Multiple Equilibria in Exchange Economies with Homothetic, Nearly Identical Preferences

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Steven Gjerstad

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, MN 55455

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Steven Gjerstad

Economic Science Laboratory
McClelland Hall 116
University of Arizona
Tucson, AZ 85721

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Abstract

For agents with identical homothetic preferences (but possibly different endowments), aggregate excess demand can be derived from maximization of a utility function of a representative agent whose endowment is the sum of the individuals' endowments. Such an economy has a unique equilibrium. In this paper, a metric ρ is defined on the set $\mathcal{P}$ of preference relations representable by CES utility functions. It is then shown that there are agents whose preference relations in $\mathcal{P}$ are arbitrarily close to one another in the metric ρ, and there are endowments for these agents, such that the resulting exchange economy has multiple Walrasian equilibria.

KEYWORDS: Exchange economies, multiple equilibria, preference aggregation, representative agent economies.

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1 Introduction

A classical result on preference aggregation, due to Antonelli [1886] and rediscovered by Gorman [1953] and Nataf [1953], demonstrates that preference aggregation is possible when agents have identical homothetic preferences. For a summary of these results, see Chipman [1974]. In the context of an exchange economy, aggregation implies that the economy has a unique Walrasian equilibrium (as in Arrow and Hahn [1971], Chapter 9).

In this paper, a parameterized class $\mathcal{E}_M$ of exchange economies with multiple Walrasian equilibria is constructed. In the class $\mathcal{E}_M$ the parameter pair (a, r) characterizes the economy. It is shown that it is possible to choose an exchange economy $\mathcal{E}(a, r) \in \mathcal{E}_M$ such that the agents in $\mathcal{E}(a, r)$ have preferences which are arbitrarily close to one another (in a given metric), and such that the economy has multiple Walrasian equilibria. This demonstrates that there are examples of preference profiles and endowment patterns which may be aggregated, but for which some small perturbations of the preference profiles lead to economies for which preferences may not be aggregated.

There are two main cases in the literature for which aggregation has been shown to be possible. In one case, all agents have homothetic preferences and proportional endowments (Chipman [1974], Theorem 4). That is, agent i has preferences represented by a homothetic utility function, and has endowment $\omega_i = \delta_i \cdot \omega$, where $\omega \in R^l_+$, $0 < \delta_i < 1$, and $\sum_{i=1}^N \delta_i = 1$. Mantel [1976] has shown that this result is sensitive to violation of the restriction of proportional endowments. He demonstrates this by showing that any function $F : R^l_+ \to R^l$ satisfying the usual properties of excess demand functions – Walras' law, continuity, boundedness from below, and the boundary condition – can be obtained as the excess demand function of an exchange economy with as few as l individuals, each with homothetic preferences, even if endowments are arbitrarily close to proportional. In the other case in which preferences can be aggregated, all agents have identical homothetic preferences (Chipman [1974], Theorem 3). (In this case there is no restriction on agents' endowments.) In this paper, it is shown that for the latter case in which agents have identical homothetic preferences, the result may be sensitive to arbitrarily small perturbations of preferences, as Mantel has shown for the former case via arbitrarily small perturbations of endowments.

The approach taken in this paper differs from the usual approach taken to demonstrate results concerning multiple equilibria in exchange economies. In that literature it is shown that any continuous function $F : R^l_+ \to R^l$ which satisfies Walras' law, boundedness from below, and a boundary condition, is the result of maximizing behavior by as few as l consumers for some specification of preferences and endowments. The traditional approach – initiated by Sonnenschein [1973, 1974] and extended by Debreu [1974] and Mas-Colell [1977] – is to begin with excess demand functions and construct preferences and endowments which generate the given excess demand. The problem addressed in this paper requires explicit determination of the distance between preference relations for an exchange economy with multiple equilibria, and as a result, it is more natural to begin with preferences and endowments as the primitive notion and treat excess demand as the derived notion.

The paper is organized as follows. In Section 2, a set $\mathcal{P} = \{\mathcal{P}(a, r) : a > 0, -\infty < r < 1\}$ of CES preferences is defined which is large enough to obtain the needed examples of multiple equilibria. A metric ρ is defined on this set of preferences in Section 3, and a closed form representation of the distance between any two members of $\mathcal{P}$ is determined for the metric ρ explicitly in terms of the parameters of the CES utility function representing preferences. In Section 4 the main result is proven: *there exist exchange economies with multiple equilibria in which the distance between the agents' preferences is arbitrarily small.* In Section 5, an experimental design is described which can test predictions of competitive equilibrium theory and theories of convergence to stable equilibria.

2 A set of economies with multiple equilibria

In this section, a set $\mathcal{E} = \{\mathcal{E}(a, r)\}$ of economies is defined, and a subset $\mathcal{E}_M \subset \mathcal{E}$ is shown to have multiple equilibria. The description of $\mathcal{E}(a, r)$ consists of agents' preferences and endowments. Market excess demand for each economy in $\mathcal{E}(a, r)$ is determined from individual excess demand, and a condition on the preference parameters (a, r) is derived which guarantees that $\mathcal{E}(a, r)$ has multiple equilibria.

Definition 1 (Preferences) Let $X = \{(x, y) : x > 0, y > 0\}$ be the consumption space for each agent. Let $\mathcal{P}$ be a class of consumers' preferences with element $\mathcal{P}(a, r) \in \mathcal{P}$ represented by the utility function[1]

$$u_{(a,r)}(x, y) = ((ax)^r + y^r)^{1/r}$$

[1]The set $\mathcal{P}$ contains all the CES utility functions, and their limits as $r \to 0$ and as $r \to -\infty$.

where $a > 0$ and $r \in (-\infty, 0) \cup (0, 1)$. For $r = 0$, let

$$u_{(a,\,0)}(x,\,y) \equiv \lim_{r \to 0} u_{(a,\,r)}(x,\,y) = a^{\frac{1}{2}}\, x^{\frac{1}{2}}\, y^{\frac{1}{2}}\,; \tag{1}$$

for $r = -\infty$, let

$$u_{(a,\,-\infty)}(x,\,y) \equiv \lim_{r \to -\infty} u_{(a,\,r)}(x,\,y) = \min\{ax,\, y\}\,.$$

Note that Equation (1) implies that for all $a > 0$, $b > 0$, $\mathcal{P}(a,\,0) \sim \mathcal{P}(b,\,0)$. In what follows, this identification will be assumed. Also, the preference relation $\mathcal{P}(a,\,r)$ and its utility representation $u_{(a,\,r)}$ are used interchangeably.

Definition 2 (The economy $\mathcal{E}(\mathbf{a},\mathbf{r})$) For each $(a,\,r) \in (0,\,\infty) \times (-\infty,\,1)$, let $\mathcal{E}(a,\,r)$ be the exchange economy with two consumers (or, alternatively, two consumer types with an equal number of each type) where the preferences and endowment of consumer A are $\mathcal{P}(a,\,r)$ and $\omega^A = (1,\,0)$ and consumer B has preferences $\mathcal{P}(b,\,s) = \mathcal{P}(a^{-1},\,r)$ and endowment $\omega^B = (0,\,1)$. Formally, $\mathcal{E}(a,\,r) = \Big((u_{(a,\,r)},\,(1,\,0)),\,(u_{(a^{-1},\,r)},\,(0,\,1))\Big)$.

Individual demand Let $S \equiv \{(p_x,\,p_y) : p_x \geq 0, p_y \geq 0,\, p_x + p_y = 1\}$. For $(p_x,\,p_y) \in S$, let $p = p_x$ and $1 - p = p_y$ so that points $(p_x,\,p_y) \in S$ are identified with $p \in [0,\,1]$. A consumer with preferences represented by $u_{(a,\,r)}$ and income m has demand[2]

$$x_{(a,\,r)}(p,\,m) = \frac{m\, a^{\frac{r}{1-r}}\,(1-p)^{\frac{r}{1-r}}}{p\,(p^{\frac{r}{1-r}} + a^{\frac{r}{1-r}}\,(1-p)^{\frac{r}{1-r}})}\,, \tag{2}$$

and

$$y_{(a,\,r)}(p,\,m) = \frac{m\, p^{\frac{r}{1-r}}}{(1-p)\,(p^{\frac{r}{1-r}} + a^{\frac{r}{1-r}}\,(1-p)^{\frac{r}{1-r}})}\,. \tag{3}$$

Excess demand From the individual demand functions for X in Equation (2) we obtain the individual excess demand functions for commodity X by substituting endowment values for m and by subtracting the agents' endowments of X. Market excess demand for X is the sum of individual excess demands for X. The excess demand for good X by consumer A as a function of p is:

$$\begin{aligned} Z_x^A(p) &= \frac{a^{\frac{r}{1-r}}\,(1-p)^{\frac{r}{1-r}}}{p^{\frac{r}{1-r}} + a^{\frac{r}{1-r}}\,(1-p)^{\frac{r}{1-r}}} - 1 \\ &= \frac{-p^{\frac{r}{1-r}}}{p^{\frac{r}{1-r}} + a^{\frac{r}{1-r}}\,(1-p)^{\frac{r}{1-r}}}\,. \end{aligned}$$

[2]Equations (2) and (3) hold in the limit as $r \to -\infty$ for $u_{(a,\,-\infty)}$.

For consumer B the excess demand for commodity X is

$$Z_x^B(p) \;=\; \frac{(1-p)^{\frac{1}{1-r}}}{p\,(a^{\frac{r}{1-r}}\,p^{\frac{r}{1-r}} + (1-p)^{\frac{r}{1-r}})}\,.$$

Market excess demand for commodity X is

$$Z_x(p) \;=\; \frac{-p^{\frac{r}{1-r}}}{p^{\frac{r}{1-r}} + a^{\frac{r}{1-r}}\,(1-p)^{\frac{r}{1-r}}} + \frac{(1-p)^{\frac{1}{1-r}}}{p\,(a^{\frac{r}{1-r}}\,p^{\frac{r}{1-r}} + (1-p)^{\frac{r}{1-r}})}\,.$$

Market equilibrium Lemmas 1 through 3 below, and Theorem 1 together show that there are parameter values $(a,\,r)$ for which the economy $\mathcal{E}(a,\,r)$ has multiple equilibria. Lemma 1 shows that for all $(a,\,r) \in (0,\,\infty)\times(-\infty,\,1)$ there is an equilibrium of $\mathcal{E}(a,\,r)$ with supporting price $p = \frac{1}{2}$. In Lemma 2 the slope of the excess demand $Z_x(p)$ is determined at this equilibrium. Lemma 3 describes a condition on the parameter values which guarantee that the slope of $Z_x(p)$ is positive at $p = \frac{1}{2}$. Theorem 1 shows that if the condition of Lemma 3 is satisfied, then the economy $\mathcal{E}(a,\,r)$ has multiple competitive equilibria.

Lemma 1 *For each $(a,\,r) \in (0,\,\infty) \times (-\infty,\,1)$, there is an equilibrium of $\mathcal{E}(a,\,r)$ with supporting price $p^* = \frac{1}{2}$ and allocation*

$$\left(x^A(\tfrac{1}{2}),\,y^A(\tfrac{1}{2})\right) = \left(\frac{a^{\frac{r}{1-r}}}{1+a^{\frac{r}{1-r}}},\,\frac{1}{1+a^{\frac{r}{1-r}}}\right)$$

and

$$\left(x^B(\tfrac{1}{2}),\,y^B(\tfrac{1}{2})\right) = \left(\frac{1}{1+a^{\frac{r}{1-r}}},\,\frac{a^{\frac{r}{1-r}}}{1+a^{\frac{r}{1-r}}}\right).$$

Proof Since $Z_x(\frac{1}{2}) = 0$, $p^* = \frac{1}{2}$ is an equilibrium price. Evaluation of the demand functions from equations (2) and (3) give the allocations. ∎

Lemma 2 *Let $\sigma = \frac{r}{1-r}$. For each $(a,\,r) \in (0,\,\infty)\times(-\infty,\,1)$, the slope of the excess demand at the equilibrium price $p^* = \frac{1}{2}$ is*

$$\frac{d}{dp}\,Z_x(p)|_{p=\frac{1}{2}} = -\,\frac{4\,[(2\,\sigma+1)\,a^\sigma+1]}{(1+a^\sigma)^2}\,. \tag{4}$$

Proof With $\sigma = \frac{r}{1-r}$,

$$Z_x(p) = \frac{-p^\sigma}{p^\sigma + a^\sigma\,(1-p)^\sigma} + \frac{(1-p)^{\sigma+1}}{p\,(a^\sigma\,p^\sigma + (1-p)^\sigma)}\,. \tag{5}$$

Then

$$\frac{d}{dp}\,Z_x(p) = -\,\frac{(1-p)^\sigma}{p}\left[\frac{\sigma\,a^\sigma\,p^\sigma}{(1-p)\,(p^\sigma + a^\sigma\,(1-p)^\sigma)^2} + \frac{(\sigma+1)\,a^\sigma\,p^\sigma + (1-p)^\sigma}{p\,(a^\sigma\,p^\sigma + (1-p)^\sigma)^2}\right]. \qquad (6)$$

Evaluation of $\frac{d}{dp}\,Z_x(p)$ at $p^* = \frac{1}{2}$ results in Equation (4). ■

Lemma 3 *Let*

$$\hat{a}(r) = \left(\frac{r-1}{r+1}\right)^{\frac{1-r}{r}}. \qquad (7)$$

Then for each $r < -1$, any exchange economy $\mathcal{E}(a,\,r)$ with $a < \hat{a}(r)$ has an excess demand function $Z_x(p)$ with positive slope at $p^* = \frac{1}{2}$.

Proof From Equation (4), at the equilibrium price $p^* = \frac{1}{2}$,

$$\tfrac{1}{4}\,(1+a^\sigma)^2\,\frac{d}{dp}\,Z_x(p)\big|_{p=\frac{1}{2}} = -[(2\,\sigma+1)\,a^\sigma + 1].$$

Since $\frac{1}{4}\,(1+a^\sigma)^2 > 0$ for all $(a,\,r) \in (0,\,\infty) \times (-\infty, 1)$, the slope of $Z_x(p)$ at $p^* = \frac{1}{2}$ is positive if and only if $(2\,\sigma+1)\,a^\sigma + 1 < 0$. Let $r < -1$. Then $\sigma < -\frac{1}{2}$ and $2\,\sigma + 1 < 0$ so

$$a^\sigma > -\,\frac{1}{2\,\sigma+1}. \qquad (8)$$

Substituting $\sigma = \frac{r}{1-r}$ into Equation (8) and solving for a shows that the slope of $Z_x(p)$ is positive at $p^* = \frac{1}{2}$ whenever $a < \hat{a}(r)$. ■

Theorem 1 *For each $r < -1$, any exchange economy $\mathcal{E}(a,\,r)$ with $a < \hat{a}(r)$ has multiple Walrasian equilibria.*

Proof For consumer A, $Z_x^A(0) = -1$. For consumer B, an application of L'Hôspital's rule shows that $\lim_{p\to 0} Z_x^B(p) = \infty$, therefore $\lim_{p\to 0} Z_x(p) = \infty$. Since $\frac{d}{dp}\,Z_x(p)|_{p=\frac{1}{2}} > 0$ and $Z_x(\frac{1}{2}) = 0$, it follows that for $p < \frac{1}{2}$ but sufficiently close to $p = \frac{1}{2}$, $Z_x(p) < 0$. It follows from the intermediate value theorem that there is a root $\tilde{p}$ of $Z_x(p)$ with $\tilde{p} \in (0,\,\frac{1}{2})$. Also, evaluation of Equations (5) and (6) shows that for $r < 0$, $\lim_{p\to 1} Z_x(p) = 0$ and $\lim_{p\to 1} \frac{d}{dp} Z_x(p) = \infty$, so $Z_x(p) < 0$ in an interval $(\alpha,\,1)$ for some $\alpha < 1$. (For an example see the graph of $Z_x(p,\,1)$ in Figure 3.) It then follows from the intermediate value theorem that there is another root $\bar{p}$ of $Z_x(p)$ with $\bar{p} \in (\frac{1}{2},\,1)$. ■

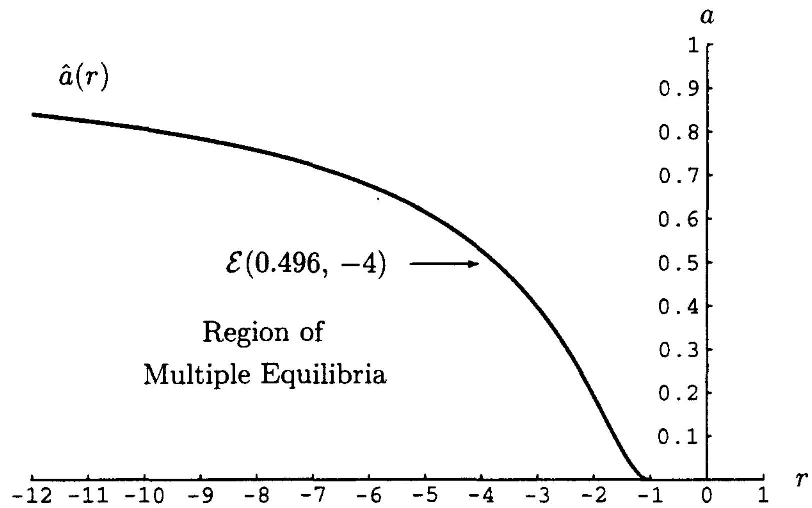

Figure 1: Region of multiple equilibria.

Example 1 (Multiple equilibria) Figure 1 shows the region in the parameter space for which the economies $\mathcal{E}(a, r)$ have multiple equilibria. As an example of an exchange economy with multiple equilibria, consider the economy $\mathcal{E}(0.496, -4)$. Since $\hat{a}(-4) = 0.5281$, this economy has multiple equilibria, by Theorem 1. An Edgeworth diagram for this economy is shown in Figure 2, and the excess demand for commodity X is shown as $Z_x(p, 1)$ in Figure 3.

Example 2 (Comparing equilibria) Although the economies $\mathcal{E}(a, r)$ have multiple equilibria for many values of (a, r), some natural comparative statics exercises are straightforward for these economies. Consider, for example, changes in equilibrium prices as a function of the endowment ω_x^A for consumer A (or consumer type A). Let

$$\mathcal{E}(a, r, \omega_x^A) = \Big((u_{(a,r)}, (\omega_x^A, 0)), (u_{(a^{-1},r)}, (0, 1))\Big).$$

The excess demand for commodity X is

$$Z_x(p, \omega_x^A) = \frac{-\omega_x^A \, p^{\frac{r}{1-r}}}{p^{\frac{r}{1-r}} + a^{\frac{r}{1-r}} (1-p)^{\frac{r}{1-r}}} + \frac{(1-p)^{\frac{1}{1-r}}}{p \left(a^{\frac{r}{1-r}} p^{\frac{r}{1-r}} + (1-p)^{\frac{r}{1-r}}\right)}.$$

Then $Z_x(p, \omega_x^A)$ is monotone decreasing in ω_x^A, $\lim_{p\to 0} Z_x(p, \omega_x^A) = \infty$, and $Z_x(1, \omega_x^A) \le 0$, so an application of Theorem 1 from Milgrom and Roberts [1994] shows that the lowest equilibrium value and the highest equilibrium value are both decreasing in ω_x^A. Figure 3

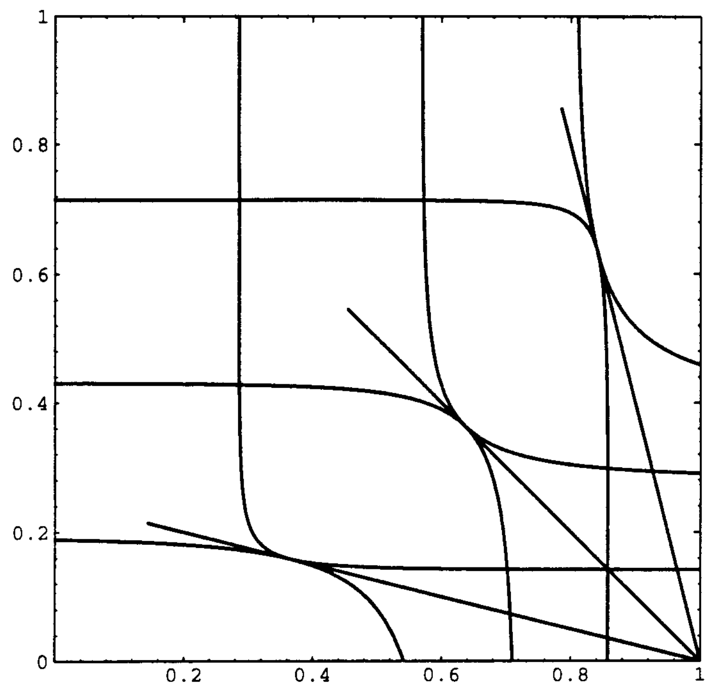

Figure 2: Edgeworth diagram for the exchange economy $\mathcal{E}(0.496, -4)$.

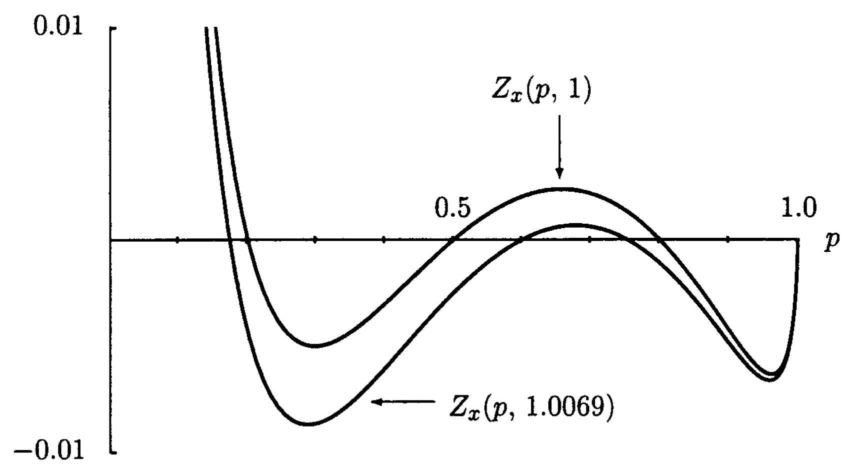

Figure 3: Excess demand for good X for $\mathcal{E}(0.496, -4, 1)$ and $\mathcal{E}(0.496, -4, 1.0069)$.

depicts an example of this. In that figure there are two excess demand curves shown, one for the economy $\mathcal{E}(0.496,\ -4,\ 1)$, and one for $\mathcal{E}(0.496,\ -4,\ 1.0069)$. The figure shows that for this example, as the endowment of X for consumer A increases, the equilibrium prices decrease at the two stable equilibria, but the equilibrium price increases at the unstable equilibrium. Figure 5 shows the graph of the equilibrium correspondence for the economies $\mathcal{E}(0.496,\ -4,\ \omega_x^A)$ with $\omega_x^A \in (0.98,\ 1.02)$. Note that the graph of this correspondence is the intersection of the two 2-dimensional manifolds $z = Z_x(p,\ \omega_x^A)$ and $z = 0$, as in Figure 4. Similarly, the excess demand functions $Z_x(p,\ 1)$ and $Z_x(p,\ 1.0069)$ shown in Figure 3 are the intersections of $z = Z_x(p,\ \omega_x^A)$ with the planes $\omega_x^A = 1$ and $\omega_x^A = 1.0069$, respectively.

The next proposition shows that the equilibrium $p^*(t)$ is increasing in t at an unstable equilibrium of a regular two commodity exchange economy when there is a parameter t such that $Z_x(p,\ t)$ is decreasing in t. (The argument is stated in terms of changes in ω_x^A, but is true for any interpretation of the parameter t with the property that $\frac{d}{dp} Z_x(p,\ t)$ exists and is positive, and that $\frac{d}{dt} Z_x(p,\ t)$ exists and is negative.)

Proposition 1 *Suppose that $\bar{p}$ supports an unstable equilibrium of the regular economy $\mathcal{E}(\bar{a},\ \bar{r},\ \bar{\omega}_x^A)$. Then there is a neighborhood $(\omega_*,\ \omega^*)$ of $\bar{\omega}_x^A$ and a function $p^* : (\omega_*,\ \omega^*) \to R_+$ such that $Z_x(p^*(w_x^A),\ \omega_x^A) = 0$ for all $\omega_x^A \in (\omega_*,\ \omega^*)$, and $\frac{d}{d\omega_x^A} p^*(\omega_x^A) > 0$.*

Proof Since $\mathcal{E}(\bar{a},\ \bar{r},\ \bar{\omega}_x^A)$ is a regular economy, there is a neighborhood $(\omega_*,\ \omega^*)$ of $\bar{\omega}_x^A$ such that every economy $\mathcal{E}(\bar{a},\ \bar{r},\ \omega_x^A)$ with $\omega_x^A \in (\omega_*,\ \omega^*)$ is also regular. Since the equilibrium of $\mathcal{E}(\bar{a},\ \bar{r},\ \bar{\omega}_x^A)$ with supporting price $\bar{p}$ is an unstable equilibrium of a regular economy, $\frac{d}{dp} Z_x(\bar{p},\ \bar{\omega}_x^A) > 0$.

Therefore, the implicit function theorem states that there is an open set $(p_0,\ p_1)$ containing $\bar{p}$, an open set $(\omega_0,\ \omega_1)$ containing $\bar{\omega}_x^A$, and a function $p^*(\omega_x^A)$ such that

$$Z_x(p^*(\omega_x^A),\ \omega_x^A) = 0$$

and

$$\frac{d}{d\omega_x^A} p^*(\omega_x^A) = -\frac{D_2\, Z_x(p^*(\omega_x^A),\ \omega_x^A)}{D_1\, Z_x(p^*(\omega_x^A),\ \omega_x^A)}.$$

The denominator of this expression is positive in $(\omega_*,\ \omega^*)$ and the numerator is everywhere negative, so that the expression is positive. ■

Note 1 Theorem 1 in Milgrom and Roberts treats only the extreme zeros of functions $f\ :\ [0,\ 1] \times \mathbf{T} \to R$. The critical feature of these zeros (under their hypotheses that

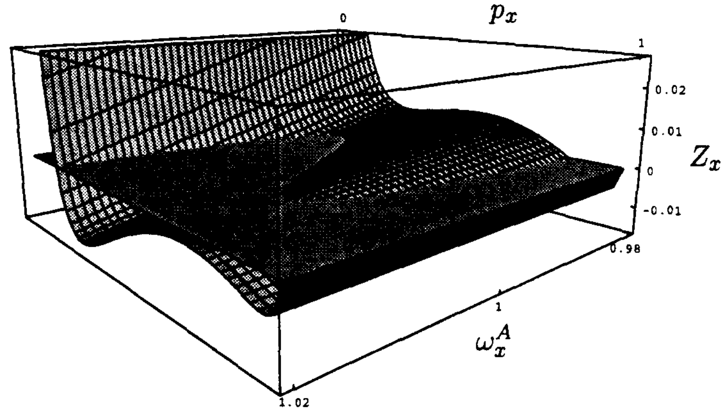

Figure 4: The excess demand function $Z_x(p_x,\, \omega_x^A)$ and the plane $Z_x = 0$.

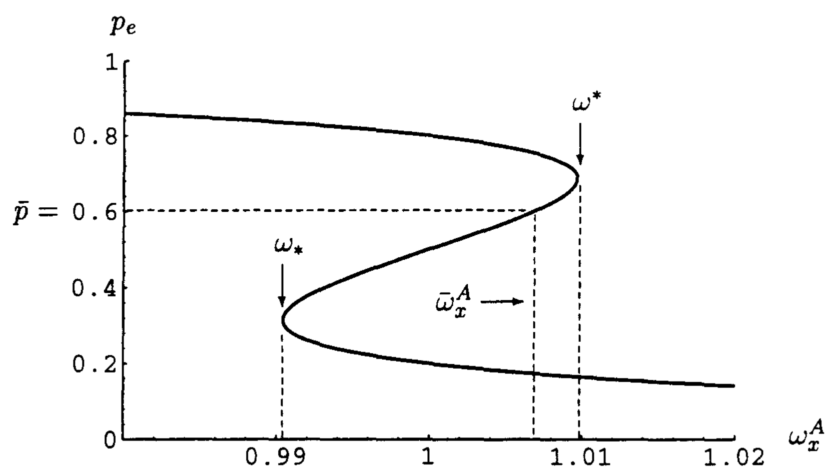

Figure 5: Equilibrium price correspondence $E[\mathcal{E}(0.496,\, -4,\, \omega_x^A)]$.

$f(0,\, t) \geq 0$ and $f(1,\, t) \leq 0$) is that these zeros occur with f crossing from above. If there are multiple zeros of $f(x,\, t)$ in $(0,\, 1)$, and if the economy is regular, then there is at least one zero $\bar{x}$ for which $f(\bar{x},\, t)$ is increasing. At these values the comparative statics conclusions are opposite those for the extreme zeros. This is apparent also in Figure 5 where along the middle sheaf of the equilibrium correspondence, the equilibrium price of X increases as the amount of X available increases.

3 A metric on preferences

In Figure 1 in the previous section, the function $\hat{a}(r)$ is shown. Economies $\mathcal{E}(a, r)$ with $a < \hat{a}(r)$ have multiple equilibria and $\lim_{r \to -\infty} \hat{a}(r) = 1$. Since the agents of each economy $\mathcal{E}(1, r)$ have identical preferences, it is apparent that in the Euclidean metric on the parameter space, there are economies with multiple equilibria for which agents' preferences are arbitrarily close to identical. Yet the Euclidean metric on the parameter space is not a metric on these preferences: each relation of the form $\mathcal{P}(a, 0)$ represents the same preferences. For this reason, the notion of 'nearness' of preferences must be defined on the utility function representing preferences, on the preference relation, or on the demand functions generated by preferences.

A topological notion of 'nearness' is inadequate for the result of this paper. For an arbitrary but fixed $\bar{r}$, a sufficiently small neighborhood of the economy $\mathcal{E}(1, \bar{r})$ will have an empty intersection with $\{(a, r) : a < \hat{a}(r)\}$. (Compare Debreu [1984], p. 188.) This eliminates use of a topological notion such as uniform convergence on compacta, which is formulated in terms of utility functions. The Hausdorff metric on preference relations, proposed by Debreu [1969], is another alternative. Debreu points out that the Hausdorff metric is not always finite. Proposition 2 at the end of this section shows that for any two distinct *homothetic* preference relations, the Hausdorff distance between them is infinite. Since the main theorem of this paper concerns agents with homothetic preferences, the Hausdorff metric is inadequate.

As a result of these considerations, in this section a metric is defined on the agents' *demand functions*. The function ρ of Definition 3 has a transparent interpretation: the preferences of two consumers are similar if, given the same choice sets, the resulting choice behavior is similar for each choice set. Following the definition, a closed form representation of the distance between any two elements of $\mathcal{P}$ is determined in Lemma 4. In Theorem 2 it is shown that ρ is a metric on $\mathcal{P}$.

Definition 3 For utility functions $u_{(a,r)}$ and $u_{(b,s)}$ and for an arbitrary fixed $m > 0$ define a function $\rho : \mathcal{P} \times \mathcal{P} \to R_+$ by

$$\rho(u_{(a,r)}, u_{(b,s)}) \equiv \max_{p \in \{\frac{1}{e+1}, \frac{e}{e+1}\}} \left(\left| \ln \frac{x_{(a,r)}(p, m)}{x_{(b,s)}(p, m)} \right| + \left| \ln \frac{y_{(a,r)}(p, m)}{y_{(b,s)}(p, m)} \right| \right).$$

Lemma 4 *The function ρ in Definition 3 takes values on $\mathcal{P} \times \mathcal{P}$ given by*

$$\rho(u_{(a,\,r)},\, u_{(b,\,s)}) = \max\left\{\left|\frac{r\,(1-\ln a)}{1-r} - \frac{s\,(1-\ln b)}{1-s}\right|, \left|\frac{r\,(1+\ln a)}{1-r} - \frac{s\,(1+\ln b)}{1-s}\right|\right\}. \tag{9}$$

Proof There are two calculations needed to verify Equation (9). The value of

$$\left|\ln\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)}\right| + \left|\ln\frac{y_{(a,\,r)}(p,\,m)}{y_{(b,\,s)}(p,\,m)}\right| \tag{10}$$

must be calculated at $p = \frac{1}{e+1}$ and then at $p = \frac{e}{e+1}$.

Calculation at $\mathbf{p = \frac{1}{e+1}}$. From Equation (2),

$$\frac{x_{(a,\,r)}(\frac{1}{e+1},\, m)}{x_{(b,\,s)}(\frac{1}{e+1},\, m)} = \frac{a^{\frac{r}{1-r}}}{b^{\frac{s}{1-s}}}\,\frac{e^{\frac{r}{1-r}}}{e^{\frac{s}{1-s}}}\,\frac{1+b^{\frac{s}{1-s}}\,e^{\frac{s}{1-s}}}{1+a^{\frac{r}{1-r}}\,e^{\frac{r}{1-r}}}. \tag{11}$$

From Equation (3),

$$\frac{y_{(a,\,r)}(\frac{1}{e+1},\, m)}{y_{(b,\,s)}(\frac{1}{e+1},\, m)} = \frac{1+b^{\frac{s}{1-s}}\,e^{\frac{s}{1-s}}}{1+a^{\frac{r}{1-r}}\,e^{\frac{r}{1-r}}}. \tag{12}$$

Assume that

$$\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)} \geq 1, \tag{13}$$

and note that for fixed $p \in (0,\,1)$ and fixed $m > 0$, this implies that

$$\frac{y_{(a,\,r)}(p,\,m)}{y_{(b,\,s)}(p,\,m)} \leq 1$$

since both (non-satiated) agents are choosing points from the same budget set. Then

$$\left|\ln\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)}\right| + \left|\ln\frac{y_{(a,\,r)}(p,\,m)}{y_{(b,\,s)}(p,\,m)}\right| = \ln\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)} - \ln\frac{y_{(a,\,r)}(p,\,m)}{y_{(b,\,s)}(p,\,m)}$$

and by Equations (11) and (12)

$$\begin{aligned}\ln\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)} - \ln\frac{y_{(a,\,r)}(p,\,m)}{y_{(b,\,s)}(p,\,m)} &= \ln\frac{a^{\frac{r}{1-r}}}{b^{\frac{s}{1-s}}} + \ln\frac{e^{\frac{r}{1-r}}}{e^{\frac{s}{1-s}}} \\ &= \frac{r\,(1+\ln a)}{1-r} - \frac{s\,(1+\ln b)}{1-s}.\end{aligned} \tag{14}$$

If

$$\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)} < 1,$$

then the value of (10) at $p = \frac{1}{e+1}$ is the same expression as in Equation (14), but has the opposite sign. In either case, the expression is positive, so the value of (10) at $p = \frac{1}{e+1}$ is

$$\left|\frac{r\,(1+\ln a)}{1-r} - \frac{s\,(1+\ln b)}{1-s}\right|.$$

Calculation at $\mathbf{p} = \frac{\mathbf{e}}{\mathbf{e+1}}$. From Equation (2),

$$\frac{x_{(a,r)}(\frac{e}{e+1}, m)}{x_{(b,s)}(\frac{e}{e+1}, m)} = \frac{a^{\frac{r}{1-r}}}{b^{\frac{s}{1-s}}} \frac{e^{\frac{s}{1-s}} + b^{\frac{s}{1-s}}}{e^{\frac{r}{1-r}} + a^{\frac{r}{1-r}}}. \tag{15}$$

From Equation (3),

$$\frac{y_{(a,r)}(\frac{e}{e+1}, m)}{y_{(b,s)}(\frac{e}{e+1}, m)} = \frac{e^{\frac{r}{1-r}}}{e^{\frac{s}{1-s}}} \frac{e^{\frac{s}{1-s}} + b^{\frac{s}{1-s}}}{e^{\frac{r}{1-r}} + a^{\frac{r}{1-r}}}. \tag{16}$$

Assume that Equation (13) holds. Then

$$\left|\ln \frac{x_{(a,r)}(p, m)}{x_{(b,s)}(p, m)}\right| + \left|\ln \frac{y_{(a,r)}(p, m)}{y_{(b,s)}(p, m)}\right| = \ln \frac{x_{(a,r)}(p, m)}{x_{(b,s)}(p, m)} - \ln \frac{y_{(a,r)}(p, m)}{y_{(b,s)}(p, m)}$$

and by Equations (15) and (16)

$$\begin{aligned} \ln \frac{x_{(a,r)}(p, m)}{x_{(b,s)}(p, m)} - \ln \frac{y_{(a,r)}(p, m)}{y_{(b,s)}(p, m)} &= \ln \frac{a^{\frac{r}{1-r}}}{b^{\frac{s}{1-s}}} - \ln \frac{e^{\frac{r}{1-r}}}{e^{\frac{s}{1-s}}} \\ &= \frac{r(-1+\ln a)}{1-r} - \frac{s(-1+\ln b)}{1-s}. \end{aligned} \tag{17}$$

If

$$\frac{x_{(a,r)}(p, m)}{x_{(b,s)}(p, m)} < 1,$$

then the value of (10) is the same expression as in Equation (17), but has the opposite sign. In either case, the expression is positive, so the value of (10) at $p = \frac{e}{e+1}$ is

$$\left|\frac{r(1-\ln a)}{1-r} - \frac{s(1-\ln b)}{1-s}\right|. \blacksquare$$

Theorem 2 *The function $\rho : \mathcal{P} \times \mathcal{P} \to R_+$ is a metric on $\mathcal{P}$.*

Proof The proof is a straightforward verification of properties 1 - 4 below, which characterize metrics.

1. For any (a, r) and (b, s), $\rho(u_{(a,r)}, u_{(b,s)}) = 0$ if and only if $(a, r) = (b, s)$ or $r = s = 0$.

   From the tangency condition for utility maximization,

$$y_{(a,r)}(p, m) = \frac{p^{\frac{r}{1-r}}}{a^{\frac{r}{1-r}}(1-p)^{\frac{r}{1-r}}} x_{(a,r)}(p, m).$$

   Similarly,

$$y_{(b,s)}(p, m) = \frac{p^{\frac{s}{1-s}}}{b^{\frac{s}{1-s}}(1-p)^{\frac{s}{1-s}}} x_{(b,s)}(p, m).$$

Then

$$\frac{y_{(a,\,r)}(p,\,m)}{y_{(b,\,s)}(p,\,m)} = \frac{b^{\frac{s}{1-s}}}{a^{\frac{r}{1-r}}}\,\frac{(1-p)^{\frac{s}{1-s}}}{(1-p)^{\frac{r}{1-r}}}\,\frac{p^{\frac{r}{1-r}}}{p^{\frac{s}{1-s}}}\,\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)}. \tag{18}$$

Note that it is not possible that both

$$\frac{x_{(a,\,r)}(p,\,m)}{x_{(b,\,s)}(p,\,m)} = 1 \quad\text{and}\quad \frac{y_{(a,\,r)}(p,\,m)}{y_{(b,\,s)}(p,\,m)} = 1$$

at both $p = \frac{1}{e+1}$ and $p = \frac{e}{e+1}$, unless $(a,\,r) = (b,\,s)$ or unless $r = s = 0$. To see this, assume that both these ratios are equal to 1. Then by Equation (18)

$$\frac{b^{\frac{s}{1-s}}}{a^{\frac{r}{1-r}}}\,\frac{(1-p)^{\frac{s}{1-s}}}{(1-p)^{\frac{r}{1-r}}}\,\frac{p^{\frac{r}{1-r}}}{p^{\frac{s}{1-s}}} = 1.$$

Then

$$\frac{b^{\frac{s}{1-s}}}{a^{\frac{r}{1-r}}}\left(\frac{1-p}{p}\right)^{\frac{s}{1-s}-\frac{r}{1-r}} = 1 \tag{19}$$

at both $p = \frac{1}{e+1}$ and $p = \frac{e}{e+1}$. Evaluation of Equation (19) at these two values of p results in

$$b^{\frac{s}{1-s}}\,e^{\frac{s}{1-s}} = a^{\frac{r}{1-r}}\,e^{\frac{r}{1-r}} \tag{20}$$

and

$$b^{\frac{s}{1-s}}\,e^{\frac{r}{1-r}} = a^{\frac{r}{1-r}}\,e^{\frac{s}{1-s}}. \tag{21}$$

Dividing Equation (20) by Equation (21) results in

$$e^{\frac{s}{1-s}-\frac{r}{1-r}} = e^{\frac{r}{1-r}-\frac{s}{1-s}}$$

so that $r = s$. For $r = s \neq 0$ Equation (21) implies that $a = b$. For $r = s = 0$, there is no restriction on a and b, but note that from Equation (1), for $r = s = 0$, all values of a and b represent the same preferences.

2. The function ρ is symmetric: $\rho(u_{(a,\,r)},\,u_{(b,\,s)}) = \rho(u_{(b,\,s)},\,u_{(a,\,r)})$.

   This follows immediately from Definition 3.

3. For any pair $(a,\,r)$ and $(b,\,s)$, $\rho(u_{(a,\,r)},\,u_{(b,\,s)})$ is finite.

   For any values of $(a,\,r)$ and $(b,\,s)$ in the relevant ranges ($a > 0$, $b > 0$, $r < 1$, $s < 1$) the value of Equation (9) is clearly finite.

4. For (a, r), (b, s), and (c, t), the triangle inequality is satisfied.

   The triangle inequality applies to quotients, and inequalities are preserved by the max function. ■

Theorem 2 shows that the function of Definition 3 is a metric on $\mathcal{P}$. Lemma 5 and Proposition 2 below show that the Hausdorff metric is ill-behaved on the set of CES utility functions, in the sense that the distance between any two distinct CES utility functions is infinite.

Let t be the metric on $X \times X$ defined by $t[(a, b), (a', b')] = |a-a'|+|b-b'|$. The Hausdorff hemi-distance $h^*[P, P']$ between sets P and P' is determined by first finding, for any point $(a, b) \in P$, $h_*[(a, b), P'] = \inf_{(a', b') \in P'} t[(a, b), (a', b')]$, and then taking the supremum over $(a, b) \in P$ to get $h^*[P, P'] = \sup_{(a, b) \in P} h_*[(a, b), P']$. The Hausdorff distance is then

$$d[P, P'] = \max \{h^*[P, P'], h^*[P', P]\}.$$

Proposition 2 shows that if P and P' are both homothetic preference relations – that is, if they can be represented by a monotonic transformation of a utility function which is homogeneous of degree 1 – then $d[P, P'] = \infty$. The proof of Proposition 2 makes use of the fact that P' is closed. This is shown in Lemma 5 below.

Lemma 5 *A preference relation P which can be represented by a continuous utility function u is closed.*

Proof Let $u : X \to R$ represent P. Let $\{(a_n, b_n)\} \subset P$ be a sequence converging to $(a, b) \in X \times X$. Assume that $(a, b) \notin P$. Then $u(b) > u(a)$. Let $\epsilon = \frac{1}{3}(u(b) - u(a))$. Then there is an N_a such that $a_n \in \{x \in X : u(x) < u(a) + \epsilon\}$ for all $n > N_a$ and there is an N_b such that $b_n \in \{x \in X : u(x) > u(b) - \epsilon\}$ for all $n > N_b$. Let $N = \max\{N_a, N_b\}$. Then for $n > N$, $u(b_n) - u(a_n) > \epsilon$. This contradicts the fact that $\{(a_n, b_n)\} \subset P$. ■

Proposition 2 *Let $P \subset X \times X$ and $P' \subset X \times X$ be distinct homothetic preference relations. Then the Hausdorff distance between P and P' is infinite.*

Proof Fix $(\bar{a}, \bar{b}) \in P$ and define $\alpha^* \equiv h_*[(\bar{a}, \bar{b}), P']$. Let $\{(a'_n, b'_n)\} \subset P'$ be a sequence of points such that $t[(\bar{a}, \bar{b}), (a'_n, b'_n)] < \alpha^* + \frac{1}{n}$. Since each point (a'_n, b'_n) lies within

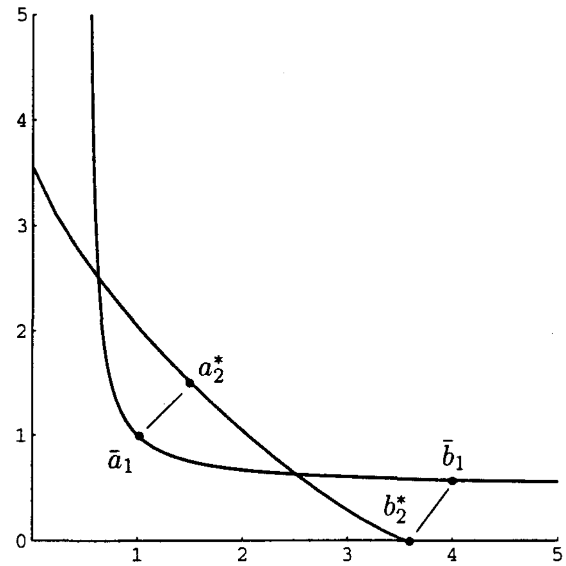

Figure 6: The element $(a_2^*,\, b_2^*)$ is the closest element of $\mathcal{P}(1,\, \frac{1}{2})$ to $(\bar{a}_1,\, \bar{b}_1) \in \mathcal{P}(1,\, -1)$.

$\alpha^* + 1$ of $(\bar{a},\, \bar{b})$, they all lie in some compact set $K \subset X \times X$. Therefore, there is a convergent subsequence $\{(a'_{n_k}, b'_{n_k})\} \subset \{(a'_n,\, b'_n)\}$. Let $(a^*,\, b^*) = \lim_{k\to\infty}(a'_{n_k}, b'_{n_k})$. Then $t[(\bar{a},\, \bar{b}),\, (a^*,\, b^*)] = \alpha^*$ and by Lemma 5, $(a^*,\, b^*) \in P'$. (See Figure 6 which shows the element of $\mathcal{P}(1,\, \frac{1}{2})$ closest to the point $(\bar{a}_1,\, \bar{b}_1) = ((1,\, 1),\, (4,\, \frac{4}{7})) \in \mathcal{P}(1,\, -1)$.)

Then for all $(a',\, b') \in P'$, $|\bar{a} - a'| + |\bar{b} - b'| \geq |\bar{a} - a^*| + |\bar{b} - b^*|$. Let $c > 0$. Then $(c \cdot \bar{a},\, c \cdot \bar{b}) \in P$ and $(c \cdot a^*,\, c \cdot b^*) \in P'$ (by homotheticity of P and P'). Assume that $(\bar{a}',\, \bar{b}') \in P'$ and

$$|c \cdot \bar{a} - \bar{a}'| + |c \cdot \bar{b} - \bar{b}'| < |c \cdot \bar{a} - c \cdot a^*| + |c \cdot \bar{b} - c \cdot b^*|.$$

Then

$$c \cdot \{|\bar{a} - c^{-1} \cdot \bar{a}'| + |\bar{b} - c^{-1} \cdot \bar{b}'|\} < c \cdot \{|\bar{a} - a^*| + |\bar{b} - b^*|\} = c \cdot \alpha^*$$

or

$$|\bar{a} - c^{-1} \cdot \bar{a}'| + |\bar{b} - c^{-1} \cdot \bar{b}'| < \alpha^*.$$

But $(c^{-1}\bar{a}',\, c^{-1}\bar{b}') \in P'$ and no element of P' is within α^* of $(\bar{a},\, \bar{b})$. So $(c \cdot a^*,\, c \cdot b^*)$ is a closest element of P' to $(c \cdot \bar{a},\, c \cdot \bar{b})$, and this distance is $c \cdot \alpha^*$. Then

$$h^*[P,\, P'] \;=\; \sup_{(a,\, b) \in P} h_*[(a,\, b),\, P']$$

$$\begin{aligned} &\geq \sup_{c>0} h_*[(c \cdot \bar{a},\, c \cdot \bar{b}),\, P'] \\ &= \sup_{c>0} c \cdot \alpha^*. \end{aligned}$$

This last expression must be either 0 or infinity. Since $\alpha^* = 0$ for all $(\bar{a}, \bar{b}) \in P$ if and only if $P = P'$, $d[P, P'] = \infty$. ■

Example 3 The examples of neighborhoods in Figure 7 provide some insight into the topology induced by the metric ρ. A straightforward calculation shows that for $r < 0$, a neighborhood of radius ϵ around the function $u_{(1,r)}$ in the metric ρ is given by

$$\begin{aligned} N_\epsilon(u_{(1,r)}) &= \left\{u_{(b,s)} : \rho(u_{(1,r)}, u_{(b,s)}) < \epsilon\right\} \\ &= \left\{u_{(b,s)} : e^{-\frac{r-s}{s(1-r)}+\epsilon\frac{1-s}{s}} < b < e^{\frac{r-s}{s(1-r)}-\epsilon\frac{1-s}{s}},\ \text{for } s \in \left(\tfrac{r-\epsilon(1-r)}{1-\epsilon(1-r)},\, r\right]\right\} \\ &\bigcup \left\{u_{(b,s)} : e^{-\frac{s-r}{s(1-r)}+\epsilon\frac{1-s}{s}} < b < e^{\frac{s-r}{s(1-r)}-\epsilon\frac{1-s}{s}},\ \text{for } s \in \left(r,\, \tfrac{r-\epsilon(1-r)}{1-\epsilon(1-r)}\right)\right\}. \end{aligned}$$

Figure 7 shows examples of four neighborhoods in the metric ρ. These are $N_{0.02}(u_{(1,-7.5)})$, $N_{0.02}(u_{(1,-4)})$, $N_{0.05}(u_{(1,-4)})$, and $N_{0.05}(u_{(1,-2)})$.

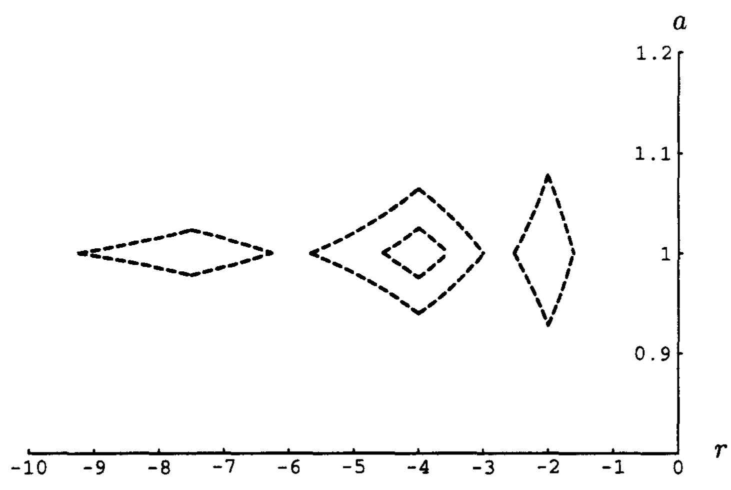

Figure 7: Examples of neighborhoods in the metric ρ.

4 The main result

Theorem 3 *There exist exchange economies with multiple equilibria in which the preferences of agents are arbitrarily close to one another in the metric ρ.*

Proof Consider the class of economies $\mathcal{E}(\hat{a}(r)^2, r)$ for $r < -1$. Since $\hat{a}(r) < 1$, $\hat{a}(r)^2 < \hat{a}(r)$, and by Theorem 1 each economy in $\mathcal{E}(\hat{a}(r)^2, r)$ has multiple equilibria.

With $a = \hat{a}(r)^2$, Equation (9) of Lemma 4 may be used to determine the distance between $u_{(\hat{a}(r)^2, r)}$ and $u_{(\hat{a}(r)^{-2}, r)}$. Since $r = s$ in the class of economies $\mathcal{E}(\hat{a}(r)^2, r)$, both terms in Equation (9) are equal, so

$$\begin{aligned} \rho(u_{(\hat{a}(r)^2, r)}, u_{(\hat{a}(r)^{-2}, r)}) &= \left|\frac{4\, r \ln \hat{a}(r)}{1-r}\right| \\ &= \left|\frac{4\, r}{1-r} \ln\left(\frac{r-1}{r+1}\right)^{\frac{1-r}{r}}\right| \\ &= 4 \ln \frac{r-1}{r+1}. \end{aligned}$$

Let $\epsilon > 0$ be given. Let

$$r^* < -\frac{e^{\epsilon/2}+1}{e^{\epsilon/2}-1}.$$

Then $\mathcal{E}(\hat{a}(r^*)^2, r^*)$ has multiple equilibria, and the preferences of the agents in the economy are within ϵ of one another in the metric ρ. ■

Note 2 In "Smooth Preferences," Debreu ([1984], p. 188) observes that "the economy with identical agents ... has a neighborhood of economies also possessing a unique equilibrium." The result of Theorem 3 and Debreu's argument are presented with different quantifiers, and reach conclusions appearing quite different, but of course there is no inconsistency. In this context Debreu's argument shows that for any given r^* there is an ϵ such that if each agent has preferences $\mathcal{P}(a, r)$ with $(a, r) \in B_\epsilon(1, r^*)$, then $\mathcal{E}(a, r)$ has a unique equilibrium. Theorem 3 shows that for any preassigned ϵ, there is an r^* and there are parameters $(a, r) \in B_\epsilon(1, r^*)$ such that $\mathcal{E}(a, r)$ has multiple equilibria.

5 An experimental design

The exchange environments described in Section 2 provide a strong test of the predictions of the theory of competitive equilibrium. In an exchange environment with the endowment pattern $\omega^A = (1, 0)$ and $\omega^B = (0, 1)$, where each trader has Cobb-Douglas utility, the slope of the excess demand function $Z_x(p)$ at the equilibrium price $p = \frac{1}{2}$ is -2. The slope of the excess demand at the stable equilibrium price $p = 0.2$ of the exchange economy $\mathcal{E}(0.496, -4)$ is -0.1335. If prices in exchange experiments adjust in response to excess demand, then an environment with multiple equilibria and relatively flat excess demand near the equilibrium provides a strong test of the theory.

Smith [1982] has described a procedure for conducting an exchange experiment. He adapts the concept of an *informationally decentralized system* (as in Hurwicz [1972]) to describe the microeconomic system created to test competitive equilibrium theory. In this formulation, an experimental market consists of an *environment* E, an *institution* $\mathcal{I}$, and an *outcome* X. Each of these is described below.

5.1 Environment

The economic environment E for a two-commodity trading experiment includes a set $\mathcal{A}$ of agents. There is a commodity space $X \times Y$ which will be taken to be a discrete approximation to $R_+ \times R_+$, where R_+ is the set of non-negative levels of consumption for commodities X and Y. Each agent $i \in \mathcal{A}$ has an objective or utility function $u^i : R^2_+ \to R$, and an endowment vector $\omega^i \in R^2_+$.

Smith [1982] describes a technique for inducing individual utility functions over two or more commodities in a market experiment. Suppose that the experimenter wants agent i to have the utility function $u^i_{(a,r)}(x, y) = ((ax)^r + y^r))^{1/r}$. This is accomplished by giving the payoff $u^i_{(a,r)}(\bar{x}^i_0, \bar{y}^i_0)$ to agent i when she holds $(\bar{x}^i_0, \bar{y}^i_0)$ at the conclusion of trading. This payoff is paid in, for example, US currency. Then agent i is indifferent between any combination (x, y) that yields a fixed income $\bar{u}$, so that

$$I_{\bar{u}} \equiv \{(x, y) \in R_+ \times R_+ : u^A_{(a,r)}(x, y) = \bar{u}\}$$

is an indifference curve for this agent. Indifference curves for other consumers in the experiment are induced similarly. Endowments are induced by giving consumers tokens which represent holdings of commodities. These tokens may be physical, electronic, or simply records kept by subjects and the experimenter. The number of tokens for agent i is $\omega^i = (\omega^i_x, \omega^i_y)$. The characteristics of agent i are therefore $e^i = (u^i, \omega^i)$ and an environment is $e = (e^i)_{i \in \mathcal{A}}$. The set of environments is E.

A parameter set The following parameter set determines an exchange market with three equilibria, and equilibrium allocations which are convenient for experimental purposes, as explained below. Let $c \cdot N = \{0, c, 2c, 3c, \ldots\}$. Let the consumption space for each agent be $X = 0.1 \cdot N \times 0.01 \cdot N$. Let the set of agents be $\mathcal{A} = \{1, 2 \ldots, 8\}$ where agents $i = 1, \ldots, 4$ have the utility function

$$u^i_{(a,r)}(x, y) = ((ax)^r + y^r)^{1/r}$$

with parameters $(a, r) = (0.36, -2.9012)$ and the endowment $\omega^i = (12, 0)$. Let agents $i = 5, \ldots, 8$ have the utility function

$$u^i_{(b,s)}(x, y) = (x^s + (by)^s)^{1/s}$$

with parameters $(b, s) = (0.36, -2.8776)$ and the endowment $\omega^i = (0, 12)$.

5.2 Institution

An institution $\mathcal{I} = (M, \beta, h)$ consists of a *message space* M, a vector $\beta = (\beta^i)_{i \in \mathcal{A}}$ of *response rules*, and a vector $h = (h^i)_{i \in \mathcal{A}}$ of *allocation functions.* The message space for agent i is M^i with elements $m^i \in M^i$. The message space is $M = \prod_{i \in \mathcal{A}} M^i$ and its elements are $m = (m^i)_{i \in \mathcal{A}} \in M$. An allocation function is $h : M \to X$, where X is the space of possible outcomes (or allocations).

Many forms of market organization (or exchange institution) are possible. One prominent exchange institution is the *call market.* This is a two-sided (buyers and sellers) version of the uniform price sealed-bid auction. In a call market, sellers submit a quantity and the minimum price at which they are willing to trade this quantity. These offers and quantities, arrayed from lowest to highest, are referred to as *revealed supply.* Buyers submit a quantity and the maximum price at which they are willing to buy that quantity. Buyers' bids, when arrayed from highest to lowest, are called the *revealed demand.* Price is then chosen at the intersection of revealed supply and revealed demand, or if this intersection is an interval, price is chosen from that interval. All buyers with bids at or above the specified price receive units at the market price from sellers with asks at or below that price.

In the case of a single commodity and unit demand, Rustichini, Satterthwaite, and Williams [1994] have shown that the call market converges to efficiency quickly as the number of traders increases, when traders' bids and offers are determined as the Bayes-Nash equilibrium of a game of incomplete information. The experimental design presented here is intended to test whether this result will hold if the number of commodities is increased to two. In the remainder of this section, the details of the call market institution are described for the Edgeworth exchange environment.

In the call market there is, in addition to the traders $i \in \mathcal{A}$, an agent 0 who sends messages $m^{0,i}$ to each trader i after receiving messages $m = (m^i)_{i\in\mathcal{A}}$ from traders. These messages $m^0 = (m^{0,i})_{i\in\mathcal{A}}$ inform each agent $i \in \mathcal{A}$ of (1) the market price, and (2) each agent's net trade. While traders $i \in \mathcal{A}$ send messages which depend on their unobserved behavioral rules β^i, the messages $m^{0,i}$ sent to traders by agent 0 are deterministic functions of $(m^i)_{i\in\mathcal{A}}$. That is, $m^{0,i} = \beta^{0,i}((m^i)_{i\in\mathcal{A}})$. The function $\beta^{0,i}$ is described below.

Market experiments typically employ repetitions of trading periods to test convergence of market outcomes to competitive equilibrium. In this case, agents send a sequence of messages. In the call market institution, traders send one message per trading period. These periods are indexed $t = 1, 2, 3, \ldots, T$, and the response rule for agent i in a stationary environment $e = (e^i)_{i\in\mathcal{A}}$ is $m^i_{t+1} = \beta^i(m^{0,i}_t, e^i)$. Agent 0 has a response rule $m^0_t = \beta^0((m^i_t)_{i\in\mathcal{A}})$.

Message space Each agent $i \in \mathcal{A}$ sends a message $m^i_t = (x^i_t, p^i_t)$ in trading period t specifying a net trade x^i_t of commodity X, and a reservation price p^i_t. If $x^i_t < 0$ then the reservation price p^i_t of trader i is interpreted as the lowest price (in units of commodity Y) at which trader i is willing to sell units of X. If $x^i_t > 0$ then this amount is interpreted as the highest price (in units of Y) that agent i is willing to pay per unit. Therefore, $M^i_t = 0.1 \cdot Z \times 0.01 \cdot N$, where $Z = \{0, 1, -1, 2, -2, \ldots\}$ and $N = \{0, 1, 2, 3, \ldots\}$ and $M_t = \prod_{i\in\mathcal{A}} M^i_t$.

Outcome function In period t, the messages $m^{0,i}_t$ sent by agent 0 reveal the outcome of trade in period t. This outcome includes the market price p^*_t and the net trade $\bar{x}^i_t$ of agent i in period t.

It is useful to partition the set $\mathcal{A}$ of agents into sets $\mathcal{A}^a$ and $\mathcal{A}^b$, where $\mathcal{A}^a$ is the set of agents submitting offers (asks) to sell units and $\mathcal{A}^b$ is the set of agents submitting bids. Let $\mathcal{A}^a = \{i_1, i_2, i_3, \ldots, i_\kappa\}$ be the set of all $i_k \in \mathcal{A}$ with

$$(1) \quad x^{i_k}_t \le 0;$$

$$(2) \quad p^{i_k}_t \le p^{i_{k+1}}_t \quad \textit{for all} \quad k \in \{1, 2, \ldots, \kappa - 1\}.$$

Condition (1) indicates that the agents in $\mathcal{A}^a$ are proposing to sell units of X (and receive units of Y). Condition (2) states that the agents in the set $\mathcal{A}^a$ are ordered with non-decreasing offer levels.

Similarly, let $\mathcal{A}^b = \{i_{\kappa+1}, i_{\kappa+2}, \ldots, i_n\}$ be the set of all $i_k \in \mathcal{A}$ with

$$(3) \quad x_t^{i_k} > 0;$$

$$(4) \quad p_t^{i_k} \geq p_t^{i_{k+1}} \textit{ for all } k \in \{\kappa+1, \kappa+2, \ldots, n-1\}.$$

Definition 4 (Revealed supply) Let $s_t^k = -\sum_{j=1}^{k} x_t^{i_j}$ for $k = \{1, 2, \ldots, \kappa\}$. The revealed market supply for commodity X in period t is $S_t(x) = p_t^{i_k} \cdot I_{(s_t^{k-1}, s_t^k]}(x)$.

Definition 5 (Revealed demand) For $k = \{\kappa+1, \kappa+2, \ldots, n\}$ let $d_t^k = \sum_{j=\kappa+1}^{k} x_t^{i_j}$. Let $D_t(x) = p_t^{i_k} \cdot I_{(d_t^{k-1}, d_t^k]}(x)$. Then $D_t(x)$ is the revealed market demand for commodity X in period t.

Let $x_t^* = \max\{x \geq 0 : D_t(x) \geq S_t(x)\}$. Then one of the following three cases occurs:

$$(1) \quad S_t(x_t^*) \leq S_t(x_t^*+) \textit{ and } D_t(x_t^*) = D_t(x_t^*+);$$

$$(2) \quad S_t(x_t^*) = S_t(x_t^*+) \textit{ and } D_t(x_t^*) \geq D_t(x_t^*+);$$

$$(3) \quad S_t(x_t^*) \leq S_t(x_t^*+) \textit{ and } D_t(x_t^*) \geq D_t(x_t^*+).$$

In the first case, as in Figure 8 (a), the market equilibrium price is $p_t^* = D_t(x_t^*)$. In the second case it is $p_t^* = S_t(x_t^*)$. In the third case, an example of which is shown in Figure 8 (b), define the market price by the midpoint of the set of CE prices:

$$p_t^* = \frac{1}{2} \cdot (\min\{D_t(x_t^*), S_t(x_t^*+)\} + \max\{S_t(x_t^*), D_t(x_t^*+)\}).$$

Net trades of good X are determined in Definition 6.

Definition 6 (Net trades) Partition the set $\mathcal{A}^a$ into the sets $\mathcal{A}^{a,1} = \{i_k \in \mathcal{A}^a : p_t^{i_k} < p_t^*\}$, $\mathcal{A}^{a,2} = \{i_k \in \mathcal{A}^a : p_t^{i_k} = p_t^*\}$, and $\mathcal{A}^{a,3} = \{i_k \in \mathcal{A}^a : p_t^{i_k} > p_t^*\}$. Partition $\mathcal{A}^b$ into $\mathcal{A}^{b,1} = \{i_k \in \mathcal{A}^b : p_t^{i_k} > p_t^*\}$, $\mathcal{A}^{b,2} = \{i_k \in \mathcal{A}^b : p_t^{i_k} = p_t^*\}$, and $\mathcal{A}^{b,3} = \{i_k \in \mathcal{A}^b : p_t^{i_k} < p_t^*\}$.

Case (1) Agent $i_k \in \mathcal{A}^{a,1} \cup \mathcal{A}^{a,2}$ sells $|x^{i_k}|$ units at price p_t^*. Agents $i_k \in \mathcal{A}^{b,1}$ purchase $x_t^{i_k}$ units at price p_t^*. Agent $i_j \in \mathcal{A}^{b,2}$ is allocated $\bar{x}_t^{i_j}$ units of X, where

$$\bar{x}_t^{i_j} = \frac{x_t^{i_j}}{\sum_{\{i_k \in \mathcal{A}^{b,2}\}} x_t^{i_k}} \cdot \left(x_t^* - \sum_{i_k \in \mathcal{A}^{b,1}} x_t^{i_k} \right).$$

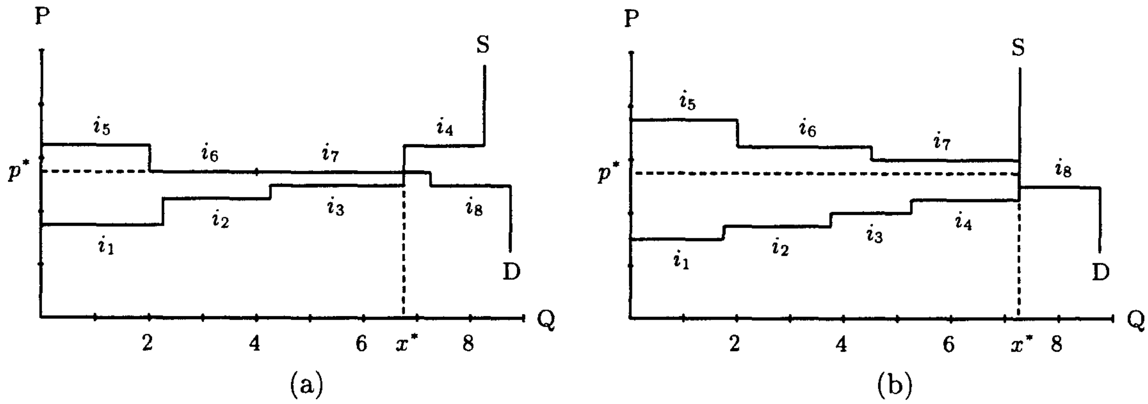

Figure 8: Revealed supply and revealed demand in a call market.

Case (2) This is analogous to case (1).

Case (3) Each agent $i_k = \mathcal{A}^{a,1}$ sells $|x_t^{i_k}|$ units and each agent $i_k = \mathcal{A}^{b,1}$ purchases $x_t^{i_k}$ units.

Response rules On the basis of previous market activity observed by agent i, and based on i's characteristics e^i, each agent $i \in \mathcal{A}$ has a response function which determines the message sent by i in the current period. This function is written $m_{t+1}^i = \beta^i(m_t^{0,i}; e^i)$.

One of two main purposes of market trading experiments is to obtain data on agents' messages m_t^i and on market outcomes in order to test theories of agent behavior. That is, for a given environment $e \in E$ and a given institution I, we can examine the messages (m_t^i) sent by agents and determine whether observed behavior is consistent with a given vector $(\beta^i(\cdot))_{i \in \mathcal{A}}$ of behavioral rules. The second main purpose of market trading experiments is to obtain evidence on the relative performance of alternative trading institutions as means of implementing competitive equilibrium – and hence Pareto optimal – allocations.

5.3 Outcome

Competitive equilibrium theory makes definite predictions about the possible prices and allocations in this market. The competitive equilibria for this economy are described below.

Walrasian equilibrium An Edgeworth diagram for this exchange experiment is shown in Figure 9. Three indifference curves are shown for consumer type A and for consumer type B. This exchange environment has three competitive equilibria. Supporting price

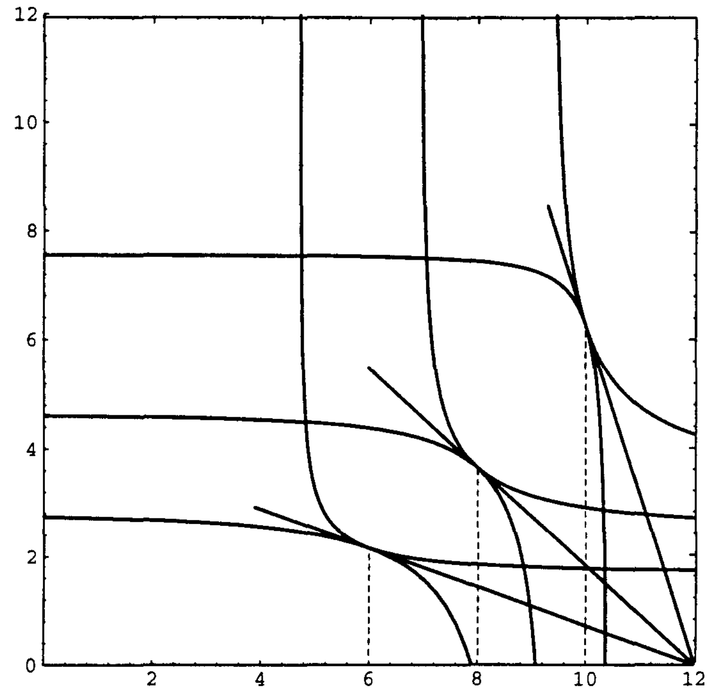

Figure 9: Edgeworth diagram for an exchange experiment.

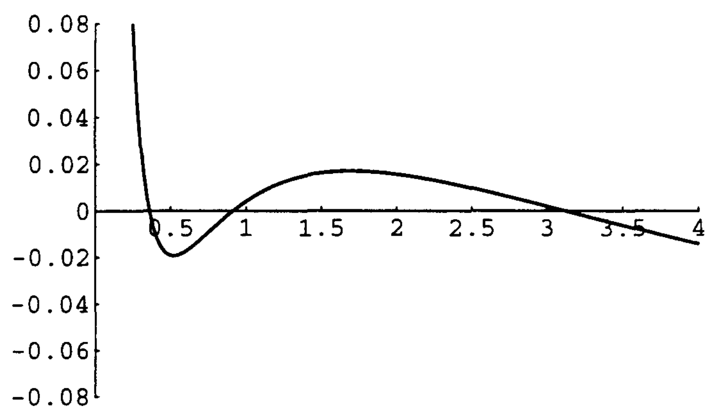

Figure 10: Excess demand graph for an exchange experiment.

lines for each of these equilibria are shown in the figure. In Figure 10, the excess demand function associated with this market is shown. The equilibrium prices for the market are approximately \$0.36, \$0.91, and \$3.13. The allocations supported by these equilibrium prices are $\{(x^A, y^A), (x^B, y^B)\} = \{(6, 2.16), (6, 9.84)\}$, $\{(8, 3.64), (4, 8.36)\}$, and $\{(10, 6.26), (2, 5.74)\}$.

The agents' parameters are chosen so that with good Y as numeraire, the equilibrium allocation of X to each consumer is an integer quantity in each of the three equilibria. As described above, the consumption space for each agent is $0.1 \cdot N \times 0.01 \cdot N$. During initial trading, it is likely that the prevailing market price will be out of equilibrium. The divisibility of quantities facilitates adjustment of price to reduce excess demand. If, for example, the market price of good X is 0.30 units of Y, then type A consumers will want to sell approximately 6.0 units of good X, but type B consumers will want to purchase approximately 6.4 units of X. Presumably, this excess demand for X will induce each type B consumer to increase their bid in order to obtain a larger fraction of the supply of X, which results in a higher payoff. If this bargaining process proceeds as theory predicts, so that price is near the competitive equilibrium price, then quantity adjustment should drive prices to competitive equilibrium prices.

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