would still give a positive payoff conditional to a win. At such a position, a bidder with budget m would not bid b if he believes that there is a positive probability for the event "others bid b". The reason is that bidding slightly higher than b would improve his chance to win by a positive number, while the sacrifice due to the slightly higher bid is negligible. The formal proof is merely an ϵ-δ version of our reasoning here. □

With Lemmas 7.1 and 7.2, we have completed the proof for the proposition. **Q.E.D.**

7.2 The Bidding Strategy for $r > \theta/(1-\theta)$

When the seller charges an interest rate $r > \theta/(1-\theta)$, what we have proved above (Proposition 7.1) says that the bids strictly increase with budgets, if bids are over budgets. Thus, it is reasonable in this case to take our solution concept as a symmetric equilibrium whose bidding strategy is continuous, strictly increasing, and piecewise differentiable in budgets. Our solution for the case $r > \theta/(1-\theta)$ turns out to be a generalization of all our results in Section 3 (in the sense that θ is no longer collapsed into 0). The claims and proofs in this section thus parallel those in Section 3, with $(1-\theta)v$ taking the role of v, and $r' := r - \theta/(1-\theta)$ as a counterpart of r. We shall use the same notation F_Y as before.

Theorem 7.1 *Let $r \in (\frac{\theta}{1-\theta}, q]$ be the interest rate chosen by the seller.*

1. *The auction game has a unique symmetric equilibrium whose bidding strategy is strictly increasing, continuous, and piecewise differentiable.*
2. *This bidding strategy $\beta : [m_0, \bar{m}] \to R$ is:*

$$\beta(m) := \begin{cases} \frac{1}{1+r}\left[v + r'm - r'\int_{m_0}^{m} \frac{F_Y(t)}{F_Y(m)}dt\right] & \text{if } m_0 \le m < m_*(r) \\ m & \text{if } m_*(r) \le m \le m^*(r) \\ (1-\theta)\left[v - \left(v - \frac{m^*(r)}{1-\theta}\right)\frac{F_Y(m^*(r))}{F_Y(m)}\right] & \text{otherwise,} \end{cases} \tag{26}$$

where $r' := r - \frac{\theta}{1-\theta}$, $m_*(r)$ is the unique non-m_0 root for the equation

$$\left(v - \frac{m_*}{1-\theta}\right) F_Y(m_*) = r' \int_{m_0}^{m_*} F_Y(t)dt, \tag{27}$$

and $m^*(r) := \max\{m_*(r), \hat{m}\}$, with $\hat{m}$ a constant given by

$$(1-\theta)v = \hat{m} + \frac{F_Y(\hat{m})}{f_Y(\hat{m})}. \tag{28}$$

The proof of the theorem is similar to that of Theorem 3.1, and it will be proved in Sections 7.2.1 (for uniqueness) and 7.2.2 (for existence). We here look at the properties of the bidding strategy. It is just a scaled down copy of the bidding strategy in Theorem 3.1 by a factor $1-\theta$.

Proposition 7.2 *Given any $r \in (\frac{\theta}{1-\theta}, q]$, the function β defined in (26) has the following properties:*

a) It is well-defined, strictly increasing, continuous and bounded from above by $(1-\theta)v$.

b) $\beta(m) > m$ if $m < m_*(r)$, $\beta(m) = m$ if $m_*(r) \le m \le m^*(r)$, and $\beta(m) < m$ otherwise.

c) $m_*(\cdot)$ is a one-to-one function over the domain $(\frac{\theta}{1-\theta}, \infty)$, and the derivative $m_*' < 0$ over $(\frac{\theta}{1-\theta}, \infty)$.

d) β is piecewise differentiable and

$$\beta'(m) = \begin{cases} \frac{r'}{1+r}\frac{f_Y(m)}{F_Y(m)^2}\int_{m_0}^{m} F_Y(t)dt & \text{if } m_0 < m < m_*(r) \\ (1-\theta)\left(v - \frac{m^*(r)}{1-\theta}\right) F_Y(m^*(r))\frac{f_Y(m)}{F_Y(m)^2} & \text{if } m^*(r) < m < \bar{m} \\ 1 & \text{if } m_*(r) < m < m^*(r). \end{cases} \tag{29}$$

Proof: Imitate the proof for Proposition 3.1 and replace the functions g and h with their generalized counterparts $\tilde{g}$ and $\tilde{h}$ defined in the following:

$$\tilde{g}(x) := \left(v - \frac{x}{1-\theta}\right) F_Y(x) - r' \int_{m_0}^{x} F_Y(t)dt, \ x \in [m_0, \bar{m}];$$

$$\tilde{h}(x) := \left(v - \frac{x}{1-\theta}\right) F_Y(x), \ x \in [m_0, \bar{m}].$$

As for the case of g and h, both $\tilde{g}$ and $\tilde{h}$ are differentiable and

$$\tilde{g}'(x) = f_Y(x)\left[v - \frac{x}{1-\theta} - (1+r)\frac{F_Y(x)}{f_Y(x)}\right], \ x \in (m_0, \bar{m});$$

$$\tilde{h}'(x) = f_Y(x)\left[v - \frac{x}{1-\theta} - \frac{1}{1-\theta}\frac{F_Y(x)}{f_Y(x)}\right], \ x \in (m_0, \bar{m}).$$

As a generalized counterpart of (6), we have

$$m_*'(r) = \frac{\int_{m_0}^{m_*(r)} F_Y}{\tilde{g}'(m_*(r))}. \tag{30}$$

The generalized version of Lemmas 3.1 and 3.2 are established:

Lemma 7.3 *For each $r \geq \frac{\theta}{1-\theta}$, there is exactly one $m_* \in (m_0, (1-\theta)v]$ such that $\tilde{g}(x) > 0$ if $x < m_*$, $\tilde{g}(x) = 0$ if $x = m_*$, and $\tilde{g}(x) < 0$ otherwise. Furthermore, $\tilde{g}'(x) < 0$ if $x \geq m_*$, and $m_* = (1-\theta)v$ if $r = \frac{\theta}{1-\theta}$.*

Proof: Use the same proof for Lemma 3.1, replacing r with r', v with $(1-\theta)v$, g with $\tilde{g}$, and ψ with ϕ. The condition $r \geq \frac{\theta}{1-\theta}$ is needed for $\tilde{g}((1-\theta)v) \leq 0$ (which guarantees the existence of m_*). $\square$

Lemma 7.4 *There is exactly one $\hat{m} \in (m_0, (1-\theta)v)$ such that $\tilde{h}'(x) > 0$ if $x < \hat{m}$, $\tilde{h}'(x) < 0$ if $x > \hat{m}$, and $\tilde{h}'(\hat{m}) = 0$.*

Proof: Use the same proof for Lemma 3.2, replacing v with $(1-\theta)v$ and h with $\tilde{g}$. $\Box$

Thus, the proposition is proved in the same manner of Proposition 3.1. **Q.E.D.**

7.2.1 Deriving the Solution (Uniqueness Proof)

This section is a proof for the uniqueness part in Theorem 7.1. It follows exactly the same structure of the uniqueness proof for Theorem 3.1 (Section 3.2). The only difference is that $\tilde{g}$ replaces g, $\tilde{h}$ replaces h, $(1-\theta)v$ plays the role of v, and r' the role of r.

Let $r \geq 0$. [13] Let β be a strictly increasing, continuous and piecewise differentiable bidding strategy that constitutes a symmetric equilibrium. With β strictly increasing, we can define, as in the proof Section 3.2,

$$V(b,m) := u(b,m,r)F_Y \circ \beta^{-1}(b), \; b \geq 0, \; m \in [m_0, \bar{m}]. \tag{31}$$

By Remark 6.1, $(1-\theta)V(b,m)$ is the expected payoff for a bidder with budget m to bid b, provided that others play the strategy β and the seller charges the interest rate r. Note that $V(\cdot, m)$ is continuous. The rest of the uniqueness proof is exactly the same as that in Section 3.2. We hence outline the main steps, each being the counterpart of a corresponding lemma in Section 3.2, and leave the detail to the reader.

1. For any $r \geq 0$,
$$\beta(m_0) = \frac{v + r'm_0}{1+r} > m_0.$$
In this step, Assumption 1' is used to prove $\beta(m_0) > m_0$ from an intermediate equation $u(\beta(m_0), m_0, r) = 0$, which is proved by the condition that the equilibrium strategy β is strictly increasong and that $u(\beta(\cdot), \cdot, r)$ is continuous.

2. If $m_0 \leq m \leq m_*$ then
$$\beta(m) = \frac{1}{1+r}\left[v + r'm - r'\int_{m_0}^{m} \frac{F_Y(t)}{F_Y(m)}dt\right].$$

[13]Note that nowhere in this proof is the condition $r \geq \frac{\theta}{1-\theta}$ needed.

3. There is exactly one $m'' \in [m_*, \bar{m})$ such that

$$\beta(m) = (1-\theta)\left[v - \left(v - \frac{m''}{1-\theta}\right)\frac{F_Y(m'')}{F_Y(m)}\right], \quad m \in [m'', \bar{m}].$$

4. If $x \geq \hat{m}$ and $\beta(x) = x$, then there is no $x' > x$ for which $\beta(x') = x'$.

5. For each $m \in [m_*, m'']$, $\beta(m) = m$.

6. $m'' = m^*$.

Summing up the above steps, we have solved the bidding strategy β. This finishes the uniqueness proof for Theorem 7.1. **Q.E.D.**

7.2.2 Verifying the Solution (Existence Proof)

This section verifies that it is an equilibrium of the auction game for each bidder to bid according to the function β defined in (26) when $r > \frac{\theta}{1-\theta}$. The proof parallels the existence proof for Theorem 3.1 (Section 3.3). With (a) of Proposition 7.2, the function $V(\cdot,\cdot)$ defined in (31) is well-defined and continuous. As in Section 3.3, we shall conduct the verification by proving the following lemmas.

Lemma 7.5 *For each $m < m_*$,*
(a) $\beta(m)$ maximizes $V(\cdot, m)$ over $[m, m_*]$,
(b) $\beta(m)$ strictly dominates any bid $b \in (m_*, m^*]$,
(c) $\beta(m)$ strictly dominates any bid $b > m^*$, and
(d) $\beta(m)$ strictly dominates any bid $b < m$.

Proof: Part (a): Follow the proof for Part (a) of Lemma 3.9. The term for $D_1V(b, m)$ is now a positive term multiplied by $r'(m - x)$ (instead of $r(m - x)$ in that proof). Part (a) then follows from the condition $r' > 0$.

Part (b): Follow the proof for Part (b) of Lemma 3.9 and use Lemma 7.3 (instead of Lemma 3.1 in that proof).

Part (c): Follow the proof for Part (b) of Lemma 3.9. The term for $D_1V(b,m)$ is now a positive term multiplied by $v + r'm - (1+r)(1-\theta)v$, which equals

$$v + r'm - \left(1 + r' + \frac{\theta}{1-\theta}\right)(1-\theta)v = r'[m - (1-\theta)v],$$

which is negative because $r' > 0$ and $m < m_* < (1-\theta)v$ ($\beta < (1-\theta)v$ by (a) of Proposition 7.2). Part (c) thus follows.

Part (d): Follow the proof for Part (d) of Lemma 3.9. The term for $D_1V(b,m)$ is now a positive term multiplied by $r'[(1-\theta)v - x]$, which is positive since $r' > 0$ and $\beta < (1-\theta)v$.

The lemma is therefore proved. □

Lemma 7.6 *For each $m > m^*$,*

(a) $V(\cdot, m)$ is constant on $[m^*, m]$, and hence $\beta(m)$ maximizes $V(\cdot, m)$ over $[m^*, m]$,

(b) any bid $b \in [m_*, m^*)$ is strictly dominated,

(c) any bid $b < m_*$ is strictly dominated, and

(d) any bid $b > m$ is strictly dominated.

Proof: Part (b): Use the proof for Part (b) of Lemma 3.10, replacing h with $\tilde{h}$. For Parts (a), (c), and (d): Follow the respective proofs for Parts (a), (c), and (d) of Lemma 3.10. The Lemma is therefore proved. □

Lemma 7.7 *For each $m \in [m_*, m^*]$,*

(a) any bid $b \in [m_*, m)$ is strictly dominated by m,

(b) any bid $b \in (m, m^*]$ is strictly dominated by m,

(c) any bid $b < m_*$ is strictly dominated, and

(d) any bid $b > m^*$ is strictly dominated.

Proof: Parts (a), (c), and (d): Follow the respective proofs for Parts (a), (c), and (d) of 3.11. For Part (b): Use the proof for Part (b) of 3.11, replacing the reference of ψ with that of ϕ. The lemma is therefore proved. □

Summing up Lemmas 7.5, 7.6, and 7.7, with $V(\cdot, m)$ a continuous function for each m, we have therefore proved that, for each budget $m \in [m_0, \bar{m}]$, $\beta(m)$ maximizes the expected payoff $V(\cdot, m)$ of any bidder with budget m, provided that others play the strategy β. In other words, we have proved that each bidder playing β is an equilibrium of the auction game. This completes the existence proof. **Q.E.D.**

7.3 The Bidding Strategy for $r < \theta/(1-\theta)$

When the seller charges an interest rate $r < \theta/(1-\theta)$, what we have proved above (Proposition 7.1) says that the bids strictly *decrease* with budgets, if bids are over budgets. Thus, we may no longer assume that a symmetric equilibrium bidding strategy is continuous, strictly increasing, and piecewise differentiable in budgets. Instead, we take our solution concept as a symmetric equilibrium whose bidding strategy is continuous and piecewise differentiable. The surprising feature of the solution in this case is that those whose budgets are lower than the expected value $(1-\theta)v$ of the object would bid higher than $(1-\theta)v$, while those whose budgets are higher than $(1-\theta)v$ would rather bid $(1-\theta)v$ than compete with the low budget bidders. An interest rate below the critical adversely selects the poorest bidder as the winner of the auction.

Theorem 7.2 *Let $r \in [0, \frac{\theta}{1-\theta})$ be the interest rate chosen by the seller.*

1. *It is an equilibrium of the auction game that each bidder bids according to the strategy $\beta : [m_0, \bar{m}] \to R$ given by*

$$\beta(m) := \begin{cases} \frac{1}{1+r}\left[v + r'm + r' \int_m^{(1-\theta)v} \left(\frac{1-F(t)}{1-F(m)}\right)^{n-1} dt\right] & \text{if } m_0 \le m \le (1-\theta)v \\ (1-\theta)v & \text{otherwise,} \end{cases} \tag{32}$$

   where $r' := r - \frac{\theta}{1-\theta}$.

2. *If $\bar{m} < \infty$, then "every bidder plays β" is the only symmetric equilibrium of the auction game whose bidding strategy is continuous and piecewise differentiable.*

Theorem 7.2 will be proved in Section 7.3.1 (for uniqueness) and 7.3.2 (for existence). We here look at the properties of the bidding strategy. In particular, Property (b) says that the poor bidders would pledge to pay for the object at a price higher than the expected value of the object!

Proposition 7.3 *Given any $r \in [0, \frac{\theta}{1-\theta})$, the function β defined in (32) satisfies the following properties:*

a) It is well-defined, continuous, strictly decreasing over $[m_0, (1-\theta)v)$, and constant from $(1-\theta)v$ on.

b) $\beta(m) > (1-\theta)v$ for all $m < (1-\theta)v$ and $\beta(m) = (1-\theta)v$ for all $m \geq (1-\theta)v$.

c) $\max \beta = \beta(m_0) < v$ and $\min \beta = (1-\theta)v$.

d) β is differentiable over the interior of its domain, and

$$\beta'(m) = \begin{cases} \frac{r'}{1+r} \frac{(n-1)f(m)}{(1-F(m))^n} \int_m^{(1-\theta)v} (1-F(t))^{n-1} dt & \textit{if } m_0 < m < (1-\theta)v \\ 0 & \textit{if } (1-\theta)v \leq m < \bar{m}. \end{cases} \tag{33}$$

Proof: Property (d) follows from directly computing the derivative of β according to (32). By (d) and the fact that $r' < 0$, β is strictly decreasing over $[m_0, (1-\theta)v)$. Thus, (a) holds. Property (a) then implies (c), where $\beta(m_0) < v$ due to (32) and $r' < 0$. For (b) we need only to prove that $\beta(m) > (1-\theta)v$ for all $m < (1-\theta)v)$. For an $m < (1-\theta)v)$, according to (32) and the fact that $r' < 0$, $\beta(m) > (1-\theta)v$ is equivalent to

$$\int_m^{(1-\theta)v} \left(\frac{1-F(t)}{1-F(m)} \right)^{n-1} dt < (1-\theta)v - m,$$

which is trivial. Thus, we have proved the proposition. **Q.E.D.**

7.3.1 Deriving the Solution (Uniqueness Proof)

Take any $r \in [0, \frac{\theta}{1-\theta})$ and let β be a symmetric equilibrium bidding strategy that is continuous and piecewise differentiable (the proposition will be vacuously true if no such β exists). We will solve for β by proving the following lemmas.

Lemma 7.8 *If $\beta(m) > m$ then there is an open interval N such that $m \in N \cap [m_0, \bar{m}]$ and β is strictly decreasing over $N \cap [m_0, \bar{m}]$.*

Proof: Let $\beta(m) > m$. Then by the continuity of β we have an open interval N such that $m \in N \cap [m_0, \bar{m}]$ and $\beta(x) > x$ for all $x \in N \cap [m_0, \bar{m}]$. With $r < \frac{\theta}{1-\theta}$, Proposition 7.1 then implies that β is strictly decreasing over N. We have hence proved the lemma. □

Lemma 7.9 *If $\bar{m} < \infty$ then* $\min \beta = (1-\theta)v$.

Proof: Let $\bar{m} < \infty$. With β a continuous function on the compactness space $[m_0, \bar{m}]$, there is a $z \in [m_0, \bar{m}]$ such that $\beta(z) = \min \beta$. Since β is an equilibrium strategy, that $\beta(z)$ is the lowest bid in equilibrium implies that $u(\beta(z), z, r) = 0$, by the same proof for (10) in Section 3.2. We further show that $\beta(z) \leq z$. This holds if $z < \bar{m}$; otherwise, Lemma 7.8 would imply that β is strictly decreasing on an interval $[z, z']$ for some $z' \in (z, \bar{m})$; but then $\beta(z)$ would not have been $\min \beta$. If $z = \bar{m}$, we also have $\beta(z) \leq z$; otherwise, the expected payoff of bidding $\beta(z)$ would be negative (since $\bar{m} > (1-\theta)v$ (Assumption 1') and $u(b, m, r) > v - b/(1-\theta)$ when $b > m$). Thus, we have $u(\beta(z), z, r) = 0$ and $\beta(z) \leq z$, which imply that $\beta(z) = (1-\theta)v$ (recall (23)). We have hence proved that $\min \beta = \beta(z) = (1-\theta)v$. The lemma is therefore proved. □

Lemma 7.10 $\beta|_{[(1-\theta)v, \bar{m}]} \equiv (1-\theta)v$.

Proof: By Lemma 7.9, we need only to prove that $\beta(m) \leq (1-\theta)v$ for all $m \geq (1-\theta)v$. This will be done by showing that a bid higher than $(1-\theta)v$ is strictly dominated for these m. Hence pick any $m \geq (1-\theta)v$. If $(1-\theta)v < b \leq m$,

$$u(b, m, r) = v - \frac{b}{1-\theta} < 0$$

by (23). Thus, such a bid yields a negative expected payoff and is strictly dominated by bidding 0. For any $b > m$,

$$u(b, m, r) = v - (1+r)b + r'm < v - (1+r-r')m = v - \frac{m}{1-\theta} \leq 0$$

by the definition of r' and the fact that $m \geq (1-\theta)v$. Thus, bidding $b > m$ is strictly dominated. It follows that $\beta(m) \leq (1-\theta)v$ for all $m \geq (1-\theta)v$. The lemma therefore follows from Lemma 7.9. □

Lemma 7.11 *For each $m \in [m_0, (1-\theta)v)$, $\beta(m) > (1-\theta)v$.*

Proof: Take any $m \in [m_0, (1-\theta)v)$. We first show that a bid $b < (1-\theta)v$ is strictly dominated: Since $(1-\theta)v = \min \beta$ (Lemma 7.9), bidding $b < (1-\theta)v$ gives 0 payoff. In contrast, bidding $(1-\theta)v$ gives a positive expected payoff, since

$$u((1-\theta)v, m, r) = v - (1+r)(1-\theta)v + r'm = -r'[(1-\theta)v - m] > 0$$

(the first equality comes from $(1-\theta)v > m$, the second from the definition of r', and the inequality comes from $r' < 0$) and the probability of being picked when the auction is tied is positive (the number n of bidders is finite). Thus, any bid lower than $(1-\theta)v$ is strictly dominated for a bidder with budget m. Now that $u((1-\theta)v, m, r) > 0$ as computed above, with $F([(1-\theta)v, \bar{m}]) > 0$ and $\beta|_{[(1-\theta)v, \bar{m}]} \equiv (1-\theta)v$ (Lemma 7.10), Lemma 7.2 implies that $\beta(m) \neq (1-\theta)v$. Therefore, we have proved that $\beta(m) > (1-\theta)v$ for all $m \in [m_0, (1-\theta)v)$. This proves the lemma. □

Lemma 7.12

$$\beta(m) = \frac{1}{1+r}\left[v + r'm + r'\int_m^{(1-\theta)v}\left(\frac{1-F(t)}{1-F(m)}\right)^{n-1} dt\right], \quad m \in [m_0, (1-\theta)v). \tag{34}$$

Proof: Denote $\beta_1 := \beta|_{[m_0,(1-\theta)v)}$. By Lemma 7.11, $\beta(x) > x$ for all $x \in (m_0, (1-\theta)v)$. Proposition 7.1 then implies that β is strictly decreasing over $(m_0, (1-\theta)v)$, and hence (by the continuity of β) strictly decreasing over $[m_0, (1-\theta)v)$. Thus, for any b in the range of β_1, the probability for b to be the highest bid is $[1 - F \circ \beta_1^{-1}(b)]^{n-1}$. Define

$$V(b, m, r) := (v - (1+r)b + r'm)(1 - F \circ \beta_1^{-1}(b))^{n-1}, \quad b \in \text{Range } \beta_1, m \in [m_0, (1-\theta)v). \tag{35}$$

By (23), $(1-\theta)V(b, m, r)$ is the expected payoff for a bidder with budget m to bid b in the range of β_1, provided that others play the strategy β. Since β is an equilibrium strategy and is piecewise differentiable, $\beta_1(m)$ satisfies the first-order-necessary-condition for the maximization of $V(\cdot, m, r)$, for all $m \in (m_0, (1-\theta)v)$ but finitely many possible exceptions (this is proved in the same way as the footnote of Lemma 3.4 of Section 3.2). The necessary condition states that

$$(1-F(m))^{n-1}\beta_1'(m)-(n-1)f(m)(1-F(m))^{n-2}\beta_1(m) = -\frac{n-1}{1+r}(v+r'm)f(m)(1-F(m))^{n-2}.$$

This being true for all $m \in (m_0, (1-\theta)v)$ but finitely many possible exceptions, with β continuous, we can solve the linear differential equation to get

$$(1-F(m))^{n-1}\beta_1(m) = \frac{1}{1+r}\int_{m_0}^{m}(v+r't)\frac{d}{dt}\left[(1-F(t))^{n-1}\right]dt + c, \ \ m \in [m_0, (1-\theta)v), \quad (36)$$

for some constant c. With β continuous, Lemma 7.10 implies that $\beta_1((1-\theta)v) = (1-\theta)v$. This, coupled with (36), determines the constant c. Plugging the value of c back into (36), we derive Equation (34) after a lengthy and yet elementary computation, where the identity $r' = r - \theta/(1-\theta)$ is repeatedly used. The lemma is hence proved. □

Summing up Lemmas 7.10 and 7.12, we have solved the bidding strategy β. This β is hence the only candidate for the symmetric equilibrium strategy of the auction game up to the restriction of continuity and piecewise differentiability. This finishes the proof for uniqueness. **Q.E.D.**

7.3.2 Verifying the Solution (Existence Proof)

This section verifies that it is an equilibrium of the auction game for each bidder to bid according to the function β defined in (32) when $r < \frac{\theta}{1-\theta}$. Since β is well-defined by Proposition 7.3, this also proves the existence part of Theorem 7.2.

We first prove that bidding $(1-\theta)v$ is optimal for any $m \geq (1-\theta)v$ given that others play the strategy β. As in the proof for Lemma 7.10, any bid higher than $(1-\theta)v$ yields negative expected payoff for any $m \geq (1-\theta)v$, while bidding $(1-\theta)v$ gives 0 payoff. With $(1-\theta)v$ being $\min \beta$, bidding lower than $(1-\theta)v$ cannot improve upon bidding $(1-\theta)v$. Thus, for any $m \geq (1-\theta)v$, $\beta(m) = (1-\theta)v$ is optimal.

We next show that $\beta(m)$ is optimal for each $m < (1-\theta)v$. Thus, pick any such m. First, any bid $b \leq (1-\theta)v$ is dominated for such an m. As shown in the proof for Lemma 7.11, bidding below $(1-\theta)v$ is strictly dominated. As the proof for Lemma 7.2 shows, bidding $(1-\theta)v$ is strictly dominated, given that β is played by other bidders. Thus, an optimal bid for m, if exists, must be greater than $(1-\theta)v$.

To complete the proof for the claim that $\beta(m)$ is optimal for m, therefore, we need only to prove that $\beta(m)$ maximizes $V(\cdot, m, r)$ over $((1-\theta)v, \beta(m_0)]$ (the range of β_1). For any $b \in ((1-\theta)v, \beta(m_0))$, i.e., b belongs to the interior of the range of β_1, $V(b, m, r)$ defined in (35) is well-defined and $(1-\theta)V(b, m, r)$ is the expected payoff for a bidder with budget m to bid b, provided that others bid according to β. Further, $V(\cdot, m, r)$ is differentiable at such b. Thus, letting $x := \beta_1(b)$ (x is well-defined since β_1 is strictly monotone),

$$D_1 V(b,m,r) = \frac{(n-1)f(x)(1-F(x))^{n-2}}{-\beta_1'(x)}\left[v - (1+r)\beta_1(x) + r'm + (1+r)\beta_1'(x)\frac{1-F(x)}{(n-1)f(x)}\right].$$

Plugging (33) and (34) into the above equation and noting that $\beta_1'(x) < 0$ ((a) of Proposition 7.3), we obtain

$$D_1 V(b,m) = (\text{a positive term}) \times r'(m-x) \begin{cases} > 0 & \text{if } x > m, \text{ i.e., } b < \beta_1(m) \\ = 0 & \text{if } x = m, \text{ i.e., } b = \beta_1(m) \\ < 0 & \text{if } x < m, \text{ i.e., } b > \beta_1(m), \end{cases}$$

since $r' < 0$ and β_1 is strictly decreasing. It follows that $\beta_1(m)$ is the maximum of $V(\cdot, m, r)$ over the range of β_1. Thus, $\beta_1(m)$ is optimal for m, provided that others bid according to β. We have hence verified that β is an equilibrium bidding strategy. This completes the existence proof of Theorem 7.2. **Q.E.D.**

7.4 The Bidding Strategy for the Interest Rate $\frac{\theta}{1-\theta}$

Theorems 7.1 and 7.2 have covered all the cases except when the seller charges the critical-value interest rate $\frac{\theta}{1-\theta}$. We shall solve this special case in this section. The solution is quite simple and is a generalization of its counterpart without default risk (Remark 3.1 in Section 3.4).

Remark 7.1 *If the seller chooses* $\frac{\theta}{1-\theta}$ *as the interest rate, then (i) each bidder bidding* $(1-\theta)v$ *is an equilibrium of the auction game and, further, (ii) the winning bid equals* $(1-\theta)v$ *under any equilibrium of the auction game.*

Proof: With $r = \frac{\theta}{1-\theta}$, a bidder's expected payoff conditional to winning is $(1-\theta)v - b$ (see (23) in Remark 6.1). Thus, (i) follows. The proof for (ii) is the same as that for (ii) in Remark 3.1. **Q.E.D.**

8 The Seller's Strategy with Default Risk

The precise solution of the auction game discovered in Section 7 gives the seller a guide to choose the interest rate to maximize his expected payoff. Using Theorems 7.1 and 7.2, as well as Remark 7.1, the first proposition in this section derives a general formula of the seller's expected payoff as a function of the interest rate he charges. Furthermore, this section provides a general guideline (combining Propositions 8.2 and 8.3) for choosing interest rates: The seller "should" (in the sense of strict dominance) charge an interest rate above the critical value $\frac{\theta}{1-\theta}$. The intuition is that an interest rate above the critical value selects the most financially capable bidder as the winner, while an interest rate below it "adversely" selects the poorest bidder as the winner, and the interest rate $\frac{\theta}{1-\theta}$ selects the winner at random.

Thus, the seller can choose his interest rate according to the following procedure:

1. Partition the domain $[\frac{\theta}{1-\theta}, q]$ of choice by a finite set $r_1, \ldots, r_k$. The more powerful the computer, the finer the partition can be.

2. For each node r_j, $j = 1, \ldots, k$, compute $m_*(r_j)$ and $m^*(r_j)$ according to (27) and (28) in Theorem 7.1; then compute the seller's expected payoff at r_j according to the formula in Proposition 8.1. [14]

3. Choose the r_j that has the highest payoff according to the above computation.

In the following, we will use the notations $F_{(1)}(x) := F(x)^n$, $F_{(Y)}(x) := F(x)^{n-1}$, and $r' := r - \frac{\theta}{1-\theta}$ as before. We will use m_* for $m_*(r)$ and m^* for $m^*(r)$. Denote $\pi(r)$ for the seller's expected payoff (before the completion of the auction game) when he charges an interest rate r.

[14] A term in (39) is multiplied by a coefficient, whose value is indeterminate between 0 and 1. This is due to the fact that we do not assume whether a winner would choose bankruptcy or not if he is indifferent between the two options. One can resolve this indeterminateness through assuming a value of the coefficient. Moreover, the problem of indeterminateness disappears when the market interest rate q is not too high, as Remark 8.1 asserts.

Proposition 8.1 *The seller's expected payoff $\pi(r)$ (before the selection of the winner) from charging $r \in [0, q]$ is the following three cases:*

***If** $r = \frac{\theta}{1-\theta}$ **then**:*

$$\pi(r) = (1-\theta)\left[v(1-\theta F((1-\theta)v) - \left(q - \frac{\theta}{1-\theta}\right)\int_{m_0}^{v} F(m)dm\right]. \tag{37}$$

***If** $r \in [0, \frac{\theta}{1-\theta})$ **then**:*

$$\begin{aligned}\pi(r) \;&=\; \left(1-\frac{q}{1+r}\right)(1-\theta)v + \left(\frac{q}{1+r}-\theta\right)m_0 + \left(\frac{q}{1+r}-\theta\right)\int_{m_0}^{(1-\theta)v}(1-F)^n \qquad (38)\\ &+ \theta(1-\theta)v(1-F((1-\theta)v))^n + nr'(1-\theta)\left(1-\frac{q}{1+r}\right)\int_{m_0}^{(1-\theta)v}(1-F)^{n-1}F.\end{aligned}$$

***If** $r \in (\frac{\theta}{1-\theta}, q]$ **then**:*

$$\begin{aligned}\pi(r) \;&=\; (1-\theta)\left(1-\frac{q}{1+r}\right)vF_{(1)}(m_*) + \left(\frac{q}{1+r}-\theta\right)\int_{m_0}^{m_*} mdF_{(1)}(m) + \chi\int_{m_*}^{m^*} mdF_{(1)}(m)\\ &+ (1-\theta)v(1-F_{(1)}(m^*)) - n(1-\theta)\left(v - \frac{m^*}{1-\theta}\right)F_Y(m^*)(1-F(m^*))\\ &- nr'(1-\theta)\left(1-\frac{q}{1+r}\right)\int_{m_0}^{m_*}(F(m_*)F_Y - F_{(1)}),\end{aligned} \tag{39}$$

where m_* and m^* are given by (27) and (28), and χ is indeterminate between 0 and 1.

Proof: Let us first derive $\pi\left(\frac{\theta}{1-\theta}\right)$. With interest rate being $\frac{\theta}{1-\theta}$, Remark 7.1 says that the winning bid $b_{(1)}$ equals $(1-\theta)v$. At this interest rate, a bidder's expected payoff conditional to winning is $(1-\theta)v - b$ regardless of whether he bids over budget or not ((see (23) in Remark 6.1); thus, a bidder's bid no longer depends on his budget. The winner's budget $m_{(1)}$ is hence a random draw from F. Therefore, the seller's expected payoff is the expected value of (24) in Remark 6.1 (the indeterminateness of payment when the winner is indifferent between payment and bankruptcy would not cause any trouble, since F is assumed to be continuous). Computing this expected value through integration by parts and using $F(m_0) =$ 0 (Assumption 2), we have (37).

We next compute $\pi(r)$ for $r \in [0, \frac{\theta}{1-\theta})$. By Theorem 7.2, the unique symmetric equilibrium of the bidding game under such an r is that each bidder bids according to the function β defined in (32). Then the seller's expected payoff $\pi(r)$ is the expected value of (24), where the winning bid $b_{(1)}$ is the highest bid when $b_{(1)} > (1-\theta)v$ and is $(1-\theta)v$ if otherwise; and the winner's budget $m_{(1)}$ is the *lowest* budget among the bidders when $b_{(1)} > (1-\theta)v$ and otherwise a random draw from $[(1-\theta)v, \bar{m}]$ according to F.

In order to compute the expected value of (24), we first calculate $\text{Prob}[b_{(1)} \leq x]$ for each x in the range of β. Denote $\beta_1 := \beta|_{[m_0,(1-\theta)v)}$. Then for any x in the range of β such that $x > (1-\theta)v)$, the event "$b_{(1)} \leq x$" is equivalent to the event "$m_i \geq \beta_1^{-1}(x)$ for each bidder $i = 1, \ldots, n$". By the independence of the bidders' budgets m_i's, we have

$$\text{Prob}[b_{(1)} \leq x] = \begin{cases} (1 - F \circ \beta_1^{-1}(x))^n & \text{if } x > (1-\theta)v \\ (1 - F((1-\theta)v))^n & \text{if } x = (1-\theta)v \\ 0 & \text{if } x < (1-\theta)v. \end{cases}$$

Thus,

$$\begin{aligned} E[b_{(1)}] &= \int_{(1-\theta)v}^{m_0} \beta_1(m) d((1-F(m))^n) + (1-\theta)v(1-F((1-\theta)v))^n; \\ E[\max\{0, b_{(1)} - m_{(1)}\}] &= \int_{(1-\theta)v}^{m_0} (\beta_1(m) - m) d((1-F(m))^n); \\ E[\chi_{[b_{(1)} < m_{(1)}]} b_{(1)}] &= (1-\theta)v(1-F((1-\theta)v))^n. \end{aligned}$$

The indeterminateness of payment when the winner is indifferent between payment and bankruptcy does not cause any trouble for the last equation, since this event, i.e., "$m_{(1)} = (1-\theta)v$", is of zero probability due to the continuity of F. Plugging these equations into the expected value of the expression (24) and switching the integrals for the term

$$\int_{m_0}^{(1-\theta)v} \int_{m}^{(1-\theta)v} (1-F(t))^{n-1} f(m) dt dm = \int_{m_0}^{(1-\theta)v} (1-F(t))^{n-1} F(t) dt$$

result in Equation (38).

Finally, we compute $\pi(r)$ for $r \in (\frac{\theta}{1-\theta}, q]$. By Theorem 7.1, the unique symmetric equilibrium of the bidding game under such an r that each bidder bids according to the strategy β defined in (26). Thus, $\pi(r)$ is the expected value of (24), with $m_{(1)}$ being the highest budget among the bidders and $b_{(1)} = \beta(m_{(1)})$ the highest bid. Note that the probability for $m_{(1)} \leq m$ is $F_{(1)}(m) := F(m)^n$. Denote $\beta_1 := \beta|_{[m_0, m_*)}$ and $\beta_2 := \beta|_{[m^*, \bar{m}]}$. We then have:

$$E[b_{(1)}] = \int_{m_0}^{\bar{m}} \beta(m) dF_{(1)}(m);$$

$$E[\max\{0, b_{(1)} - m_{(1)}\}] = \int_{m_0}^{m_*} (\beta_1(m) - m) dF_{(1)}(m);$$
$$E[\chi_{[b_{(1)} < m_{(1)}]} b_{(1)}] = \chi \int_{m_*}^{m^*} m dF_{(1)}(m) + \int_{m^*}^{\bar{m}} \beta_2(m) dF_{(1)}(m).$$

Note that $\chi_{[b_{(1)}<m_{(1)}]}$ affects the last equation, since it is possible that $m^* > m_*$ (then the event that the winner is indifferent between payment and bankruptcy, i.e., "$m_{(1)} \in [m_*, m^*]$", has positive probability). Plugging these equations into the expected value of the expression (24) results in Equation (39). Thus, the proposition is proved. **Q.E.D.**

The following remark says that m_* and m^* collapse into one point when the financing cost q is not too high. Thus, the indeterminateness problem due to χ when $r > \frac{\theta}{1-\theta}$ disappears for q sufficiently low.

Remark 8.1 *Suppose that F satisfies Assumption (18) and that $q < \frac{1+\theta}{1-\theta}$. Then, for any $r \in (\frac{\theta}{1-\theta}, q]$, $m_*(r) = m^*(r)$ and $m_*(r'') = m^*(r'')$ for all r'' sufficiently close to r.*

Proof: By the definition of m^*, we need only to prove that $m_*(r) > \hat{m}$ for any $r \in (\frac{\theta}{1-\theta}, q]$. To do that, we merely use the same proof for $m_*(r) > \hat{m}$ in the proof of Remark 4.2, replacing r in that proof with $(1-\theta)r'$. The proof works due to Assumption 18) and $(1-\theta)r' < 1$ (since $q < \frac{1+\theta}{1-\theta}$). **Q.E.D.**

The following proposition says that charging interest rate lower than $\theta/(1-\theta)$ is strictly dominated when there are sufficiently many bidders.

Proposition 8.2 *Suppose that $\frac{\theta}{1-\theta} < q \leq 1$. Then for any $r_1 \in (\frac{\theta}{1-\theta}, q]$ and any $r_2 \in [0, \frac{\theta}{1-\theta})$*

$$\lim_{n\to\infty} (\pi(r_1) - \pi(r_2)) > (1-\theta)q[(1-\theta)v - m_0] + \theta m_0 > 0.$$

Proof: Take any $r_1 \in (\frac{\theta}{1-\theta}, q]$ and any $r_2 \in [0, \frac{\theta}{1-\theta})$. Since $\pi(r_1)$ and $\pi(r_2)$ are both continuous in n (see (38) and (39)), the limits $\lim_{n\to\infty} \pi(r_1)$ and $\lim_{n\to\infty} \pi(r_2)$ exist. By the condition $q \leq 1$, $1 - q/(1+r_2) \geq 0$; with $r_2 < \frac{\theta}{1-\theta}$, $r_2' := r_2 - \frac{\theta}{1-\theta} < 0$. By (38), we have

$$\begin{aligned}\pi(r_2) \leq & \left(1 - \frac{q}{1+r_2}\right)(1-\theta)v + \left(\frac{q}{1+r_2} - \theta\right) m_0 \\ & + \left(\frac{q}{1+r_2} - \theta\right) \int_{m_0}^{(1-\theta)v} (1-F)^n + \theta(1-\theta)v(1 - F((1-\theta)v))^n.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \pi(r_2) \leq \left(1 - \frac{q}{1+r_2}\right)(1-\theta)v + \left(\frac{q}{1+r_2} - \theta\right) m_0. \tag{40}$$

With the condition $\frac{\theta}{1-\theta} < q \leq 1$, we have $1 - q/(1+r_1) \geq 0$ and $q/(1+r_1) \geq q/(1+q) > \theta$. Thus, (39) implies that

$$\begin{aligned}
\pi(r_1) &\geq (1-\theta)v(1-F_{(1)}(m^*)) - n(1-\theta)\left(v - \frac{m^*}{1-\theta}\right) F_Y(m^*)(1-F(m^*)) \\
&\quad - nr_1'(1-\theta)\left(1-\frac{q}{1+r_1}\right)\int_{m_0}^{m_*}\int_{m_0}^{m} F_Y(t)f(m)dtdm \\
&> (1-\theta)v(1-F((1-\theta)v)^n) - n(1-\theta)vF((1-\theta)v)^{n-1} \\
&\quad - 2r_1'(1-\theta)\left(1-\frac{q}{1+r_1}\right)F((1-\theta)v)\int_{m_0}^{(1-\theta)v} F(t)^{n-1}dt,
\end{aligned} \tag{41}$$

where the second inequality comes from the fact that $m^* \leq (1-\theta)v$ ((a) of Proposition 7.2) and the computation

$$\int_{m_0}^{m_*}\int_{m_0}^{m} F_Y(t)f(m)dtdm = \int_{m_0}^{m_*}(F(m_*)F_Y - F_{(1)}) < 2F(m_*)\int_{m_0}^{m_*} F_Y.$$

As $n \to \infty$, all the terms on the right-hand-side of Inequality (41) vanish, except for $(1-\theta)v$ (To see that $n\int_{m_0}^{(1-\theta)v} F(t)^{n-1}dt \to 0$, compute

$$\begin{aligned}
\frac{d}{dn}\left[n\int_{m_0}^{(1-\theta)v} F(t)^{n-1}dt\right] &= \int_{m_0}^{(1-\theta)v} F(t)^{n-1}(1+n\ln F(t))dt \\
&< (1+n\ln F((1-\theta)v))\int_{m_0}^{(1-\theta)v} F(t)^{n-1}dt,
\end{aligned}$$

which is negative for n sufficiently large; in the inequality we have used the Assumption 1').
Thus,

$$\lim_{n\to\infty} \pi(r_1) \geq (1-\theta)v.$$

Coupled with (40), this inequality implies that

$$\begin{aligned}
\lim_{n\to\infty}(\pi(r_1) - \pi(r_2)) &\geq (1-\theta)v - \left[(1-\theta)\left(1-\frac{q}{1+r_2}\right)v + \left(\frac{q}{1+r_2} - \theta\right)m_0\right] \\
&= \frac{q}{1+r_2}[(1-\theta)v - m_0] + \theta m_0 \\
&> (1-\theta)q[(1-\theta)v - m_0] + \theta m_0,
\end{aligned}$$

where the last inequality follows from $r_2 < \theta/(1-\theta)$ and Assumption 1'. The right-hand-side of the inequality is positive again by Assumption 1'. We have hence proved the proposition.
Q.E.D.

The next proposition says that charging interest rate at $\theta/(1-\theta)$ is strictly dominated by the interest rates slightly higher than $\theta/(1-\theta)$.

Proposition 8.3 *Suppose that* $\frac{\theta}{1-\theta} < q \leq 1$. *Then for all* $r' > 0$ *sufficiently small,* $\pi\left(\frac{\theta}{1-\theta} + r'\right) > \pi\left(\frac{\theta}{1-\theta}\right)$.

Proof: We need only to prove that, for each $r' > 0$, $\pi(\frac{\theta}{1-\theta} + r') > X + O(r')$ for some positive term X independent of r' (see the footnote in the proof of Remark 4.1 for a definition of $O(\cdot)$). We will use the following fact from calculus.

Lemma 8.1 *If* $\zeta : R \to R$ *is continuously differentiable over a neighborhood of* $a \in R$, *then there exists an* $\eta > 0$ *such that*

$$\zeta(x) = \zeta(a) + O(x-a), \;\; x \in [a-\eta, a+\eta].$$

Proof: By the definition of $O(\cdot)$, we need only to find a finite number M such that $\zeta(x) - \zeta(a) \to M(x-a)$ as $x \to a$. With ζ continuously differentiable over a neighborhood of a, it is continuously differentiable on a closed interval $[a-\eta, a+\eta]$ for some $\eta > 0$. Let $M := \max\{|\zeta'(t)| : t \in [a-\eta, a+\eta]\}$, which exists by the continuity of ζ' on the compact set $[a-\eta, a+\eta]$. Pick any $x \in [a-\eta, a+\eta]$. Let $\epsilon > 0$. Choose $\delta \leq \epsilon/M$. With the mean-value theorem for derivatives, it is straightforward to prove that $|x-a| < \delta$ implies $|\zeta(x) - \zeta(a) - M(x-a)| < \epsilon$. This proves the lemma. □

Let $r' > 0$ and $r := r' + \theta/(1-\theta)$. Then Equation (39) applies. Delete the third term on the right-hand-side of the equation and call the rest truncated sum. Then the truncated sum does not exceed $\pi(r)$. Note that the last term of this truncated sum is r' multiplied by a finite number (since $m_* \leq (1-\theta)v$ and F is a bounded function), and hence it is $O(r')$. Look at the second last term of the truncated sum. It has the factor $(v - \frac{m^*}{1-\theta})F_Y(m^*)$, which does not exceed $r' \int_{m_0}^{m^*} F_Y$ (since $\tilde{g}(m^*) \leq \tilde{g}(m_*) = 0$). Thus, the absolute value of the second last term of the truncated sum is again $O(r')$. Therefore, we have

$$\begin{aligned} \pi(r) \;\; \geq \;\; & (1-\theta)\left(1 - \frac{q}{1+r}\right) vF_{(1)}(m_*) + \left(\frac{q}{1+r} - \theta\right) \int_{m_0}^{m_*} m dF_{(1)}(m) \\ & + (1-\theta)v(1 - F_{(1)}(m^*)) + O(r'). \end{aligned}$$

With every term on the right-hand-side of this inequality continuously differentiable with respect to $r \neq 0$ ($m_*(r)$ is so by (c) of Proposition 7.2), we apply Lemma 8.1 to these terms around the point $r_0 = \theta/(1-\theta)$ (note $m_*(\frac{\theta}{1-\theta}) = (1-\theta)v$ by (27)). Thus, we have

$$\pi(r) \geq (1-\theta)v - \theta(1-\theta)vF_{(1)}((1-\theta)v) - (q(1-\theta)-\theta)\int_{m_0}^{(1-\theta)v} F_{(1)} + O(r').$$

Applying (37), we then have

$$\begin{aligned}\pi(r) - \pi\left(\frac{\theta}{1-\theta}\right) &\geq \theta(1-\theta)v(F((1-\theta)v) - F_{(1)}((1-\theta)v)) \\ &\quad + (q(1-\theta)-\theta)\int_{m_0}^{(1-\theta)v}(F - F_{(1)}) + O(r').\end{aligned}$$

Since $n > 1$ by assumption, the first two terms on the right-hand-side are both positive constants independent of r'. We have hence proved the proposition. **Q.E.D.**

As we have known by now, a change in interest rate within the "reasonable" range $(\frac{\theta}{1-\theta}, q]$ has two opposite effects on the expected payoff to the seller: It changes the seller's share of the financing cost to one direction and drives the expected winning bid to the other direction. From the analyst's viewpoint, it would be interesting to have a formula that computes the combination of the two opposite effects. The following remark provides such a formula.

Remark 8.2 *Suppose that F satisfies Assumption (18) and that $\frac{\theta}{1-\theta} < q < 1$. Then the seller's expected payoff π is differentiable at all $r \in (\frac{\theta}{1-\theta}, q]$ and for any such r*

$$\begin{aligned}\frac{\pi'(r)}{n(1-\theta)} &= \frac{q}{n(1+r)^2}\int_{m_0}^{m_*(r)}(r'F(m_*(r))F_Y + F_{(1)}) \\ &\quad - \left[1 - \frac{q}{(1-\theta)(1+r)^2}\right]\int_{m_0}^{m_*(r)}(F(m_*(r))F_Y - F_{(1)}) \\ &\quad - \left[(1-F(m_*(r)))\tilde{h}'(m_*(r)) + \frac{\theta}{1-\theta}m_*(r)F_Y(m_*(r))f(m_*(r))\right]\frac{dm_*(r)}{dr},\end{aligned} \tag{42}$$

where $\frac{dm_*(r)}{dr}$ is given by (30).

Proof: Pick any $r \in (\frac{\theta}{1-\theta}, q]$. By the given hypotheses, Lemma 8.1 applies. Thus, $m_*(r'') = m^*$ for all r'' sufficiently close to r. Plug $m^* = m_*(r)$ into Equation (39) (where we write m_*

for $m_*(r)$ for simplicity). This new equation holds over a neighborhood of r, since $m_* = m^*$ over a neighborhood of r. Thus, we can differentiate this new equation with respect to r. We then obtain (42) after several steps of arithmetic and using (27) twice. This proves the proposition. **Q.E.D.**

9 Appendix: The Proofs of Lemmas 7.1 and 7.2

9.1 The Proof of Lemma 7.1

Let β be as specified by the hypothesis. Denote Prob[win | b] for the probability for the event that a bidder who bids b wins the auction, provided that others bid according to the strategy β. Define

$$V(b, m, r) := u(b, m, r)\text{Prob[win} \mid b\text{]},\ b \geq 0,\ m \in [m_0, \bar{m}]. \tag{43}$$

Thus, $(1-\theta)V(b, m, r)$ is the expected payoff for a bidder with budget m to bid b under interest rate r, provided that others play the strategy β.

We shall prove that $\beta|_N$ weakly decreases if $r < \frac{\theta}{1-\theta}$, and the proof for the other case is analogous. Hence take any $r \in [0, \frac{\theta}{1-\theta})$ and denote $r' := r - \frac{\theta}{1-\theta}$. Let us temporarily assume the following:

For any $x \in N$ there is a neighborhood $N(x) \subset N$ of x such that

$$[\ x > x' \Rightarrow \beta(x) \leq \beta(x')\ ] \text{ and } [\ x < x' \Rightarrow \beta(x) \geq \beta(x')\ ] \tag{44}$$

for any $x' \in N(x)$.

Then we can pass this local assertion to its global counterpart by a standard compactness argument: Take any $m, m' \in N$ such that $m < m'$. For each $x \in [m, m']$, the temporary assumption claims that there is an open interval $N(x)$ containing x such that (44) holds for all $x' \in N(x)$. With $\{N(x) : x \in [m, m']\}$ being an open cover for the compact space $[m, m']$, we can extract a finite subcover $\{N(x_k) : k = 1, \ldots, l\}$ such that

$$m = x_1 < x_2 < \cdots < x_l = m'.$$

For each $k = 1, \ldots, l-1$, pick a $y_k \in N(x_k) \cap N(x_{k+1})$ such that $x_k < y_k < x_{k+1}$. We then have

$$m = x_1 < y_1 < x_2 < y_2 < \cdots < y_{l-1} < x_l = m'.$$

Repeatedly using (44) for $2(l-1)$ times, we have $\beta(m) \geq \beta(m')$. Since m and m' are arbitrarily chosen from N such that $m < m'$, β is weakly decreasing over N. Therefore, the lemma will be proved if the temporary assumption is true.

Thus, let us prove the claim that we temporarily assumed. Take any $x \in N$. Let $\epsilon := (\beta(x)-x)/2$. Since $\beta(x) > x$, $\epsilon > 0$. With β continuous over N, there are neighborhoods N_1 and N_2 of x such that

$$x' \in N_1 \Longrightarrow |\beta(x') - \beta(x)| \leq \epsilon$$

and

$$x' \in N_2 \Longrightarrow (\beta(x') - x') - (\beta(x) - x) \geq -\epsilon.$$

Let $N(x) := N_1 \cap N_2$. Then by the definition of ϵ

$$x' \in N(x) \Longrightarrow [\ |\beta(x') - \beta(x)| \leq \epsilon \text{ and } \beta(x') - x' \geq \epsilon\ ]. \tag{45}$$

Take any $x' \in N(x)$. Since β is an equilibrium strategy, we have

$$V(\beta(x), x, r) \geq V(\beta(x'), x, r) \text{ and } V(\beta(x'), x', r) \geq V(\beta(x), x', r). \tag{46}$$

By (45), we have $\beta(x') > x'$, $\beta(x) \geq \beta(x') - \epsilon \geq x'$, and $\beta(x') \geq \beta(x) - \epsilon > x$. Coupled with $\beta(x) > x$ and (23), these inequalities imply that Equations (46) are equivalent to the following two inequalities:

$$(v - (1+r)\beta(x) + r'x)\text{Prob}[\text{win}|\beta(x)] \geq (v - (1+r)\beta(x') + r'x)\text{Prob}[\text{win}|\beta(x')];$$

$$(v - (1+r)\beta(x') + r'x')\text{Prob}[\text{win}|\beta(x')] \geq (v - (1+r)\beta(x) + r'x')\text{Prob}[\text{win}|\beta(x)].$$

Summing the two inequalities, we have Inequality (25), as stated in Section 7.1. This, with $r' = r - \theta/(1-\theta) < 0$, implies that

$$(x - x')(\text{Prob}[\text{win}|\beta(x)] - \text{Prob}[\text{win}|\beta(x')]) \leq 0.$$

Since Prob[win$|\cdot$] is strictly increasing on the range of β (because F is strictly increasing and β continuous), this inequality implies (44), as desired. Thus, the statement we temporarily assumed is indeed true. It follows that the lemma is proved. **Q.E.D.**

9.2 The Proof of Lemma 7.2

Let β be as specified by the hypothesis. Define $V(\cdot,\cdot,r)$ by (43). For each bidder i, denote

$$\beta_Y(m_{-i}) := \max\{m_j \in [m_0, \bar{m}] : j \neq i\}.$$

To bidder i, $\beta_Y(m_{-i})$ is then the highest bid submitted by other bidders than i. Since m_j's are identically and independently distributed according to F, $\beta_Y(m_{-i})$ (as a random variable) is identical across all i's. We abuse the notation and write β_Y for $\beta_Y(m_{-i})$.

Suppose that there were a subset E of $[m_0, \bar{m}]$ such that $F(E) > 0$, $\beta|_E \equiv b$ for some b, and $u(b, m, r) > 0$ for some $m \in E$. With $F(E) > 0$, $\text{Prob}[\beta_Y = b] > 0$. Denote $A := \text{Prob}[\beta_Y = b]$. With $A > 0$ and $u(b, m, r) > 0$, we can choose an $\epsilon > 0$ sufficiently small such that

$$\epsilon < \frac{Au(b, m, r)}{2[\text{Prob}[\beta_Y < b] + A]}.$$

Since $u(\cdot, m, r)$ is continuous, there is a $\delta > 0$ so small that $u(b + \delta, m, r) > u(b, m, r) - \epsilon$. But then a bidder with budget m would rather bid $b + \epsilon$ than $\beta(m) = b$:

$$\begin{aligned}
V(b, m, r) &\leq u(b, m, r)\left(\text{Prob}[\beta_Y < b] + \frac{A}{2}\right) \\
&< [u(b + \delta, m, r) + \epsilon]\left(\text{Prob}[\beta_Y < b] + \frac{A}{2}\right) \\
&< u(b + \delta, m, r)(\text{Prob}[\beta_Y < b] + A) \\
&\leq V(b + \delta, m, r),
\end{aligned}$$

where the factor $1/2$ in the first inequality comes from the assumption that there are at least 2 bidders and ties ($\beta_Y = b$) are broken by a random draw with equal probability, the second inequality comes from the choice of δ, and the third results from the choice of ϵ. Therefore, the existence of such a set E would contradict the fact that β is an equilibrium strategy. It follows that such a set cannot exist. We have thus proved the proposition. **Q.E.D.**

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