GENERIC 4 x 4 TWO PERSON GAMES HAVE AT MOST 15 NASH EQUILIBRIA

by

Andrew McLennan and In-Uck Park

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, MN 55455

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by

Andrew McLennan

Department of Economics
University of Minnesota
271 19th Avenue South
Minneapolis, MN 55455

mclennan@atlas.socsci.umn.edu

and

In-Uck Park

Department of Economics
University of Bristol
8 Woodland Road
Bristol BS8 1TN, U.K.

ecip@ssa.bristol.ac.uk

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Abstract

The maximal generic number of Nash equilibria for two person games in which the two agents each have four pure strategies is shown to be 15. In contrast to Keiding (1995), who arrives at this result by computer enumeration, our argument is based on a collection of lemmas that constrain the set of equilibria. Several of these pertain to any common number d of pure strategies for the two agents. *Journal of Economic Literature* Classification Number C72.

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1. Introduction

Consider the two person game in which each agent has d pure strategies and the payoffs are $(1,1)$ on the diagonal (with respect to some orderings of the agents' pure strategies) and $(0,0)$ elsewhere. For any nonempty set of pure strategies for agent 1 there is an equilibrium in which agent 1 assigns equal probability to all elements of this set and agent 2 mixes equally on her corresponding pure strategies. All such equilibria are *regular* (Harsanyi (1973)) implying that nearby games (in the space of pairs of utility functions) have at least as many equilibria, namely $2^d - 1$.

Quint and Shubik (1994) initiated the study of the conjecture that there is no pair of utilities for which there are more than $2^d - 1$ regular equilibria, showing that this is the case when $d = 3$. The problem has been studied by Keiding (1995), who used electronic enumeration of cases, together with Corollary 4.2 below, to show that the conjecture holds in the case $d = 4$. Recently von Stengel (1997) constructed an ingenious sequence of examples (one for each d) which are counterexamples for $d = 6$ and $d \geq 8$.

In this paper we present a variety of results bearing on the problem, some of which are general, while others are specific to the case $d = 4$. Collectively these results allow a different proof for the case $d = 4$, which does not depend on automated calculations.

The remainder of this section describes related literature.

Fixing strategy spaces, McKelvey and McLennan (1996) characterize the maximal (as the payoffs are varied) number of regular totally mixed equilibria. McLennan (1996) shows that the maximal number of pure equilibria, for generic payoffs, is the number of pure strategy vectors (that is, the product of the cardinalities of the agents' pure strategy sets) divided by the maximal number of pure strategies possessed by any agent. A lower bound on the maximal number of regular equilibria, of all sorts, is obtained by combining this with the theorem of Gul, Pearce, and Stacchetti (1993), who use index theory to show that, for generic payoffs, the number of mixed equilibria cannot be less than one fewer than the number of pure equilibria.

Extending earlier results of Dresher (1970) and Powers (1990), Stanford (1993) obtains a formula for the probability that a "randomly selected" payoff vector will have exactly m pure strategy equilibria. (The assumed distribution on payoff vectors has all agents' payoffs statistically independent, with each agent assigning equal probability to all possible orderings of the pure strategies.) Stanford (1994) extends this analysis to symmetric two person games, differentiating between symmetric and asymmetric equilibria.

2. Problem Formulation

Let S and T be nonempty finite sets of *pure strategies* for agents 1 and 2, respectively, with the same number of elements. Let $d = |S| = |T|$. (Throughout we denote the cardinality of a set X by $|X|$.) We identify the elements of S with the standard unit basis vectors in $\mathbb{R}^S$. Let H_S be the affine hull of these points, i.e., the affine plane consisting of the points in $\mathbb{R}^S$ whose coordinates sum to one. For any $A \subset S$ let $\Delta(A)$ be the convex hull of A, let $\Delta^\circ(A)$ be its interior (relative to the affine hull of A) and for $\sigma \in \Delta(S)$ let supp σ be the *support* of σ, i.e., the subset $A \subset S$ such that $\sigma \in \Delta^\circ(A)$.

The payoffs for agent 2 are given by affine functions $u_t : H_S \to \mathbb{R}$ for the various $t \in T$. Here $u_t(\sigma)$ is the expected payoff agent 2 receives when she plays t and agent 1 follows the (generalized) mixed strategy σ. Let

$$P = \{ (\sigma, u) \in \Delta(S) \times \mathbb{R} : u \geq u_t(\sigma) \text{ for all } t \in T \}.$$

Then P is a polyhedron that is, geometrically, a sawed off prism that extends "to heaven." A possibility for P in the case $S = \{A, B, C\}$ and $T = \{a, b, c\}$ is shown in Figure 1.

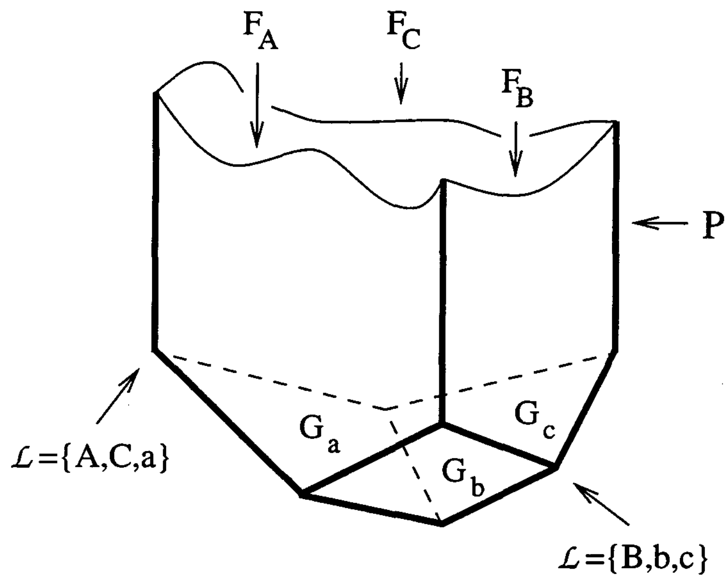

Figure 1

The facets of P are:

(a) F_s, for $s \in S$, where F_s is the vertical facet of P, called a *heavenly facet*, along which the probability of s is zero;

(b) $G_t = \{ (\sigma, u) \in P : u = u_t(\sigma) \}$, for $t \in T$. (This may be empty.)

Let

$$\Phi_s = \{ (\sigma, u) \in H_S \times I\!R : \sigma(s) = 0 \} \quad \text{and} \quad \Gamma_t = \{ (\sigma, u_t(\sigma)) \in H_S \times I\!R : \sigma \in H_S \}$$

be the hyperplanes that bound F_s and G_t respectively.

Symmetrically, identify the elements of T with the standard unit basis vectors in $I\!R^T$, and let H_T be the affine hull of these points. Define $\Delta(B)$ and $\Delta^\circ(B)$ for $B \subset T$ and supp τ for $\tau \in \Delta(T)$ as above. For each $s \in S$ let $u_s : H_T \to I\!R$ be an affine function representing the payoff for player 1, and let

$$Q = \{ (\tau, u) \in \Delta(T) \times I\!R : u \geq u_s(\tau) \text{ for all } s \in S \}.$$

Let the facets of Q be the F_t (with bounding hyperplane Φ_t) and G_s (with bounding hyperplane Γ_s) as above.

The set of *labels* of a point $(\sigma, u) \in H_S \times I\!R$ is

$$\mathcal{L}(\sigma, u) = \{ s \in S : (\sigma, u) \in \Phi_s \} \cup \{ t \in T : (\sigma, u) \in \Gamma_t \}.$$

Similarly, the set of labels of a point $(\tau, u) \in H_T \times I\!R$ is

$$\mathcal{L}(\tau, u) \;=\; \{\, t \in T : (\tau, u) \in \Phi_t \,\} \cup \{\, s \in S : (\tau, u) \in \Gamma_s \,\}.$$

A *Nash equilibrium* is a pair $(\sigma, \tau) \in \Delta(S) \times \Delta(T)$ with

$$\mathcal{L}(\sigma, \max_{t \in T} u_t(\sigma)) \cup \mathcal{L}(\tau, \max_{s \in S} u_s(\tau)) \;=\; S \cup T.$$

That is, each pure strategy is either unused or optimal (or both).

We are interested in generic utilities. Here the important consequence of genericity is that, for any proper subset $A \subset S$ and any subset $B \subset T$, the dimension of $(\bigcap_{s \notin A} \Phi_s) \cap (\bigcap_{t \in B} \Gamma_t)$ is

$$d - (d - |A|) - |B| \;=\; |A| - |B|,$$

and, similarly, $(\bigcap_{t \notin B} \Phi_t) \cap (\bigcap_{s \in A} \Gamma_s)$ is $(|B| - |A|)$-dimensional. Note, in particular, that one of these sets must be empty whenever $|A| \neq |B|$.

When $|A| = |B|$ the unique elements v and w of the intersections

$$\{v\} = (\bigcap_{s \notin A} \Phi_s) \cap (\bigcap_{t \in B} \Gamma_t) \quad \text{and} \quad \{w\} = (\bigcap_{t \notin B} \Phi_t) \cap (\bigcap_{s \in A} \Gamma_s)$$

are called *virtual vertices*. Note that $\mathcal{L}(v) = (S \backslash A) \cup B$ and $\mathcal{L}(w) = A \cup (T \backslash B)$, so that $\mathcal{L}(v) \cup \mathcal{L}(w) = S \cup T$. For each virtual vertex in $v \in H_S \times I\!R$ there is a unique virtual vertex in $w \in H_T \times I\!R$ such that $\mathcal{L}(v') \cup \mathcal{L}(w') = S \cup T$. In this situation we say that v and w are *virtually complementary.*

A *vertex* is a virtual vertex that is an element of P or Q. Let V and W be the sets of vertices in P and Q respectively. A *complementary pair* is a pair of vertices, one from V and one from W, that are virtually complementary. We say that a single vertex $v \in V$ or $w \in W$ is *complementary* if it is a member of such a complementary pair.

Let π denote the projections $H_S \times I\!R \to H_S$ and $H_T \times I\!R \to H_T$. If $v \in V$, then $v = \big(\pi(v), \max_{t \in T} u_t(\pi(v))\big)$, and similarly for $w \in W$, so that $(\pi(v), \pi(w))$ is a Nash equilibrium if v and w constitute a complementary pair. Conversely, if (σ, τ) is a Nash equilibrium, then $(\sigma, \max_{t \in T} u_t(\sigma))$ and $(\tau, \max_{s \in S} u_s(\tau))$ are complementary.

Remark: By perturbing a facet F_s of P and a facet F_t of Q, one can obtain simple bounded polyhedra (polytopes) that each have one additional vertex, but whose other vertices are "the same" as those in P and Q, in the sense of having the same d-tuples of labels. The two new vertices are complementary. In this way we achieve an instance of the form of the problem analyzed by Keiding (1995): if $J_1, \ldots, J_{2d}$ and $K_1, \ldots, K_{2d}$ are halfspaces of $\mathbb{R}^d$, each vertex v of $\tilde{P} = \bigcap_{i=1}^{2d} J_i$ is labelled with the indices of the halfspaces whose boundaries contain v, the vertices of $\tilde{Q} = \bigcap_{j=1}^{2d} K_j$ are labelled in similar fashion, and $\tilde{P}$ and $\tilde{Q}$ are *simple*—each vertex has exactly d labels—what is the maximal number of pairs (v, w) consisting of a vertex of $\tilde{P}$ and a vertex of $\tilde{Q}$ that are complementary in the sense that the two sets of labels encompass all the indices $1, \ldots, 2d$?

We now sketch the construction used to show that Keiding's framework is not more general than ours. Identify $\mathbb{R}^d$ in the description above with a d-dimensional affine subspace H of $\mathbb{R}^{d+1}$ that does not contain the origin. Let v and w be complementary vertices of $\tilde{P}$ and $\tilde{Q}$. Let $L \subset H$ be an affine hyperplane (relative to H) that separates v from $\tilde{P}$ in the strict sense that v is contained in L while $\tilde{P}\backslash\{v\}$ is contained in one of the open half spaces of $H\backslash L$. Let $\hat{H}$ be a hyperplane in $\mathbb{R}^{d+1}$ that is parallel to, but different from, the hyperplane $\text{span}(L)$ of $\mathbb{R}^{d+1}$ that contains L. Let $\rho : H\backslash L \to \hat{H}$ be defined by requiring $\rho(x)$ to be the unique scalar multiple of x that is contained in $\hat{H}$. Let $\hat{P} = \rho(\tilde{P})$. Transform $\tilde{Q}\backslash\{w\}$ similarly to arrive at $\hat{Q}$. The details (which are numerous) of the verification that $\hat{P}$ and $\hat{Q}$ satisfy the description of the problem used here, up to affine transformation, are left to the reader. For a detailed discussion of the properties of the transformations used in this construction we recommend Ziegler (1995, §2.6).

In passing from Keiding's framework to ours, a particular complementary pair is singled out arbitrarily, with the facets intersecting at the vertices in this pair becoming the heavenly facets of our formulation. This seems artificial, in that it creates an asymmetry that does not exist in the original formulation, but we have found that it induces a rich and useful structure. The *level* of a virtual vertex of P is the number of its labels that are members of S. Let V_i be the set of vertices of P in level i. Then V_{d-1} is the set of vertices that project to elements of S; these vertices are said to be *pure* since they correspond to pure strategies. Also, $V_0 = \bigcap_{t \in T} \Gamma_t$ is a singleton whose unique element is called the *geocenter* of P and is denote by g^P. The geocenter of P is a vertex of P if and only if $\pi(g^P) \in \Delta(S)$.

The notation for the other agent is symmetric: the level of a virtual vertex of Q is the number of its labels that are contained in T, W_i is the set of vertices of Q in level i, and the geocenter g^Q is the unique element of $W_0 = \bigcap_{s \in S} \Gamma_s$. Indeed, our results will be usually stated as applying to P, but should be understood as applying equally to Q.

For the most part our argument proceeds by identifying various subsets of V which necessarily contain a certain number of elements that are not complementary. The structure of our framework explained above provides useful information in this process. For instance, a pair of complementary vertices are in the same level. Also, since a vertex is complementary to at most one vertex, the number of complementary vertices of level i is at most $\min\{|V_i|, |W_i|\}$.

3. The Decomposition of the Simplex by Best Responses

We now describe a different geometric presentation, in $H_S \approx \mathbb{R}^{d-1}$, of the information constituting the problem. For the case of $d = 4$ this presentation can be visualized, and is the most intuitive formulation for many purposes.

For each $t \in T$ let C_t be the set of $\sigma \in H_S$ that have t as a best response. Clearly C_t is a convex cone emanating from $\pi(g^P)$. We have $\bigcup_{t \in T} C_t = H_S$, essentially because T is finite. The sets C_t determine the best response correspondence, and thus all the other information relevant to the problem.

For $t \in T$ consider the line $l_t = \pi(\bigcap_{t' \neq t} \Gamma_{t'}) \subset H_S$ consisting of those mixed strategies such that all pure strategies for agent 2 other than t have the same expected utility. This line is divided into two rays by $\pi(g^P)$. Let

$$r_t = \{\, \sigma \in l_t : \; u_{t'}(\sigma) \geq u_t(\sigma) \text{ for } t' \neq t \,\}$$

be the closed ray consisting of the points for which all elements of $T \backslash \{t\}$ are best responses. (See Figure 2.)

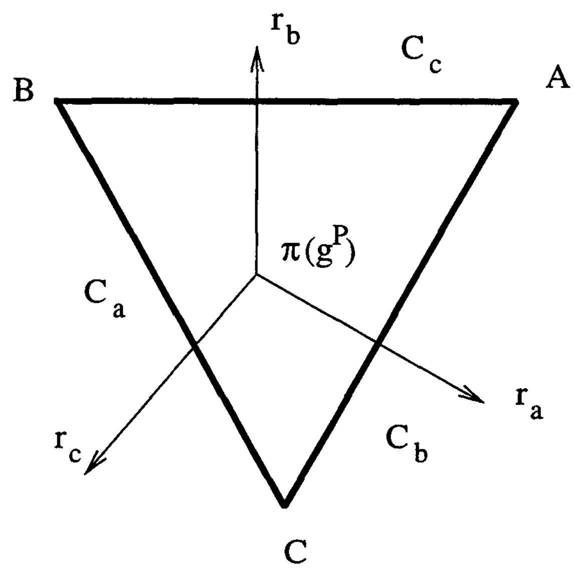

Figure 2

Lemma 3.1: *For each $t \in T$,*

$$\mathrm{con}\big(\bigcup_{t' \neq t} r_{t'}\big) = C_t \quad \textit{and} \quad \bigcap_{t' \neq t} C_{t'} = r_t.$$

(Here con *is the convex hull operator.) For any hyperplane in H_S containing $\pi(g^P)$, each of the two open halfspaces bounded by the hyperplane contains at least one open ray $r_t \backslash \{\pi(g^P)\}$.*

Proof: Clearly $\mathrm{con}(\bigcup_{t' \neq t} r_{t'}) \subset C_t$. At a point $\sigma' \in r_{t'}$ agent 2's best responses include all pure strategies except t', so t is the unique best response to any strict convex combination of points from the various sets $r_{t'} \backslash \{\pi(g^P)\}$. The set of such convex combinations is the interior of $\mathrm{con}(\bigcup_{t' \neq t} r_{t'})$. The line segment between such a convex combination and any point $\sigma \in H_S \backslash \mathrm{con}(\bigcup_{t' \neq t} r_{t'})$ intersects the boundary of $\mathrm{con}(\bigcup_{t' \neq t} r_{t'})$ at a point which has at least one best response, say t^*, that is different from t. At every point in the interior of $\mathrm{con}(\bigcup_{t' \neq t} r_{t'})$ the value of u_t is greater than the value of u_{t^*}, and the two functions must be equal at the point on the boundary, so $u_{t^*}(\sigma) > u_t(\sigma)$ since u_t and u_{t^*} are affine. Thus there are no points outside of $\mathrm{con}(\bigcup_{t' \neq t} r_{t'})$ that have t as a best response, i.e., $C_t \subset \mathrm{con}(\bigcup_{t' \neq t} r_{t'})$.

That $\bigcap_{t' \neq t} C_{t'} = r_t$ follows immediately from the definitions.

If $H \subset H_S$ is a closed halfspace with $\pi(g^P)$ on its boundary that contains all of the rays r_t, then H also contains all the cones C_t, from which we could infer that points outside H had no best responses, which is absurd. ∎

Let

$$D_S = \{\, (\sigma, \max_{t \in T} u_t(\sigma)) : \sigma \in \Delta(S) \,\}$$

be the *lower envelope* of P. Since $\max_t u_t(\sigma)$ is a continuous function of σ, the restriction of π to D_S is a homeomorphism of D_S and $\Delta(S)$. For $A \subset S$ let $D_A = D_S \cap (\Delta(A) \times I\!R)$. The equivalence of the perspective presented here with the one described in the last section can now be expressed by the equation

$$\pi\big(D_A \cap \bigcap_{t \in B} G_t\big) = \Delta(A) \cap \bigcap_{t \in B} C_t.$$

4. Results for All Dimensions

With the hope of building a useful tookit, we have tried to find versions of our methods that are applicable to all d. The cases in which this effort has succeeded are described in this section.

4.1. Cliques

Our genericity assumption implies that P and Q are *simple*: each vertex is a member of exactly d facets. Since each vertex v of P has exactly d labels, say $\ell_1, \ldots, \ell_d$, the intersection of the planes associated with any $d-1$ of these labels, say $\ell_2, \ldots, \ell_d$, is a 1-dimensional affine subspace of $H_S \times I\!R$ that intersects P in a line segment, called *an edge,* that has v as one endpoint. The other endpoint is described as the *neighbor of v reached along the edge obtained by dropping ℓ_1.* In general two vertices of P are *adjacent* or *neighbors* if they are the two endpoints of an edge. Clearly two vertices are adjacent if and only if they have $d-1$ labels in common. Since P is simple, each vertex is an endpoint of exactly d edges of P. For pure vertices one of the edges is unbounded, leading to "heaven," while $d-1$ of them lead to adjacent vertices. All other vertices are adjacent to precisely d other vertices.

Virtual vertices $v_1, \ldots, v_k$ $(k \geq 2)$ constitute a *virtual clique* if any two of them have $d-1$ labels in common. One way that $v_1, \ldots, v_k$ might constitute a virtual clique is if there is some set of $d-1$ labels that they all have in common; in this case we say that $v_1, \ldots, v_k$ is a *linear virtual clique.* Another possibility is that there is a set of $d+1$ labels, with each $\mathcal{L}(v_i)$ a d-element subset of this set, when we say that $v_1, \ldots, v_k$ is a *simplicial virtual clique.*

Lemma 4.1: *Any virtual clique is either linear or simplicial.*

Proof: A virtual clique with two elements is of both types, clearly, so we may assume that the virtual clique has three distinct elements, say v_1, v_2, and v_3. First suppose that $\mathcal{L}(v_3) \cap \mathcal{L}(v_1) \subset \mathcal{L}(v_1) \cap \mathcal{L}(v_2)$. Since the two sets have the same number of elements, they must be equal, and in fact

$$\mathcal{L}(v_3) \cap \mathcal{L}(v_1) = \mathcal{L}(v_3) \cap \mathcal{L}(v_2) = \mathcal{L}(v_1) \cap \mathcal{L}(v_2) \cap \mathcal{L}(v_3) = \mathcal{L}(v_1) \cap \mathcal{L}(v_2).$$

Any other member of the virtual clique must also share these labels since that is the only way that it can share $d-1$ labels with each of v_1, v_2, and v_3. Thus the virtual clique is linear.

The other possibility is that $\mathcal{L}(v_3)$ contains the unique element of $\mathcal{L}(v_1) \backslash \mathcal{L}(v_2)$. In order to have $d-1$ elements in common with $\mathcal{L}(v_2)$ it must also contain the unique element of $\mathcal{L}(v_2) \backslash \mathcal{L}(v_1)$, and it does not contain some member of $\mathcal{L}(v_1) \cap \mathcal{L}(v_2)$, which implies that there is no $(d-1)$-element subset of $\mathcal{L}(v_1) \cup \mathcal{L}(v_2)$ that is a subset of each of $\mathcal{L}(v_1)$, $\mathcal{L}(v_2)$, and $\mathcal{L}(v_3)$, so the labels of any other member of the virtual clique must be contained in $\mathcal{L}(v_1) \cup \mathcal{L}(v_2)$. ∎

Let $w_1, \ldots, w_k$ be the virtual vertices that are complementary to $v_1, \ldots, v_k$. If two d-element subsets of $S \cup T$ have $d-1$ elements in common, then their complements are also d-element subsets of $S \cup T$ with $d-1$ common elements. Thus $w_1, \ldots, w_k$ is a virtual clique if $v_1, \ldots, v_k$ is, and conversely by symmetry.

Lemma 4.2: *If $v_1, \ldots, v_k$ is a linear (simplicial) virtual clique, then the complementary virtual vertices are a simplicial (linear) virtual clique.*

Proof: If $v_1, \ldots, v_k$ is a linear virtual clique, then $(\bigcap_i \mathcal{L}(v_i))^c$ has $d+1$ elements, and each $\mathcal{L}(w_i) = \mathcal{L}(v_i)^c$ is a d-element subset of it, so $w_1, \ldots, w_k$ is simplicial. If $v_1, \ldots, v_k$ is simplicial, then each $\mathcal{L}(w_i)$ is a d element superset of $(S \cup T) \setminus \bigcup_i \mathcal{L}(v_i)$, so $w_1, \ldots, w_k$ is linear. ∎

If all elements of a virtual clique are, in fact, vertices, then the collection is said to be a *clique.* Now all the virtual vertices of a linear virtual clique are contained in the affine line defined by the $d-1$ common labels, and if there are any vertices among them, they must be the two endpoints of the line segment that is the intersection of this line with P (or Q). Thus:

Lemma 4.3: *A linear clique has exactly two elements.*

For any clique, those complementary virtual vertices that happen to be vertices constitute a clique, so we also have:

Lemma 4.4: (cf. Lemma 4 of Keiding (1995)) *If $v_1, \ldots, v_k \in V$ constitute a clique, then at most two of $v_1, \ldots, v_k$ are complementary.*

If, as per the Remark, we had perturbed a face F_s of P and a face F_t of Q to make P and Q bounded, thereby adding an additional vertex to each, we could apply the Corollary to cliques containing these new vertices. In place of this we have:

Lemma 4.5: *If $v_1, \ldots, v_k$ are pure vertices of P that constitute a clique, then at most one of $v_1, \ldots, v_k$ is complementary.*

Proof: Each pure vertex has $d-1$ labels in S and a single label in T, so to be a clique $v_1, \ldots, v_k$ must have a common label in T, say t. The unique pure vertex of Q that has all labels in T except t is complementary to at most one of $v_1, \ldots, v_k$. ∎

Keiding's method in showing that there are at most 15 equilibria is to apply these results on cliques to a computer generated list of all simple polytopes in $I\!R^4$ that have at most eight facets but have more that 16 vertices. The results in the remainder of this section, and in the next section, give conditions under which certain vertices are not complementary. Except as noted, these results do not seem to be consequences of the clique principle. We have not succeeded in constructing an argument using only the result on cliques, even though Keiding's analysis indicates that such an argument should exist.

4.2. Almost Completely Mixed Strategies

Consider a vertex v with a single label s in S, so that $v \in V_1$. If the geocenter is a vertex, it is adjacent to v, and must be the vertex reached along the edge obtained by dropping s. Otherwise the vertex reached along the edge obtained by dropping s is another element of V_1.

For $s \in S$ let λ_s be the number of elements of V_1 that have label s, and are consequently in F_s. Let $\vec{\lambda}$ be the d-tuple whose components are the integers λ_s, $s \in S$. When the geocenter is a vertex, the adjacent vertices are precisely the d elements of V_1. Thus:

Lemma 4.6: *If g^P is a vertex, then $\sum_{s \in S} \lambda_s = d$.*

If g^P is not a vertex, then a heavenly facet F_s is said to be an *exiting facet* if the geocenter and P are in the same halfspace of the two defined by Φ_s. Otherwise F_s is said to be an *entering facet.* Let S_{in} and S_{out} be the sets of $s \in S$ such that F_s is entering or exiting, respectively. When the geocenter is not a vertex, each ray beginning at the geocenter and passing through a vertex in V_1 must intersect P in a line segment with the endpoint closest to the geocenter in an entering facet and the other endpoint in an exiting facet. Equivalently, proceeding from $\pi(g^P)$ along a ray r_t that passes through $\Delta(S)$, we first pass through $\Delta(S-\{s\})$ for some s such that F_s is an entering facet, then exit through $\Delta(S-\{s'\})$ for some s' such that $F_{s'}$ is an exiting facet. Thus we describe $\Delta(S-\{s\})$ as an *entering facet* of $\Delta(S)$ if $\pi(g^P)$ and $\Delta(S)$ are on oppossite sides of the hyperplane of H_S that contains $\Delta(S-\{s\})$, and otherwise $\Delta(S-\{s\})$ is an *entering facet.*

Lemma 4.7: *If g^P is not a vertex, then $\sum_{s \in S_{\text{in}}} \lambda_s = \sum_{s \in S_{\text{out}}} \lambda_s \leq d-1$. Thus there are at most $2d-2$ elements of V_1.*

Proof: Each ray r_t that intersects $\Delta(S)$ defines exactly two vertices of level 1, one in an entering facet and one in an exiting facet, so the vertices of level 1 are divided equally between the two types of facets. Since g^P is not a vertex, $\pi(g^P)$ is not in $\Delta(S)$, hence can be separated from it by a hyperplane, and the halfspace defined by this halfspace that does not contain $\Delta(S)$ must contain one of the rays r_t, by Lemma 3.1. So, at most $d-1$ of these rays intersect $\Delta(S)$. ∎

Lemma 4.8: *For each s, $\lambda_s \leq d-1$.*

Proof: Separate $\Delta(S\backslash\{s\})$ from $\pi(g^P)$ with a hyperplane and proceed as above. ∎

We now note a particular application of the clique principle.

Lemma 4.9: *For each s there are at most two complementary vertices in $V_0 \cup (V_1 \cap F_s)$.*

Proof: Any vertex in $V_1 \cap F_s$ has, as labels, s and all the elements of T but one, so any two such vertices are adjacent. If the geocenter is a vertex, it is adjacent to every vertex in V_1, so $V_0 \cup (V_1 \cap F_s)$ is a clique and the claim follows from Corollary 4.4. ∎

Lemma 4.10: *If $d \geq 4$ is even, there are at most $2d-3$ complementary vertices in $V_0 \cup V_1$.*

Proof: If g^P is a vertex, Lemma 4.6 implies that $V_0 \cup V_1$ has exactly $d+1 \leq 2d-3$ elements. So we may suppose that g^P is not a vertex, and also, by symmetry, that g^Q is not a vertex. By virtue of Lemma 4.7, we assume that $\sum_s \lambda_s = 2d-2$. If $\lambda_s \geq 3$ for some s, then Lemma 4.9 implies that some element of $V_1 \cap F_s$ is not complementary, so we may assume that $\lambda_s \leq 2$ for all s. We may also assume that $\vec{\lambda} = (2,2,\ldots,2,1,1)$, since the other possibility, $\vec{\lambda} = (2,2,\ldots,2,2,0)$, is inconsistent with Lemma 4.7 because $d-1$ is odd. Thus, for each s there is a vertex $v \in V_1 \cap F_s$, and this vertex is complementary to a vertex, say w, whose labels include $S\backslash\{s\}$. Therefore $\pi(w)$ is in the intersection of the ray r_s with $\Delta(T)$, and in particular this intersection is nonempty. It follows that all d rays r_s must intersect $\Delta(T)$, which is impossible: $\pi(g^Q)$ is not in $\Delta(T)$ and can be separated from it by a hyperplane, after which Lemma 3.1 implies that some r_s does not intersect $\Delta(T)$. ∎

4.3. Mixtures Over All But Two Pure Strategies

Now consider a vertex $v \in V_2$. It has two labels in S, say s and s', so there are two edges leading out of it obtained by dropping these labels. Each of the two edges terminates either at an element of V_1 or at an element of V_2, and according to which case pertains we say that the edge is *anchored* or *floating* respectively. The edge obtained by dropping s' is contained in F_s.

For each $s \in S$ let ι_s be the number of floating edges contained in F_s. Now there are λ_s vertices in $V_1 \cap F_s$, and each such vertex has $d-1$ neighbors reached along edges obtained by dropping labels in T. Any two vertices in $V_1 \cap F_s$ are adjacent, so precisely $\lambda_s - 1$ of these edges lead to other elements of $V_1 \cap F_s$, and the remaining $d - \lambda_s$ lead to elements of V_2. Every element of $V_2 \cap F_s$ is an endpoint of an edge, anchored or floating, obtained by dropping the other label in S. Thus:

Lemma 4.11: *For each s, the number of elements of $V_2 \cap F_s$ is $2\iota_s + \lambda_s(d - \lambda_s)$.*

Each element of V_2 is in precisely two facets F_s, so summing over s yields:

Lemma 4.12: *V_2 has $\frac{1}{2}\sum_s\big(2\iota_s + \lambda_s(d-\lambda_s)\big)$ elements.*

4.4. Floating Walls

Much of the analysis specific to $d = 4$ will be focused on the four facets of $\Delta(S)$. These facets are two dimensional, so in this context a floating edge divides the containing facet into two parts. The generalization of this idea to higher dimensions involves the potentially dividing object having codimension one. For distinct $t, t' \in T$ and $\emptyset \neq A \subset S$, the set $D_A \cap G_t \cap G_{t'}$ is called a *wall.*

Note that $\Gamma_t \cap \Gamma_{t'}$ is an affine plane of codimension 2 in $H_S \times I\!R$. Since Γ_t and $\Gamma_{t'}$ are graphs of affine functions from H_S to $I\!R$, there cannot be two points in $\Gamma_t \cap \Gamma_{t'}$ that differ only in their final component, so $\pi(\Gamma_t \cap \Gamma_{t'})$ is an affine plane of codimension one in H_S. Our genericity assumption implies that, for any nonempty $A \subset S$, $\pi(\Gamma_t \cap \Gamma_{t'}) \cap \Delta^\circ(A)$ is either empty or has codimension 1 in $\Delta^\circ(A)$, in which case $\Delta^\circ(A) \backslash \pi(\Gamma_t \cap \Gamma_{t'})$ has two connected components. The idea we are interested in is expressed most directly by saying that $D_A \cap G_t \cap G_{t'}$ is a *floating wall* if $\Theta = D_A \backslash \Theta$ has two connected component, but there are many equivalent formulations:

Lemma 4.13: For distinct $t, t' \in T$ and $\emptyset \neq A \subset S$ let $\Theta = D_A \cap G_t \cap G_{t'}$. If $\Theta \neq \emptyset$ the following are equivalent:

(a) Θ is a floating wall;

(b) $\Delta(A)\backslash\pi(\Theta)$ has two connected components;

(c) $\Delta(A) \cap \pi(G_t \cap G_{t'}) = \Delta(A) \cap \pi(\Gamma_t \cap \Gamma_{t'})$;

(d) all the vertices of Θ are in V_{d-2}.

Proof: Since the restriction of π to D_A is a homeomorphism onto $\Delta(A)$, (a) and (b) are equivalent. Given our remarks above, it is clear that (c) implies (b). In general, whenever a simplex is cut into two pieces by the removal of a affine set of one lower dimension, restoring any point restores the connectedness of the set, so (b) implies (c). Thus (a)–(c) are equivalent.

Suppose (c) holds. By genericity, all the vertices in $D_A \cap \Gamma_t \cap \Gamma_{t'}$ are mapped by π to vertices of $\Delta(A) \cap \pi(\Gamma_t \cap \Gamma_{t'})$, so they are all in V_{d-2}. Thus (c) implies (d). Conversely, if (c) does not hold let S be the boundary of $\Delta(A) \cap \pi(G_t \cap G_{t'})$ in $\Delta(A) \cap \pi(\Gamma_t \cap \Gamma_{t'})$. Then S is nonempty, and the vertices of $\pi^{-1}(S) \cap D_A$ cannot lie in V_{d-2}. Thus (c) and (d) are equivalent. ∎

For $s \in S$ we write E_s in place of $D_{S\backslash\{s\}}$. Let θ_s be the number of floating walls in E_s. We have the following relationship between λ_s and θ_s.

Lemma 4.14:

(a) If $\lambda_s \geq 2$, then $\theta_s = 0$.

(b) If $\lambda_s = 1$, then $\theta_s \leq 1$.

(c) If $\lambda_s = 0$, then $\theta_s \leq d - 1$.

Proof: (a) Observe that each element of $V_1 \cap E_s$ is on the boundary of $d-1$ of the sets $G_t \cap E_s$, and each group of $d-1$ sets of the form $G_t \cap E_s$ has at most one point in common. In particular, if $V_1 \cap E_s$ has two or more elements, then every wall contains at least one of them, so no wall is floating.

(b) Suppose $v \in V_1 \cap E_s$, and let t be the unique element of T such that $v \notin G_t$. Any floating wall in E_s must be of the form $\Theta' = E_s \cap G_t \cap G_{t'}$ and if $\Theta'' = E_s \cap G_t \cap G_{t''}$ is another floating wall, then Θ' and Θ'' divide E_s into three pieces. Since $G_t \cap E_s$ has both walls on its boundary, $G_{t'} \cap E_s$ and $G_{t''} \cap E_s$ must be disjoint, a contradiction of the assumption that they both contain v.

(c) Any collection of k floating walls divides E_s into $k+1$ regions. Each such region contains the interior of at least one of the sets $G_t \cap E_s$, and the interior of such a set is connected and cannot be part of more than one region. Thus the number of floating walls is not greater than the number of elements of T less one. ∎

4.5. Separation

The point made in this subsection is a consequence of the convexity of each facet G_t. For $s \in S$ ($t \in T$) let $v^s = (s, \max_t u_t(s))$ ($w^t = (t, \max_s u_s(t))$) denote the corresponding pure vertex of P (Q). For any particular pair of pure strategies $s, s' \in S$, $D_{\{s,s'\}}$ will consist of a sequence of touching edges proceeding from $v^{s'}$ to v^s. Each of these edges projects to a line segment in $\Delta(\{s, s'\})$ on whose interior there is a unique best response. In this way the ordering of the edges, as one proceeds along $D_{\{s,s'\}}$ from $v^{s'}$ to v^s, induces a finite sequence $t_1, \ldots, t_K$ called the *best response chain going from s' to s*. Each $t \in T$ appears at most once in a best response chain because each G_t is convex.

Lemma 4.15: *For any $s, s', s'' \in S$, the elements of T that appear in the best response chain going from s' to s, and also in the best response chain going from s'' to s, have the same order in the two chains.*

Proof: Suppose that t and t' appear in both chains. We may assume that t is not the last element of either chain, since then it would be the best response to s, hence the last element of both chains. We may also assume that t is not the first term in at least one of the two chains, say the one going from s' to s. We can now see that $\pi(D_{\{s,s',s''\}} \cap G_t)$ is a convex polygon in $\Delta(\{s, s', s''\})$ that includes portions of both $\Delta(\{s, s'\})$ and $\Delta(\{s, s''\})$, but does not include either s or s'. Therefore $D_{\{s,s',s''\}} \backslash G_t$ has at least two connected components. For any $t' \in T$, the relative interior of $D_{\{s,s',s''\}} \cap G_{t'}$ is convex (in particular, connected), hence is contained in one of these components. The desired result follows easily. ∎

Let $v_0 = v^{s'}, v_1, \ldots, v_{K-1}, v_K = v^s$ be the vertices of $D_{\{s,s'\}}$, in order. The sequence $v_0, \ldots, v_K$ is called the *vertex chain going from s' to s*. If $(t_1, \ldots, t_K)$ is the best response chain going from s' to s, the unique best response to v_0 is t_1, the unique best response to v_K is t_K, and for $j = 1, \ldots, K-1$ the best responses to v_j are t_j and t_{j+1}. We need the following two results on complementary vertices in vertex chains.

Lemma 4.16: *If t_1, t_2, t_3 and v_0, v_1, v_2, v_3 are the best response chain and the vertex chain going from s to s', then at least one of the vertices v_0, v_1, v_2, v_3 is not complementary.*

Proof: Suppose that v_0, v_1, v_2, v_3 are all complementary, with complementary vertices $w_0, w_1, w_2, w_3 \in W$. Then $w_0 = w^{t_1}$ and $w_3 = w^{t_3}$, and s and s' are the best responses to t_1 and t_3 respectively. In addition, w_1 (w_2) is the vertex in $D_{\{t_1,t_2\}}$ ($D_{\{t_3,t_2\}}$) for which $\{s, s'\}$ is the best response set, so s and s' (s' and s) are the first and the second member, respectively, of the best response chain going from t_1 to t_2 (from t_3 to t_2). Lemma 4.15 asserts that this is impossible. ∎

Lemma 4.17: *Let $t_1, \ldots, t_K$ and $v_0, \ldots, v_K$ be the best response chain and the vertex chain going from s to s', where $K \geq 3$. If v_0, v_1 and v_2 are complementary, then s' appears immediately after s in the best response chain going from t_3 to t_2.*

Proof: Let w_0, w_1 and w_2 be the complementary vertices to v_0, v_1 and v_2, respectively. Then $w_0 = w^{t_1} \in D_{\{t_1,t_2\}} \cap G_s$, and

$$\{w_1\} = D_{\{t_1,t_2\}} \cap G_{s'} \cap G_s \quad \text{and} \quad \{w_2\} = D_{\{t_2,t_3\}} \cap G_{s'} \cap G_s.$$

In particular, the edge between w_1 and w_2 is a floating wall $D_{\{t_1,t_2,t_3\}} \cap G_{s'} \cap G_s$ that divides $D_{\{t_1,t_2,t_3\}}$ into two components: the component that contains the relative interior of $D_{\{t_1,t_2,t_3\}} \cap G_{s'}$ also contains w^{t_2}, and the other contains the relative interior of $D_{\{t_1,t_2,t_3\}} \cap G_s$ and $\{w^{t_1}, w^{t_3}\}$. Hence, moving along $D_{\{t_2,t_3\}}$ from w^{t_3}, the best response is s on the edge immediately before w_2 and is s' on the edge immediately after w_2. ∎

4.6. Oddness

Lemma 4.18: *There are an odd number of Nash equilibria.*

Proof: Cf. Shapley (1974). ∎

5. The Four Dimensional Case

In this section we present two lemmas that seem to be particular to the case $d = 4$. Now V is partitioned into the sets $V_0 = \{g^P\}$, V_1, V_2, and V_3, where V_3 is the set of pure vertices. The elements of V_1 and V_2 will now be called *low* and *high* vertices, respectively. As we noted earlier, for this dimension a floating edge in F_s is a floating wall in E_s and vice versa, so $\theta_s = \iota_s$; henceforth we shall refer only to ι_s.

Lemma 5.1: *If $\lambda_s = 0$, then two of the pure or high vertices in F_s are not complementary.*

Proof: If $\iota_s = 0$, so that there are no high vertices in F_s, then, by Lemma 4.5, two of the three pure vertices of F_s are noncomplementary.

Suppose $\iota_s \geq 1$. By hypothesis, all vertices in F_s are either pure or high, so every wall of F_s is floating. Each floating wall Θ divides $E_s \backslash \Theta$ into two pieces, and each other floating wall Θ' is contained in one or the other of these pieces. It is obvious (we leave the combinatoric details to motivated readers) that the floating walls can be ordered, say from "left" to "right," in such a way that, for each Θ, all the floating walls to the left of Θ are contained in one of the two components of $E_s \backslash \Theta$ and all the floating walls to the right of Θ are contained in the other component.

Consider an extreme (say the leftmost) floating wall Θ in this ordering, and let C be the closure of the component of $E_s \backslash \Theta$ that does not contain any other floating walls. Then C contains either one or two pure vertices. When C contains one pure vertex, it and the two endpoints of Θ constitute a clique, hence one of them must be noncomplementary. If C contains two pure vertices, then Lemma 4.5 requires that one of the two is noncomplementary. Note that the two pure vertices are *not* in Θ.

The result is now clear. If there are two or more floating walls, then the two extremes of the ordering are distinct, the sets corresponding to C above are disjoint, and in each we can find a noncomplementary vertex. When there is a single floating wall Θ, the two analogues of C have Θ as their intersection, but in addition to having a noncomplementary vertex in each analogue of C, we know that one analogue has two pure vertices, which cannot lie in Θ, one of which is noncomplementary. ∎

Lemma 5.2: *If $\lambda_s = \iota_s = 1$, then at least two of the vertices in F_s are not complementary.*

Proof: For concreteness we let $S = \{A, B, C, D\}$ and $T = \{a, b, c, d\}$, with $s = D$, so that $E_s = D_{\{A,B,C\}}$. Let v be the low vertex in E_D. There are three edges emanating from v that are obtained by dropping labels of v that are in T. Let v_1, v_2 and v_3 be the other endpoints of these edges. These vertices must be high, since v is the only low vertex in E_D. Let v_4 and v_5 be the two endpoints of the floating edge.

We begin by considering the possibility that $v_1, v_2 \in D_{\{A,B\}}$. Then v, v_1 and v_2 constitute a clique, so one of them is not complementary (Lemma 4.4). In addition, another noncomplementary vertex exists on the side of the floating edge that does not contain v. Either this side contains one pure vertex, in which case the two endpoints of the floating edge and this pure vertex form another clique, or this side contains two pure vertices which are adjacent so that Lemma 4.5 applies.

The choice of A, B, v_1 and v_2 was arbitrary, so (after relabelling) we may assume that $v_1 \in \Delta(\{A, C\})$, $v_2 \in \Delta(\{A, B\})$ and $v_3 \in \Delta(\{B, C\})$. There is no loss of generality in assuming that $v_4 \in \Delta(\{A, B\})$ and $v_5 \in \Delta(\{B, C\})$. (See Figure 3.)

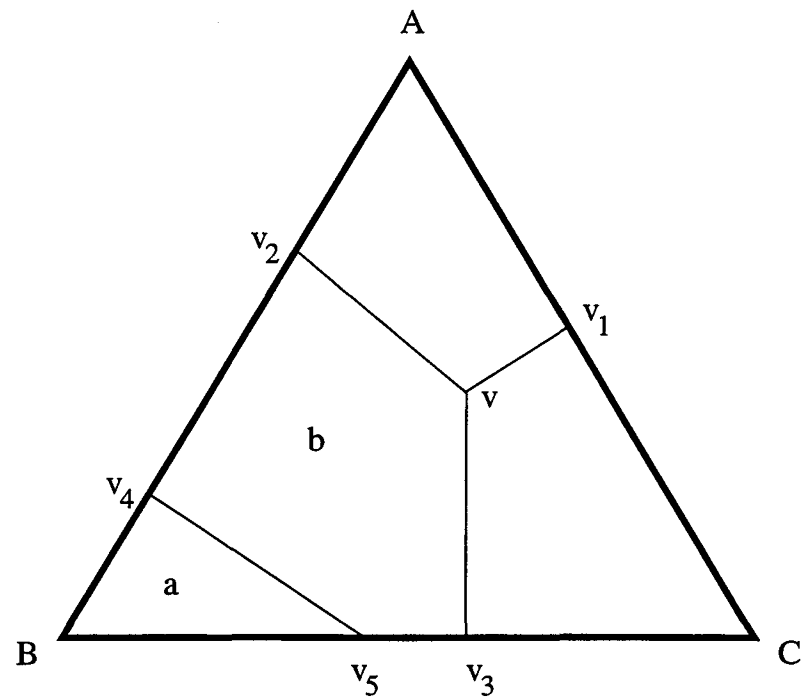

Figure 3

Now Lemma 4.16 implies the existence of a noncomplementary vertex in each of

$\{A, v_2, v_4, B\}$ and $\{C, v_3, v_5, B\}$, so the claim can fail only if B is the only noncomplementary vertex in either chain. Without loss of generality assume that a is the best response to B, and that b is the best response in the polygon in $\Delta(\{A, B, C\})$ with vertices v, v_2, v_3, v_4, and v_5. Then, Lemma 4.17 applied to the vertex chain (A, v_2, v_4, B) implies that B appears immediately after A in the best response chain going from a to b. Analogously, Lemma 4.17 applied to (C, v_3, v_5, B) implies that B appears immediately after C in the same best response chain. This is a contradiction, completing the proof, because B appears only once in the best response chain and so cannot have both A and C as the immediate predecessor. ∎

6. Conclusion of the Argument

Using the resources developed above, we have enough to establish the desired result, as we now demonstrate.

Theorem: *If $d = 4$, there are at most 15 Nash equilibria.*

Proof: We begin with an overview of our methodology. Let Δ be a variable that is 1 if g^P is a vertex of P, and 0 otherwise. Let N be the number of vertices of P that are not complementary. There are always 4 pure vertices, so the results of the last section (Lemma 4.12, in particular) imply that a counterexample occurs precisely when

$$\Delta + \sum_s \lambda_s + \frac{1}{2}\sum_s \big(2\iota_s + \lambda_s(4 - \lambda_s)\big) + 4 - N \geq 17$$

which simplifies to

$$\Delta + \frac{1}{2}\sum_s \big(2\iota_s + \lambda_s(6 - \lambda_s)\big) - N \geq 13 \tag{1}$$

We show, for each possible tuple $\vec{\lambda}$ that is consistent with Lemmas 4.6, 4.7, and 4.8, that the inequality (1) is impossible because N is sufficiently large. We start with the next observation that follows immediately from Lemma 4.14:

$$2\iota_s + \lambda_s(6 - \lambda_s) \leq \begin{cases} 2 \times 3 = 6 & \text{if } \lambda_s = 0 \\ 2 + 5 = 7 & \text{if } \lambda_s = 1 \\ 2 \times 4 = 8 & \text{if } \lambda_s = 2 \\ 3 \times 3 = 9 & \text{if } \lambda_s = 3 \end{cases}$$

or, equivalently, $2\iota_s + \lambda_s(6 - \lambda_s) \leq 6 + \lambda_s$. Summing over s, we have

$$\sum_s (2\iota_s + \lambda_s(6 - \lambda_s)) \leq 24 + \sum_s \lambda_s$$

Hence, (1) implies

$$\Delta + \frac{1}{2}(\sum_s \lambda_s) - N \geq 1 \tag{2}$$

First, we consider the case that $\lambda_{\hat{s}} = 0$ for some $\hat{s} \in S$. In this case two of the pure or high vertices are not complementary by Lemma 5.1, so that $N \geq 2$. If $\Delta = 0$, therefore, Lemma 4.7 implies that (2) holds only if $\sum_s \lambda_s = 6$ and $N = 2$, i.e., there is no additional noncomplementary vertex. This is impossible because, if $\sum_s \lambda_s = 6$, Lemma 4.10 says that there is a *low* noncomplementary vertex and we must conclude $N \geq 3$. If $\Delta = 1$, on the other hand, $\sum_s \lambda_s = 4$ by Lemma 4.6 and so, $\lambda_{s'} \geq 2$ for some s' because $\lambda_{\hat{s}} = 0$. Then, either g^P or one low vertex in $F_{s'}$ is not complementary by Lemma 4.9. Again, $N \geq 3$ and (2) cannot hold. This completes the proof for the case that $\lambda_{\hat{s}} = 0$.

It remains to consider the possibility that $\lambda_s \geq 1$ for all $s \in S$. The three possible $\vec{\lambda}$ allowed by Lemma 4.6 and 4.7 are investigated separately.

Case A: $\vec{\lambda} = (1, 1, 1, 1)$.

In this case inequality (1) reduces to

$$\Delta + \sum_s \iota_s - N \geq 3 \tag{3}$$

for which we need $\sum_s \iota_s \geq 2$. If so, Lemma 4.14 (b) implies $\lambda_s = \iota_s = 1$ for some s and Lemma 5.2 implies $N \geq 2$. Since $\sum_s \iota_s > 4$ is impossible by Lemma 4.14 (b), it follows that (3) holds only if $\Delta = 1$, $N = 2$ and $\sum_s \iota_s = 4$ so that $\lambda_s = \iota_s = 1$ for all s. This is impossible: each F_s contains at least two noncomplementary vertices by Lemma 5.2, and since the intersection of the four facets F_s is null, it follows that $N \geq 3$.

Case B: $\vec{\lambda} = (2, 2, 1, 1)$.

From Lemma 4.6 we infer that $\Delta = 0$, so (1) reduces to

$$\sum_s \iota_s - N \geq 0 \tag{4}$$

Lemma 4.10 implies that some low vertex is not complementary, so we need $\sum_s \iota_s \geq 1$ for (4). In addition, we know $\sum_s \iota_s \leq 2$ from Lemma 4.14. If $\sum_s \iota_s = 1$, Lemma 5.2 implies $N \geq 2$, violating (4). If $\sum_s \iota_s = 2$, applying Lemma 5.2 to the two facets with $\lambda_s = \iota_s = 1$, we conclude:

(∗) For two $s \in S$, F_s contains two noncomplementary vertices, and in order for these vertices to be shared by the two facets they must be either pure or high.

Since we already had a low noncomplementary vertex, we deduce that $N \geq 3$, again violating (4).

Case C: $\vec{\lambda} = (3, 1, 1, 1)$.

Again Lemma 4.6 implies that $\Delta = 0$, so (1) reduces to

$$\sum_s \iota_s - N \geq 1 \tag{5}$$

Lemma 4.10 implies that some low vertex is noncomplementary, so we need $\sum_s \iota_s \geq 2$ for (5). In addition, we know $\sum_s \iota_s \leq 3$ from Lemma 4.14. In case $\sum_s \iota_s = 2$ or 3, Lemma 5.2 implies (∗) and we conclude that $N \geq 3$, violating (5). ∎

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