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Abstract: Classical economic agents perform arbitrarily complex operations on arbitrarily complex magnitudes (real numbers). By contrast, real world agents have bounds on their abilities to perceive, think about, calculate with, and communicate magnitudes. There are many ways to model agents with bounded abilities, and here we mention two — one through bounds on computational abilities, and one through bounds on descriptive or definitional abilities.

In both cases, we propose a "uniformity principle" constraining in a parallel fashion both the magnitudes (prices, quantities, etc.) and the operations (to perceive, evaluate, choose, communicate, etc.) that agents can use. We focus on the definitional bounds, deferring computational bounds to other papers (1996a,b). The languages allowed are those of ordered rings, and certain expansions; the structures are those of real closed ordered fields, and corresponding expansions.

It is not obvious that a theory of definable economies is possible, since there may not be any definable structures that are reasonably close to the classical one. And even if such structures existed, it is not obvious that the classical theorems of economics would hold in them.

Our two main conclusions are positive: In many interesting cases mathematical structures do exist with definability–bounded agents. Furthermore, many classical theorems of economic theory survive in a definable context: existence of demand and utility functions, existence of competitive equilibria, First and

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Second Welfare Theorems, characterization of aggregate excess demand, etc.

Our proofs rely on theorems of mathematical logic (completeness (Tarski), model completeness (A. Robinson, Wilkie), o-minimality (van den Dries, Pillay and Steinhorn, Wilkie)) that allow us to establish existence of definable models and to transfer classical theorems to a definable framework.

Although superficially different, the concepts underlying (Blume and Zame, 1992) are fundamentally close to the ones we use here.

JEL Classification: D11, D51, C62

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Waitress. How much pizza would you like?

Homo Œconomicus, Jr. $\boxed{}$.

Waitress. How much?

Homo Œconomicus, Jr. I don't have a word for the number.

Waitress. Would you like a drink with that?

Homo Œconomicus, Jr. $\boxed{}$.

Waitress. Well?

Homo Œconomicus, Jr. I can't figure it out.

Junior has neither the communication abilities nor the decision making abilities of his famous father. But then who does?

I Bounding the Classical Model: a uniformity principle

Classical models assume that individual agents can perceive, calculate with, and communicate magnitudes (numbers) of arbitrary complexity. Furthermore, they assume that agents can have preferences, utility functions, technologies, and production functions of arbitrary complexity, which operate on those magnitudes.

It is natural to question the realism of such models. Agents may have limited perception, cognition, analysis, decision making, and communications. (Cf. Simon (1959) and others.) There may be several reasons for these limitations. The intrinsic abilities of the agent may be limited. The time allowed may be limited. The linguistic framework may be limited: agents would require an unrealistic language (with uncountably many names) if they are to communicate arbitrary real numbers as prices or quantities, and they would need more than continuumly many names if they are to communicate about arbitrary relations or functions.[1]

Constraining our theory so that agents have bounds on the magnitudes and operations they can use raises important questions for theory — general equilibrium theory, as well as individual agent theory. Do solutions exist for an agent's maximization problems? Do the general equilibrium welfare theorems still hold? Does a competitive equilibrium exist? Of course the answers will depend on the

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[1] **Leonid Hurwicz notes (private communication) that Jacob Marschak made similar observations at a Session on Welfare Economics at the September 1976 Atlantic City meeting of the American Economic Association. Cf. Hurwicz and Marschak (1985, footnote †).**

particular type of constraints that we impose. In this paper and the companion papers (Richter and Wong, 1996a,b) we obtain some answers for two distinct types of constraints — computability bounds and definability bounds.

There will not be a single theory of bounded rationality, applicable in all situations. As with non-perfect competition, there will be many theories, appropriate to various contexts. This paper and (1996a,b) describe several such theories, and we can organize their ideas by the following tree description.

At the root of the tree is a *uniformity principle*, by which the same type of bounds are to be imposed on the complexity of magnitudes and on the complexity of (preference and technology) relations and (utility and production) functions.

From this root principle there are at least two branches to the next tree level — one imposing a definability bound, and the other imposing a computability bound. Analogously, the definability bound imposes a language, and requires that all magnitudes, relations, and functions be definable in that language. The computability bound requires that all magnitudes, relations, and functions be computable.

Each of these two nodes — the definability node and the computability node — has several branches to the third tree level. The branches from the definability node are obvious, since there are many different languages, of different expressive powers, that might be natural in many contexts. The branches from the computability node are also obvious, since there are many plausible notions of computability. The plurality of branches from each of the two nodes means there is a possibility of a rich analysis of complexity in bounded rationality models. In one case it arises from the complexity of the languages agents can use. In the other case it arises from the complexity of the computations agents can perform.(2) In this paper and in (1996a,b) we begin a study of only the most basic branches, using the most basic notions of definability and computability.

Section II provides an intuitive outline of the two types of constraints, and mentions some relationships between them. In the rest of the paper we focus on definability.

Section III motivates several languages (e.g., $\mathcal{L}_0$, $\mathcal{L}_K$, and $\mathcal{L}_{\text{exp}}$) in which agents can define magnitudes and operations. (These will be the language of ordered rings, or some extensions.) A formal definition of definable agents and definable economies is stated in terms of such languages.

Section IV states one of our main results: many classical theorems of eco-

(2) It has long been recognized in mathematical logic and also in computer science that there are important relationships between complexity measures in the two approaches.

nomic theory survive in a definable context. We illustrate this fact, first by showing the adequacy of the languages like $\mathcal{L}_0$, $\mathcal{L}_K$, and $\mathcal{L}_{\exp}$ for stating definable versions of several classical theorems. And then we apply theorems of mathematical logic (using completeness, model completeness, and o-minimality techniques) to show that several classical results transfer to our definable context: existence of definable preference maximizers on definable competitive budget sets, existence of definable utility representations for definable preference relations,[3] a revealed preference characterization of demand functions of definable consumers (on finite sets), the classical First and Second Welfare Theorems for definable economies, existence of definable competitive equilibrium for definable economies, and a characterization of definable aggregate excess demand functions.

Section V provides proofs for the theorems of Section IV.

Section VI proves our second main result: the definable structures required by our uniformity principle do exist.

II Usability

To address complexity doubts about the realism of the classical models, it is useful to restrict our models to use only some particular subset of magnitudes and some particular subsets of relations and operations (preferences, utility functions, technologies, production functions).

There are several different views on what the "usable" magnitudes and operations might be. That is to be espected. There is wide consensus on what constitutes competitive equilibrium behavior, but no single theory of behavior outside equilibrium — there are infinitely many ways behavior can fail to be competitive. In the same way, there are many natural notions of "usable" magnitudes and operations. We do not expect "one size fits all" in this arena.

What we impose in this paper, and in the companion papers (Richter and Wong, 1996a,b), is the uniformity principle: the same kind of bounds should apply to both the magnitudes and the operations that agents can perform with them.

We highlight two views of why agents might have usability limitations. The definability view focuses on linguistic complexity. (I can't use a number if I have no name for it.) The computability view focuses on computational complexity. (I can't use a number if I can't calculate with it.)

[3] We present this in Richter and Wong (1996c).

II.1 Definability. One notion of usability is based on the idea that I can't use a number for communication if I don't have a name for it or a way of describing it. Perhaps I can't even perceive it, and maybe I can't calculate with it if I can't talk to myself about it. Similarly, I might call a preference or function usable, only if I can describe it (to myself if it is my preference relation or utility function, or to others if I am part of a team effort in a firm).

This suggests identifying usability with definability. Then we must be precise about the language in which the numbers, relations, and functions are to be defined. As we demonstrate in Section III.1 below, we are led naturally to the first order predicate calculus language of ordered rings, at least as a minimal language. Here the only numbers definable are the algebraic numbers, and the only relations and functions are the semialgebraic ones.

A more powerful language, would be an expansion of the language of ordered rings, with names for some specific numbers — for example, the non-algebraic numbers π and e might be important for agents in some models. (The language might have infinitely many such additional names, but if there are only countably many of them, it avoids imposing an uncountable, unrealistic language on our agents.) This leads to a larger class of usable (definable) numbers.

An even more powerful language would extend the language with new functions. For example, the exponential function $e^{(\cdot)}$ might be important in some models involving discounting.

As we will see below, many interesting economic concepts in consumer theory, producer theory, and general equilibrium theory can be expressed by a formula in a first order predicate calculus with appropriate names for magnitudes and operations.

Approaching usability by specifying a language opens the door to analysis of complexity through the already well developed study of complexity of sentences or formulas describing the magnitudes and operations.

II.2 Computability. An alternative notion of usability is based on the notion that I can't use a number if I can't use it in calculations — for example if I can't tell, in a routine, algorithmic fashion, whether it is bigger or smaller than any given rational number. And a utility function isn't usable if I can't tell, in a routine way, whether its value at some point is bigger or smaller than any given rational number. And a preference relation is not usable if, when x is preferred to y, I can't tell, in a routine way, whether x is preferred to y.

To make this notion precise, we must clarify what we mean by routine, algorithmic calculations. We use the standard approach of general recursive functions, Turing machines, etc.

One definition of routine calculation would be simply that there exists an approximation by rational numbers that can be calculated in a routine way. For example, we might say that a real number x is usable if there is a Turing machine that generates a sequence of rational numbers that approximates the given number. The difficulty with this notion of usability, is that, just knowing that the sequence converges to some real, we may not know how fast it converges, so at any given time we don't know whether we are near to or far from the real.

A more demanding notion of usability would require, in addition, that the rate of convergence can be determined in a routine way, by a Turing machine. Thus we might require that a Turing machine gives the sequence of decimal approximations to x. This notion of usability corresponds to the computable real numbers developed by Turing (1936), the computable functions, and the computable relations developed by Moschovakis (1964a,b) and others.

Although the main emphasis of this approach is computability, one consequence is also beneficial in the linguistic context. There are only countably many magnitudes allowed, and only countably many functions. As in the definability approach, we thus avoid requiring a language with more than countably many terms for agent communication concerning prices and quantities. In particular, all magnitudes and operations can be coded as natural numbers.

Of course there are many degrees of computability one could consider — computability by finite automata, by primitive recursive Turing machines, by polynomial time Turing machines, by general recursive Turing machines, etc.

Relationships. In this paper we will examine the definability approach to bounded rationality. The computability approach will be examined in other papers (Richter and Wong, 1996a,b). Here we mention a few of the relationships.

In order to compare the definability and computability approaches, we must specify a language for the first approach, and a notion of computability for the second. For example, in the definability approach we might admit only names for magnitudes that can be defined in terms of the symbols 0 and 1, and admit only functions that can be defined in terms of the arithmetic operations plus $+$ and times $\cdot$, and admit only relations that can be defined in terms of the greater-than relation $>$ and equality $=$. In other words, our language might be the language of ordered fields. In this language, the only definable real numbers turn out to be the algebraic numbers.

And our notion of computability might be Turing's notion of computable real numbers, mentioned above. Since every algebraic number is a computable real, the computability approach is more general than the definability approach, as it allows our agents to use a larger class of magnitudes.

Of course, it might be natural to include some specific names for some additional real numbers in our language, or perhaps some additional relations or functions. In that case the two approaches might overlap, with neither set of magnitudes including the other.

Even when the language is restricted to that of ordered fields, the inclusion relationship does not apply to relations or functions. For example, it will turn out that, while the lexicographic ordering on $\mathbb{R}_c \times \mathbb{R}_c$ is definable, it is not computable. (It is not continuous on $\mathbb{R}_c \times \mathbb{R}_c$. But computable relations, as defined in (Richter and Wong, 1996a), are necessarily continuous.)

From another point of view, the computability approach can be seen as imposing a calculation requirement on top of a definability concept. Since the computability conditions we consider are all related to Turing machines, and since Turing machines can be described (via Gödel numbers) in the language of arithmetic with plus (+) and times ($\cdot$), the magnitudes, relations, and functions can also be described in that language. This allows a different notion of definability (what might be called "definability with respect to a notation system" (cf. Moschovakis, 1964a)). And we can imagine the agents of our economic models doing calculations and communicating about prices and quantities in terms of these names, interpreted through Gödel numbers.

III Definable Economies

The rest of this paper interprets usability to mean definability, for agents of bounded rationality. All magnitudes, all relations, and all functions that the agents can use are definable (in a language to be specified). Whereas classical economics assumes agents can use (in perceiving, communicating, analyzing, making decisions, etc.) arbitrary real number magnitudes and arbitrary functions and relations on real numbers, our bounded rationality imposes a uniform linguistic restriction on the magnitudes, functions and relations that agents can use.

Although we depart from conventional economic modeling, our main theme is that all is not lost from what we have learned in the conventional context: many results translate into the new framework.

To understand what numbers the agents in our models can talk about, we must describe their languages.

III.1 Which Language? A language consists of a set of symbols (names) and a set of formulas (meaningful strings of symbols). Different languages are appropriate for modeling agents in different contexts, so we describe three types: A) a basic language; B) languages with additional constant symbols; and C) languages with additional function symbols.

A) A basic language.

We consider languages whose basic symbol sets satisfy these minimal properties (S.a–f):

S.a) (Variables) There must be names (symbols) for abstract variables v_0, $v_1, v_2, \cdots, u_0, u_1, u_2, \cdots$, so that agents can perform abstract analysis, e.g. involving prices and commodities.

S.b) (Constants) There must be names for numbers 0 and 1. (Agents can mention the absence of something, as well as a single unit of something.) Sometimes they can mention other magnitudes. If they live, for example, in a world of standard real numbers, they may have a name for the number π. If their world includes nonstandard reals, then they may have a name for some infinitely large number, or for some infinitesimal. Thus, depending on the environment they live in, and on their language abilities, we may allow their language a set $\{c_k : k \in K\}$ of other constant symbols.

S.c) (Comparability relations) There must be names for the equality $=$ and greater-than $>$. (Agents can compare relevant magnitudes (profits, utilities, expenditures, incomes, etc.) in making their decisions.)

S.d) (Calculation functions) There must be names for the simple calculation operations of addition $+$ and multiplication $\cdot$.(4) (Agents can perform common economics calculations: calculating total or average expenditure or cost, etc.)

S.e) (Logical connectives) There must be names for the logical connectives of negation $\neg$ and conjunction $\wedge$ that agents commonly use in economics analyses.(5)

(4) The usual calculation operations of subtraction $-$, and division $/$ can be obtained through $+$ and $\cdot$.

(5) The usual logical operation of conjunction $\wedge$ and implication $\rightarrow$ can be obtained through the operations $\neg$ and $\vee$.

S.f) (Quantifiers) There must be the universal quantifiers $\forall x$.[6] (Such quantifiers are required, for example, when agents think about profit-maximizing outputs, utility-maximizing bundles, etc.)

We assume that our agents can use only those magnitudes and operations that they can define in a language built up from those basic ingredients as formulas in a first order predicate calculus. Such a construction is standard, and the Appendix provides an outline. See (Enderton, 1972), (Chang and Keisler, 1990), and other texts for details.

We are focusing, then, on a language we can denote by $\mathcal{L}_0 = (+, \cdot, >, 0, 1)$.[7]

B) Languages with additional constants.

In the language $\mathcal{L}_0$ an agent can talk of the integers 0, 1, $1+1$, -1, $\sqrt{2}$, etc. But if she lives in a classical world of real numbers, there are many she cannot talk of, that may be important to her. For example, π, e, $2^{\sqrt{2}}$, and infinitely many others cannot be mentioned in $\mathcal{L}_0$.

So we might consider supplementing $\mathcal{L}_0$ with a constant symbol c_r for certain real numbers r. Thus the language $\mathcal{L}_\pi = (+, \cdot, >, 0, 1, c_\pi)$ has a new constant, c_π. The intuition is that agents can refer to the number π, in addition to all the numbers they can define using $+$, $\cdot$, $>$, 0, 1, and the logical connectives and quantifiers.

More generally, we might consider supplementing the language $\mathcal{L}_0$ with a constant symbol c_k for every k in some set K. We denote the language by $(\mathcal{L}_0, c_k)_{k \in K}$, or $\mathcal{L}_K = (+, \cdot, >, 0, 1, c_k)_{k \in K}$. If K is empty, then $\mathcal{L}_K$ is the same as $\mathcal{L}_0$.

C) Languages with additional functions.

Again, if agents live in a classical world of real numbers, they may want to talk about certain functions. For example, the language $\mathcal{L}_{\exp} = (+, \cdot, > 0, 1, \exp)$ allows agents to talk about the exponential function. More generally, language $\mathcal{L}_J = (+, \cdot, >, 0, 1, f_j)_{j \in J} = (\mathcal{L}_0, f_j)_{j \in J}$ allows them to talk about the functions f_j; and $\mathcal{L}_{JK} = (+, \cdot, >, 0, 1, f_j, c_s)_{j \in J, k \in K} = (\mathcal{L}_0, f_j, c_k)_{j \in J, k \in K}$ permits them to talk about the functions f_j and the constants c_k.

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[6] The usual existential quantifiers $\exists x$ can be defined through $\forall x$ and $\neg$.

[7] This is the first order predicate language of ordered rings.

First order languages. With bounded rationality and "usability" as our guide, we restrict attention to languages of the simple types $\mathcal{L}_0$, $\mathcal{L}_K$, $\mathcal{L}_J$, and $\mathcal{L}_{JK}$ described above. All these languages are "simple" by virtue of being first order languages. In particular we do not consider second order languages (allowing quantification over relation or function symbols) or other more complex languages.

Having described the (bounded) richness of the language our agents can use, we now describe the structures, or "worlds" in which they live, and about which they talk. First, we specify axioms ensuring properties similar to those of classical real structures. Among all the structures satisfying the axioms, however, will be many that are *nonstandard* — i.e., not the classical real structures of conventional economic theory. Indeed, the weaker the language and axioms, the larger will be the class of nonstandard structures. In our second step, we will limit our attention those that are definable — i.e., for which all elements are definable. This definability requirement will satisfy our uniformity principle, but the structures will generally not be the standard ones.

III.2 Appropriate Structures: axioms for nonstandard structures. Our uniformity principle requires that all "usable" elements and quantities of our economies — consumption bundles, endowment vectors, prices, etc. — be definable. This will force use to consider some unusual structures.[8] Although we have to consider nonclassical structures and quantities, our economic intuition expects them to retain many properties of classical models: $x + y = y + x$, $x \cdot y = y \cdot x$, etc. We expect them to be "close" to the classical structures, even though they differ. To describe "closeness," we will specify axioms that the nonclassical structures must obey. The family of structures satisfying these axioms will include the standard classical ones, but many others as well. We will use the inclusive term *nonstandard* to refer to all of them.

Although definability will force us to work with nonstandard structures, our intuition still expects them to be "close" to classical structures. In this subsection we will use simple axioms to describe how close. In III.3 we will describe our definability requirement.

Why not rational numbers? At the very beginning, it is natural to ask whether we could restrict attention to a structure containing just the rational numbers. A superficial answer would say "no," because very simple consumer

[8] For example, if the language $\mathcal{L}$ has only countably many terms, then only countably many elements are definable. Then we cannot use the classical models based on the standard continuum.

and producer problems would not be solvable. For example, if a consumer with preference $\succcurlyeq$ given by the utility function

$$u(x_1, x_2) = x_1^3 + x_2^3 \tag{1}$$

faces income $m = 1$ and prices $p_1 = 1$ and $p_2 = 2$, then the demand for x_1 is the irrational $2^{2/3}/(2^{2/3}+1)$.[9] At a deeper level, of course, that is an unsatisfactory answer. When the real world does not contain solutions in the mathematical model we have chosen, that does not mean we should adjust our model simply for mathematical convenience. Our task, after all, is to construct a model of the real world, not to construct a world for our model.

A better answer is that in the real world people can and do talk about and work with some numbers that are not rational. If, for example, I am purchasing materials to construct a walking path in the shape of a right triangle with legs of rational length, then I must purchase material for the hypotenuse of irrational length.

So we must work with more appropriate nonstandard structures. We will describe them through axioms. While the language $\mathcal{L}$ itself determines the form of its structures, the axioms will determine the finer properties — whether a root exists for the equation $x^2 = 1 + 1$, etc.

The "appropriate" nonstandard structures will be closely related to classical real number structures $\mathcal{R} = (\mathbb{R}, +, \cdot, >, 0, 1, \ldots,)$. After all, that is where our basic notions of addition, multiplications, and even numbers, come from. But if the "appropriate" structures are to be definable in a reasonable (i.e., countable) language, then they can't be too close to the (uncountable) real structures. So we will start with real-like structures, and describe "appropriate" structures that diverge in ways that allow definability as well as some more generality. As already noted, these structures, the worlds in which agents live, will be closely tied to their languages — if infinitesimals exist in their world, for example, agents may have names for some of them.

We consider three types of structures, appropriate to the three types of language complexity that agents can use.

A) Basic structures: real closed fields.

We wanted a language in which agents could discuss 0 and 1, addition and multiplication, and comparison by $>$. For the same reasons, we want them to have the corresponding objects, operations, and relations available in their world.

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(9) Cf. Varian (1992), p. 112.

So we describe that world by a structure $\mathcal{A} = (A, +^{\mathcal{A}}, \cdot^{\mathcal{A}}, >^{\mathcal{A}}, 0^{\mathcal{A}}, 1^{\mathcal{A}})$. Here the *universe* A is some nonempty set, and $+^{\mathcal{A}}$ and $\cdot^{\mathcal{A}}$ are each binary operations on A; and $>^{\mathcal{A}}$ is a binary relation on A; and $0^{\mathcal{A}}$ and $1^{\mathcal{A}}$ are specific elements (*constants*) of A.[10]

We assume these operations, relations, and constants, $+^{\mathcal{A}}$, $\cdot^{\mathcal{A}}$, $>^{\mathcal{A}}$, $0^{\mathcal{A}}$, and $1^{\mathcal{A}}$ obey the usual laws of arithmetic; in particular, we assume that $\mathcal{A}$ is an ordered field.[11]

Our agents will want to solve some simple equations: For every $a, b \in A$, there should be a $c \in A$ such that $a = b \cdot c$. And every positive "number" a in A should have a square root, an element b of A such that $b \cdot b = a$. Of course we don't expect every polynomial equation to be solvable — certainly not $a \cdot a = -1$. However, in order that our agents can perform calculations that arise naturally in economic activities, polynomials equations of *odd* degree should have solutions:

$$a_0 x^{2k+1} + a_1 x^{2k} + \cdots + a_{2k+1} x + a_{2k+2} = 0. \tag{2}$$

Putting all these requirements together, then, "appropriate" structures for $\mathcal{L}_0$ are the *real closed ordered fields* $\mathcal{A} = (A, +, \cdot, >, 0, 1)$. [12] We may abbreviate this to "real closed field," and denote its axioms by T_{rcf}.

From algebra, we know there are many different real closed fields. The system of algebraic numbers is the base for a denumerable example — a least example. The standard reals $\mathcal{R} = (\mathbb{R}, +, \cdot, >, 0, 1)$ is the prototype of them all — an uncountable example. And there are infinitely many examples "in between" these two. Furthermore, there are nonstandard models of the reals, providing many more examples, of arbitrarily large cardinalities.

B) Real closed fields with additional constants.

Our second level of complexity allowed agents a richer language, with names for particular magnitudes. The notion of "appropriate" structure becomes more complex.

The language $\mathcal{L}_\pi = (+, \cdot, >, 0, 1, c_\pi)$ had a new constant, c_π, so agents could refer to the number π. So "appropriate" structures will now be of the form $\mathcal{A}_\pi = (A, +, \cdot, >, 0, 1, \pi)$, where the underlying reduct $\mathcal{A}_\pi \restriction \mathcal{L}_0 = (A, +, \cdot, >, 0, 1)$ is a

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[10] **For simplicity, we usually employ the same symbols $+$, $\cdot$, $>$, and 1 for both the symbols in language $\mathcal{L}_0$ and there interpretations in $\mathcal{L}_0$-structure $\mathcal{A}$, writing $\mathcal{A} = (A, +, \cdot, >, 0, 1)$.**

[11] **Cf. the Appendix.**

[12] **See the Appendix.**

real closed field, as again dictated by Section III.2.

But "appropriate" means even more than that. When agents talk about π in a structure other than the classical structure $\mathcal{R}$, what properties of "π-ness' should the object interpreted as π have? An "appropriate" structure will be one in which the interpretation of π has all the properties agents can describe in their language $\mathcal{L}_\pi$, and which π has in the prototype model, $\mathcal{R}$.

The real closed field axioms T_0, for example, do not guarantee that $22/7 > \pi$. So for appropriateness we want to adjoin the axiom:

$$\underbrace{1+\cdots+1}_{22 \text{ terms}} > (\underbrace{1+\cdots+1}_{7 \text{ terms}})\cdot\pi, \tag{3}$$

to the axioms T_0. And even that is not enough. To capture the full intuition of π-ness, we take as axioms all the sentences of the language $\mathcal{L}_\pi$ that are true in the prototype classical structure $\mathcal{R}_\pi = (\mathbb{R}, +, \cdot, >, 0, 1, \pi)$. Thus we take as axioms for an "appropriate" structure the set $\mathrm{Th}(\mathcal{R}_\pi)$ of sentences. (As the set of all $\mathcal{L}_\pi$-sentences true in $\mathcal{R}_\pi$, this includes all the real closed field axioms.)

In place of $\mathcal{L}_\pi$, another example is the language $\mathcal{L}_\eta = (+, \cdot, > 0, 1, c_\eta)$, where the constant symbol c_η is interpreted as an infinitesimal nonstandard real in a prototype structure $\mathcal{R}^\star = (\mathbb{R}^\star, +, \cdot, >, 0, 1, \eta)$, with infinitesimal η. Then the "appropriate" structures $\mathcal{B}_\eta$ will be those that satisfy all sentence of $\mathrm{Th}(\mathcal{R}\star)$, i.e. $\mathcal{B}_\eta \vDash \mathrm{Th}(\mathcal{R}^\star)$. They include, for example, all the sentences:

$$c_\pi > \underbrace{1+\cdots+1}_{n \text{terms}}, \tag{4}$$

for each integer n, as well as the axioms for real closed fields.

More generally, when the agents' language is $\mathcal{L}_K = (+, \cdot, >, 0, 1, c_k)_{k\in K}$, where $\{c_k : k \in K\}$ is some set of individual constant symbols, these constants derive their interest from some prototype structure. In the case of $\mathcal{L}_\pi$, that prototype was the real structure $\mathcal{R}_\pi$. In general, we assume the prototype is some real closed field $\mathcal{A}_K = (A, +, \cdot, >, 0, 1, a_k)_{k\in K}$ with constants. Then the "appropriate" structures $\mathcal{B}_K = (B, +, \cdot, >, 0, 1, a_k)_{k\in K}$ are those in which the interpretations $c_k^{\mathcal{B}_K}$ of the constants obey the same $\mathcal{L}_K$-axioms as in the prototype: $\mathcal{B}_K \vDash \mathrm{Th}(\mathcal{A}_K)$.

C) Real closed fields with additional constants and functions.

Our third level of complexity allowed agents a still richer language, with names for certain functions. For example, the language $\mathcal{L}_{\exp} = (+, \cdot, > 0, 1, \exp)$ has a name for the exponential function. As with constants, now the notion of "appropriate" structure must capture the properties of the exponential function in some prototype structure.

A natural prototype is the real exponential field $\mathcal{R}_{\exp} = (\mathbb{R}, +, \cdot, >, 0, 1, \exp)$. Let $T_{\exp}$ be the set of all $\mathcal{L}_{\exp}$ sentences that are true in $\mathcal{R}_{\exp}$: $T_{\exp} = \mathrm{Th}(\mathcal{R}_{\exp})$. Of course that will include the sentences:

$$\begin{aligned} &\forall v_1 \forall v_2 (\exp(v_1) \cdot \exp(v_2) = \exp(v_1 + v_2)) \\ &\exp(0) = 1 \\ &\forall v_1 \forall v_2 (v_1 < v_2 \rightarrow \exp(v_1) > \exp(v_2)), \end{aligned} \tag{5}$$

and many others as well.[13]

More generally, the agents' language is $\mathcal{L}_{JK} = (+, \cdot, >, 0, 1, f_j, c_s)_{j \in J, k \in K}$, with additional function and constant symbols. We assume there is a prototype classical structure $\mathcal{R}_{JK} = (\mathbb{R}, +, \cdot, >, 0, 1, f_j, c_k)_{j \in J, k \in K}$, the standard real field with additional functions and constants. The "appropriate" nonstandard[14] structures $\mathcal{A}_{JK} = (A, +, \cdot, >, 0, 1, f_j, a_k)_{j \in J, k \in K}$ will be those for which $\mathcal{A}_{JK} \models \mathrm{Th}(\mathcal{R}_{JK})$, so by completeness[15] of $\mathrm{Th}(\mathcal{R}_{JK})$ the structures $\mathcal{A}_{JK}$ and $\mathcal{R}_{JK}$ satisfy the same $\mathcal{L}_{JK}$-sentences, i.e., $\mathcal{A}_{JK} \equiv \mathcal{R}_{JK}$.

III.3 Definability. The philosophy of Section II suggests that agents have definable endowments and definable preferences; indeed, they live in environments containing only definable elements. To describe this, we bring together languages and structures.

DEFINITION 1. We say that a formula $\Phi(v)$ of $\mathcal{L}$ *defines* a set X in an $\mathcal{L}$-structure $\mathcal{A} = (A, \dots)$, in the language $\mathcal{L}_K$ if

$$X = \{a \in A : \mathcal{A}_K \models \Phi(\frac{a}{v})\}, \tag{6}$$

where $\mathcal{A} \models \Phi(a/v)$ means that the element $a \in A$ satisfies the formula $\Phi(v)$ in the structure $\mathcal{A}$.[16] If the set X is a singleton $\{a\}$, then we say that the formula $\Phi(v)$ *defines* the element a in $\mathcal{A}$. Of course, the universe A of $\mathcal{A}$ is always definable (by the formula $1 = 1$, for example).

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(13) Cf. Dahn and Wolter (1983).

(14) Our terminology includes the classical structures as special cases of the nonstandard ones.

(15) See the Appendix.

(16) Cf. the Semantics section of the Appendix.

In a similar way we have the notion of a *definable* binary relation R in a structure $\mathcal{A}$:

$$R = \{(a,b) \in A \times A : \mathcal{A} \models \Phi(\frac{a}{v_1}, \frac{b}{v_2})\}, \tag{7}$$

and similarly for n-ary relations, for any n, or for any subsets of A^n. We say that a function is *definable* if its graph is a definable relation.

In particular, in any structure $\mathcal{A} = (A, \dots)$, a binary relation $\succcurlyeq$ on a definable set $X \subseteqq A^l$ will be $\mathcal{L}$-definable if there is some formula $\Phi(v_1, \dots, v_{2l})$ such that:

$$x \succcurlyeq y \iff \mathcal{A} \models \Phi(\frac{x_1}{v_1}, \dots, \frac{x_l}{v_l}, \frac{y_1}{v_{l+1}}, \dots, \frac{y_l}{v_{2l}}) \tag{8}$$

for all $x = (x_1, \dots, x_l)$ and $y = (y_1, \dots, y_l)$ in X.

We are especially interested in environments where all elements are definable.

DEFINITION 2. For any $\mathcal{L}$-structure $\mathcal{A}$, we let $\delta(\mathcal{A})$ be the restriction[17] of $\mathcal{A}$ to its $\mathcal{L}$-definable elements. We say that an $\mathcal{L}$-structure $\mathcal{A} = (A, \dots)$ is a *definable structure* for the language $\mathcal{L}$ if all elements of A are $\mathcal{L}$-definable, i.e., if $\delta(\mathcal{A}) = \mathcal{A}$.

REMARK 1. a) In the language $\mathcal{L}_0$, there are only countably many definable elements in any structure $\mathcal{A} = (A, +, \cdot, >, 0, 1)$, since there are only countably many formulas in $\mathcal{L}_0$. The same is true if we expand $\mathcal{L}_0$ by constants to $\mathcal{L}_K$ and correspondingly expand $\mathcal{A}$ to $\mathcal{A}_K$, where K is a countable set. By concentrating on such definable magnitudes, then, we can avoid requiring that agents have unrealistically many names in their language.

b) For any real closed ordered field $\mathcal{A} = (A, +, \cdot, >, 0, 1)$ and for the language $\mathcal{L}_0$, the set A_d of definable elements is (up to isomorphism) the set $\mathbb{R}_d$ of real algebraic numbers.[18] Thus the ordered field $\mathcal{R}_d = (\mathbb{R}_d, +, \cdot, >, 0, 1)$ of algebraic real numbers is the smallest real closed field, being contained in every real closed field (up to isomorphism).

c) More generally, if $\mathcal{A}_K = (A, +, \cdot, >, 0, 1, a_k)_{k \in K}$ is any real closed ordered field with constants, and if A_d is the set of elements of A definable in $\mathcal{L}_K$, then

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(17) The Appendix shows that a unique such restriction exists.

(18) It is well known that the restriction $\mathcal{D}$ of $\mathcal{R}$ to the set of algebraic reals is a real closed ordered field, so by Tarski's transfer principle, Proposition 13 in the Appendix, $\mathcal{D} \equiv \mathcal{R}$. Furthermore, all algebraic reals are clearly $\mathcal{L}_0$-definable elements of $\mathcal{D}$, so $\mathcal{D}$ is a definable structure ($\delta(\mathcal{D}) = \mathcal{D}$). Then it follows from Proposition 5(c) in the Appendix that $\mathcal{D}$ is isomorphic to the set of definable reals, i.e., $\mathcal{D} \cong \delta(\mathcal{R})$.

the structure $\mathcal{A}_d = (A_d, +, \cdot, >, 0, 1, a_k)_{k \in K}$ obtained by restricting $\mathcal{A}$ to A_d is also a real closed ordered field. (See Proposition 4 in the Appendix.)

d) Since $I\!R_d$ is the set of real algebraic numbers, all $\mathcal{L}_0$-definable reals are computable reals, but the converse is false (cf. Richter and Wong (1996a, Section 3, Item ii)).

e) We have noted that the least upper bound property does not hold in most real closed ordered fields. However, every nonempty $\mathcal{L}_0$-*definable* set does have the least upper bound property. (This follows easily from Proposition 13 below.)

f) Many of the functions used in applied economics are $\mathcal{L}_0$-definable, $\max\{x, y\}$, $x \cdot y$, $\sqrt{x}$, polynominals with definable coefficients, etc.

g) The nature of $\mathcal{L}_0$-definable sets, relations, and functions is easily described. Due to a basic result of Tarski (1951) (Proposition 14 in the Appendix), we know that the definable sets (including singletons, relations, and functions) are the semialgebraic sets, i.e. those defined by finite unions of finite intersections of sets where polynomials are either positive or zero. See Proposition 14 below.

h) Although $\mathcal{L}_0$ and $\mathcal{L}_K$ allow us to talk about any specific integers 0, 1, $1 + 1$, -1, etc., they do not allow us to talk about the *set* of all integers in the (ordered field) structures $\mathcal{A}$ and $\mathcal{A}_K$ that we use for representing economic environments. To express "if x is an integer, ..." would require a more powerful language. Similarly, in real closed ordered fields, $\mathcal{L}_0$ and $\mathcal{L}_K$ do not allow us to talk about the *set* of all rational numbers, although they do allow us to talk about any specific ones.

III.4 Definable Appropriate Structures. Now we bring together the two aspects of "appropriate" structures — axioms and definability.

To keep close to classical systems, the appropriate structures (Section IV.2) to start with are $\mathcal{L}$-structures $\mathcal{R} = (I\!R, +, \cdot, >, 0, 1, \ldots,)$ that are standard real field expansions. To allow more freedom, we will consider "nonstandard" versions, i.e., $\mathcal{L}$-structures $\mathcal{A} = (A, +, \cdot, >, 0, 1, \ldots)$ with the same first order properties, or axioms: $\mathcal{A} \equiv \mathcal{R}$. To allow still more freedom, we will consider expansions $\mathcal{A}' = (\mathcal{A}, a_k)_{k \in K}$, with corresponding language $\mathcal{L}' = (\mathcal{L}, c_k)_{k \in K}$. We will call any such structure $\mathcal{A}'$ an *expansion by constants of a nonstandard real model*.

Next, to satisfy Section III.3's definability prescription, we seek $\mathcal{L}'$-structures $\mathcal{D}$ that *definably model* $\mathcal{A}'$:

i) $\mathcal{D} \equiv \mathcal{A}'$, so $\mathcal{D}$ is "close" to $\mathcal{A}'$ (which in turn is an expansion by constants of something close to $\mathcal{R}$);

ii) $\mathcal{D}$ is definable.

These are the structures in which we want to model economics.

It is not obvious that such structures $\mathcal{D}$ exist — nonstandard models simultaneously "close" to $\mathcal{R}$ and definable; the answer will depend on the initial structure $\mathcal{R}$, and we postpone our results on that issue to Section VI. Now we describe economics in nonstandard structures, assuming they exist.

III.5 Nonstandard and Definable Agents and Economies. We start with an $\mathcal{L}$-structure $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$ that is an expansion of a real closed ordered field. This is our nonstandard structure.

We formalize a *consumer* as a quadruple $(\mathcal{A}, X, \succcurlyeq, \omega)$, where $X \subseteqq A^l$ is the set of *feasible consumption* bundles, and $\succcurlyeq$ is a preference relation on X, and $\omega \in X$ is an *endowment* vector. By a *preference relation* we mean a reflexive, transitive, and total binary relation. We represent by $\succ$ the asymmetric part of $\succcurlyeq$.

DEFINITION 3. Let $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$ be an ordered field expansion, and let $\succcurlyeq$ be a preference relation on a set $X \subseteqq A^l$.

a) $\succcurlyeq$ is *upper semicontinuous* if: for all $x, y \in X$ with $x \succ y$, there exists a positive $\epsilon \in A$ such that $x \succ y'$ for all $y' \in X$ with $\|y' - y\| < \epsilon$.

b) $\succcurlyeq$ is *lower semicontinuous* if: for all $x, y \in X$ with $x \succ y$, there exists a positive $\epsilon \in A$ such that $x' \succ y$ for all $x' \in X$ with $\|x' - x\| < \epsilon$.

c) $\succcurlyeq$ is *continuous* if it is both upper and lower semicontinuous.

d) $\succcurlyeq$ is *locally insatiable* if: for all $x \in X$ and all positive $\epsilon \in A$, there exists a $y \in X$ with $\|y - x\| < \epsilon$ and $y \succ x$.

e) $\succcurlyeq$ is *(weakly) monotone* if: for all $x, y \in X$ with $x \gg y$, one has $x \succ y$.

f) $\succcurlyeq$ is *strictly monotone* if: for all $x, y \in X$ with $x \geqq y$ and $x \neq y$, one has $x \succ y$.

g) $\succcurlyeq$ is *convex* if: for all $x, y, z \in X$ with $x \succcurlyeq z$ and $y \succcurlyeq z$, one has $tx + (1-t)y \succcurlyeq z$ for all $t \in A$ with $0 \leqq t \leqq 1$ and $tx + (1-t)y \in X$.

h) $\succcurlyeq$ is *strictly convex* if: for all $x, y, z \in X$ with $x \succcurlyeq z$ and $y \succcurlyeq z$ and $x \neq y$, one has $tx + (1-t)y \succ z$ for all $t \in A$ with $0 < t < 1$ and $tx + (1-t)y \in X$.

For realism, our philosophy (Section II.1) requires that, X, $\succcurlyeq$, and ω be definable.

DEFINITION 4. Let $\mathcal{L} = (+, \cdot, >, 0, 1, \dots)$, and let the $\mathcal{L}$-structure $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$ be a real closed ordered field expansion. Then a consumer $(\mathcal{A}, X, \succcurlyeq, \omega)$ is a *weakly definable consumer* if the set X, the relation $\succcurlyeq$, and the vector ω are all definable in the structure $\mathcal{A}$ in the language $\mathcal{L}$.

REMARK 2. a) If the set X and the preference relation $\succcurlyeq$ are $\mathcal{L}$-definable for any such language $\mathcal{L}$, then all the properties (a)–(h) in Definition 3 are $\mathcal{L}$-definable (cf. Lemma 1).

b) Many preference relations are $\mathcal{L}_0$-definable. These include, for example, the weakly greater than relation $\geqq$, the lexicographic preference, the preferences corresponding to the utility functions $\min\{x, y\}$, polynomials $g(x)$ with definable real coefficients, the function in (1), etc.

c) $\mathcal{L}_0$-definable preferences need not be continuous, and they need not be "computable" (in the sense of (Richter and Wong, 1996a)). Nor do computable or continuous preferences need to be $\mathcal{L}_0$-definable.[19]

Weak definability of a consumer $(\mathcal{A}, X, \succcurlyeq, \omega)$ does not require that all elements of A be definable,[20] so it does not fully achieve our bounded rationality requirements for realism. The uniformity principle (Section I) mandates restricting magnitudes (the elements of A) in the same way that we restrict relations (the set X and the preference $\succcurlyeq$) and functions. When we assume further that the structure $\mathcal{A}$ contains only definable elements then we say that the consumer is a *definable consumer*. (Sometimes for emphasis we say *strongly definable consumer*.)

Putting consumers together yields an economy.

DEFINITION 5. An *economy* $E = ((\mathcal{A}, X_1, \succcurlyeq_1, \omega_1), \dots, (\mathcal{A}, X_m, \succcurlyeq_m, \omega_m))$ is a list of consumers. It is a *weakly definable economy* if each consumer $(\mathcal{A}, X_i, \succcurlyeq_i, \omega_i)$ is a weakly definable consumer. It is a *definable economy* if each consumer is

(19) However, continuity and definability together do imply computability. More precisely, consider a real closed ordered field $\mathcal{R}_c = (\boldsymbol{R}_c, +, \cdot, >, 0, 1, r_k)_{k \in K}$ of computable real numbers. Let $\succcurlyeq$ be a definable on $\boldsymbol{R}^l_{c+}$. Then the set $\mathcal{P} = \{(x, y) \in \boldsymbol{R}^l_{c+} \times \boldsymbol{R}^l_{c+} : x \succ y\}$ is a semialgebraic set in $\boldsymbol{R}^l_{c+} \times \boldsymbol{R}^l_{c+}$ (see Proposition 14). If $\succcurlyeq$ is continuous, then the semialgebraic set $\mathcal{P}$ is open, so $\mathcal{P} = \cup_{m=1}^{M} \cap_{k=1}^{K_m} \{z \in \boldsymbol{R}^l_{c+} \times \boldsymbol{R}^l_{c+} : P_{mk}(z) > 0\}$, where the $P_{mk}(z)$ are computable polynomials. Thus $\mathcal{P}$ is a listable set (in the sense of Moschovakis (1964)), so, equivalently, $\succ$ is computable.

(20) The real number π, for example, is not $\mathcal{L}_0$-definable in the classical model $\mathcal{R} = (\boldsymbol{R}, +, \cdot, >, 0, 1)$. In fact, since there are only countably many $\mathcal{L}_0$-formulas, most elements of $\boldsymbol{R}$ are not definable.

a definable consumer.[21] (Sometimes for emphasis we say *strongly definable economy.*)

IV Some Classical Theorems in Definable Worlds

Once we allow nonstandard environments, as we must,[22] it is no longer true that classical economic theorems will all be true. For example, in a countable structure $\mathcal{A}$, the system $(A, |\cdot|^{\mathcal{A}})$ will not be a complete complete metric space, so the usual economic theorems about existence of maximizing vectors, existence of competitive equilibrium, and so forth, will fail.

However, the same bounded rationality–definability considerations that forced us into nonstandard structures, also restricts our attention to definable sets (consumption sets) and relations (preference relations). In this section we show that this restriction re-established many of the classical conclusions, even in nonstandard environments.

In other words, if we simultaneously extend to nonstandard structures and restrict to definable sets and relations, the classical results remain.

IV.1 Weakly Definable Economies. Choosing from both individual microeconomics, and from general equilibrium theory, we illustrate the survival of several classical theorems under definability constraints. These theorems are also true, a fortiori, under strong definability, since then the hypotheses are special cases of the weak definability hypotheses.[23]

Weakly definable preference maximizers. We begin with a maximizer existence theorem for upper semicontinuous weakly definable preferences. The result ensures that the important notion of demand can be used in our nonstandard definability framework, even though the commodity space is not a complete metric space and the budgets are not compact.

In what follows, for any $(p, w) \in A^l \times A$, we define the budget set $B(p, w) = \{x \in A^l : x \in X \ \ \& \ \ p \cdot x \leqq w\}$. Since the consumption set X is definable, the set $B(p, w)$ is clearly definable.

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[21] In particular, then, $\mathcal{A}$ is a definable structure.

[22] Cf. footnote 8.

[23] I.e., the special case when the underlying structure is $\mathcal{L}$-definable.

And for any preference $\succcurlyeq$ on A^l, we say that x is a $\succcurlyeq$-*maximizer* over $B(p,w)$ if $x \in B(p,w)$ and $x \succcurlyeq y$ for all $y \in B(p,w)$. Again, this is a definable notion.

THEOREM 1 (EXISTENCE OF PREFERENCE MAXIMIZERS).
Let $\mathcal{A} = (A, +, \cdot, >, 0, 1, \ldots, a_k)_{k \in K}$ be an $\mathcal{L}$-structure that is an expansion by constants of a nonstandard real model.

Let X be a definable subset of A^l that is closed in A^l and bounded below, and let $\succcurlyeq$ be an upper semicontinuous definable preference on X. Suppose $(p,w) \in A^l_{++} \times A_+$ and $B(p,w) \neq \emptyset$. Then there exists a $\succcurlyeq$-maximizer over the budget set $B(p,w)$.

REMARK 3. Let $\mathcal{A} = (A, +, \cdot, >, 0, 1, \ldots)$ be a definable structure as in the theorem, and let X be the nonnegative orthant A^l_+. Suppose $\succcurlyeq$ is a definable preference on X. Then:

a) If $\succcurlyeq$ is upper semicontinuous and strictly convex, then it is clear from Theorem 1 that a unique $\succcurlyeq$-maximizer $h(p,w)$ exists on every budget $B(p,w)$ for $(p,w) \in A^l_{++} \times A_+$. It is also clear that $h(p,w)$ is a definable function of (p,w).

b) If $\succcurlyeq$ is also continuous, then h is continuous. (This will be justified in Section V.)

c) If $\succcurlyeq$ is locally insatiable, then $h(p,w)$ is exhaustive (i.e., $p \cdot h(p,w) = w$ for all $(p,w) \in A^l_{++} \times A_+$).

Testing weakly definable rationality. We ask whether weak definability imposes any restrictions on the observed data of rational consumer behavior, beyond the classical Strong Axiom requirement (cf. Matzkin and Richter (1991)).

Consider an agent whose definable preference $\succcurlyeq$ on A^l_+ is continuous, monotone, and strictly convex. Then for any definable set $Y \subseteq A^l_{++} \times A_+$, it follows from Remark 3 that the agent has a definable, continuous, and exhaustive demand function h on Y. Certainly h also satisfies the Strong Axiom of Revealed Preference (cf. Richter (1966)): the binary relation W on A^l_+ is acyclic (i.e. $xWyW \cdots WzWx$ is impossible), where xWy means $x \neq y$ & $x = h(p,w)$ & $y \in B(p,w)$ for some $(p,w) \in Y$. Thus definability, exhaustiveness, and the Strong Axiom property are necessary for h to be a demand function of such and agent. Is there any other requirement? Or is the converse true?

If X is finite, nothing else is required. (The general converse remains open.) This result can be obtained by carrying over the result of Matzkin and Richter (1991, Theorem 1) to our definable context. This is possible, since their argument holds in any ordered field (see Richter and Wong (1996b, Proof of Theorem 4, and Footnote 10)).

THEOREM 2 CHARACTERIZATION OF DEFINABLE RATIONALITY (CF. MATZKIN AND RICHTER (1991)). *Let $\mathcal{A}$ be an $\mathcal{L}$-structure that is an expansion by constants of a nonstandard real model.*

Let $Y \subseteqq A^l_+ \times A_+$ be definable and finite. If a definable function $h : Y \to A^l_+$ is exhaustive and satisfies the Strong Axiom of Revealed Preference, then there is an $\mathcal{L}$-definable preference $\succcurlyeq$ on A^l_+ such that:

i) $h(p, w) = \{x \in B(p, w) : \forall y_{y \in B(p,w)} x \succcurlyeq y\}$ *for all* $(p, w) \in Y$,

ii) $\succcurlyeq$ *has an $\mathcal{L}$-definable, strictly monotone, and strictly concave utility representation* $U : A^l_+ \to A$.

General equilibrium. To study general equilibrium theory with definable agents, we again work with a real closed ordered field extension $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$, and a corresponding first order predicate language $\mathcal{L}$. Assuming there are l commodities, the commodity space is A^l, and the price space is A^l_+. For simplicity we ignore production activities, focusing on weakly definable economies in the sense of Definition 5.

We now state the usual general equilibrium definitions. Given an economy $E = \{(\mathcal{A}, X_i, \succcurlyeq_i, \omega_i)\}_{i=1}^m$, a *allocation* is an m-tuple $(x_1, \dots, x_m)$ where each $x_i \in X_i$. The allocation $(x_1, \dots, x_m)$ is *feasible* if $\sum_{i=1}^m x_i \leqq \sum_{i=1}^m \omega_i$. An allocation is *Pareto optimal* if it is feasible and if there does not exist any feasible allocation $(y_i, \dots, y_m)$ such that:

P.1) $y_i \succcurlyeq_i x_i$ for all i, and

P.2) $y_i \succ_i x_i$ for some i.

An *equilibrium* for E is a tuple $(\bar{p}, (\bar{x}_1, \dots, \bar{x}_m))$ consisting of a price vector $\bar{p} \in A^l_+$ and an allocation $(\bar{x}_1, \dots, \bar{x}_m) \in X_1 \times \cdots \times X_m$, such that:

E.1) $\bar{x}_i$ is a $\succcurlyeq_i$-maximizer on $B(\bar{p}, \bar{p} \cdot \omega_i)$ for all $i = 1, \cdots, m$,

E.2) $\sum_{i=1}^m \bar{x}_i \leqq \sum_{i=1}^m \omega_i$ and $\bar{p} \cdot (\sum_{i=1}^m \omega_i - \sum_{i=1}^l \bar{x}_i) = 0$.

For simplicity of exposition, we will focus the following discussion on the special case where every consumption set X_i is the nonnegative orthant A^l_+.

First and Second Welfare Theorems. The classical First Welfare Theorem asserts the efficiency of equilibrium for economies. This assertion is also true in weakly definable economies.

Theorem 3 (First Welfare Theorem).
Let $\mathcal{A}$ be an $\mathcal{L}$-structure that is an expansion by constants of a nonstandard real model.

Let $E = \{(\mathcal{A}, A^l_+, \succcurlyeq_i, \omega_i)\}_{i=1}^m$ be a weakly definable economy, and $(\bar{p}, (\bar{x}_1, \ldots, \bar{x}_m))$ be an equilibrium for E. If each $\succcurlyeq_i$ is locally insatiable, then the equilibrium allocation $(\bar{x}_1, \ldots, \bar{x}_m)$ is Pareto optimal.

The next theorem asserts the sustainability of Pareto optima by a price system.

Theorem 4 (Second Welfare Theorem).
Let $\mathcal{A}$ be an $\mathcal{L}$-structure that is an expansion by constants of a nonstandard real model.

Let $E = \{(\mathcal{A}, A^l_+, \succcurlyeq_i, \omega_i)\}_{i=1}^m$ be a weakly definable economy, and let $(\bar{x}_1, \ldots, \bar{x}_m)$ be a Pareto optimal definable allocation. Assume each definable preference $\succcurlyeq_i$ is continuous, convex and (weakly) monotone, and $\bar{x}_i \gg 0$. Then there exists a price $\bar{p}$ and a vector $(\bar{\omega}_i)_{i=1}^m$ of endowments with $\sum_{i=1}^m \bar{\omega}_i \leqq \sum_{i=1}^m \omega_i$, and such that $(\bar{p}, (\bar{x}_1, \ldots, \bar{x}_m))$ is an equilibrium for the economy $\bar{E} = \{(\mathcal{A}, A^l_+, \succcurlyeq_i, \bar{\omega}_i)\}_{i=1}^m$.

Existence of equilibrium. The next theorem provides sufficient conditions for definable economies to have definable equilibria.

Theorem 5 (Existence of Equilibrium).
Let $\mathcal{A}$ be an $\mathcal{L}$-structure that is an expansion by constants of a nonstandard real model.

Let $E = \{(\mathcal{A}, A^l_+, \succcurlyeq_i, \omega_i)\}_{i=1}^m$ be a weakly definable economy. Assume each $\succcurlyeq_i$ is continuous, convex and (weakly) monotone, and $\omega_i \gg 0$. Then there exists an equilibrium for E.

This positive result for definability-bounded rationality contrasts with the nonexistence result (Richter and Wong, 1996a, Corollary 2) for computability-bounded rationality.

As in classical economies, counterexamples to Theorems 4 and 5 are available if we drop continuity, convexity, monotonicity, or strict positivity.

Characterization of definable aggregate excess demand functions. As we did for individual consumers, we seek to characterize rationality of aggregate demand functions. In particular, we provide a definability-bounded version of the Sonnenschein-Mantel-Debreu characterization (cf. Debreu (1974)).

Let $E = \{(\mathcal{A}, A^l_+, \succcurlyeq_i, \omega_i)\}_{i=1}^l$ be a weakly definable economy where

$$\text{for all } i = 1, \cdots, l, \text{ the preference } \succcurlyeq_i \text{ is continuous, strictly monotone and strictly convex, and the endowment } \omega_i \gg 0. \tag{9}$$

Let $S_A = \{p \in A^l : p \gg 0 \text{ \& } ||p|| = 1\}$. From Remark 3 we can define the *excess demand function* $f : S_A \to A^l$ of E, where

$$f(p) = \sum_{i=1}^{l} \{x \in B(p, p \cdot \omega_i) : \forall y_{y \in B(p,p\cdot\omega)} x \succcurlyeq_i y\} - \sum_{i=1}^{l} \omega_i, \tag{10}$$

and the function f is definable, continuous, and satisfies

$$\text{WL) (Walras' Law) } p \cdot f(p) = 0 \text{ for all } p \in S_A. \tag{11}$$

In the classical environment, the Sonnenschein-Mantel-Debreu Theorem (Debreu (1974, Theorem)) showed that the properties (WL) and continuity characterize aggregate demand functions on the ϵ-trimmed portion of the price sphere. We have the following definability version.

THEOREM 6 (CHARACTERIZATION OF EXCESS DEMAND FUNCTIONS) .
Let $\mathcal{A}$ be an $\mathcal{L}$-structure that is an expansion by constants of a nonstandard real model.

Let $0 < \epsilon \in A$, and $S_{A,\epsilon} = \{p \in S_A : \forall i_{\ i=1,\cdots,l}\ p_i \geqq \epsilon\}$. If a function $f : S_{A,\epsilon} \to A^l$ is definable, continuous and satisfies (WL), then $f = \tilde{f}|_{S_{A,\epsilon}}$ for some excess demand function $\tilde{f}$ of some weakly definable economy $E = \{(\mathcal{A}, A^l_+, \succcurlyeq_i, \omega_i)\}_{i=1}^l$ satisfying (9).

IV.2 Strong Definability. Theorems 1–6 do not meet our bounded rationality–definability goals. While the relations are definable, the underlying elements (whose vectors represent endowments, prices, and consumption bundles) need not be. Invoking the uniformity principle, we ask what happens when the environment is required to be a *definable* structure $\mathcal{A}$, i.e., when the agents and the economy are strongly definable. Of course the theorems remain true when we strengthen their hypotheses.

COROLLARY (THEOREMS 1′–6′). *Theorems 1–6 hold when $\mathcal{A}$ is a definable structure.*

While Theorems 1–6 are more general, the Corollary is more interesting from a bounded rationality point of view. If we call the strong definability versions Theorems 1′–6′, then Theorem 1′ asserts existence of a *definable* maximizer; theorem 3′ asserts existence of a *definable* price vector; Theorem 5′ asserts existence of *definable* equilibrium prices and commodities; etc.

Theorems 1′–6′ follow immediately from their weak counterparts, but they are only interesting if they are not vacuous — an issue mentioned above in Section III.4 and addressed below in Section VI.

V Proofs

As much as possible, our approach will be "conservative," making use of known economics theorems for classical economies. Because the known theorems are true in numerous other, non-classical structures, we can simply "transfer" many of them to the structures that interest us, in particular to definable structures.

Proofs by transfer. We must be careful about what sentences we use in our proofs by transfer. It would not be useful to require that *all* sentences transfer. If we considered sentences of a second order language (allowing existential and universal quantification over relation and function variables), such sentences would completely describe our structures up to isomorphism; then we could only transfer their truth to other classical structures.

So we seek transferrability only for a more limited class of sentences (i.e., for a weaker language). The smaller we make the class of sentences for which transfer is required, the larger will be the class of non-classical, nonstandard structures to which they transfer.

We cannot choose too small a class of sentences, however, or they won't be able to express the classical theorems, and the transfer approach will fail.

The balance is somewhat delicate. On the one hand, all the Theorems 1–6 are second order: "For all preferences ... .". On the other hand, second order languages prevent us, as noted above, from transferring sentences from classical to non-classical models.

At first it seems impossible for proofs by transfer to avoid using second order languages, the the theorems themselves are second order. Theorem 1, for example, says in effect, "for all definable subsets X of $A^l \cdots$." We can avoid the problem and stay with first order languages, however, by making use of the fact that, in each instance of Theorems 1–6, the set X and the relation $\succcurlyeq$ are, by hypothesis, definable by first order formulas, say by $\phi(v_{11},\ldots,v_{1X})$ and $\psi(v_{11},\ldots,v_{1W})$, respectively. So in each instance we can talk about X or $\succcurlyeq$ through statements involving $\phi(v_{11},\ldots,v_{1X})$ or $\psi(v_{11},\ldots,v_{1W})$. Our proofs by transfer will avoid using a second order language, then, by transferring the classical theorems instance by instance.

The languages we are talking about now — the sentences to be transferred, are the ones our proofs will use. Our earlier discussions of language (Section III.1) concerned the "usable" languages for economic agents. There is a relationship between them. While we want our proof language to be as weak as possible, (so we can transfer classical theorems to as large a class of structures as possible), our "instance by instance" proof approach requires expressibility of the "usable" preferences and sets in the proof language. Thus we will take as our proof language the agents' "usable" language itself.

A definability lemma. For our proofs by transfer, we must verify that, in each instance of each theorem, all elements, sets, and relations are definable in the agents' language $\mathcal{L}$. To make clear that this can be done, we break the exposition into simple parts, noting that certain fragments of those sentences — closedness of sets, maximization of relations over a set, etc. — can be expressed by first order sentences of $\mathcal{L}$. In fact they can be expressed in a uniform manner, in the parametric expressibility sense of the Appendix. We collect these observations in Lemma 1 below.

LEMMA 1 (PARAMETRICALLY EXPRESSIBLE NOTIONS).
Let $\mathcal{L} = (+, \cdot, >, 0, 1, \dots)$ be an expansion of the language $\mathcal{L}_0$ of ordered fields. Let $Z(u_1, \dots, u_n)$ be a parametric 1-set, let $Y(u_1, \dots, u_n)$ be a parametric k-set, and let $R(u_1, \dots, u_n)$ be a parametric binary relation on $Y(u_1, \dots, u_n)$. Then the following notions are parametrically expressible in $\mathcal{L}$:

a) *"$Y(u_1, \dots, u_n)$ is closed"*
b) *"$Y(u_1, \dots, u_n)$ is open"*
c) *"$Y(u_1, \dots, u_n)$ is empty"*
d) *"$Y(u_1, \dots, u_n)$ is nonempty"*
e) *"$R(u_1, \dots, u_n)$ is reflexive"*
f) *"$R(u_1, \dots, u_n)$ is transitive"*
g) *"$R(u_1, \dots, u_n)$ is total"*
h) *"$R(u_1, \dots, u_n)$ is asymmetric"*
i) *"$R(u_1, \dots, u_n)$ is negatively transitive"*
j) *"$x(u_1, \dots, u_n)$ is an $R(u_1, \dots, u_n)$-greatest element of $Y(u_1, \dots, u_n)$"*
k) *"$(v_1, \dots, v_k)$ is a member of $B(z_1, \dots, z_k, z_0)$"*
l) *"v_1 is the supremum of $Z(u_1, \dots, u_n)$".*

PROOF SKETCH. To verify (a), let $\phi(v_1,\ldots,v_k,u_1,\ldots,u_n)$ define the parametric n-set $Y(u_1,\ldots,u_n)$. Let the formula $\psi(u_1,\ldots,u_n)$ be:

$$\begin{aligned}&\forall v_1\cdots\forall v_k(\neg\phi(v_1,\ldots,v_k,u_1,\ldots,u_n)\rightarrow\\ &\qquad\exists\epsilon(\epsilon>0\wedge\forall z_1\cdots\forall z_k(\text{``}\sqrt{((z_1-v_1)^{\frac{1}{2}}+\cdots+(z_k-v_k)^{\frac{1}{2}})}\text{''}<\epsilon \\ &\qquad\qquad\rightarrow\neg\phi(z_1,\ldots,z_k,u_1,\ldots,u_n)))],\end{aligned} \tag{12}$$

where the quoted phrase is replaced by an obvious proper definition in $\mathcal{L}$. The other expressibility claims are proved as easily. Q.E.D.

Proof of Theorem 1. Each instance of Theorem 1 is well known in the classical standard real context, and we take advantage of that; using elementary equivalence we can transfer it to the world $\mathcal{A}$, which by hypothesis is a nonstandard real model. In particular, it is enough to verify the claim for each instance of X and $\succcurlyeq$. We sketch the details.

For simplicity, we assume $X = A^l_+$. Since the preference $\succcurlyeq$ on A^l_+ is weakly $\mathcal{L}$-definable in $\mathcal{A}$, there is a formula $\phi(v_1,\cdots,v_{2l},c_1,\cdots,c_n)$ of $\mathcal{L}$ defining $\succcurlyeq$ in $\mathcal{A}$. So $\succcurlyeq$ has an interpretation $\succcurlyeq^{\mathcal{B}}$ in every model $\mathcal{B}$ of Th($\mathcal{A}$).

Replacing the constant symbols c_i by variable symbols u_i, the formula $\phi(v_1,\cdots,v_{2l},u_1,\cdots,u_n)$ of $\mathcal{L}$ yields a parametric relation $R(u_1,\cdots,u_n)$. So if $a_1,\ldots,a_n\in A$ are the interpretations in $\mathcal{A}$ of the constant symbols $c_1,\ldots,c_n$, then

$$\succcurlyeq \;=\; R^{\mathcal{A}}(\tilde{u}_1,\ldots,\tilde{u}_n), \tag{13}$$

i.e. $x\succcurlyeq y \;\Leftrightarrow\; (x,y)\in R^{\mathcal{A}}(\frac{a_1}{u_1},\ldots,\frac{a_n}{u_n})$ for all $x,y\in A^l_+$.

Using Lemma 1, Remark 2(a), and the formula ϕ of $\mathcal{L}$, we can construct a formula $\Gamma_1(u_1,\ldots,u_n)$ of $\mathcal{L}$ expressing the hypothesis of Theorem 1, i.e., that $R(u_1,\ldots,u_n)$ is reflexive, transitive, total, and upper semicontinuous: in every such structure $\mathcal{B}$, and for all $b_1,\ldots,b_n\in B$,

$$\begin{aligned}&\mathcal{B}\models\Gamma_1(\frac{b_1}{u_1},\ldots,\frac{b_n}{u_n})\\ &\qquad\text{if and only if}\\ &R^{\mathcal{B}}(\frac{b_1}{u_1},\ldots,\frac{b_n}{u_n}) \text{ is an upper semicontinuous preference on the set } B^l_+.\end{aligned} \tag{14}$$

Using Lemma 1, Remark 2(a), and the formula $\phi(v_1,\cdots,v_{2l},u_1,\ldots,u_n)$, we can also construct a formula $\Gamma_2(u_1,\ldots,u_n)$ of $\mathcal{L}$ expressing the conclusion of

Theorem 1, i.e., that there exists a $R^{\mathcal{B}}(\frac{b_1}{u_1}, \ldots, \frac{b_n}{u_n})$-maximizer over every budget $B(p, w)$:[24] for every such structure $\mathcal{B}$,

$$
\begin{aligned}
&\mathcal{B} \models \Gamma_2(\frac{b_1}{u_1}, \ldots, \frac{b_n}{u_n}) \\
&\qquad \text{if and only if} \\
&\text{for every positive } p \in B^l \text{ and every nonnegative } w \in B, \\
&\text{there exists a } R^{\mathcal{B}}(\frac{b_1}{u_1}, \ldots, \frac{b_n}{u_n})\text{-maximizer over } B(p, w).
\end{aligned}
\tag{15}
$$

Let Ξ be the sentence $\forall u_1 \cdots \forall u_n(\Gamma_1(u_1, \ldots, u_n) \to \Gamma_2(u_1, \ldots, u_n))$ of $\mathcal{L}$. Then the interpretation of Ξ in any structure $\mathcal{B} = (B, +, \cdot, >, 0, 1, f_j, a_k)_{j \in J, k \in K}$ is that if $R^{\mathcal{B}}$ is an upper semicontinuous preference on B^l_+, then for every $(p, w) \in B^l_{++} \times B_+$ there exists a $R^{\mathcal{B}}$-maximizer on $B(p, w)$. Since this is known to be true in any classical standard real model $\mathcal{R}$ of the language $\mathcal{L}$, by elementary equivalence it transfers to the nonstandard real model $\mathcal{A}$. ∎

Proof of Remark 3(b) and Theorems 3, 4, and 5. The results are well known in the classical context. Using Lemma 1, each instance of the remark and the theorems can be expressed in a first order language, as in our proof of Theorem 1. Then by elementary equivalence we transfer them to our nonstandard models $\mathcal{A}$. We omit the details. Q.E.D.

Instead of proving them by transfer, Theorems 3 and 5 could be proved directly by imitating in the nonstandard structure $\mathcal{A}$ the usual classical proofs. Indeed, that is how we will prove the remaining Theorems, 2 and 6, which are not easily expressed as first order statements, even for specific instances.

Proof of Theorem 6. We do not try to express Theorem 6 directly in a first order language. Instead, we follow Debreu (1974) in decomposing the given definable function f into l individual demand functions f^i. Debreu's decomposition uses simple operations such as orthogonal projection, multiplication, etc., which can be expressed in a first order manner, so the functions f^i are definable.

For each individual demand f^i, Debreu constructs a "continuous" family of "indifference" surfaces, yielding a preference rationalization $\succcurlyeq_i$ of f^i. Again, a careful examination of his construction verifies that it uses only simple operations

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[24] In other words, $\Gamma_2(u_1, \ldots, u_n)$ expresses:
$\forall(p, w)_{p \gg 0 \;\&\; w \geqq 0} \exists x_{x \in B(p,w)} \forall y_{y \in B(p,w)} x R(u_1, \ldots, u_n) y$.

such as taking infima, taking convex closures, etc., which can be expressed in a first order manner. So the preference rationalization $\succcurlyeq_i$ will be definable.

Proof of Theorem 2. We carry over the proof technique of Matzkin and Richter (1991, Theorem 1) to our weakly definable context.

Since Y is a finite definable set, all its elements $(p^1, w^1), \cdots, (p^k, w^k)$ in $A^l_{++} \times A_+$ are definable. Let the definable elements $x^i = h(p^i, w^i)$, where $h : Y \to A_+$ is weakly definable, exhaustive and satisfies the Strong Axiom. As noted in Richter and Wong (1996b, footnote 10), Lemma 1 in Matzkin and Richter (1991) can be carried over to any ordered field, hence to our weakly definable ordered field $(A, +, \cdot, >, 0, 1, \dots)$; so there exist numbers $\mu^1, \lambda^1, \cdots, \mu^k, \lambda^k \in A$ satisfying their finite system of linear inequalities (3.3(a-d)) with the weakly definable parameters $p^1, x^1, \cdots, p^k, x^k$. We pick such μ^i and λ^i. Then we pick any definable $T \in A$. It is easily verified that we can define $\bar{\varepsilon}$ to be the supremum of the set of positive $\varepsilon \in A$ satisfying their (4.15) and (4.19), and $\bar{\varepsilon}$ is finite and positive. Finally, we pick any definable positive $\varepsilon < \bar{\varepsilon}$ in A, and define a utility function $U : A^l_+ \to A$ by their (4.18). As in their proof (or alternatively by Tarski's transfer principle (cf. Proposition 13)) it is easily verified that the definable function U is continuous, strictly monotone, and strictly concave on A^l_+, and such that $h(p, w)$ uniquely maximizes U on $B(p, w)$.
Q.E.D.

VI From Weak Definability to Strong Definability

We now address the question of whether Theorems 1′–6′ are vacuous (cf. Sections III.4 and IV.2). To model our economies, we considered a language $\mathcal{L} = (+, \cdot, >, 0, 1, \dots)$ and a corresponding structure $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$. In order to be an appropriate structure for the economic analysis, Section III.2 required that $\mathcal{A}$ be an expansion of a real closed ordered field. Moreover, Section II mandated that the structure $\mathcal{A}$ should also be definable. Is it possible, then, to find a structure $\mathcal{D}$ that resembles the prototype classical field but is also definable? I.e., can we find a structure $\mathcal{D}$ such that

i) $\mathcal{D} \equiv \mathcal{A}$

ii) $\mathcal{D}$ is definable

are both true? In other words, is there a structure $\mathcal{D}$ definably modeling $\mathcal{A}$?

For our most basic language $\mathcal{L}_0 = (+, \cdot, >, 0, 1)$ (Section III.1(A)) and our basic real closed ordered field structures $\mathcal{A} = (A, +, \cdot, >, 0, 1)$ (Section III.2(A)) the answer is "yes." For then the ordered field of real algebraic numbers is definable (in fact, it is $\delta(\mathcal{A})$); and it is another real closed field, so Tarski's completeness theorem for real closed fields (Proposition 13 in the Appendix) guarantees that both fields satisfy the same first order sentences: $\delta(\mathcal{A}) \equiv \mathcal{A}$. This structure satisfies both our requirements — definability and closeness to the classical structure.

We now generalize that result to more complicated structures, allowing certain expansions by functions and constants. According to Proposition 5 in the Appendix, if there exists a structure $\mathcal{D}$ definably modeling $\mathcal{A}$, then we can take $\mathcal{D} = \delta(\mathcal{A})$ since then $\delta(\mathcal{A})$ is definable and $\delta(\mathcal{A}) \preccurlyeq \mathcal{A}$. Conversely, Proposition 6 shows that if $\delta(\mathcal{A}) \preccurlyeq \mathcal{A}$, then not only does $\delta(\mathcal{A})$ model $\mathcal{A}$ (so $\delta(\mathcal{A}) \equiv \mathcal{A}$), but it definably models $\mathcal{A}$ ($\delta(\delta(\mathcal{A})) = \delta(\mathcal{A})$).

So our search for $\mathcal{D}$ amounts to finding conditions under which $\delta(\mathcal{A}) \preccurlyeq \mathcal{A}$. We give two sets of conditions under which this occurs, one based on model completeness, and one based on o-minimality of the theory of $\mathcal{A}$.

VI.1 Model completeness. A structure whose theory is model complete can be definably modeled[25] under the following conditions.

THEOREM A (MODEL COMPLETENESS TRANSFER THEOREM). *Let the $\mathcal{L}$-structure $\mathcal{R} = (\mathbb{R}, +, \cdot, > 0, 1, f_j, r_k)_{j\in J, k\in K}$ be an expansion by functions and constants of the real closed ordered field of real numbers, such that:*

a) Th($\mathcal{R}$) is model complete;

b) the functions f_j are continuous.

Suppose that $\mathcal{A}$ is a nonstandard model of $\mathcal{R}$, i.e., $\mathcal{A} \equiv \mathcal{R}$. If $\mathcal{A}' = (\mathcal{A}, a'_{k'})_{k'\in K'}$ is an expansion of $\mathcal{A}$ by constants, with language $\mathcal{L}' = (\mathcal{L}, c'_{k'})_{k'\in K'}$. Then:

i) $\mathcal{D} = \delta(\mathcal{A}')$ is an $\mathcal{L}'$-structure definably modeling $\mathcal{A}'$;

ii) $D \preccurlyeq \mathcal{A}'$; in particular, $\mathcal{D} \equiv \mathcal{A}'$;

consequently $\mathcal{D}$ is a real closed ordered field expansion.

iii) Every definable model of Th($\mathcal{A}'$) is isomorphic to $\mathcal{D}$.

PROOF. Part (i) follows from part (ii) and the definition $\mathcal{D} = \delta(\mathcal{A}')$.

Part (ii) follows from the hypotheses and from Proposition 10's conclusion that $\delta(\mathcal{A}') \preccurlyeq \mathcal{A}'$. Then Proposition 9(a) implies that $\delta((\mathcal{A}, a_{k'})_{k'\in K'})) \preccurlyeq (\mathcal{A}, a_{k'})_{k'\in K'}$.

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Part (iii) follows from Part (i) and Proposition 5(c). Q.E.D.

VI.2 O-minimality. By Proposition 14, every $\mathcal{L}_0$-definable set in a real closed field $\mathcal{A} = (A, +, \cdot, >, 0, 1)$ is a semialgebraic set — a finite union of "polynomial" sets. In particular, every definable subset of A is a finite union of points and intervals. This property was emphasized by van den Dries (1984), and later abstracted by Pillay and Steinhorn (1986) who called any $\mathcal{L}$-structure $\mathcal{A} = (A, >, \cdots)$ in which $>$ is a linear (asymmetric, transitive, total) order on A, an *o-minimal* ("order-minimal") structure if every parametrically $\mathcal{L}$-definable subset of A is a finite union of points and $>$-intervals.

Alternatively, in any ordered structure $\mathcal{A} = (A, >, \ldots)$, one can define an *o-minimal system on* $\mathcal{A}$ to be a sequence $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \ldots$ such that each $\mathcal{S}_n$ is a collection of subsets of A^n with the special properties listed in the Appendix.

O-minimal systems have been actively studied in recent years ((Pillay and Steinhorn, 1986), (Marker, 1996), (van den Dries (1984, 1996)), (Wilkie, 1991b), and their references). They are a basis for the study of "tame topology," and new results in model theory. Applications to game theory and economics have been made by Blume and Zame (1989, 1991) (see Section VI.3).

From one point of view, the approach through definability and the approach through o-minimality are very similar. If one considered the "base" sets, the sets in $\mathcal{S}_1$, to be "definable," then the axioms for an o-minimal system amount to closure of the system under definability: the complement (corresponding to negation) of a set in the system is also in the system; so is the union (disjunction) of two sets in the system; and so is the projection (existential quantification).

The close relationship of the two approaches is illustrated by two further facts. First, every real closed ordered field $\mathcal{A} = (A, +, \cdot, >, 0, 1)$ is o-minimal for $\mathcal{L}_0$. In the opposite direction, every o-minimal ordered field expansion is an expansion of a real closed ordered field (Proposition 16 in the Appendix).

On the other hand, there are model complete theories of real closed field expansions that are not o-minimal. And there are o-minimal expansions that are not model complete. A recent result of Wilkie (1991b) shows that the theory of the expansion $(\mathbb{R}, +, \cdot, >, 0, 1, \exp)$ is both model complete and o-minimal.

Theorem A justified the economics results, Theorems $1'$–$6'$, by showing that definable equivalents exist, at least for certain structures. It was based on the existence of certain kinds of model complete expansions of real closed fields. Now we give an alternative justification, based on o-minimal expansions.

THEOREM B (O-MINIMALITY TRANSFER THEOREM).
Let $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$ *be an o-minimal expansion of an ordered field, and let* $\mathcal{L}$ *be its language. Then:*

i) $\mathcal{D} = \delta(\mathcal{A})$ *is an* $\mathcal{L}$*-structure definably modeling* $\mathcal{A}$*;*

ii) $\mathcal{D} \preccurlyeq \mathcal{A}$*; in particular,* $\mathcal{D} \equiv \mathcal{A}$*;*

consequently $\mathcal{D}$ *is a real closed ordered field expansion.*

iii) Every definable model of $Th(\mathcal{A})$ *is isomorphic to* $\mathcal{D}$*.*

PROOF. Part (i) follows from part (ii) and the definition $\mathcal{D} = \delta(\mathcal{A})$. Part (ii) follows from Proposition 15 in the Appendix, and part (iii) from Proposition 5. Q.E.D.

VI.3 Theorems for Definable Economies. Theorems A and B show that Theorems $1'$–$6'$ are not vacuous. The model completeness route of Theorem A shows that definable structures of the desired kind exist, if we start with certain kinds of structures — expansions by constants and continuous functions of model complete real structures. These include, of course, the basic models, simple real closed ordered fields. It is not known how extensive this class is, although several such structures are known. The most interesting one is the real ordered field extended by the exponential function, which has recently been shown to be model complete (Wilkie, 1991b).

The o-minimality route of Theorem B also shows existence of the desired kind of structures, this time for o-minimal expansions of ordered fields. Again, the class of such structures is not fully known. It includes, of course, the basic example of real closed ordered fields. It also includes the standard real ordered field with exponential function as again shown recently by Wilkie (1991b).

Relationship to Blume and Zame. As noted above, Blume and Zame (1989, 1992) have applied o-minimal systems to game theory and economics. While our work was commenced before we learned of their work, their papers predate ours, and it may be useful to indicate how their economics paper (1992) relates to the present one.

A motivation for the Blume-Zame paper was the search for preferences, other than Debreu's C^2 preferences (1970), that still ensure generic local determinacy of competitive equilibrium prices. We were motivated, on the other hand, to describe magnitudes and preferences that were "realistic" in a bounded rationality sense.

While Blume and Zame are interested in a classical real model, our bounded rationality objective, with its uniformity principle, forced us to consider definable economies — in particular, nonclassical strongly definable economies.

Given their motivation, it is natural that Blume and Zame focus on topological properties of preferences. On the other hand, our focus on definability concepts arises naturally from our bounded rationality viewpoint. While they use o-minimal preferences to address their topological concerns, for our definability goals we find it useful to use both model completeness and o-minimality tools.[26] [27]

In another paper (1996c), we have used o-minimality techniques to extend one of Blume and Zame's subsidiary results, by showing that upper semicontinuous definable preference relations in o-minimal systems have definable upper semicontinuous utility functions.

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(26) Though see our comments elsewhere about the close relationships between the tools.

(27) It is straightforward to prove Debreu's differentiable-type local determinacy result in our definability framework (even without reference to o-minimality). For definable economies, the set X of endowments for which equilibrium prices exist is a definable set, so it is easy to express the fact that, the complement of X is nowhere dense in A^l — i.e., for all $\varepsilon > 0$, its closure contains no ε-balls. The expressibility allows transfer of Debreu's classical result to any elementarily equivalent structure — whether the structure is weakly definable, or strongly definable (perhaps obtained through o-minimality via Theorem B) or through model completeness (via Theorem A)).

In a similar way, Blume and Zame's result can be extended to our framework (with reference to o-minimality).

Appendix

We collect some of the basic definitions and notation of the first order predicate calculus. More precise and complete discussions should be consulted in standard mathematical logic texts, such as (Enderton, 1972), (Chang and Keisler, 1990), (Ebbinghaus, Flum, and Thomas, 1994), and (van Dalen, 1994). We also state as Propositions several results, many of which will be found in the standard texts.

To describe a language $\mathcal{L}$, we need to specify its symbols, its syntax (formulas and proofs), and its semantics (interpretations).

Symbols and Syntax.

Our languages contain (individual) *constant* symbols c_i, (individual) *variable symbols* v_i, relation (predicate) symbols P_i, *equality* symbol $=$, *function* symbols f_i, *connective* symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, *universal quantifier* symbols $\exists v_i$ and *existential quantifiers* $\exists v_i$, and *parentheses* (and). In our examples, 0 and 1 will be constants; there may also be others. We use $\exists_{=1} v\phi(v, v_1, \ldots, v_n)$ as a shorthand for $\exists v \exists v_1 \cdots \exists v_n \wedge \forall w(\phi(w, v_1, \ldots, v_n) \rightarrow w = v))$.

For each relation symbol P and each function symbol f there is an *arity*, a positive integer n describing the number of "arguments" the predicate or function uses. We then say that P is an n-ary relation and f is an n-ary function. In our applications, addition ($+$) and multiplication ($\cdot$) will be binary (i.e., 2-ary) functions, and $>$ will be a binary relation.

The syntax of the language describes the strings of symbols that are called (well formed) formulas of the language. The definition of a formula is built up inductively, as follows.

The *terms* of the language will be used to represent specific objects, or to range over objects in our universe. They are defined inductively, as variables v_i, or constants c_i, or formulas $f_i(t_1, \ldots, t_n)$ where f_i is an n-ary function and $t_1, \ldots, t_n$ are terms. In our applications, for example, $1 + 1$ will be a term.

The *atomic formulas* of $\mathcal{L}$ are either strings like $t_1 = t_2$, for some terms t_1 and t_2, or they are strings of the form $P_i(t_1, \ldots, t_n)$ for some n-ary predicate P_i and some terms $t_1, \ldots, t_n$. In our applications, for example, $1+1 > 1+1+0+1$ will be a formula.

The *formulas* of $\mathcal{L}$ are strings defined inductively, as either

i) atomic formulas,

ii) strings of the form $(\phi \wedge \psi)$, $(\neg\phi)$, $(\phi \rightarrow \psi)$, or $(\phi \leftrightarrow \psi)$, for some formulas ϕ and ψ

iii) strings of the form $((\exists v)\phi)$ or $((\forall v)\phi)$ for some variable v and some formula ϕ.

In the cases (iii), we say that the variable v is *bound* by the quantifiers.

Thus, for example if P_8 is a binary predicate, and if f_7 is a unary function, then $P_8(v_5, v_1)$ and $(\neg((\exists v_2) f_7(v_2) = c_4))$ are formulas; $P_1)\exists v_3 \neg$ is not a formula. Usually we will drop parentheses from our formulas when there is no danger of confusion.

We say that a variable v is a *free variable* in a formula ϕ if there is an occurrence of v in ϕ at which v is not bound by any quantifier. If a formula ϕ has no free variables, then we call ϕ a *sentence*. For any formula ϕ, we will sometimes write something like $\phi(v_1, \ldots, v_n)$ to denote that the free variables of ϕ (if there are any) lie in $\{v_1, \ldots, v_n\}$.

Sometimes we start with a language $\mathcal{L}$ and then expand it to a language with more symbols. If we adjoin new function symbols f_j for $j \in J$, we obtain a new language $(\mathcal{L}, f_j)_{j\in J}$. If we also adjoin new constant symbols c_k for $k \in K$, we have the language $(\mathcal{L}, f_j, c_k)_{j\in J, k\in K}$. In each case we call the new language an *expansion* of $\mathcal{L}$, and consider the original language $\mathcal{L}$ to be the *reduct* of the expansion obtained by restricting it: $\mathcal{L} = (\mathcal{L}, f_j, c_k)_{j\in J, k\in K} \restriction \mathcal{L}$.

Semantics.

Having described the formulas of $\mathcal{L}$, we now interpret them.

Formulas ϕ of $\mathcal{L}$ will be interpreted within structures for the language $\mathcal{L}$. Thus, for example, if $\mathcal{L}$ has no constant symbols, no function symbols, and has just one predicate symbol P, which is 2-ary, then we will consider structures $\mathcal{A} = (A, R)$, where A is a set and R is a binary relation on A. More generally, if the constants of the language are c_k for $k \in K$, and if the predicates of the language are P_i for $i \in I$, where each P_i is r_i-ary, and if the function symbols of the language are f_j for $j \in J$, where each f_j is r_j-ary, then we say that a structure $\mathcal{A} = (A, R_i, F_j, a_k)_{i\in I, j\in J, k\in K}$ is a *structure for the language* $\mathcal{L}$ if each a_k is an element of A, each R_i is an r_i-ary relation on A, and each F_j is an r_j-ary function on A. Sometimes, to make the connection between the language and the structure very explicit, we may write $\mathcal{A} = (A, P_i^{\mathcal{A}}, f_j^{\mathcal{A}}, c_k^{\mathcal{A}})_{i\in I, j\in J, k\in K}$, making it clear that the relation $P_i^{\mathcal{A}}$ is the interpretation in $\mathcal{A}$ of the predicate symbol P_i, the function $f_j^{\mathcal{A}}$ is the interpretation $\mathcal{A}$ of the function symbol f_j, and the element $c_k^{\mathcal{A}}$ is the interpretation in $\mathcal{A}$ of the constant symbol c_k.

When we denote the structure by a script letter $\mathcal{A}$, we usually denote its universe by the corresponding roman letter: $\mathcal{A} = (A, \dots)$. Occasionally we write "$a \in \mathcal{A}$" as a shorthand for "$\mathcal{A} = (A, \dots)$ and $a \in A$."

Just as we considered expansions $(\mathcal{L}, f_j, c_k)_{j \in J, k \in K}$ of languages, we consider expansions of structures. Given a structure $\mathcal{A}$, if we adjoin new functions f_j and constants a_k, we obtain an *expansion* $(\mathcal{A}, f_j, c_k)_{j \in J, k \in K}$, whose reduct $(\mathcal{A}, f_j, c_k)_{j \in J, k \in K} \restriction \mathcal{A}$ is $\mathcal{A}$.

Certain structures are of particular importance in this paper. A structure $\mathcal{A} = (A, +, \cdot, >, 0, 1)$ is an *ordered field* if: for all k and for all $x, y, z, x_1, \dots, x_k \in A$,

$$
\begin{array}{ll}
x + (y + z) = (x + y) + z & \text{(+ is associative)} \\
x + y = y + x & \text{(+ is commutative)} \\
x + 0 = x & \text{(additive identity)} \\
x + w = 0 \text{ for some } w & \text{(additive inverse)} \\
x \cdot (y \cdot z) = (x \cdot y) \cdot z & (\cdot \text{ is associative}) \\
x \cdot y = y \cdot x & (\cdot \text{ is commutative}) \\
x \cdot 1 = x & \text{(multiplicative identity)} \\
x \cdot w = 1 \text{ for some } w & \text{(multiplicative inverse)} \\
0 \neq 1 & \text{(nontriviality)} \\
x > y \Rightarrow \text{ not } y > x & (> \text{ is asymmetric}) \\
x > y \text{ and } y > z \Rightarrow x > z & (> \text{ is transitive}) \\
x = y \text{ or } x > y \text{ or } y > z & (> \text{ is total}) \\
x_1^2 + \cdots + x_k^2 \neq -1 & \text{(formally real).}
\end{array}
\tag{16}
$$

And $\mathcal{A}$ is a *real closed ordered field* if, in addition: for all odd n and all $x, x_1, \dots, x_n \in A$,

$$
\begin{array}{l}
x > 0 \Rightarrow x = w^2 \text{ for some } w \\
\qquad \text{(positive elements have square roots)} \\
x_n \cdot w^n + x_{n-1} \cdot w^{n-1} + \cdots + x_1 \cdot w + x_0 = 0 \text{ for some } w \\
\qquad \text{(polynomials of odd degree have roots).}
\end{array}
\tag{17}
$$

It is intuitively clear how we will interpret sentences of $\mathcal{L}$ in any structure $\mathcal{A}$ for $\mathcal{L}$. First, each constant c_k gets interpreted as the element $a_k \in A$ (we may write this interpretation as $a_k = c_k^{\mathcal{A}}$), each relation symbol P_i gets interpreted as the relation R_i on A (we may write this interpretation as $R_i = P_i^{\mathcal{A}}$), and each function symbol f_j gets interpreted as the function F_j on A (we may write this interpretation as $F_j = f_j^{\mathcal{A}}$). Then, all the logical connectives, $\neg$, $\wedge$, $\rightarrow$, $\exists v_0$, etc., get interpreted in the natural way — as negation, conjunction, implication, existential quantification, etc. For example, if ϕ is the sentence $\forall v_3 \exists v_2 P(v_2, v_3, c)$, for some constant symbol c, then we interpret ϕ in $\mathcal{A}$ as meaning that, for every a_3 in A, there is some a_2 in A such that $P^{\mathcal{A}}(a_2, a_3, c^{\mathcal{A}})$ holds in $\mathcal{A}$. If that is the case, then we say that ϕ is true in the structure $\mathcal{A}$, or that $\mathcal{A}$ is a *model* of ϕ, and we write:

$$\mathcal{A} \models \phi \tag{18}$$

To interpret formulas ϕ that may have free variables, consider the formula $Q(v_2, v_4)$ for some predicate Q, whose only free variables are among $\{v_2, v_4\}$. We say that any elements a and b of A *satisfy* $P(v_2, v_4)$ in $\mathcal{A}$ if $Q^{\mathcal{A}}(a, b)$ holds in $\mathcal{A}$ — i.e., if, when we interpret v_2 as a and v_4 as b, then (a, b) belongs to the relation $Q^{\mathcal{A}} \subseteqq A \times A$. Then we write $\mathcal{A} \models P(\frac{a}{v_2}, \frac{b}{v_4})$ or $\mathcal{A} \models P[a, b]$.

More generally, it is clear how to generalize the definition of satisfaction

$$\mathcal{A} \models \phi(\frac{a_1}{v_1}, \ldots, \frac{a_n}{v_n}) \tag{19}$$

to apply to all formulas $\phi(v_1, \ldots, v_n)$ and all structures $\mathcal{A}$ for the language $\mathcal{L}$ and all $a_1, \ldots, a_n \in A$.

If (19) holds for all elements $a_1, \ldots, a_n$ of A, then we say that the formula ϕ is *true* in $\mathcal{A}$, or that $\mathcal{A}$ is a *model* of the formula ϕ. If Γ is a set of formulas of $\mathcal{L}$, and if $\mathcal{A} \models \phi$ for every formula ϕ in Γ, then we say that $\mathcal{A}$ is a *model* of Γ, or that all of Γ is *true* in $\mathcal{A}$, and we write $\mathcal{A} \models \Gamma$.

It is important to note that $\models$ and the structures $\mathcal{A}$ and elements $a_1, \ldots, a_n$ are not symbols of the language L. The statement (19) is a statement in our metalanguage (English), discussing the language $\mathcal{L}$ and a structure $\mathcal{A}$.

If Γ is a set of formulas of $\mathcal{L}$, and ϕ is a formula of $\mathcal{L}$, then we say that Γ *semantically implies* ϕ, or that ϕ is a *semantic consequence* of Γ, if every model of Γ is also a model of ϕ. Then, overworking the notation, we write $\Gamma \models \phi$.

We can now define an important notion of semantic equivalence. We say that two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent* if they satisfy the same $\mathcal{L}$-sentences, i.e., for every $\mathcal{L}$-sentence ϕ, we have: $\mathcal{A} \models \phi$ if and only if $\mathcal{B} \models \phi$. Then we write $\mathcal{A} \equiv \mathcal{B}$. this is clearly an equivalence relation on the class of all $\mathcal{L}$-structures.

A *theory* in $\mathcal{L}$ is a set of $\mathcal{L}$-sentences. A set T of *axioms* of a theory T in $\mathcal{L}$ is a set Γ of $\mathcal{L}$-sentences γ such that $\{\gamma : \Gamma \models \gamma\} = T$, and then we say that T is the *theory of* Γ. We say that T is *consistent* in $\mathcal{L}$ if it has at least one model —

equivalently, if for all $\mathcal{L}$-sentences σ, either $\sigma \notin T$ or $\neg\sigma \notin T$. It is *complete* in $\mathcal{L}$ if $\sigma \in T$ or $\neg\sigma \in T$.

For any $\mathcal{L}$-structure $\mathcal{A}$, by the *theory of* $\mathcal{A}$ we mean the set of $\mathcal{L}$-sentence that are true in $\mathcal{A}$: $\mathrm{Th}(\mathcal{A}) = \{\gamma : \mathcal{A} \models \gamma\}$. For any structure $\mathcal{A}$, its theory $\mathrm{Th}(\mathcal{A})$ is clearly complete.

If $\mathcal{A} = (A, \dots)$ and $\mathcal{B} = (B, \dots)$ are $\mathcal{L}$-structures we say that $\mathcal{A}$ is a *substructure* of $\mathcal{B}$ if $A \subseteqq B$ and if:

i) Each n-ary relation $R^{\mathcal{A}}$ of $\mathcal{A}$ is the restriction to A of the corresponding relation $\mathcal{R}^{\mathcal{B}}$; i.e., $R^{\mathcal{A}} = A^n \cap R^{\mathcal{B}}$;

ii) Each m-placed function $f^{\mathcal{A}}$ of $\mathcal{A}$ is the restriction to A of the corresponding function $f^{\mathcal{B}}$; i.e., $f^{\mathcal{A}} = f^{\mathcal{B}} \restriction_{A^m}$.

iii) Each constant of $\mathcal{A}$ is the corresponding constant of $\mathcal{B}$.

Then we write $\mathcal{A} \subseteq \mathcal{B}$, and we call $\mathcal{A}$ a *restriction* of $\mathcal{B}$, and $\mathcal{B}$ and *extension* of $\mathcal{A}$. Clearly $\mathcal{A} \subseteq \mathcal{B}$ & $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A} = \mathcal{B}$.

Note that we use $=$, rather than simply $\subseteqq$, in stating properties (i) and (ii) above. Thus it may not be true that, for a given structure, every subset of its universe can be the universe of a substructure. In particular, the substructure needs to be closed under certain operations:

PROPOSITION 1.
Let $\mathcal{B} = (B, \dots)$ be an $\mathcal{L}$-structure and let $A \subseteqq B$. Then there is a substructure $\mathcal{A} = (A, \dots)$ of $\mathcal{B}$ with universe A if and only if:

for every constant symbol c of $\mathcal{L}$, we have $c^{\mathcal{B}} \in A$;

for every $m > 0$, every m-placed function symbol f of $\mathcal{L}$, and all $a_1, \dots, a_n \in A$, we have $f^{\mathcal{B}}(a_1, \dots, a_n) \in A$.

And then there is a unique such substructure.

We also use a stronger semantic notion than elementary equivalence. It concerns formulas — not just sentences. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-structures, we say that $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$, i.e. $\mathcal{A} \subseteq \mathcal{B}$, and if for every $\mathcal{L}$-formula $\phi(v_1, \dots, v_n)$, for some n, and for all $a_1, \dots, a_n \in A$, we have

$$\mathcal{A} \models \phi[a_1, \dots, a_n] \quad \text{if and only if} \quad \mathcal{B} \models \phi[a_1, \dots, a_n]. \tag{20}$$

Then we write $\mathcal{A} \preccurlyeq \mathcal{B}$. (The context and symbol orientation will prevent confusion with the preference notation $\succcurlyeq$ of Sections III.5 and IV.) Note that:

$$\text{If } \mathcal{A} \preccurlyeq \mathcal{B} \text{ then } \mathcal{A} \equiv \mathcal{B}. \tag{21}$$

We use this characterization of $\preccurlyeq$:

PROPOSITION 2 (TARSKI-VAUGHT CRITERION FOR ELEMENTARY SUBSTRUCTURES).
Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. Then $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if:

i) $\mathcal{A} = (A, \dots)$ is a substructure of $\mathcal{B} = (B, \dots)$, i.e. $\mathcal{A} \subseteq \mathcal{B}$;

ii) for all $\mathcal{L}$-formulas $\phi(v_0, v_1, \dots, v_n)$ and all $a_1, \dots, a_n \in A$,
if $\mathcal{B} \models \exists v_0 \phi(v_0, a_1/v_1, \dots, a_n/v_n)$ then $\mathcal{B} \models \phi(a/v_0, a_1/v_1, \dots, a_n/v_n)$ for some $a \in A$.

We say that $\mathcal{L}$-structures $\mathcal{A} = (A, \dots)$ and $\mathcal{B} = (B, \dots)$ are *isomorphic*, and write $\mathcal{A} \cong \mathcal{B}$, if there is a function $f : A \to B$ such that:

i) f is one to one, and onto B;

ii) for every n-ary relation symbol R of $\mathcal{L}$, and all $a_1, \dots, a_n \in A$:

$$R^{\mathcal{A}}(a_1, \dots, a_n) \quad \text{if and only if} \quad R^{\mathcal{B}}(f(a_1), \dots, f(a_n)); \tag{22}$$

iii) for every n-ary function symbol F of $\mathcal{L}$, and all $a_0, a_1, \dots, a_n \in A$:

$$\begin{aligned} a_0 &= F^{\mathcal{A}}(a_1, \dots, a_n) \\ &\quad \text{if and only if} \\ f(a_0) &= F^{\mathcal{B}}(f(a_1, \dots, f(a_n)); \end{aligned} \tag{23}$$

iv) for all constant symbols c of $\mathcal{L}$:

$$f(c^{\mathcal{A}}) = c^{\mathcal{B}}. \tag{24}$$

PROPOSITION 3.
For any $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.[28]

We mention three notions of "smallness" for structures. They are not essential to our main objective — definable structures equivalent to classical structures; but as Proposition 5 will show, all the structures we seek must be small. So we mention these notions very briefly.

---

[28] The converse is not generally true.

First, a model $\mathcal{A}$ of a theory T is called a *minimal* model for T if $\mathcal{A}$ has no proper elementary substructure that is a model of T. In other words, if $\mathcal{B} \preccurlyeq \mathcal{A}$, then $\mathcal{B} \not\models T$.

Second, a model $\mathcal{A}$ of a theory T is called a *prime* model of T if A is elementarily embeddable in every model of T. In other words, if $\mathcal{B}$ is a model of T, then there is an isomorphism $\mathcal{B} \equiv \mathcal{C}$ for some $\mathcal{C} \preccurlyeq \mathcal{B}$.

Third, and strongest of all, a structure $\mathcal{A}$ is called *atomic* if, in a certain sense, its elements satisfy only those properties that the complete theory $\mathrm{Th}(\mathcal{A})$ forces it to have.[29] What do we mean by "properties"? If $a_1, \ldots, a_k \in \mathcal{A}$, then the set of formulas $\phi(v_1, \ldots, v_n)$ that they satisfy describe a "complete k-type" — a set of $\mathcal{L}$-defined properties. If we start from the other end, with a set $\Phi(v_1, \ldots, v_n)$ of $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_k)$, we can ask whether there is any k-tuple $(a_1, \ldots, a_k) \in \mathcal{A}$ that realizes them all: $\phi(a_1/v_1, \ldots, a_k/v_k)$. If so, we say that $\mathcal{A}$ *realizes the type* $\Phi(v_1, \ldots, v_n)$, and if not we say that $\mathcal{A}$ *omit the type* $\Phi(v_1, \ldots, v_n)$.

To obtain a "small" model of the complete theory $\mathrm{Th}(\mathcal{A})$, we want to omit as many types as possible. Of course we can't omit a type $\Phi(v_1, \ldots, v_k)$ from a model of $\mathrm{Th}(\mathcal{A})$ if there is a formula $\gamma(v_1, \ldots, v_k)$ with:

$$\begin{aligned} &\mathrm{Th}(\mathcal{A}) \models \exists v_1 \cdots \exists v_k \gamma(v_1, \ldots, v_k) \\ &\mathrm{Th}(\mathcal{A}) \models \forall v_1 \cdots \forall v_k \gamma(v_1, \ldots, v_k) \to \phi(v_1, \ldots, v_k) \end{aligned} \tag{25}$$

for all $\phi(v_1, \ldots, v_k) \in \Phi(v_1, \ldots, v_k)$. What atomicity of $\mathcal{A}$ says is that $\mathcal{A}$ omits all types except these.

Definability.

If $\mathcal{A} = (A, \ldots)$ is a structure for the language $\mathcal{L}$, and if $X \subseteqq A^n$ for some n, then we say that X is *$\mathcal{L}$-definable* if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$ such that:

$$X = \{(a_1, \ldots, a_n) \in A^n : \mathcal{A} \models \phi(\frac{a_1}{v_1}, \ldots, \frac{a_n}{v_n})\}. \tag{26}$$

If also $X = \{a\}$ for some $a \in A$, then we say that a is *$\mathcal{L}$-definable* in $\mathcal{A}$.

It follows from Proposition 1 that, for any structure $\mathcal{A}$, the set of $\mathcal{L}$-definable elements in A constitute the universe of a unique substructure. We denote by

[29] For a precise definition, see (Chang and Keisler, 1992, p. 97) or (Hodges, 1994, p. 336.

$\delta(\mathcal{A})$ this restriction of $\mathcal{A}$ to its definable elements.[30] (Our notation does not specify the language $\mathcal{L}$ since it is implicit in the structure $\mathcal{A}$.) Thus

$$\delta(\mathcal{A}) \subseteq \mathcal{A}, \tag{27}$$

hence

$$\delta(\delta(\mathcal{A})) \subseteq \delta(\mathcal{A}). \tag{28}$$

The opposite conclusion need not hold. It will hold, however, when $\mathcal{A}$ is a definable structure, in which case $\delta(\mathcal{A}) = \mathcal{A}$; and then

$$\delta(\delta(\mathcal{A})) = \delta(\mathcal{A}). \tag{29}$$

PROPOSITION 4.
Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

If $\mathcal{A}$ is an expansion of a real closed ordered field, then $\delta(\mathcal{A})$ is also an expansion of a real closed ordered field.

PROOF. $\mathcal{D} = \delta(\mathcal{A})$ is formally real, since $D \subseteqq A$ so any violation $a_1^2 + \cdots + a_n^2 = -1$ of the formally real property with the a_i in D would also be a violation of that property in A. And $\mathcal{D}$ has square roots for positive elements, since $D \subseteqq A$ and A has unique positive square roots for positive elements, so we can *define* unique positive square roots in A: if $\phi(v_0)$ defines positive a_0 in $\mathcal{A}$, then we can use the formula:

$$\begin{aligned} &\forall v_0(\phi(v_0) \wedge v_1 > 0 \wedge v_1 \cdot v_1 = v_0 \wedge \forall v_2(v_2 \cdot v_2 = v_0 \rightarrow \\ &\qquad (v_1 < v_2 \vee v_1 = v_2))). \end{aligned} \tag{30}$$

Finally, polynomials in $\mathcal{D}$ of odd degree have definable roots, since $D \subseteqq A$ implies that polynomials in $\mathcal{D}$ are polynomials in $\mathcal{A}$, and polynomials of odd degree in $\mathcal{A}$ have (finitely many) roots, so we can *define* a root in D as the least one in A. Q.E.D.

The next proposition does not prove that definable equivalents exist for a given structure, but it shows some important properties they would have to have.

---

[30] **This differs from the definitions of definable closure in Pillay and Steinhorn (1986, p. 569) and Hodges (1994, p. 134), where parameters from $\mathcal{A}$ are allowed in ϕ, and where the definable closure is always with respect to a given superstructure. In particular, $\delta(\delta(\mathcal{A})$ need not equal $\delta(\mathcal{A})$ under our definition, while it always does under theirs.**

PROPOSITION 5 (DEFINABLE STRUCTURES THEOREM).
Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

If $\mathcal{D}$ is an $\mathcal{L}$-structure and if:

a) $\mathcal{D} \equiv \mathcal{A}$

b) $\mathcal{D}$ *is a definable structure (i.e.,* $\delta(\mathcal{D}) = \mathcal{D}$*),*

then for any $\mathcal{L}$-structure $\mathcal{B} \equiv \mathcal{A}$:

c) $\mathcal{D} \cong \delta(\mathcal{B})$ *by a unique isomorphism*

d) $\delta(\mathcal{B}) \preccurlyeq \mathcal{B}$

e) $\delta(\mathcal{B})$ *is a definable structure (i.e.,* $\delta(\delta(\mathcal{B})) = \delta(\mathcal{B})$*).*

f) $\delta(\mathcal{B})$ *is "small": it is*

an atomic structure for $Th(\mathcal{B}) = Th(\mathcal{A})$,

a minimal model for $Th(\mathcal{B}) = Th(\mathcal{A})$,

a prime model for $Th(\mathcal{B}) = Th(\mathcal{A})$.

REMARK 4. Since an elementary submodel of a structure $\mathcal{A}$ is elementarily equivalent to $\mathcal{A}$, it follows from the proposition that, if we are interested in finding a structure $\mathcal{D}$ definably modeling $\mathcal{A}$, then we might as well choose as $\mathcal{D}$ the structure $\delta(\mathcal{A})$. If anything works, that will.

PROOF OF PROPOSITION 5. Because we are are not aware of any published proof of this theorem, in quite this form, we will give below a detailed proof of the main conclusions.

However, the results are closely related to some in the literature, and can be proved by straightforward modifications of proofs of related theorems. For example, Theorems 8.1, 8.2, and Corollary 8.5 in (Kaye, 1991, pp. 91-95) are proved for the special case where the model $\mathcal{A}$ is a model of Peano arithmetic. It is straightforward to modify the proofs given there, replacing that specific assumption with the assumption that $\mathcal{A}$ is any structure with a definable equivalent $\mathcal{D}$.

In particular, our conclusion (d) is similar to Kaye's Theorem 8.1; our (c), (d), and the primality part of (f) are similar to Kaye's Theorem 8.2; and the atomicity conclusion in (f) is analogous to Kaye's Corollary 8.5; as he remarks (1991, pp. 94–95), he avoids the usual omitting types theorem (with its restriction to countable languages and the ability to omit only countably many types) by working with a Peano model "and its very special feature of having these 'pointwise definable models'."

Although proofs can be written along those lines, we provide details here, in order to make this paper self contained.

To prove (c), we define an isomorphism $f : \mathcal{D} \to \delta(\mathcal{A})$ as follows. Let $a \in \mathcal{D}$. By (b) a is definable, so for some $\mathcal{L}$-formula ϕ:

$$\mathcal{D} \models \exists_{=1} v_0 \phi(v_0) \quad \text{and} \quad \mathcal{D} \models \phi(\frac{a}{v_0}). \tag{31}$$

Then by (a), $\mathcal{A} \models \exists_{=1} v_0 \phi(v_0)$, so there is a unique $b \in \mathcal{A}$ with $\mathcal{A} \models \phi(b/v_0)$; and b is definable, i.e., $b \in \delta(\mathcal{A})$. We can define $f(a)$ as this element that ϕ defines in $\mathcal{A}$:

$$f(a) = b. \tag{32}$$

To see that this is a proper definition, not depending on the choice of the formula ϕ that defines $\mathcal{A} \in \mathcal{D}$, suppose that ϕ_1 and ϕ_2 define the same $a \in \mathcal{D}$: (31) holds for both ϕ_1 and ϕ_2 as ϕ. Then

$$\mathcal{D} \models \forall v_0(\phi_1(v_0) \leftrightarrow \phi_2(v_0)), \tag{33}$$

so by (a) we have:

$$\begin{aligned} &\mathcal{A} \models \exists_{=1} v_0 \phi_1(v_0) \\ &\mathcal{A} \models \exists_{=1} v_0 \phi_2(v_0) \\ &\mathcal{A} \models \forall v_0(\phi_1(v_0) \leftrightarrow \phi_2(v_0)). \end{aligned} \tag{34}$$

Thus both ϕ_1 and ϕ_2 define the same element in $\mathcal{A}$.

We omit the straightforward proofs, using (a) and (b), that f is one to one, that f is onto $\delta(\mathcal{A})$, and that f is an isomorphism.

To prove (d), we apply the Tarski-Vaught criterion, Proposition 2: suppose $b_1, \ldots, b_n \in \delta(\mathcal{A})$ and

$$\mathcal{A} \models \exists v_0 \phi(v_0, \frac{b_1}{v_1}, \ldots, \frac{b_n}{v_n}). \tag{35}$$

We must show that:

$$\begin{aligned} &\mathcal{A} \models \phi(\frac{a_0}{v_0}, \frac{b_1}{v_1}, \ldots, \frac{b_n}{v_n}) \\ &\qquad\qquad \text{for some } a_0 \in \delta(\mathcal{A}). \end{aligned} \tag{36}$$

Since the b_i belong to $\delta(\mathcal{A})$, suppose they are defined in $\mathcal{A}$ by formulas $\psi_i(w_i)$:

$$\mathcal{A} \models \exists_{=1} w_i \psi(w_i) \wedge \psi(\frac{b_i}{w_i}). \tag{37}$$

Then from (35) we have:

$$\begin{aligned}\mathcal{A} \models \exists_{=1} w_1 \cdots \exists_{=1} w_n (\psi_1(w_1) \wedge \cdots \wedge \psi_n(w_n)) \\ \wedge \forall w_1 \cdots \forall w_n (\psi_1(w_1) \wedge \cdots \psi_n(w_n) \to \exists v_0 \phi(v_0, w_1, \ldots, w_n)),\end{aligned} \tag{38}$$

so by (a) the same is true for $\mathcal{D}$:

$$\begin{aligned}\mathcal{D} \models \exists_{=1} w_1 \cdots \exists_{=1} w_n (\psi_1(w_1) \wedge \cdots \wedge \psi_n(w_n)) \\ \wedge \forall w_1 \cdots \forall w_n (\psi_1(w_1) \wedge \cdots \psi_n(w_n) \to \exists v_0 \phi(v_0, w_1, \ldots, w_n)).\end{aligned} \tag{39}$$

Thus there exist unique $b_i' \in \mathcal{D}$ with:

$$\mathcal{D} \models \exists_{=1} w_i \psi_i(w_i) \wedge \psi_i(\frac{b_i'}{w_0}), \tag{40}$$

and for these b_i' there exists an $a_0' \in \mathcal{D}$ such that:

$$\mathcal{D} \models \phi(\frac{a_0'}{v_0}, \frac{b_1'}{w_1}, \ldots, \frac{b_n'}{w_n}). \tag{41}$$

By hypothesis (b) there is a formula $\psi_0(w_0)$ defining a_0' in $\mathcal{D}$:

$$\mathcal{D} \models \exists_{=1} w_0 \psi_0(w_0) \wedge \psi_0(\frac{a_0}{w_0}). \tag{42}$$

From (41, (42), and (40) we see:

$$\begin{aligned}\mathcal{D} \models \exists_{=1} w_1 \cdots \exists_{=1} w_n (\psi_1(w_1) \wedge \cdots \wedge \psi_n(w_n) \wedge \\ \exists_{=1} w_0 \psi_0(w_0) \wedge \exists v_0 \phi(v_0, w_1, \ldots, w_n),\end{aligned} \tag{43}$$

so by (a) the same is true for $\mathcal{A}$:

$$\begin{aligned}\mathcal{A} \models \exists_{=1} w_1 \cdots \exists_{=1} w_n (\psi_1(w_1) \wedge \cdots \wedge \psi_n(w_n) \wedge \\ \exists_{=1} w_0 \psi_0(w_0) \wedge \exists v_0 \phi(v_0, w_1, \ldots, w_n).\end{aligned} \tag{44}$$

Then for the element $a_0 \in \delta(\mathcal{A})$ that is defined in $\mathcal{A}$ by $\psi_0(w_0)$, it follows form (37) that:

$$\mathcal{A} \models \phi(\frac{a_0}{v_0}, \frac{b_1}{v_1}, \ldots, \frac{b_n}{v_n}). \tag{45}$$

Conclusion (e) follows from (d) and Proposition 6 below.

In conclusion (f), the primality follows directly from conclusions (c) and (d). We omit proofs of the minimality and atomicity, as they are lengthier, and the smallness aspects are not central to our main theme. Q.E.D.

PROPOSITION 6. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure, and suppose*

$$\delta(\mathcal{A}) \preccurlyeq \mathcal{A}. \tag{46}$$

Then $\delta(\delta(\mathcal{A})) = \delta(\mathcal{A})$. (So $\delta(\mathcal{A})$ definably models $\mathcal{A}$.)

PROOF. From (28) we have $\delta(\delta(\mathcal{A})) \subseteqq \delta(\mathcal{A})$, so we need only prove that $\delta(\mathcal{A}) \subseteqq \delta(\delta(\mathcal{A}))$. Suppose $a \in \delta(\mathcal{A})$; it suffices to prove that $a \in \delta(\delta(\mathcal{A}))$. Since $a \in \delta(\mathcal{A})$, we know that a is $\mathcal{L}$-definable in $\mathcal{A}$, say $\mathcal{A} \vDash \phi(a/v_0) \wedge \exists_{=1} v_0 \phi(v_0)$. Then by (46) we have $\delta(\mathcal{A}) \vDash \phi(a/v_0) \wedge \exists_{=1} v_0 \phi(v_0)$, so a is $\mathcal{L}$-definable in $\delta(\mathcal{A})$, hence $a \in \delta(\delta(\mathcal{A}))$. Q.E.D.

Parametric definability.

Now we generalize the notion of definability by allowing parameters. Suppose $\phi(v_1, \ldots, v_n, u_1, \ldots, u_m)$ is an $\mathcal{L}$-formula. Then every $\mathcal{L}$-structure $\mathcal{A}$ and m-vector $(b_1, \ldots, b_m) \in A^m$ determine an m-set:

$$X^{\mathcal{A}}(b_1, \ldots, b_m) = \{(a_1, \ldots, a_n) \in A^n : \phi(\frac{a_1}{v_1}, \ldots, \frac{a_n}{v_n}, \frac{b_1}{u_1}, \ldots, \frac{b_m}{u_m})\}. \tag{47}$$

We say that $X^{\mathcal{A}}(b_1, \ldots, b_m)$ is *parametrically definable*. For each such formula $\phi(v_1, \ldots, v_n, u_1, \ldots, u_m)$ we thus have a map from structures $\mathcal{A}$ and elements $b_1, \ldots, b_m$ to m-sets $X^{\mathcal{A}}(b_1, \ldots, b_m)$. We denote this map by $X(u_1, \ldots, u_m)$, and call it a *parametric m-set*.

If every value $X^{\mathcal{A}}(b_1, \ldots, b_m)$ of a parametric m-set $X(u_1, \ldots, u_m)$ has a property P (e.g., closedness), then we say that $X(u_1, \ldots, u_m)$ *has property P*. And for any given parametric m-set $X(u_1, \ldots, u_m)$, we say that its possession of a property P is *parametrically expressible* in $\mathcal{L}$ if there is an $\mathcal{L}$-formula $\psi(w_1, \ldots, w_m)$ such that: for every $\mathcal{A}$ and all $b_1, \ldots, b_m \in A$,

$$\text{if} \quad \mathcal{A} \vDash \psi(\frac{b_1}{w_1}, \ldots, \frac{b_m}{w_m}) \quad \text{then} \quad X^{\mathcal{A}}(b_1, \ldots, b_m) \text{ has property } P. \tag{48}$$

Thus property P holds in each realization of the parametric m-set $X(u_1, \ldots, u_m)$. We generalize this in obvious ways to properties P that relate parametric sets, or relate parametric sets to elements, etc.

Model completeness.

We make use of A. Robinson's notion of model completeness for a theory, a weaker notion than elimination of quantifiers. An $\mathcal{L}$-theory T is called *model complete* (in $\mathcal{L}$) if for all $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ that are models of T, if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preccurlyeq \mathcal{B}$.

PROPOSITION 7.
If T is a theory in the language $\mathcal{L}$ that admits elimination of quantifiers in $\mathcal{L}$, then T is model complete in $\mathcal{L}$.

PROPOSITION 8 (CHARACTERIZATION OF MODEL COMPLETENESS).
Let T be a consistent theory of $\mathcal{L}$. The following are equivalent:

i) T is model complete in $\mathcal{L}$.

ii) For every $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$ there is an existential $\mathcal{L}$-formula $\psi(v_1, \ldots, v_n)$ that is T-equivalent: $T \models \psi(v_1, \ldots, v_n) \leftrightarrow \phi(v_1, \ldots, v_n)$.

PROPOSITION 9 (EXPANSION BY CONSTANTS).
Let $\mathcal{B}$ be an $\mathcal{L}$-structure, and let $(\mathcal{B}, b_k)_{k\in K}$ be an expansion by constants, with corresponding language $(\mathcal{L}, c_k)_{k\in K}$. Then:

a) If $\delta(\mathcal{B}) \preccurlyeq \mathcal{B}$, then $\delta((\mathcal{B}, b_k)_{k\in K}) \preccurlyeq (\mathcal{B}, b_k)_{k\in K}$.

b) If Th($\mathcal{B}$) is model complete in $\mathcal{L}$, then Th($(\mathcal{B}, b_k)_{k\in K}$) is model complete in $(\mathcal{L}, c_k)_{k\in K}$.

PROPOSITION 10.
Let $\mathcal{R} = (\mathbb{R}, +, \cdot, >, 0, 1, f_j, r_k)_{j\in J, k\in K}$ be an expansion of the real field, with language $\mathcal{L} = (+, \cdot, >, f_j, c_k)_{j\in J, k\in K}$. Suppose that Th($\mathcal{R}$) is model complete and the functions f_j are continuous. If $\mathcal{B}$ is an $\mathcal{L}$-structure with $\mathcal{B} \models$ Th($\mathcal{R}$), then $\delta(\mathcal{B}) \preccurlyeq \mathcal{B}$ and $\delta(\mathcal{B})$ is a definable structure.

The conclusion of Proposition 10 seems stronger than required for our economics applications (Theorem A): the definable structure $\delta(\mathcal{B})$ satisfies the elementary substructure property $\delta(\mathcal{B}) \preccurlyeq \mathcal{B}$, which is stronger than our elementary equivalence requirement $\delta(\mathcal{B}) \equiv \mathcal{B}$. It follows from Proposition 5 however, that the stronger property follows from the weaker one in our context.

PROOF OF PROPOSITION 10. Suppose Th($\mathcal{R}$) is model complete and $\mathcal{B} \models$ Th($\mathcal{R}$), so $\mathcal{B}$ is a nonstandard model of $\mathcal{R}$; then Th($\mathcal{B}$) is also model complete. Defining $\mathcal{D} = \delta(\mathcal{B}) = (D, +, \cdot, >, 0, 1, f_j, c_k)_{j\in J, k\in K}$, we want to show $\mathcal{D} \preccurlyeq \mathcal{B}$. Because $\mathcal{D} \subseteq \mathcal{B}$, it suffices[31] to show that for every formula $\phi(v_0, v_1, \ldots, v_n)$ of

[31] By the Tarski-Vaught criterion, Proposition 2.

$\mathcal{L}$, and for all $a_1, \ldots, a_n \in D$,

$$\begin{aligned} &\text{if } \mathcal{B} \models \exists v_0 \phi(v_0, \frac{a_1}{v_1}, \ldots, \frac{a_n}{v_n}), \\ &\text{then } \mathcal{B} \models \phi(\frac{a_0}{v_0}, \frac{a_1}{v_1}, \ldots, \frac{a_n}{v_n}) \text{ for some } a_0 \in D. \end{aligned} \tag{49}$$

Because $\mathrm{Th}(\mathcal{B})$ is model complete in $\mathcal{L}$, it follows from Proposition 8(ii) that there is an *existential* $\mathcal{L}$-formula $\gamma(v_1, \ldots, v_n)$ that is $\mathrm{Th}(\mathcal{B})$-equivalent to $\phi(v_0, v_1, \ldots, v_n)$:

$$\mathrm{Th}(\mathcal{B}) \models \phi(v_0, v_1, \ldots, v_n) \leftrightarrow \gamma(v_0, v_1, \ldots, v_n). \tag{50}$$

In particular, $\gamma(v_0, v_1, \ldots, v_n)$ has the form $\exists w_1 \cdots \exists w_m \zeta(v_0, v_1, \ldots, v_n, w_1, \ldots, w_m)$ where ζ is a quantifier free formula of $\mathcal{L}$. As noted in Proposition 11, every quantifier free formula of $\mathcal{L}$ is $\mathrm{Th}(\mathcal{R})$-equivalent (hence $\mathrm{Th}(\mathcal{B})$-equivalent) to an existentially quantified polynomial equality. In particular, we can rewrite (50) in the following form (for brevity, we assume that only a single function, *exp*, is involved, so we use the special example (58)):

$$\begin{aligned} \mathrm{Th}(\mathcal{B}) \models\ &\phi(v_0, v_1, \ldots, v_n) \leftrightarrow \\ &\exists z_1 \cdots \exists z_k p(v_0, v_1, \ldots, v_n, z_1, \ldots, z_k, \exp(z_1), \ldots, \exp(z_k)) = 0 \end{aligned} \tag{51}$$

for some $\mathcal{L}$-definable polynomial in the terms $v_1, \ldots, v_n, z_1, \ldots, z_k, \exp(z_1), \ldots, \exp(z_k)$. According to the hypothesis of (49), then, we are assuming that

$$\mathcal{B} \models p(\frac{a_0}{v_0}, \frac{a_1}{v_1}, \ldots, \frac{a_n}{v_n}, \frac{b_1}{z_1}, \ldots, \frac{b_k}{z_k}, \exp(\frac{b_1}{z_1}), \ldots, \exp(\frac{b_k}{z_k})) = 0 \tag{52}$$

for some $a_0, b_1, \ldots, b_k \in B$; and we want to prove that there exist such elements $a_0, b_i \in D$ — i.e., such a_0 and b_i in B that are $\mathcal{L}$-definable. To see that such b_i do exist first note that the statement:

> "for all $a_0, a_1, \ldots, a_n$, if there exist $a_0, b_1, \ldots, b_k$ satisfying $p(a_0, a_1, \ldots, a_n, b_1, \ldots, b_k, \exp(b_1), \ldots, \exp(b_k)) = 0$, then there exists a lexicographically least such $(a_0, b_1, \ldots, b_k)$"

is expressible in the language $\mathcal{L}$, say by the sentence Φ. Since the polynomial p is continuous and the function *exp* is continuous, it is clear (by elementary classical analysis) that Φ is true in the classical model $\mathcal{R}$; i.e., $\mathcal{R} \models \Phi$. Since $\mathcal{B} \models \mathrm{Th}(\mathcal{R})$, the sentence Φ is also true in the nonstandard model $\mathcal{B}$:

$$\mathcal{B} \models \Phi. \tag{53}$$

We can let $\Psi(v_0, z_1, \ldots, z_k)$ be an $\mathcal{L}$-formula expressing:

"$p(a_0, a_1, \ldots, a_n, z_1, \ldots, z_k, exp(z_1), \ldots, exp(z_k)) = 0$,
and among all $z_1, \ldots, z_k$ satisfying that, this $(z_1, \ldots, z_k)$ is lexicographically least."

Then by (52) and (53) the formula $\Psi(v_0, z_1, \ldots, z_k)$ defines a particular vector $a_0, b_1, \ldots, b_k$ satisfying the equality in (52). So these $a_0, b_1, \ldots, b_k$ belong to D. Q.E.D.

REMARK 5. Under the hypotheses of Proposition 10, it follows from the conclusion and from Proposition 5(c) that $\delta(\mathcal{R})$ is a *prime* model of the theory $\mathrm{Th}(\mathcal{R})$: it is elementarily embedded in (i.e., isomorphic to an elementary submodel of) every model of the theory.

Quantifier free formulas.

When formulas are quantifier free, they often have valuable model theoretic properties. We note several of them for real closed fields and expansions.

First, when the underlying structure is an expansion of a real closed ordered field, then quantifier free formulas are equivalent to polynomial equalities.

PROPOSITION 11 (REDUCTION OF QUANTIFIER FREE FORMULAS).
Let $\mathcal{R} = (R, +, \cdot, > 0, 1, f_j, r_k)_{j \in J, k \in K}$ be an expansion of the real field by functions f_j and constants r_k, and let $\mathcal{L}_{JK}$ be its language. Suppose $\psi(u_1, \ldots, u_l)$ is a quantifier free formula of $\mathcal{L}_{JK}$. Then for some k there exist variables $z_1, \ldots, z_k$ and an $\mathcal{L}_{JK}$-definable polynomial $p(u_1, \ldots, u_l, z_1, \ldots, z_k)$ such that:

$$\begin{aligned} Th(\mathcal{R}) \vDash &\psi(u_1, \ldots, u_l) \leftrightarrow \\ &\exists z_1 \cdots \exists z_k p(u_1, \ldots, u_l, z_1, \ldots, z_k) = 0. \end{aligned} \tag{54}$$

PROOF SKETCH. For simplicity, we prove this only for a simple case, in which the structure is $\mathcal{R} = (\mathbb{R}, +, \cdot, >, 0, 1, \exp, r_k)_{k \in K}$, where *exp* is the exponential function. The general result can be proved in a similar way. 1 Because ζ is quantifier free, it is a Boolean combination of atomic formulas; so by standard arguments of logic, it can be written as a disjunction of conjunctions of formulas like:

$$\begin{aligned} &t(u_1, \ldots, u_l) = 0 \\ &t(u_1, \ldots, u_l) > 0 \\ &\neg t(u_1, \ldots, u_l) = 0 \\ &\neg t(u_1, \ldots, u_l) > 0 \end{aligned} \tag{55}$$

for some terms $t(u_1, \ldots, u_l)$ of $\mathcal{L}_{\exp, K}$. Using the properties of $\mathrm{Th}(\mathcal{R})$, we can introduce new variables z_i and write all of these in equivalent equality form using existential formulas: $t > 0$ is equivalent to $\exists z(z \cdot z \cdot t - 1 = 0)$, and $\neg t > 0$ is equivalent to $\exists z(t + z \cdot z = 0)$. Furthermore, terms using composite applications of the exponential function can be reduced to single applications, again through existential statements: for example $\exp(v_1 \cdot \exp(v_2)) = x$ is equivalent to $\exists z(\exp(z) = x \wedge z = v_1 \cdot \exp(v_2))$. Then bringing all the existential quantifiers to the beginning, we obtain a formula like:

$$\exists z_1 \cdots \exists z_k \bigwedge_{j=1}^{J} \bigvee_{i=1}^{I} \alpha_{ij}, \tag{56}$$

where each α_{ij} is of the form

$$p_{ij}(u_1, \ldots, u_l, z_1, \ldots, z_k, \exp(z_1), \ldots, \exp(z_k)) = 0 \tag{57}$$

for some polynomials p_{ij} (with $\mathcal{L}_K$-definable coefficients) in the terms $u_1, \ldots, u_l, z_1, \ldots, z_k, \exp(z_1), \ldots, \exp(z_k)$. Finally, we can replace the disjunction of conjunctions of the several polynomial equalities α_{ij} in (56) by a single polynomial equality

$$p(u_1, \ldots, u_l, z_1, \ldots, z_k, \exp(z_1), \ldots, \exp(z_k)) = 0. \tag{58}$$

That is because any conjunction like $p_1 = 0 \wedge p_2 = 0$ is equivalent to the single polynomial condition $p_1 \cdot p_1 + p_2 \cdot p_2 = 0$; and any disjunction like $p_1 = 0 \vee p_2 = 0$ is equivalent to the single polynomial condition $p_1 \cdot p_2 = 0$. Since $\exp(\cdot)$ is part of the language $\mathcal{L}_{\exp,K}$, we can rewrite (58) as:

$$p(u_1, \ldots, u_l, z_1, \ldots, z_k) = 0. \tag{59}$$

Q.E.D.

Tarski (1948,1951) showed that the theory of real closed order fields admits elimination of quantifiers:

PROPOSITION 12 (ELIMINATION OF QUANTIFIERS (TARSKI))).
Let T_0 be a set of first order axioms for the theory of real closed ordered fields. For every formula $\phi(v_1, \ldots, v_n)$ of $\mathcal{L}_0$, there is a quantifier free formula $\psi(v_1, \ldots, v_n)$ with the same free variables.

$$T_0 \models \phi(v_1, \ldots, v_n) \leftrightarrow \psi(v_1, \ldots, v_n). \tag{60}$$

As a corollary of quantifier elimination, Tarski (1951) obtained a transfer principle for real closed ordered fields.

Proposition 13 (Completeness, Transfer Principle (Tarski, 1951)). *Let* $\mathcal{A} = (A, +, \cdot, >, 0, 1)$ *and* $\mathcal{B} = (B, +, \cdot, >, 0, 1)$ *be two real closed ordered fields, and let* Γ *be a sentence of the language* $\mathcal{L}_0$. *If* Γ *is true in* $\mathcal{A}$, *then* Γ *is true in* $\mathcal{B}$.

As another corollary of quantifier elimination, Tarski (1951) characterized the definable sets in real closed ordered fields as the semialgebraic sets:

Proposition 14 (Definability Principle (Tarski, 1951, footnote 13)).

Let $\mathcal{A}_K = (A, +, \cdot, >, 0, 1, a_k)_{k \in K}$ *be a real closed ordered field. For any* k, *a set* $Y \subseteqq A^k$ *is definable in* $\mathcal{L}_K$ *if and only if* Y *is a semialgebraic set, i.e.,* Y *has the following form:*

$$Y = \{(x_1, \dots, x_j : f^i(x_1, \dots, x_k) = 0 \quad \& \quad g^j(x_1, \dots, x_k) > 0 \\ \text{for all } i = 1, \dots I \text{ and for all } j = 1, \dots J\} \tag{61}$$

for some I, *some* J, *and some polynomials* f^i *and* g^j, *whose coefficients are integers or in* $\{a_k : k \in K\}$.

O-minimality.

We make use of the o-minimality (order-minimality) notion introduced by Pillay and Steinhorn, following van den Dries.

By an *interval* in an $\mathcal{L}$-structure $\mathcal{A} = (A, >, \dots)$ we mean a set of one of these forms, for $a, b \in A \cup \{-\infty, \infty\}$:

$\{x \in A : \mathcal{A} \vDash a > x > b\}$, with $a > b$

$\{x \in A : \mathcal{A} \vDash a \geqq x > b\}$, with $a > b$

$\{x \in A : \mathcal{A} \vDash a > x \geqq b\}$, with $a > b$

$\{x \in A : \mathcal{A} \vDash a \geqq x \geqq b\}$ with $a \geqq b$.

And we say that $\mathcal{A}$ is *o-minimal* if every parametrically definable subset of A is a finite union intervals. A first order theory T of $\mathcal{L}$ is called o-minimal if every model of T is o-minimal.

An alternative approach to o-minimality is through collections of subsets, as follows. If $\mathcal{A} = (A, >)$ is a nonempty linearly ordered set without endpoints, then an *o-minimal structure on* $\mathcal{A}$ is a sequence $\{\mathcal{S}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$:[32]

[32] Cf. (van den Dries, 1996b). This is basically the approach used in the economics paper by Blume and Zame (1992).

1) $\mathcal{S}_n$ is a boolean algebra of subsets of A^n, with $A^n \in \mathcal{S}_n$.

2) $X \in \mathcal{S}_n \Rightarrow X \times A$ and $A \times X$ belong to $\mathcal{S}_n$.

3) $\{(x_1, \dots, x_n) \in A^n : x_i = x_j\} \in \mathcal{S}_n$ for $1 \leqq i < j \leqq n$.

4) $X \in \mathcal{S}_{n+1} \Rightarrow \pi(X) \in \mathcal{S}_n$, where $\pi : A^{n+1} \to A^n$ is given by $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.

5) $\{x\} \in \mathcal{S}_1$ for each $x \in A$, and $\{(x, y) \in A^2 : x < y\} \in \mathcal{S}_2$.

6) (Minimality axiom) Every set in $\mathcal{S}_1$ is a finite union of intervals and points.

PROPOSITION 15.
Let the $\mathcal{L}$-structure $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$ be a real closed ordered field expansion. If $\mathcal{A}$ is o-minimal for $\mathcal{L}$, then $\delta(\mathcal{A}) \preccurlyeq \mathcal{A}$ for $\mathcal{L}$.

REMARK 6. It follows from Propositions 15, 6, and 5 that the theory of any o-minimal expansion of a real closed ordered field has a prime model.

PROOF OF PROPOSITION 15. (Following the proof of Pillay and Steinhorn (1986, Lemma 3.2).) To apply the Tarski-Vaught criterion, Proposition 2, suppose that $\phi(v_0, v_1, \dots, v_n)$ is an $\mathcal{L}$-formula and:

$$\mathcal{A} \vDash \exists v_0 \phi(v_0, \frac{a_1}{v_1}, \dots, \frac{a_n}{v_n}) \qquad \text{for some } a_1, \dots, a_n \in \delta(A). \tag{62}$$

We must show:

$$\mathcal{A} \vDash \phi(\frac{a_0}{v_0}, \frac{a_1}{v_1}, \dots, \frac{a_n}{v_n}) \qquad \text{for some } a_0 \in \delta(A). \tag{63}$$

For that it suffices to find an $\mathcal{L}$-definable element of:

$$X = \{x \in A : \mathcal{A} \vDash \phi(\frac{x}{v_0}, \frac{a_1}{v_1}, \dots, \frac{a_n}{v_n})\}. \tag{64}$$

Since each a_i is definable, clearly X is definable. And X is nonempty, by (62). Then the o-minimality of $\mathcal{A}$ implies that X is a union of finitely many intervals (including singletons), whose endpoints are clearly definable. Then it is easy to pick a definable element from X.

For example, if any of the intervals contains a least point, then we can $\mathcal{L}$-define an element of X as the least element of X that has nearby greater X-elements, but no nearby lesser X-elements. And if one of the intervals contains a greatest point, we can analogously $\mathcal{L}$-define an element of X.

It remains to find a definable element of X when all the intervals are open. First, if $X = (-\infty, +\infty)$, so $X = A$, then 0 is a definable element. Second, if

$X = (\bar{a}, +\infty)$, then $\bar{a}+1$ is a definable element of X. An analogous argument holds for $X = (-\infty, \bar{a})$. Finally, if none of these cases holds, there must be a first nonempty open interval $(\bar{a}, \bar{\bar{a}}) \subseteqq X)$ for some $\bar{a}, \bar{\bar{a}} \in A$. These endpoints are definable, so their midpoint $(\bar{a}+\bar{\bar{a}})/2$ is a definable member of X. Q.E.D.

PROPOSITION 16 (PILLAY AND STEINHORN).
Let $\mathcal{A} = (A, +, \cdot, >, 0, 1, \dots)$ be an o-minimal ordered field extension. Then $\mathcal{A}$ is a real closed field expansion.

PROOF. Cf. Pillay and Steinhorn (1986, Theorem 2.3) and Marker (1996, Theorem 5.6). Q.E.D.

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