EXCESS DEMAND FUNCTIONS, EQUILIBRIUM PRICES AND EXISTENCE OF EQUILIBRIUM

by

Kam-Chau Wong

Discussion Paper No. 283, October 1995

Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, MN 55455

Excess Demand Functions, Equilibrium Prices, and Existence of Equilibrium

Kam-Chau Wong†

Department of Economics
Chinese University of Hong Kong
Shatin, Hong Kong
E-mail: kamchauwong@cuhk.hk
Phone: (852) 2609-7053
Fax: (852) 2603-5805

September 6, 1995

Abstract: For continuous excess demand functions, the existing literature (e.g. Sonnenschein [1972, 1973], Mantel [1974], Debreu [1974], Mas-Colell [1977], etc.) achieves a complete characterization only when the functions are defined on special subsets of positive prices. In this paper, we allow the functions to be defined on a larger class of price sets, (allowing, for example the closed unit simplex, including its boundary). Besides characterizing excess demands for a larger class of economies, it is also a useful tool for proving other results. It allows us to characterize the equilibrium price set for a larger class of economies. It also permits extending Uzawa's observation [1962], by showing that Brouwer's Fixed-Point Theorem is implied by the Arrow-Debreu Equilibrium Existence Theorem ([1954], Thm. I.).

JEL Classification: D51, D11, D50

Keywords: Excess demand, Exchange economy, Equilibrium, Fixed-point

† This paper was written while I was visiting the Department of Economics of the University of Minnesota in the summer of 1995; the hospitality of the department is gratefully acknowledged. I also wish to thank Professor M. Richter for his helpful comments and suggestions on earlier versions of this paper.

1. Introduction

In the existing literature of characterizing continuous excess demand functions (Sonnenschein [1972, 1973], Mantel [1974], Debreu [1974], Mas-Colell [1977], etc.), a complete characterization is achieved only for the special case where the functions are defined on compact subsets of positive prices.[1] By weakening the preference requirement of monotonicity to insatiability, we are able to characterizing a larger class of excess demand functions, even those defined at boundary points, where some prices are zero. In particular, Theorem 1 below gives a complete characterization for any continuous excess demand function defined on any compact subset of nonnegative nonzero prices, including, e.g. the closed unit simplex. This extension has applications in several areas. We give two examples.

The first application is a characterization of equilibrium price sets. In the series of papers by Sonnenschein [1972, 1973] and Mas-Colell [1977], it has been established that any non-empty compact set in the *relative interior* of the closed unit simplex is indeed the set of equilibrium prices for some exchange economy (with monotone preferences). By using Theorem 1, we drop the "interior" restriction, showing that any non-empty compact set in the closed simplex is the set of equilibrium prices for some exchange economy (with insatiable preferences).

The second application concerns the connection between two very fundamental theorems — Brouwer's Fixed-Point Theorem and the Arrow-Debreu Equilibrium Existence Theorem ([1954], Thm. I). Uzawa [1962] proved that Brouwer's Theorem is equivalent[2] to the equilibrium existence

---

[1] McFadden et al. [1974] considered not-necessarily-continuous excess demand functions on the set of positive prices.

[2] In a formal sense, all mathematical theorems are equivalent, since they can be derived from the same set of axioms for mathematics. But in the present context, we use "equivalent" in the usual informal sense, meaning that there is a fairly direct proof

theorem *for excess demand functions* defined on the closed unit simplex. By Theorem 1, it follows that the latter is implied[3] by the Arrow-Debreu Equilibrium Existence Theorem *for economies.* Consequently, Brouwer's Theorem is implied by the Arrow-Debreu Theorem. This formalizes Debreu's observation[4] that "the proof of existence of a competitive equilibrium requires mathematical tools of the same power as a fixed point theorem" (Debreu [1982], p. 720).

The proof of Theorem 1 also has some interest of its own. It uses modifications of the methods of both McFadden et al. [1974] (for the decomposition of the given function into individual excess demand functions)[5] and Debreu [1974] (for the construction of preferences for the individual excess demand functions). It sheds some light on the complementary nature of the basic ideas of these two different methods.

2. Statement of Results

Let[6] $\mathbb{R}^l_+ = \{p \in \mathbb{R}^l : p \geq 0\}$, $\Delta = \{p \in \mathbb{R}^l_+ : \sum_{i=1}^l p_i = 1\}$, $\overset{o}{\Delta} = \{p \in \Delta : p \gg 0\}$, $S = \{p \in \mathbb{R}^l : p \gg 0 \;\&\; \|p\| = 1\}$, and $S_\epsilon = \{p \in S : p_i \geq \epsilon$ for every $i\}$ for every real number $\epsilon > 0$.

We will consider any *exchange economy* $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ such

---

of one from the other.

[3] See Footnote 3 above.

[4] Cf. also Sonnenschein [1973], p. 352 and Sonnenschein [1982], p. 691.

[5] I.e. we use oblique projections, as given in McFadden et al. [1974], rather than the orthogonal projection of Debreu [1974]. The reason is that the former method allows us to handle situations in which a continuous excess function is defined at boundary points.

[6] For any $x, y \in \mathbb{R}^l$, we write $x \geq y$ to mean $x_i \geq y_i$ for every $i = 1, \cdots, l$, and write $x \gg y$ to mean $x_i > y_i$ for every $i = 1, \cdots, l$. Also, $\|\cdot\|$ denotes the Euclidean norm.

that

$$\text{for all } i,\ \omega_i \gg 0 \in \mathbb{R}^l_+ \text{ and } \succeq_i \text{ is a continuous, insatiable, strictly convex preference relation on the consumption space } \mathbb{R}^l_+. \tag{1}$$

Let $G \subseteq \mathbb{R}^l_+ \backslash \{0\}$. A function $\zeta : G \to \mathbb{R}^l$ is an *excess demand function* if ζ is continuous and such that:

WL) $\quad p \cdot \zeta(p) = 0 \quad$ for every $p \in G$,

H) $\quad \zeta(p) = \zeta(\lambda p) \quad$ for every $p \in G$ and every $\lambda > 0$ with $\lambda p \in G$,

BB) $\quad \zeta(G) + q \gg 0 \quad$ for some $q \gg 0 \in \mathbb{R}^l$.

It is well-known that for any function $\zeta : G \to \mathbb{R}^l$, if there exists an exchange economy $E = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ satisfying (1) and such that

$$\begin{aligned} \{\zeta(p)\} = \sum_{i=1}^{l} \{x \in \mathbb{R}^l_+ :\ & [p \cdot x \le p \cdot \omega_i] \\ & \& \ (\forall y \in \mathbb{R}^l_+)[(p \cdot y \le p \cdot \omega_i) \Rightarrow (x \succeq_i y)]\} - \sum_{i=1}^{l} \{\omega_i\} \end{aligned} \tag{2}$$

for all $p \in G$, then ζ is continuous and satisfies (WL), (H) and (BB), i.e. ζ is an excess demand function. A partial converse is provided by the Sonnenschein-Mantel-Debreu Theorem (Debreu [1974]), which shows that for any $\epsilon > 0$, any excess demand function $\zeta : S_\epsilon \to \mathbb{R}^l$, there exists an exchange economy $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ satisfying (1) and

$$\text{every } \succeq_i \text{ is monotone ,} \tag{3}$$

and such that (2) holds for all $p \in S_\epsilon$.[7] By dropping (3), we obtain the following result, which considers excess demand functions defined on any compact subset of $\mathbb{R}^l_+ \backslash \{0\}$ (including e.g. the whole closure of S, Δ, etc.), rather than just S_ϵ.

[7] Different variants of the theorem were obtained by McFadden et al. [1974], Mas-Colell [1977], Mantel [1979] and others.

Theorem 1 (Characterization of Excess Demand Functions). *Let K be a compact subset of $\mathbb{R}^l_+ \backslash \{0\}$. Then for any $\zeta : K \to \mathbb{R}^l$ to be an excess demand function, it is necessary and sufficient that there exists an exchange economy $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ satisfying (1) and such that (2) holds for all $p \in K$.*[8]

Proof. The sufficiency is clear. And the necessity follows immediately from Lemma 1 and Proposition 1 in Section 3. Q.E.D.

We will now discuss two applications of Theorem 1. The first one concerns a characterization of equilibrium price sets. For any exchange economy $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$, a *competitive equilibrium* is a tuple $(\bar{p}, (\bar{x}_i)_{i=1}^l)$ such that $\bar{p} \in \mathbb{R}^l_+$, $\bar{x}_i \in \mathbb{R}^l_+$ for all $i = 1, \cdots, l$, and

E.1) $(\sum_{i=1}^l \bar{x}_i - \sum_{i=1}^l \omega_i) \leq 0$ and $\bar{p} \cdot (\sum_{i=1}^l \bar{x}_i - \sum_{i=1}^l \omega_i) = 0$,

E.2) $(\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i)$ & $(\forall y \in \mathbb{R}^l_+)[(\bar{p} \cdot y \leq \bar{p} \cdot \omega_i) \Rightarrow (\bar{x}_i \succeq_i y)]$ for all $i = 1, \cdots, l$.

A *competitive equilibrium price* for $\mathcal{E}$ is a vector $\bar{p} \in \mathbb{R}^l_+$ such that $(\bar{p}, (\bar{x}_i)_{i=1}^l)$ is a competitive equilibrium for $\mathcal{E}$ for some vector $(\bar{x}_i)_{i=1}^l$. We use $E_{\mathcal{E}}$ to denote the set of competitive equilibrium prices for the economy $\mathcal{E}$ in Δ.

Following the work of Sonnenschein [1972, 1973], Mas-Colell [1977] found a necessary and sufficient condition for any set K in the relative interior $\overset{o}{\Delta}$ of Δ to be the set of equilibrium prices for an exchange economy $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ satisfying (1) and (3), namely that K is a nonempty compact set. In Theorem 2 below, we relax the requirement (3) for economies, but we obtain a more general characterization condition for equilibrium price sets by dispensing with the "interior" restriction.

[8] By (H), it is clear that Theorem 1 also covers the case where $K = \mathbb{R}^l_+ \backslash \{0\}$.

Theorem 2 (Characterization of Equilibrium Price Sets). *A set $K \subseteq \Delta$ is a non-empty compact set if and only if $K = E_{\mathcal{E}}$ for some exchange economy $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ satisfying (1).*

Proof. The "if" part is well-known. To show the "only if" part, it is clear that if $\zeta : \Delta \to \mathbb{R}^l$ is an excess demand function and $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ is an exchange economy such that (2) holds for every $p \in \Delta$, then we have: $E_{\mathcal{E}} = E_\zeta$, where $E_\zeta = \{p \in \Delta : \zeta(p) \leq 0\}$. By Theorem 1, it suffices to construct an excess demand function $\zeta : \Delta \to \mathbb{R}^l$ with $K = E_\zeta$. To do this, we will modify the methods given in Uzawa [1962, p. 61], in Mas-Colell [1977, Corollary 1] and Mas-Colell [1985, p. 195]. First, we pick any $\bar{p} \in K$. Then we define a function $\xi : \Delta \to \mathbb{R}^l$ by $\xi(p) = \bar{p} - (\bar{p} \cdot p / p \cdot p)p$. It is clear that ξ is an excess demand function. It is also easily verified that for every $p \in \Delta$, one has: $\xi(p) \leq 0$ if and only if $p = \bar{p}$. Finally, we define a function $\zeta : \Delta \to \mathbb{R}^l$ by $\zeta(p) = \lambda_p \xi(p)$, where $\lambda_p = \min_{\tilde{p} \in K} ||\tilde{p} - p||$. Then clearly the function ζ is an excess demand such that $E_\zeta = K$, and we are done. Q.E.D.

We now discuss our second application of Theorem 1. As emphasized by Sonnenschein [1972, 1973] and Debreu [1982], characterizations of excess demand functions are useful in studying relationships between general proofs of existence of competitive equilibrium for exchange economies and fixed-point theorems.

Uzawa [1962] proved that Brouwer's Fixed-Point Theorem is equivalent to the following existence theorem:

Statement A (cf. Gale [1955], Nikaido [1956] & Debreu [1956]). *For every excess demand function $\zeta : \Delta \to \mathbb{R}^l$, there exists an equilibrium $\bar{p} \in \Delta$, i.e. $\zeta(\bar{p}) \leq 0$.*

However, Uzawa's result only shows that the existence of fixed-points is equivalent to the existence of equilibrium for excess demand functions defined on $\triangle$. But how does equilibrium for excess demand functions defined on $\triangle$ relate to equilibrium for *economies*? This has remained an open question, which the existing literature (e.g. the Sonnenschein-Mantel-Debreu Theorem, Mas-Colell's Theorem [1977], etc.) does not answer because its excess demand functions are defined only on certain subsets of $\overset{o}{\triangle}$, not on $\triangle$. However, our Theorem 1 does cover the situation.

Theorem 3 (Existence of Fixed-Points and Equilibrium Existence).

i) Brouwer's Fixed-Point Theorem is equivalent to Statement A.

ii) Statement A is implied by the Arrow-Debreu Equilibrium Existence Theorem ([1954], Thm. I).

Proof. Assertion (i) was shown by Uzawa [1962]. And (ii) follows immediately from Theorem 1. Q.E.D.

Therefore, by combining (i) and (ii) in Theorem 3, we see that Brouwer's Theorem is implied by the Arrow-Debreu Theorem.

3. Technical details.

Here we will establish the two results (Lemma 1 and Proposition 1) which we have used in the proof of Theorem 1. The following Lemma 1 extends demands from "small" to large price sets, permitting us to focus on the special case where an excess demand function ζ is defined on all of $\mathbb{R}^l_+\backslash\{0\}$.

Lemma 1 (Extension of Excess Demand Functions). *Let K be a non-empty compact subset of $\mathbb{R}^l_+\backslash\{0\}$, and $\zeta : K \to \mathbb{R}^l$ be an excess demand function. Then there exists an excess demand function $\tilde{\zeta} : \mathbb{R}^l_+\backslash\{0\} \to \mathbb{R}^l$ such that $\tilde{\zeta}|_K = \zeta$.*

Proof. By (H), it clearly suffices to consider the case where $K \subseteq \Delta$. By Tietze's Extension Theorem (cf. Munkres [1975], p. 212), there exists a continuous function $\bar{\zeta} : \Delta \to \mathbb{R}^l$ such that $\bar{\zeta}|_K = \zeta$. Then we define a function $\tilde{\zeta} : \Delta \to \mathbb{R}^l$ by $\tilde{\zeta}(p) = \bar{\zeta}(p) - (p \cdot \bar{\zeta}(p)/p \cdot p)p$. It is clear that $\tilde{\zeta}$ is an excess demand function such that $\tilde{\zeta}|_K = \zeta$. Finally, we extend the function $\tilde{\zeta}$ to the whole space $\mathbb{R}^l_+\backslash\{0\}$ by defining $\tilde{\zeta}(p) = \tilde{\zeta}(p/\sum_{i=1}^l p_i)$ for any nonnegative nonzero $p \notin \Delta$. This extension is clearly an excess demand function whose restriction to K is ζ. Q.E.D.

By Lemma 1, the following proposition then establishes the necessity part of Theorem 1.

Proposition 1 (Characterization of Excess Demand Functions defined on $\mathbb{R}^l_+\backslash\{0\}$). *Let $\zeta : \mathbb{R}^l_+\backslash\{0\} \to \mathbb{R}^l$ be an excess demand function. Then there is an exchange economy $\mathcal{E} = \{(\succeq_i, \omega_i, \mathbb{R}^l_+)_{i=1}^l\}$ satisfying (1) and such that (2) holds for every $p \in \mathbb{R}^l_+\backslash\{0\}$.*

To characterize excess demand functions on compact subsets of positive prices, Debreu [1974] used "orthogonal" projection to obtain individual preferences $\succeq_i$ and endowments ω_i. By contrast, in order to permit boundary prices p (i.e. some $p_i = 0$), we will use the "oblique" projection as in McFadden et al. [1974]. Thus, we will normalize the price set $\mathbb{R}^l_+\backslash\{0\}$ to the set P, the intersection of $\mathbb{R}^l_+$ with the sphere of radius 2 $||q||$ centered at $-q$, i.e.

$$P = \{p \in \mathbb{R}^l_+ : (p+q) \cdot (p+q) = 4q \cdot q\}, \tag{4}$$

where q is given in (BB). In particular, we will decompose any given excess demand function ζ on P into individual "oblique" excess demand functions $\phi_1, \cdots, \phi_l : P \to \mathbb{R}^l$. By a function $\varphi : P \to \mathbb{R}^l$ being an *oblique excess demand function*,[9] we mean that there exist a vector $a \gg 0 \in \mathbb{R}^l$ and a continuous function $\beta : P \to \mathbb{R}$ such that:

$$\begin{aligned} &\text{a) } \beta(p) > 0, \\ &\text{b) } \varphi(p) = \beta(p)h(p), \end{aligned} \tag{5}$$

for every $p \in P$, where $h(p)$ is the oblique projection along the direction of $p+q$ of a on the tangent space of p (see. Figure 1 below), i.e.[10]

$$h(p) = a - \frac{p \cdot a}{p \cdot (p+q)}(p+q). \tag{6}$$

Then we can apply the rationalization result for oblique excess demand functions given below in Lemma 3.

Proof of Proposition 1. Let $\zeta : \mathbb{R}^l_+ \backslash \{0\} \to \mathbb{R}^l$ be an excess demand function. We pick any vector $q \gg 0$ given in (BB), and consider the set P defined in (4). Then we can pick vectors $a_1, \cdots, a_l \gg 0 \in \mathbb{R}^l$ such that $\{a_1, \cdots, a_n\}$ is linearly independent and such that for every $p \in P$, $\zeta(p) + p + q$ is contained in the interior of the convex cone spanned by $\{a_1, \cdots, a_l\}$ (see Figure 1 below, cf. also Mas-Colell [1977, p.125].) For every $p \in P$, we can write $\zeta(p) + p + q = \sum_{i=1}^l \beta_i(p) a_i$, where $\beta_i(p) > 0$, in a unique fashion. It is clear that the functions $\beta_i(p)$ are continuous. It is

---

9 This notion is a modification of the function defined in McFadden et. al. [1974], p. 364, Equation (2). There ζ is defined only on the relative interior of P, β is a not-necessarily-continuous function, and a is only required to be nonnegative nonzero (i.e. not necessarily positive). Also, when $q = 0$ and $a = e_i$ the positive unit vector on the i coordinate axis of $\mathbb{R}^l$, then $h(p)$ given in (6) becomes Debreu's orthogonal projection function $b^i(p) = e_i - ((p \cdot e_i)/(p \cdot p))p$.

10 Actually, $h(p)$ is well-defined for every $p \in \mathbb{R}^l$ with $(p+q) \cdot (p+q) = 4q \cdot q$, by the following fact: $*$) $(p+q) \cdot (p+q) = 4q \cdot q$ implies $p \cdot (p+q) > 0$. To see ($*$), consider any such a p. Then we have: $p \cdot (p+q) = (q+q) \cdot (q+q) - q \cdot (p+q) = (q+q) \cdot (q+q) - (q+q) \cdot (p+q) + q \cdot (p+q) \geq q \cdot (p+q)$. Therefore, $p \cdot (p+q) \geq (1/2)(p \cdot (p+q) + q \cdot (p+q)) = (1/2)(p+q) \cdot (p+q) = 2q \cdot q > 0$.

also clear that these a_i can be chosen so that for every i, conditions (A-B) in Lemma 3 below holds with $a = a_i$.

Then consider the oblique excess demand functions $\phi_i : P \to I\!R^l$ defined by $\phi_i(p) = \beta_i(p)h_i(p)$, where $h_i(p)$ is defined in (6) with $a = a_i$ (see Figure 1 below). By Lemma 3 below, there exists an exchange economy $\mathcal{E} = \{(\succeq_i, \omega_i, I\!R^l_+)_{i=1}^l\}$ satisfying (1) and such that for every $i = 1, \cdots, l$ and every $\in P$, equation (8) below holds with $\varphi = \phi_i$, $\omega = \omega_i$ and $\succeq = \succeq_i$. Also, by an easy linear algebra argument (cf. McFadden et al. [1974], p. 365), it follows that $\zeta(p) = \sum_{i=1}^l \phi_i(p)$ for all $p \in P$, and so (2) holds for very $p \in P$. Then by (H), we have: (2) holds for every $p \in I\!R^l_+ \backslash \{0\}$.

Q.E.D.

We now show, as claimed in our proof of Lemma 2, that an appropriate choice of the oblique projection parameters q and a yields an endowment and preference relation rationalizing individual oblique excess demands on P.

Lemma 3 (Rationalization of Individual Oblique Excess Demand Functions on P). *Let $q, a \gg 0 \in I\!R^l$, and define P by (4). Let $\beta : P \to I\!R$ be a continuous function satisfying (5.a). Let $\varphi : P \to I\!R^l$ be the oblique excess demand function defined by (5.b). Assume:*

A) $\quad a \cdot a = 4q \cdot q, \qquad$ *and*

B) $\quad a - q \notin I\!R^l_+$.

Then there exist a vector $\omega \gg 0 \in I\!R^l$ and a preference relation $\succeq$ on $I\!R^l_+$ such that

$$\begin{array}{l} \succeq \text{ is continuous, strictly convex and insatiable (indeed} \\ \quad x + \lambda a \succ x \text{ for every } \lambda > 0 \in I\!R \text{ and every } x \in I\!R^l_+), \end{array} \tag{7}$$

and

$$\begin{aligned} \{\varphi(p)\} = \{x \in I\!R^l_+ : \ & [p \cdot x \leq p \cdot \omega] \\ & \& \ (\forall y \in I\!R^l_+)[(p \cdot y \leq p \cdot \omega) \Rightarrow (x \succeq y)]\} - \{\omega\} \end{aligned} \tag{8}$$

for every $p \in P$.

Lemma 3 requires an (insatiable) rationalization $\succeq$ on all of P. Debreu [1974] showed how to construct (monotone) preferences rationalizing his individual "orthogonal" excess demand functions on compact subsets S_ϵ of S. To apply the spirit of Debreu to our context, we will prove the following Lemma 4, where we obtain rationalizations on compact subsets, but of a larger space

$$Q = \{p \in I\!R^l : (p+q)\cdot(p+q) = 4q\cdot q \text{ and } p\cdot a \geq 0\}, \tag{9}$$

which contains all of P (see Fig. 2 below). (Cf. Footnotes 15 and 18.) Also, there we extend the consumption space from $I\!R^l_+$ to $I\!R^l$, and construct (insatiable) preferences defined on $I\!R^l$, not just $I\!R^l_+$. As we will see, Lemma 3 follows as a consequence of Lemma 4.

Lemma 4 (Rationalization of Individual Oblique "Excess Demand Functions" on $\mathbf{Q}_\epsilon$). *Let $q, a \gg 0 \in I\!R^l$, and define Q by (9). Let $\hat{\alpha} = \max_{p\in Q} p\cdot a$. Let $\beta : Q \to I\!R$ be a continuous function satisfying (5.a) on Q. Let $\varphi : Q \to I\!R$ be the corresponding oblique "excess demand" function, i.e. φ is defined by (5.b).*[11] *Let $\epsilon > 0 \in I\!R$ and define*

$$Q_\epsilon = \{p \in Q : \epsilon \leq p\cdot a \leq \hat{\alpha} - \epsilon\}. \tag{10}$$

Then there exist a preference relation $\succeq$ on $I\!R^l$ such that:

$$\begin{aligned} &i)\ \succeq \text{ is continuous and strictly convex,} \\ &ii)\ x + \lambda a \succ x \text{ for every } \lambda > 0 \in I\!R \text{ and every } x \in I\!R^l, \end{aligned} \tag{11}$$

and

$$\{\varphi(p)\} = \{x \in B_p : x \succeq y \text{ for every } y \in B_p\} \tag{12}$$

[11] **By Footnote 10, $h(\cdot)$ is well-defined on Q, and so does $\varphi(\cdot)$.**

for every $p \in Q_\epsilon$, where $B_p = \{x \in I\!R^l : p \cdot x \leq 0\}$.

Before proving Lemma 4, we apply it to prove Lemma 3.

Proof of Lemma 3. Since Lemma 4 obtains a preference relation only for sets Q_ϵ, we first ensure that P is contained in some Q_ϵ. By an easy geometry argument, it is clear that (A) implies that the vector a maximizes uniquely the function $\tilde{p} \mapsto \tilde{p} \cdot a$ on the sphere $\{\tilde{p} \in I\!R^l : \tilde{p} \cdot \tilde{p} = 4q \cdot q\}$, and therefore the vector $\hat{p} = a - q$ maximizes uniquely the function $p \mapsto p \cdot a$ on Q. By assumption (B), $\hat{p} \notin P$, and hence: $p \cdot a < \hat{p} \cdot a = \hat{\alpha}$ for every $p \in P$. Also, since $a \gg 0$ and $P \subseteq I\!R^l_+ \backslash \{0\}$, we have: $0 < p \cdot a$ for every $p \in P$. Then it follows that $P \subseteq Q_\epsilon$ for every sufficiently small $\epsilon > 0$.

Now, let $\beta : P \to I\!R$ and $\varphi : P \to I\!R^l$ be as given in Lemma 3. It is clear that we can pick a continuous function $\bar{\beta} : Q \to I\!R$ such that $\bar{\beta}|_P = \beta$ and $\bar{\beta}(p) > 0$ for every $p \in Q$. We define a function $\bar{\varphi} : Q \to I\!R^l$ by $\bar{\varphi}(p) = \bar{\beta}(p)h(p)$, and we pick any $\epsilon > 0$ with $P \subseteq Q_\epsilon$. By Lemma 4, there exists a preference relation $\bar{\succeq}$ on $I\!R^l$ satisfying (11) and such that $\{\bar{\varphi}(p)\} = \{x \in B_p : x \bar{\succeq} y \text{ for every } y \in B_p\}$ for every $p \in Q_\epsilon$, and hence $\{\varphi(p)\} = \{x \in B_p : x \bar{\succeq} y \text{ for every } y \in B_p\}$ for every $p \in P$. Then we pick any $\omega \gg 0 \in I\!R^l$ with $\varphi(P) + \omega \gg 0$, and it is clear that the translation $\succeq$ of $\bar{\succeq}$ from $I\!R^l$ to $I\!R^l_+$ defined by $x \succeq y \iff x - \omega \bar{\succeq} y - \omega$ satisfies (7) and such that (8) holds for every $p \in P$. Q.E.D.

It remains to prove Lemma 4. As mentioned earlier, our proof follows the spirit of Debreu's proof [1974]: our steps (stages (3.1-7) below) parallel his. However, we require modifications so that we can rationalize oblique excess demand functions (rather than orthogonal ones) (cf. Footnote 14) and allow boundary prices or even negative prices (cf. Footnote 19).

As in Debreu [1974], we endow the set $\mathcal{L}$ of non-empty, closed subsets of $\mathbb{R}^l$ with the topology of closed convergence (see Hildenbrand [1974], p.15). From here on, continuity of a function from any subset of $\mathbb{R}$ into $\mathcal{L}$ is always understood with respect to that topology.[12]

To illustrate the ideas of our proof of Lemma 4, consider the following facts [13]:

Facts (Indirect Rationalization). *For every $p, r \in Q$:*

$$\begin{aligned} &a)\ p \cdot h(p) = 0. \\ &b)\ \text{if } r \neq p \text{ and } 0 < r \cdot a \geq p \cdot a, \text{ then } r \cdot h(p) > 0. \end{aligned} \tag{13}$$

Proof. Fact (13.a), which states that $h(p)$ is in the tangent space of p, holds as h being a projection function. To prove Fact (13.b), we apply the arguments in McFadden et al. [1974], p.364, as follows. Consider any pair of distinct $r, p \in Q$. By definition (6), we have:
$r \cdot h(p) = r \cdot a - p \cdot a(r \cdot (p+q)/p \cdot (p+q))$. By $(*)$ in Footnote 10, we have: $p \cdot (p+q) > 0$. Also, $r \cdot (p+q) - p \cdot (p+q) = (r+q) \cdot (p+q) - (p+q) \cdot (p+q) < 0$. Therefore, we have: $r \cdot h(p) > 0$, and so (13.b) holds.

Q.E.D.

Fact (13.b) suggests that the function $p \mapsto -p \cdot a$ plays a role similar to an indirect utility function for the oblique projection h.[14] For example, by viewing $-r \cdot a$ and $-p \cdot a$ as the indirect utilities of r and p, and $h(p)$ as the bundle purchased under p, one can interpret (13.b) as follows: if $p \neq r$ and the indirect utility at r is less than or equal to that at p, then what

[12] For any $X \subseteq \mathbb{R}^l$, the interior of X is denoted by Int$X$, and the boundary of X is denoted by ∂X.

[13] Fact (13.b) has its antecedent in McFadden et al. [1974], p. 364. There the price set is the relative interior of P, rather than Q, and a is any nonnegative nonzero vector (i.e. not necessarily positive).

[14] By contrast, Debreu [1974] used $p \mapsto -p \cdot e_i$ as the indirect utility function for his orthogonal projection function $b^i(p) = e_i - ((p \cdot e_i)/(p \cdot p))p$, where e_i is the i-unit vector.

is purchased under p is not affordable at r.[15] Similar intuition can also be applied to the function $\varphi = \beta h$ as given in the Lemma 4, where $\beta > 0$ (as in (5.a)).

Corresponding to the "indirect utility" function $p \mapsto -p \cdot a$, we will define the indirect weakly-preferred sets $U_t = \{p \in Q : p \cdot a \leq t\}$, the indirect indifference sets $V_t = \{p \in Q : p \cdot a = t\}$, and the indirect weakly-worsen sets $L_t = \{p \in Q : p \cdot a \geq t\}$, where $t \in [0, \hat{\alpha}]$. (See Figure 2 below.) Actually, in our proof of Lemma 4, we will transform the sets U_t into a profile of (direct) weakly-preferred sets M_t which constitutes to a preference on $\mathbb{R}^l$ satisfying (11) and rationalizing φ (i.e. (12) holds) on Q_ϵ.

Each (direct) weakly-preferred set M_t we will construct is corresponding to the indirect weakly-preferred set U_t and rationalizes φ, therefore it must satisfy: i) M_t is weakly above the budget set B_p for every $p \in L_t$, and ii) for every $p \in V_t$, the set M_t intersects with B_p only at $\varphi(p)$. This motivates our use of the convex[16] cone $L_t^* = \{x \in \mathbb{R}^l : p \cdot x \geq 0 \text{ for every } p \in L_t\}$, which is the set of all commodity bundles which are either unaffordable or just-affordable at every $p \in L_t$. Thus (i) simply states: i') $M_t \subseteq L_t^*$. Also, it can be verified from (14) below and that for every $t \in (0, \hat{\alpha})$ and $p \in V_t$, the convex cone L_t^* (which by definition is weakly above B_p) intersects with B_p only at the ray $\{\lambda h(p) : \lambda \geq 0\}$. Therefore, (ii) simply means: ii') $M_t \cap \partial L_t^* = \varphi(V_t)$. (See Figure 2 below; ref. (15.a) below; cf. Debreu [1974], p.18, paragraph 3.)

---

[15] This intuition fails at any pair of boundary points p, r of Q, where $p \cdot a = r \cdot a = 0$. In particular, for any such a pair of p, r, we have: $h(p) = h(r) = a$, and $r \cdot h(p) = p \cdot h(r) = 0$. By this fact, and by picking β with $\beta(p) \neq \beta(r)$, one can construct an oblique "excess demand" function $\tilde{\varphi}$ violating the Weak Axiom of Revealed Preference at p, r, and so it is impossible to rationalize $\tilde{\varphi}$ on the whole set Q by any preference $\succeq$. This explains why in Lemma 4, we can obtain rationalization only on Q_ϵ, rather than the whole set Q.

[16] By (13) and (14) below, it can be verified that L_t^* is indeed a strictly convex cone (i.e. any ray in ∂L_t^* passes through 0) for every $t \in (0, \hat{\alpha})$. This observation, however, will not be used in our proof of Lemma 4, because we will use (13) and (14) directly.

We now characterize the interior and boundary of L_t^*. This result is useful for constructing the sets M_t.

Facts (Characterization of $\mathrm{Int}L_t^*$ **and** ∂L_t^***).**

a) For every $t \in [0, \hat{\alpha}]$,

$$\mathrm{Int}L_t^* = \{x \in I\!R^l : p \cdot x > 0 \text{ for all } p \in L_t\}. \tag{14.a}$$

b) For every $t \in (0, \hat{\alpha})$ *and every* $x \in I\!R^l$,

$$\begin{array}{l} i)\ \text{if } x = 0, \text{ then } x \in \partial L_t^*, \text{ and} \\ ii)\ \text{if } x \neq 0, \text{ then one has: } x \in \partial L_t^* \text{ if and only if} \\ \qquad x = \lambda h(p) \text{ for some } \lambda > 0 \text{ and some } p \in V_t. \end{array} \tag{14.b}$$

Proof. To show part (a), consider any $t \in [0, \hat{\alpha}]$ and any $x \in I\!R^l$. It is easily verified that if $p \cdot x > 0$ for every $p \in L_t$, then $x \in \mathrm{Int}L_t^*$. Conversely, let $x \in \mathrm{Int}L_t^*$. Suppose $p \cdot x = 0$ for some $p \in L_t$; then for any sufficiently small $\sigma > 0$, we have: $x - \sigma p \in \mathrm{Int}L_t^* \subseteq L_t^*$ and $p \cdot (x - \sigma p) = -\sigma p \cdot p < 0$; and a contradiction is derived. Thus we must have: $p \cdot x > 0$ for every $p \in L_t$. And (14.a) holds.

To show part (b), it is clear that (14.b.i) is immediate from (14.a). It remains to show (14.b.ii) for any $t \in (0, \hat{\alpha})$ and any nonzero $x \in I\!R^l$. First, suppose $x = \lambda h(p)$ for some $\lambda > 0 \in I\!R$ and some $p \in V_t \subseteq L_t$. Notice that for every $r \in L_t$, we have: $r \cdot a \geq t = p \cdot a$, and so $r \cdot x = \lambda r \cdot h(p) \geq 0$ by (13). Therefore, $x \in L_t^*$. Then by (13.a) and (14.a), we have: $x \in \partial L_t^*$.

Now, suppose $x \in \partial L_t^*$. By (14.a), there exists a $p \in L_t$ with $p \cdot x = 0$. We will show that: A) $p \in V_t$ and B) there exists $\lambda > 0$ with $x = \lambda h(p)$. To show (A), we suppose by contradiction $p \notin V_t$, i.e. $p \cdot a > t$. Then consider the function $F(\gamma, \sigma) = (\gamma p - \sigma x + q) \cdot (\gamma p - \sigma x + q) - 4q \cdot q$, which has $F(1, 0) = 0$, and $\frac{\partial F(1,0)}{\partial \gamma} = 2p \cdot (p + q) > 0$ by fact $(*)$ in Footnote 10. By the Implicit Function Theorem (cf. Rudin [1976], p. 224), we can pick a $\tilde{\sigma} > 0$ small enough and a $\tilde{\gamma}$ close enough to 1 so that $F(\tilde{\gamma}, \tilde{\sigma}) = 0$ and

$(\tilde{\gamma}p - \tilde{\sigma}x) \cdot a > t$. Then we have the contradiction : $\tilde{\gamma}p - \tilde{\sigma}x \in L_t$ and $(\tilde{\gamma}p - \tilde{\sigma}x) \cdot x = -\tilde{\sigma}x \cdot x < 0$. Thus have shown (A).

To show (B), observe that $p \cdot x = 0 \leq \tilde{p} \cdot x$ for every $\tilde{p} \in L_t$. Then p minimizes $\tilde{p} \cdot x$ subject to: i) $\tilde{p} \cdot a = t$ and ii) $(\tilde{p}+q) \cdot (\tilde{p}+q) = 4q \cdot q$. The first order condition (cf. El-Hodiri [1970], p.27) of this minimization problem yields $\lambda, \mu \in I\!R$ with $x = \lambda a + 2\mu(p+q)$. Since $0 = p \cdot x = \lambda p \cdot a + 2\mu p \cdot (p+q) = 0$, we have: $x = \lambda(a - (p \cdot a / p \cdot (p+q))(p+q)) = \lambda h(p)$. It remains to show that $\lambda > 0$. First, since $x \neq 0$, we have $\lambda \neq 0$. Second, we suppose by contradiction $\lambda < 0$, then we can pick any $r \neq p$ in L_t, and so by (13.b) we have: $r \cdot x = \lambda r \cdot h(p) < 0$, contradicting to the fact that $x \in L_t^*$. Thus we have $\lambda > 0$, and hence (B) holds. This completes our proof for (14.b).

Q.E.D.

We now give several useful topological properties for the cones L_t^*. First, for every $t \in [0, \alpha]$, since L_t is compact, it follows that L_t^* is closed. Then by the Krein-Milman (cf. Royden [1988], p. 242), it follows from (14) that L_t^* is indeed the convex cone spanned by $h(V_t)$. Also, it is clear that each V_t is compact; and it is easily verified (e.g. using the Implicited Function Theorem) that the function $t \mapsto V_t$ is continuous on $[0, \hat{\alpha}]$. Therefore, the function $t \mapsto L_t^*$ continuous on $(0, \hat{\alpha})$ into $\mathcal{L}$.[17]

We now present our proof of Lemma 4.

Proof of Lemma 4. Let a, q, Q, $\hat{\alpha}$, β and φ be as given in Lemma 4. Consider any $\epsilon > 0$. Notice that the claim of the lemma is trivial when Q_ϵ is empty. We now consider the non-trivial case, i.e. Q_ϵ is non-empty. We will construct a family $\{M_t : t \in (-\infty, \infty)\}$ of (direct) weakly-preferred sets (defined in (25) and (27) below) which satisfies the following properties

[17] **Also, by using (13) and (14), it can be shown the following hereditary property: if $t' < t$, then $L_{t'}^* \backslash \{0\} \subset \mathrm{Int} L_t^*$. This property, however, is not required in our proof of Lemma 4.**

(see Fig. 2 below):

$$
\begin{array}{ll}
\text{a) (Rationality) for every } t \in [\epsilon, \hat{\alpha} - \epsilon]\text{:} & \\
\quad \text{i) } M_t \subseteq L_t^*, & \\
\quad \text{ii) } M_t \cap \partial L_t^* = \varphi(V_t), & \\
\text{b) (Closedness) every } M_t \text{ is closed,} & \\
\text{c) (Graduality) the function } t \mapsto M_t \text{ is continuous,} & \\
\text{d) (Open Hereditary) if } t' < t\text{, then } M_{t'} \subseteq \mathrm{Int} M_t, & (15) \\
\text{e) (Strict Convexity) every } M_t \text{ is strictly convex,} & \\
\text{f) (Insatiability) } \mathrm{Int} M_t \ni x + \lambda a \text{ for every } \lambda > 0 \in I\!R, & \\
\quad \text{every } x \in M_t\text{, and every } M_t, & \\
\text{g) (Progressiveness) for every } x \in I\!R^l\text{, there exist} & \\
\quad t, t' \in (-\infty, \infty) \text{ with } x \in M_t \text{ and } x \notin M_{t'}. &
\end{array}
$$

Before constructing these M_t, we define a preference relation $\succeq$ on $I\!R^l$ by $x \succeq y \iff (\forall t \in (-\infty, \infty))[y \in M_t \Rightarrow x \in M_t]$. By (15.b-c) and (15.g), it is easily verified that the preference $\succeq$ has $\{M_t : t \in (-\infty, \infty)\}$ as its family of weakly-preferred sets, $\{\partial M_t : t \in (-\infty, \infty)\}$ as its family of indifferent sets, and $\{\mathrm{Int} M_t : t \in (-\infty, \infty)\}$ as its family of strictly-preferred sets. Then it follows clearly from (15.b-g) that $\succeq$ satisfies (11).

We now show that $\succeq$ rationalizes φ on Q_ϵ, i.e. (12) holds for every $p \in Q_\epsilon$. Consider any $p \in Q_\epsilon$, and let $t = p \cdot a$. We need to show that: A) $M_{t'} \cap B_p = \emptyset$ for every $t' < t$, and B) $M_t \cap B_p = \{\varphi(p)\}$. To see (A), consider any $t' < t$. By (15.a.i) and (15.d), we have: $M_{t'} \subseteq \mathrm{Int} M_t \subseteq \mathrm{Int} L_t^*$, and so by (14.a), we have: $p \cdot y > 0$ (hence $y \notin B_p$) for every $y \in M_{t'}$. Hence (A) holds. To see (B), by (15.a.ii), (15.d) and (14.a) we have: $p \cdot y > 0$ for every $y \in M_t \backslash \varphi(V_t)$. Also, by (13.b) and (5.b), we have: $p \cdot y > 0$ for every $y = \varphi(p')$ with $p \neq p' \in V_t$. Then we have: $M_t \cap B_p \subseteq \{\varphi(p)\}$. Also, since $p \in V_t$, by (15.a.ii) we have $\varphi(p) \subseteq M_t$. Notice that $\varphi(p) \in B_p$. Thus we have: $M_t \cap B_p \supseteq \{\varphi(p)\}$. Hence (B) follows. And so (12) holds for every $p \in Q_\epsilon$.

As mentioned earlier, we will obtain the profile of weakly-preferred sets M_t from the profile of indirect weakly-preferred sets U_t by making use of the sets L_t^*. This will be done along the lines of Debreu [1974], with appropriate modifications (e.g. see Footnote 19).

(3.1 Indirect Stage: picking U_t, where $t \in [\epsilon, \hat{\alpha} - \epsilon]$) We begin by listing the following properties for the family $\{U_t : t \in [\epsilon, \hat{\alpha} - \epsilon]\}$ of indirect weakly-preferred sets:

$$\begin{array}{ll} \text{a) (Compactness) every } U_t \text{ is compact,} & \\ \text{b) (Graduality) the function } t \mapsto U_t \text{ is continuous,} & \quad (16) \\ \text{c) (Hereditary) if } t' < t, \text{ then } U_{t'} \subset U_t, & \end{array}$$

Facts (16.a) and (16.c) are obvious. And (16.b) is easily verified (e.g. by using the Implicited Function Theorem).[18]

(3.2 Direct Stage: transforming U_t to D_t, where $t \in [\epsilon, \hat{\alpha} - \epsilon]$) To transform the indirect preference to a direct preference, as the first step we define $D_t = \varphi(U_t)$ for every $t \in [\epsilon, \hat{\alpha} - \epsilon]$ (see Figure 3 below). Then the sets D_t satisfy the following properties:

$$\begin{array}{ll} \text{a) (Rationality) for every } t, & \\ \quad \text{i) } D_t \subseteq L_t^*, & \\ \quad \text{ii) } D_t \cap \partial L_t^* = \varphi(V_t), & \\ \text{b) (Compactness) every } D_t \text{ is compact,} & \quad (17) \\ \text{c) (Graduality) the function } t \mapsto D_t \text{ is continuous,} & \\ \text{d) (Hereditary) if } t' < t, \text{ then:} & \\ \quad \text{i) } D_{t'} \subseteq D_t, \text{ and} & \\ \quad \text{ii) } D_{t'} \subseteq \mathrm{Int} L_t^*. & \end{array}$$

Facts (17.b-c) are clear from (16.a-b). And (17.d.i) holds by (16.c). To prove the rest of them, consider any $t \in [\epsilon, \hat{\alpha} - \epsilon]$ and any $x \in D_t = \varphi(U_t)$. Then $x = \beta(p)h(p)$ for some $p \in U_t$. To see (17.a.i), for every $r \in L_t$, since $r{\cdot}a \geq t \geq p{\cdot}a$, it follows from (13) that $r{\cdot}x = \beta(p)r{\cdot}h(p) \geq 0$. Thus, $x \in L_t^*$, and so (17.a.i) holds. We now show (17.d.ii). Suppose $x \in D_{t'}$ for some $t' < t$. By repicking p, we can assume $p \in U_{t'}$, and so $p \cdot a = t' < t \leq r \cdot a$

[18] **By contrast, the graduality property is not necessary for the profile of sets $U_t \cap P$. This explains why we have to extend our price set from P to Q.**

for every $r \in L_t$. By (14.b), we have: $r \cdot x = \beta(p) r \cdot h(p) > 0$ for every $r \in L_t$, and so $x \in \mathrm{Int} L_t^*$ by (14.a). Thus (17.d.ii) holds. It remains to show (17.a.ii). Suppose $p \in V_t$. Then $p \cdot x = \beta(p) p \cdot h(p) = 0$, and so: $x \in \partial L_t^*$ by (14.b). Thus, we have: $D_t \cap \partial L_t^* \supseteq \varphi(V_t)$. Also it follows from (17.d.ii) that $D_t \cap \partial L_t^* \subseteq \varphi(V_t)$. Therefore (17.a.ii) follows.

(3.3 Open Hereditary Stage: transforming D_t to E_t where $t \in [\epsilon, \hat{\alpha} - \epsilon]$.) The sets D_t satisfy only the property of hereditary, but not the open hereditary. To obtain this property, we will expand each D_t to a union of closed balls $\bar{N}_{t,x}$ centered at $x \in D_t$, while maintaining rationality, graduality and compactness. (See Figure 4 below.)

In defining the radii of the closed balls $\bar{N}_{t,x}$, we will make use of the function $d : [\epsilon, \hat{\alpha} - \epsilon] \times \mathbb{R}^l \to \mathbb{R}$ defined by $d(t, x) = \min\{\|x - y\| : y \in \cup_{p \in L_t} B_p\}$. Since the sets L_t are compact, it follows that the sets $\cup_{p \in L_t} B_p$ are closed, and so the function d is well-defined. Also, it is easy to see that $d(t, x)$ is non-increasing in t. Moreover, observe that each $\cup_{p \in L_t} B_p$ is the closure of the complement of L_t^*, then by the continuity of the mapping $t \mapsto L_t^*$, it follows that the function d is continuous.

We now define the closed ball $\bar{N}_{t,x} = \{y \in \mathbb{R}^l : \|y - x\| \leq (t/\hat{\alpha}) d(t, x)\}$ for every $x \in D_t$ and every $t \in [\epsilon, \hat{\alpha} - \epsilon]$. We define the set $E_t = \cup_{x \in D_t} \bar{N}_{t,x}$ for every $t \in [\epsilon, \hat{\alpha} - \epsilon]$. Then the sets E_t satisfy:

a) (Rationality) for every t,
   i) $E_t \subseteq L_t^*$,
   ii) $E_t \cap \partial L_t^* = \varphi(V_t)$,
b) (Compactness) every E_t is compact,
c) (Graduality) the function $t \mapsto E_t$ is continuous,
d) (Open Hereditary) if $t' < t'$, then $E_{t^*} \subseteq \mathrm{Int} E_t$.

(18)

Facts (18.b-c) follows clearly from (17.b-c) and the continuity of the function d. To show (18.d), let $t' < t$. For any $x \in D_{t'}$, we have: $x \in D_t$ by (17.d.i). Then it clearly suffices to show that $\bar{N}_{t',x} \subseteq \mathrm{Int} \bar{N}_{t,x}$. By

(17.d.ii) we have $0 < d(t,x)$. Since $d(t',x) \leq d(t,x)$ and $t' < t$, we have: $\frac{t'}{\hat{\alpha}}d(t',x) < \frac{t}{\hat{\alpha}}d(t,x)$. Therefore, we have: $\bar{N}_{t',x} \subseteq \text{Int}\bar{N}_{t,x}$; and so (18.d) holds. To show (18.a), consider any t and $x \in D_t$. It is clear that $\bar{N}_{t,x} \subseteq L_t^*$, and so $E_t = \cup_{x \in D_t} \bar{N}_{t,x} \subseteq L_t^*$, i.e. (18.a.i) holds. It remains to show (18.a.ii). Since $E_t \supseteq D_t$, by (17.a.ii) we have: $E_t \cap \partial L_t^* \supseteq \varphi(V_t)$. We now show that $E_t \cap \partial L_t^* \subseteq \varphi(V_t)$. To see this, consider any $y \in E_t \cap \partial L_t^*$. Then $y \in \bar{N}_{t,x}$ for some $x \in D_t$. Suppose by contradiction that $x \in \text{Int}L_t^*$. Then we clearly would have: $\bar{N}_{t,x} \subseteq \text{Int}L_t^*$, and so: $\text{Int}L_t^* \supseteq \bar{N}_{t,x} \ni y \in \partial L_t^*$, a contradiction. Therefore, we must have: $x \in \partial L_t^*$; so $d(t,x) = 0$, and hence: $\bar{N}_{t,x} = \{x\}$. Therefore, we have: $\partial L_t^* \ni y = x \in D_t$; and so $y \in \varphi(V_t)$ by (17.a.ii). Thus, we have: $E_t \cap \partial L_t^* \subseteq \varphi(V_t)$. Hence (18.a.ii) holds.

(3.4 Convexity Stage: transforming E_t to F_t, where $t \in [\epsilon, \hat{\alpha} - \epsilon]$.) Since E_t may not be convex, we let F_t be the convex hull of E_t. (See Figure 5 below.) Then the sets F_t satisfy:

$$\begin{array}{ll}
\text{a) (Rationality) for every } t, & \\
\quad \text{(i) } F_t \subseteq L_t^*, & \\
\quad \text{(ii) } F_t \cap \partial L_t^* = \varphi(V_t), & \\
\text{b) (Compactness) every } F_t \text{ is compact,} & \qquad (19) \\
\text{c) (Graduality) the function } t \mapsto F_t \text{ is continuous,} & \\
\text{d) (Open hereditary) if } t' < t, \text{ then } F_{t'} \subseteq \text{Int}F_t, & \\
\text{e) (Convexity) every } F_t \text{ is convex.} &
\end{array}$$

Facts (19.b-d) follow clearly from (18.b-d). And (19.a.i) follows from (18.a.i) and the fact that the cones L_t^* are convex. To see (19.a.ii), consider any $t \in [\epsilon, \hat{\alpha} - \epsilon]$. Since $F_t \supseteq E_t$, by (18.a.ii) we have: $F_t \cap \partial L_t^* \supseteq \varphi(V_t)$. To show that $F_t \cap \partial L_t^* \subseteq \varphi(V_t)$, consider any $x \in F_t \cap \partial L_t^*$. By the definition of F_t, we have: $x = \sum_{i=1}^n \lambda_i x_i$ for some n, some real numbers $\lambda_1, \cdots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i$, and some vectors $x_1, \cdots x_n \in E_t \subseteq L_t^*$. Since $x \in \partial L_t^*$, we have: $x_1, \cdots, x_n \in \partial L_t^*$. By (18.a.ii), there exist $p_1, \cdots, p_n \in V_t$ such that $x_i = \varphi(p_i)$ for all i. We claim that $p_1 = \cdots = p_n$. Suppose not, then it follows from (13.b) that for every $r \in L_t^*$, we have: $0 <$

$\beta_i(p_i)r \cdot h(p_i) = r \cdot \varphi(p_i) = r \cdot x_i$ for some i. Also, for every i, since $x_i \in L_t^*$, we have: $r \cdot x_i \geq 0$. Therefore, we have: $r \cdot x = \sum_{i=1}^n \lambda_i r \cdot x_i > 0$ for every $r \in L_t$; and so $x \in \mathrm{Int}L_t^*$ by (14.a). Thus we have derived the contradiction: $\mathrm{Int}L_t^* \ni x \in \partial L_t^*$. Therefore, we must have: $p_1 = \cdots = p_n$; and so $x = x_1 = \cdots = x_n \in \varphi(V_t)$. Hence $F_t \cap \partial L_t^* \supseteq \varphi(V_t)$. And (19.a.ii) holds.

(3.5 Insatiability Stage: transforming F_t to G_t, where $t \in [\epsilon, \hat{\alpha} - \epsilon]$.) Since the sets F_t does not satisfy the instability property. To obtain this property, we will adding them with a convex cone A spanned by a closed ball centered at a, while maintaining the properties of rationality, graduality and convexity.[19] (See Figure 6 below.) Of course, this leads us to weaken the property of compactness to closedness.

The closed ball we pick can be any $\bar{N}_a = \{x \in I\!R^l : ||x - a|| < \delta\}$ with $\delta > 0 \in I\!R$ and such that $p \cdot x > 0$ for every $p \in L_\epsilon$ and every $x \in \bar{N}_a$. We then define the cone $A = \{\lambda x : \lambda \geq 0 \in I\!R, x \in \bar{N}_a\}$, which is clearly closed and convex, and has non-empty interior. Also, for any $z \in A$, any $p \in L_t$ and any $t \in [\epsilon, \hat{\alpha} - \epsilon]$, we have: i) $p \cdot z \geq 0$, and ii) $p \cdot z > 0$ if $z \neq 0$. Then by (i) we have: $A \subseteq L_t^*$; and by (ii) we have: $z \in \mathrm{Int}L_t^*$ for every non-zero $z \in A$. Also, it is easy to see that $z + \lambda a \in \mathrm{Int}A$ for every $z \in A$ and every $\lambda > 0 \in I\!R$.

For every $t \in [\epsilon, \hat{\alpha} - \epsilon]$, we define $G_t = F_t + A$. Then the sets G_t satisfy

---

19 Debreu [1974] made use of $I\!R^l_+$ to obtain monotone weak-preferred sets. Our convex cone A shall be a proper subset of $I\!R^l$ in order to maintain rationality property for allowing "non-positive" prices p (i.e. $p_i < 0$ for some i) in L_t. (Cf. the proof of (20.a.i) given below.)

the following properties:

$$\begin{array}{ll}
\text{a) (Rationality) for every } t, & \\
\quad \text{i) } G_t \subseteq L_t^*, & \\
\quad \text{ii) } G_t \cap \partial L_t^* = \varphi(V_t), & \\
\text{b) (Closedness) every } G_t \text{ is closed,} & \\
\text{c) (Graduality) the function } t \mapsto G_t \text{ is continuous,} & (20) \\
\text{d) (Open Hereditary) if } t' < t, \text{ then } G_{t'} \subseteq \mathrm{Int} G_t, & \\
\text{e) (Convexity) every } G_t \text{ is convex,} & \\
\text{f) (Instability) } x + \lambda a \in \mathrm{Int} G_t \text{ for every } \lambda > 0, & \\
\qquad \text{every } x \in G_t \text{ and every } G_t. &
\end{array}$$

Fact (20.b) holds because of (19.b) and the fact that A is closed. Facts (20.c-d) follow clearly from (19.c-d). Fact (20.e) follows clearly from (19.e) and the fact that A is convex. Fact (20.a.i) follows from the fact that each L_t^* is a convex cone containing E_t and A. It remains to show (20.f) and (20.a.ii). Consider any t and any $x \in G_t$. Then $x = y + z$ for some $y \in F_t$ and some $z \in A$. To see (20.f), consider any $\lambda > 0$. Then we have: $z + \lambda a \in \mathrm{Int} A$, and so $x + \lambda a = y + z + \lambda a \in F_t + \mathrm{Int} A \subseteq \mathrm{Int} G_t$. Thus (20.f) holds. To see (20.a.ii), since $G_t \supseteq F_t$, by (19.a.ii) we have: $G_t \cap \partial L_t^* \supseteq \varphi(V_t)$. We now show that $G_t \cap \partial L_t^* \subseteq \varphi(V_t)$. Suppose $x \in \partial L_t^*$. Then we have: $z = 0$, because otherwise, we would have $z \in \mathrm{Int} L_t^*$, and so $x = y + z \in \mathrm{Int} L_t^*$. Therefore, we have: $F_t \ni y = x \in \partial L_t^*$. Then by (19.a.ii), we have: $x \in \varphi(V_t)$. Thus, we have shown that $G_t \cap \partial L_t^* \subseteq \varphi(V_t)$. Hence (20.a.ii) holds.

(3.6 Strict Convexity Stage: transforming G_t to M_t, where $t \in [\epsilon, \hat{\alpha} - \epsilon]$.) The sets G_t are not strictly convex. To obtain this property, we will transform them to the sets M_t as given in the beginning of this proof of the lemma, where $t \in [\epsilon, \hat{\alpha} - \epsilon]$. To do this, we pick any vector $\hat{p} \in Q$ with $\hat{p} \cdot a = \hat{\alpha}$ and let $T_{\hat{p}}$ be the tangent space of $\hat{p}$, i.e. $T_{\hat{p}} = \{y \in \mathbb{R}^l : \hat{p} \cdot y = 0\}$. We will define strictly concave functions $\mu_t : T_{\hat{p}} \to \mathbb{R}$, and will define $M_t = \{y + sa : y \in T_{\hat{p}} \text{ and } \mu_t(y) \leq s \in \mathbb{R}\}$. (See Figure 7 below, cf. also (26) below.)

To define the functions μ_t, we first define two functions as follows. For every $t \in [\epsilon, \hat{\alpha} - \epsilon]$ and every $y \in T_{\hat{p}}$, let $\gamma_t(y)$ be the least s such that $y + sa \in G_t$, and let $\lambda_t(y)$ be the least s such that $y + sa \in L_t^*$. (See Figure 7 below.) We claim: i) $0 \leq \lambda_t(y)$, ii) $\lambda_t(y) \leq \gamma_t(y)$, and iii) $\gamma_t(y) < \infty$. To see (iii), it suffices to find any s with $y + sa \in G_t$. To do this, we can pick any $x \in F_t$ and pick any $s \in \mathbb{R}$ with $y - x \in s(\bar{N}_a - a)$, where $\bar{N}_a$ is given in the second paragraph in Stage 3.5. Then we have: $y + sa = x + sz$ for some $z \in \bar{N}_a$, and so $y + sa \in F_t + A = G_t$. Hence (iii) holds. To see (i), by (iii) we have: $y + sa \in G_t \subseteq L_t^*$ for some $s \in \mathbb{R}$. Also, notice that for every $s < 0$ we have: $\hat{p} \cdot (y + sa) = s\hat{p} \cdot a = s\hat{\alpha} < 0$, and so: $y + sa \notin L_t^*$ because $\hat{p} \in L_t$. Therefore (i) follows. Finally, (ii) is clear from (20.a.i). Thus the claim is established.

We now list four useful properties for the functions $(t, y) \mapsto \lambda_t(y)$ and $(t, y) \mapsto \gamma_t(y)$. First, they are continuos, by (20.c) and by the continuity of the mapping $t \mapsto L_t^*$. Second, each $\lambda_t(\cdot)$ and $\gamma_t(\cdot)$ is a convex function on $T_{\hat{p}}$ for every t, because each G_t and L_t^* is a convex set. Third, for every t and every $y \in T_{\hat{p}}$, since the sets L_t^* and G_t are closed, it follows that $y + \lambda_t(y) \in \partial L_t^*$ and $y + \gamma_t(y) \in \partial G_t$. Fourth, for every t and every $y \in T_{\hat{p}}$, by the previous property and (20.a.ii), we have: $\lambda_t(y) = \gamma_t(y)$ if and only if $y + \lambda_t(y) \in \varphi(V_t)$.

We pick a continuous convex function ρ defined on $\{(s, \tilde{s}) \in \mathbb{R}_+^2 : s \leq \tilde{s}\}$ (e.g. the one given in the appendix of Debreu [1974]) such that:

$$
\begin{array}{ll}
\text{(a) } \rho \text{ is strictly increasing in each variable,} & \\
\text{(b) } \rho(s, s) = s \text{ for every } s \in \mathbb{R}_+, & \\
\text{(c) if } s \neq \tilde{s} \text{ and/or } s' \neq \tilde{s}', \text{ then } \rho \text{ is strictly convex on} & \\
\quad\ \text{the segment } [(s, \tilde{s}), (s', \tilde{s}')]. &
\end{array} \tag{21}
$$

Then we define $\mu_t(y) = \rho(\lambda_t(y), \gamma_t(y))$ for every $t \in [\epsilon, \hat{\alpha}]$ and every $y \in T_{\hat{p}}$. Clearly, the function $(t, y) \mapsto \mu_t(y)$ is continuous.

We now show that $\mu_t(\cdot)$ is a strictly convex function for every $t \in [\epsilon, \hat{\alpha}]$.

To see this, consider any $t \in [\epsilon, \hat{\alpha} - \epsilon]$, any pair of distinct vectors $y, y' \in T_{\hat{p}}$, and any $\tau \in (0,1)$. Then:

$$
\begin{aligned}
&\mu_t(\tau y + (1-\tau)y') \\
&\quad = \rho(\lambda_t(\tau y + (1-\tau)y'), \gamma_t(\tau y + (1-\tau)y')) \\
&\quad \leq \rho(\tau\lambda_t(y) + (1-\tau)\lambda_t(y'), \tau\gamma_t(y) + (1-\tau)\gamma_t(y')) && (22) \\
&\quad \leq \tau\rho(\lambda_t(y), \gamma_t(y)) + (1-\tau)\rho(\lambda_t(y'), \gamma_t(y')) && (23) \\
&\quad = \tau\mu_t(y) + (1-\tau)\mu_t(y').
\end{aligned}
$$

Therefore, it suffices to show that either (22) or (23) is a strict inequality. There are three possible cases, as follows.

First, suppose $\lambda_t(y) = \lambda_t(y')$. Notice that $y + \lambda_t(y)a, y' + \lambda_t(y')a \in \partial L_t^*$. By (14.b), there exist real numbers $s, s' \geq 0$ and vectors $p, p' \in V_t$ such that $y + \lambda_t(x)a = sh(p)$ and $y' + \lambda_t(x')a = s'h(p')$. We claim that $p \neq p'$. Suppose not. Then we have: $(s - s')h(p) = (y - y')$, and hence: $(s - s')\hat{p} \cdot h(p) = \hat{p} \cdot (y - y') = 0$. Notice that $s \neq s'$ because $y \neq y'$. Therefore we have: $\hat{p} \cdot h(p) = 0$, contradicting to (13.b) because $a \cdot \hat{p} = \hat{\alpha} > t = p \cdot a$. Thus we must have: $p \neq p'$. Similarly, it can be shown that $s, s' > 0$. Then it follows from (13.b) that for every $\tilde{p} \in L_t$, we have: $0 < \tilde{p} \cdot (\tau sh(p) + (1-\tau)s'h(p')) = \tilde{p} \cdot (\tau y + (1-\tau)y' + (\tau\lambda_t(x) + (1-\tau)\lambda_t(y'))a)$. Therefore, by (14.a), we have: $\tau y + (1-\tau)y' + (\tau\lambda_t(x) + (1-\tau)\lambda_t(y'))a \in \mathrm{Int}L_t^*$. Then it follows from the definition of $\lambda_t(\tau y + (1-\tau)y')$ that we have: $\lambda_t(\tau y + (1-\tau)y') < \tau\lambda_t(y) + (1-\tau)\lambda_t(y')$. Then by (21.a) and the convexity of the function $\gamma_t(\cdot)$, it follows that the inequality (22) is strict.

Second, suppose $\lambda_t(y) \neq \lambda_t(y')$, and suppose $\lambda_t(y) = \gamma_t(y)$ and $\lambda_t(y') = \gamma_t(y')$. Then there exist $p, p' \in V_t$ such that: $y + \lambda_t(y)a = \varphi(p)$ and $y' + \lambda_t(y')a = \varphi(p')$. We claim that $p \neq p'$. Suppose not. Then we have: $y + \lambda_t(y)a = y' + \lambda_t(y')a$; and hence have the contradiction: $0 = \hat{p}(y - y') = (\lambda_t(y) - \lambda_t(y'))\hat{p} \cdot a = (\lambda_t(y) - \lambda_t(y'))\hat{\alpha} \neq 0$. Thus, we must have: $p \neq p'$.

By applying the arguments used in the last paragraph, it follows that the inequality (22) is strict.

Third, suppose $\lambda_t(y) \neq \lambda_t(y')$ and suppose $\lambda_t(y) \neq \gamma_t(y)$ and/or $\lambda_t(y') \neq \gamma_t(y')$. Then $(\lambda_t(y), \gamma_t(y)) \neq (\lambda_t(y'), \gamma_t(y'))$, and so (21.c) implies that the inequality (23) is strict. Thus we have shown that $\mu_t(\cdot)$ is a strictly convex function for every $t \in [\epsilon, \hat{\alpha} - \epsilon]$.

As mentioned earlier, we define:

$$M_t = \{y + sa : y \in T_{\hat{p}},\ s \geq \mu_t(y)\} \qquad \text{for every } t \in [\epsilon, \hat{\alpha} - \epsilon]. \tag{24}$$

We now show that:

$$\{M_t : t \in [\epsilon, \hat{\alpha} - \epsilon]\} \text{ defined by (24) satisfies (15.a-g).} \tag{25}$$

Properties (15.b-c) are satisfied because $T_{\hat{p}}$ is closed and the function $(t, y) \mapsto \mu_t(y)$ is continuous. Property (15.e) will follow immediately from the fact that each $\mu_t(\cdot)$ is strictly convex, once we have shown (15.f). Our proof for (15.f) requires the following decomposition fact:

each $x \in I\!R^l$ has uniquely a vector $\pi_x \in I\!R^l$ and a $s \in I\!R$ such that: $x = \pi_x + s_x a$; in particular, one has: $s_x = \frac{\hat{p} \cdot x}{\hat{p} \cdot a}$ and $\pi_x = x - s_x a$. (26)

Thus π_x is the projection of x along the direction of a on $T_{\hat{p}}$. To see (26), since $\hat{p} \cdot a = \hat{\alpha} > 0$, and so π_x and s_x are well-defined. By algebra calculation, it is easy to see that $x = \pi_x + s_x a$. To see the uniqueness of the decomposition, let $x = y + sa$, where $y \in T_{\hat{p}}$ and $s \in I\!R$. Then have: $\hat{p} \cdot x = \hat{p} \cdot y + s\hat{p} \cdot a = s\hat{p} \cdot a$. Therefore, $s = s_x$ and so $y = x - s_x a = \pi_x$.

We now show that $\{M_t : t \in [\epsilon, \hat{\alpha} - \epsilon]\}$ satisfies (15.f). Consider any $t \in [\epsilon, \hat{\alpha} - \epsilon]$ and any $x \in M_t$. Let $x = y + sa$, where $y \in T_\epsilon$ and $s \geq \mu_t(y)$. Consider any $\lambda > 0$. We need to show that the vector $x' = x + \lambda a \in \mathrm{Int} M_t$. By (26), x' has a unique decomposition: $x' = \pi_{x'} + s_{x'} a$, where $\pi_{x'} \in T_{\hat{p}}$ and

$s_{x'} \in \mathbb{R}$ are as defined in (26). Since $y+(s+\lambda)a = x+\lambda a = x' = \pi_{x'}+s_{x'}a$, we have: $\pi_{x'} = y$ and $s_{x'} = s+\lambda > \mu_t(y) = \mu_t(\pi'_x)$, and so $x' \in M_t$. We now show that x' is indeed an interior point of M_t. Notice that the mappings $\tilde{x} \mapsto s_{\tilde{x}}$ and $\tilde{x} \mapsto \mu_t(\pi_{\tilde{x}})$ are continuous on $\mathbb{R}^l$; therefore, we can pick any open ball $\tilde{N} \ni x'$ such that $s_{\tilde{x}} > \mu_t(\pi_{\tilde{x}})$ for every $\tilde{x} \in \tilde{N}$. Then we have: $x' \in \tilde{N} = \{\pi_{\tilde{x}} + s_{\tilde{x}}a : \tilde{x} \in \tilde{N}\} \subset \{\pi_{\tilde{x}} + sa : s \geq \mu_t(\pi_{\tilde{x}}), \text{ and } \tilde{x} \in \tilde{N}\} \subset \{y + sa : s \geq \mu_t(y), \text{ and } y \in T_{\hat{p}}\} = M_t$. Thus, $x' \in \text{Int}M_t$. And we have shown that $\{M_t : t \in [\epsilon, \hat{\alpha} - \epsilon]\}$ satisfies (15.f)

We now show that $\{M_t : t \in [\epsilon, \hat{\alpha} - \epsilon]\}$ satisfies (15.d). Consider any $t, t' \in [\epsilon, \hat{\alpha} - \epsilon]$ with $t' < t$. Then we have: $\lambda_{t'}(\cdot) \geq \lambda_t(\cdot)$, and $\gamma_{t'}(\cdot) > \gamma_t(\cdot)$ by (20.d). By (21.a), we have: $\mu_{t'}(\cdot) > \mu_t(\cdot)$. Now consider any $x' \in M_{t'}$, and let $x' = y + sa$, where $y \in T_{\hat{p}}$ and $s \geq \mu_{t'}(y)$. Then we have: $x' = x+(s-\mu_t(y))a$, where $x = y+\mu_t(y)a \in M_t$. Since $s-\mu_t(y) > s-\mu_{t'}(y) \geq 0$, by (15.f) we have: $x' \in \text{Int}M_t$. Therefore, $\{M_t : t \in [\epsilon, \hat{\alpha} - \epsilon]\}$ satisfies (15.d).

It remains to show (15.a). Consider any $t \in [\epsilon, \hat{\alpha} - \epsilon]$ and any $x \in M_t$. Let $x = y + sa$, where $y \in T_{\hat{p}}$ and $s \geq \mu_t(y)$. Since $\gamma_t(y) \geq \lambda_t(y)$, by (21.a) we have: $\mu_t(y) \geq \lambda_t(y)$; and so $s \geq \lambda_t(y)$. Therefore, we have: $p \cdot x = p \cdot (y + \lambda_t(y)a) + (s - \lambda_t(y))p \cdot a \geq 0$ for every $p \in L_t$; and so $x \in L_t^*$. Thus (15.a.i) holds.

To show (15.a.ii), consider any $t \in [\epsilon, \hat{\alpha} - \epsilon]$ and any $x \in \mathbb{R}^l$. Let π_x and s_x be as given in (26). Then $x = \pi_x + s_x a$. It is clear that $x \in \partial L_t^*$ if and only if $s_x = \lambda_t(\pi_x)$. We claim that: I) $x \in \varphi(V_t)$ implies $x \in M_t \cap \partial L_t^*$, and II) $x \in M_t \cap \partial L_t^*$ implies $x \in \varphi(V_t)$. To show (I), suppose $x \in \varphi(V_t)$. By (20.a), we have $\pi_x + s_x a = x \in L_t^* \cap G_t$ and so: $s_x \geq \lambda_t(\pi_x), \gamma_t(\pi_x)$. Then we have: $\lambda_t(\pi_x) = s_x \geq \gamma_t(\pi_x) \geq \lambda_t(\pi_x)$, and so: $s_x = \lambda_t(\pi_x) = \gamma_t(\pi_x)$. By (21.b), we have: $s_x = \mu_t(\pi_x)$; and so $x = \pi_x + s_x a \in M_t$. Thus, we have: $x \in M_t \cap \partial L_t^*$. Hence (I) holds. To show (II), suppose $x \in M_t \cap \partial L_t^*$. Since

$x \in M_t$, we have: $x = y + sa$ for some $y \in T_{\hat{p}}$ and $s \in I\!R$ with $s \geq \mu_t(y)$. By (26), we have: $\pi_x = y$ and $s_x = s$; and so $s_x \geq \mu_t(\pi_x)$. Since $x \in L_t^*$, we have: $\lambda_t(\pi_x) = s_x$. We assert that $\gamma_t(\pi_x) = s_x$. Suppose not. Then we have $\gamma_t(\pi_x) > s_x = \lambda_t(\pi_x)$; and so we have: $\mu_t(\pi_x) > \lambda_t(\pi_x) = s_x$ by (21.a-b). Therefore, we have the contradiction: $s_x \geq \mu_t(\pi_x) > \lambda_t(\pi_x) = s_x$. Thus we must have: $\gamma_t(\pi_x) = s_x$. Therefore, $x = \pi_x + s_x a = \pi_x + \gamma_t(\pi)a \in G_t$. Then we have: $x \in L_t^* \cap G_t$, and so $x \in \varphi(V_t)$ by (20.aii). Thus (II) holds. Then by (I) and (II), we have: $M_t \cap \partial L_t^* = \varphi(V_t)$, and therefore (15.a.ii) holds.

(3.7 Progressiveness Stage: defining M_t for every $t \notin [\epsilon, \hat{\alpha}-\epsilon]$.) To complete the construction of a profile $\{M_t : t \in (-\infty, \infty)\}$ as given in the beginning of this proof for Lemma 4, we define

$$M_t = \begin{cases} M_\epsilon + (\epsilon - t)a & \text{for every } t \in (-\infty, \epsilon), \\ M_t = M_{\hat{\alpha}-\epsilon} + (\hat{\alpha} - \epsilon - t)a & \text{for every } t \in (\hat{\alpha} - \epsilon, \infty), \end{cases} \qquad (27).$$

where $M_{\hat{\epsilon}}$ and $M_{\hat{\alpha}-\epsilon}$ are defined in (24). It is easy to see that the family $\{M_t : t \in (-\infty, \infty)\}$ of sets defined by (24) and (27) satisfies (15.a-f). We now show that it also satisfies (15.g). Consider any $x \in I\!R^l$, which by (26) has a unique decomposition $x = \pi_x + s_x a$. By the definitions (24) and (27), it is easily verified that: I) for every $t \geq \hat{\alpha} - \epsilon$, if $s \geq \mu_{\hat{\alpha}-\epsilon}(\pi_x) + (\hat{\alpha} - \epsilon) - t$, then $\pi_x + sa \in M_t$; and II) for every $t' \leq \epsilon$, if $s < \mu_\epsilon(\pi_x) + (\epsilon - t)$, then $\pi_x + sa \notin M_{t'}$. Therefore, we have: $x \in M_t$ for every $t \geq \max\{\hat{\alpha} - \epsilon, \mu_{\hat{\alpha}-\epsilon}(\pi_x) + (\hat{\alpha} - \epsilon) - s_x\}$; and $x \notin M_{t'}$ for every $t' < \min\{\epsilon, \mu_\epsilon(\pi_x) + \epsilon - s_x\}$. Hence (15.g) holds.

Thus we have constructed a family $\{M_t : t \in (-\infty, \infty)\}$ as claimed in the beginning of this proof for Lemma 4, and we have shown that it satisfies (15.a-g).

Q.E.D.

References

Arrow, K.J. and Debreu, G. 1954. Existence of an equilibrium for a competitive economy. *Econometrica* **22:** 265-90.

Brouwer, L.E.J. 1910. Über ein eindeutige, stegie Transformationen von Flächen in sich. *Mathematische Annalen* **69:** 176-80.

Debreu, G. 1956. Market equilibrium. *Proceedings of the National Academy of Sciences of the U.S.A.*, **42:** 876-878.

Debreu, G. 1974. Excess demand functions. *Journal of Mathematical Economics,***1:** 15-21.

Debreu, G. 1982. Existence of competitive equilibrium. Ch. 15 in *Handbook of Mathematical Economics* Vol. 2, edited by K.J. Arrow and M.D. Intrilligator. Amsterdam: North-Holland.

El-Hodiri, M. A. 1970. *Constrained Extrema: Introduction to the Differentiable Case with Economic Applications.* Springer-Velag.

Gale, D. 1955. The law of supply and demand. *Mathematica Scandinavica,* **3:** 155-169.

Hildenbrand, W. 1974. *Core and Equilibria of a Large Economy.* New Jersey: Princeton University Press.

Mantel, R. 1974. On the characterization of aggregate excess demand. *Journal of Economic Theory* **12:** 197-202.

Mantel, R. 1979. Homothetic preferences and community excess demand functions. *Journal of Economic Theory* **12:** 197-202.

Mas-Colell, A. 1977. On the equilibrium price set of an exchange economy. *Journal of Mathematical Economics* **4:** 117-126.

Mas-Colell, A. 1985. *The Theory of General Economic Equilibrium: A Differentiable Approach.* New York: Cambridge University Press.

McFadden, D., A. Mas-Colell, R. Mantel, and M.K. Richter 1974. A characterization of community excess demand functions. *Journal of*

Economic Theory **9:** 361-374.

Munkres, J.M. 1975. *Topology, A First Course.* New Jersey: Prentice Hall, Inc.

Nikaido, H. 1956. On the classical multilateral exchange problem. *Metroeconomica,* **8:** 135-145.

Rodyen, H.L. 1988. *Real Analysis.* 3rd edition. New York: Macmillan.

Rudin, W. 1976. *Principles of Mathematical Analysis.* 3rd. edition, 1986. McGrawHill.

Sonnenschein, H. 1972. Market excess demand functions. *Econometrica,* **40:** 549-563.

Sonnenschein, H. 1973. Do Walras identity and continuity characterize the class of community excess demand functions? *Journal of Economic Theory* **6:** 345-354.

Uzawa, H. 1962. Walras' existence theorem and Brouwer's fixed-point theorem. *Economic Studies Quarterly,* **13:** 59-62.

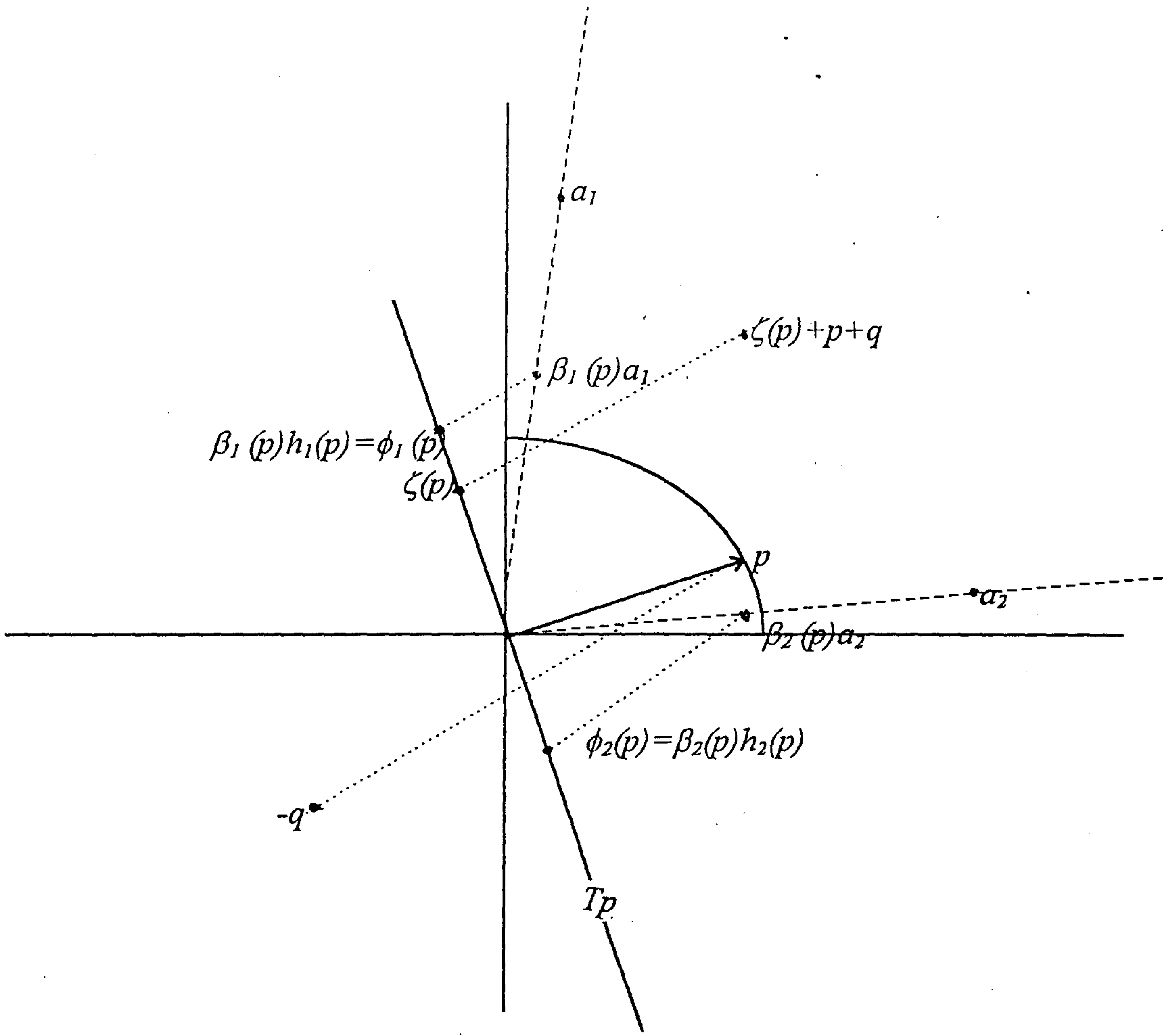

Figure 1: Oblique Decomposition.

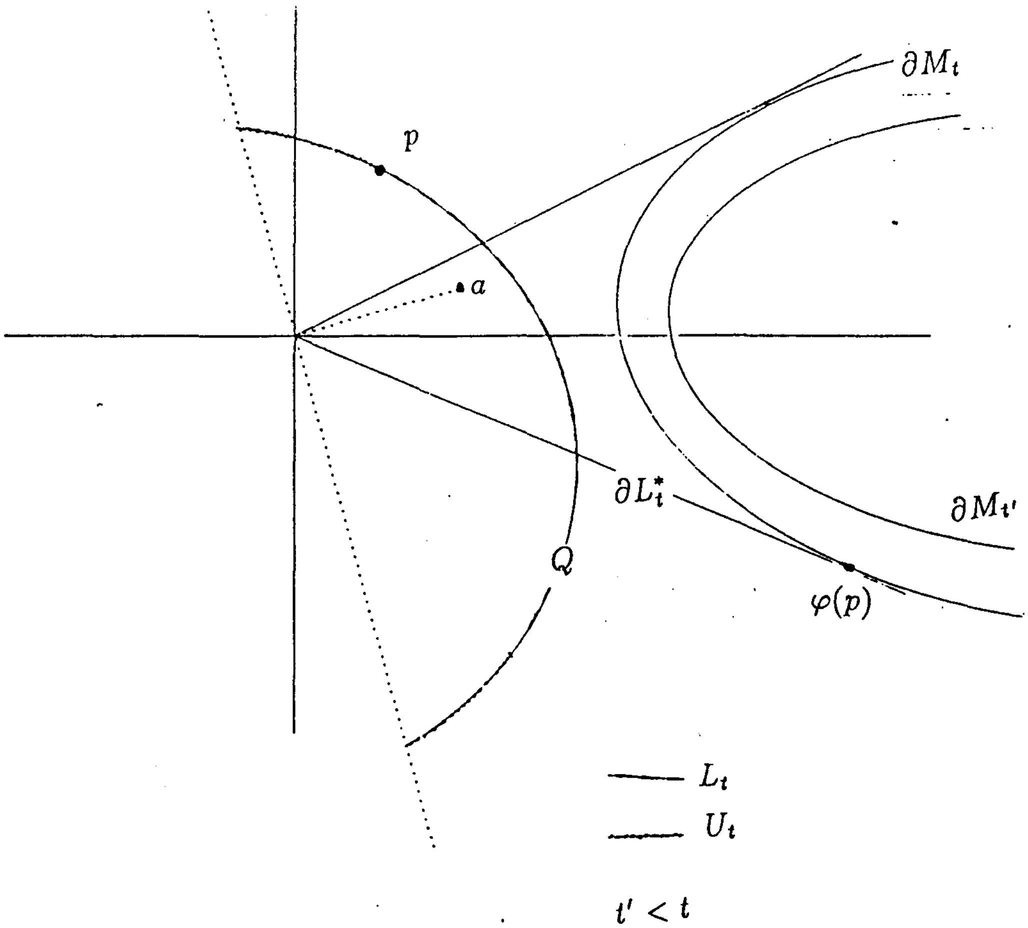

Figure 2: Indirect Preference and Direct Preference.

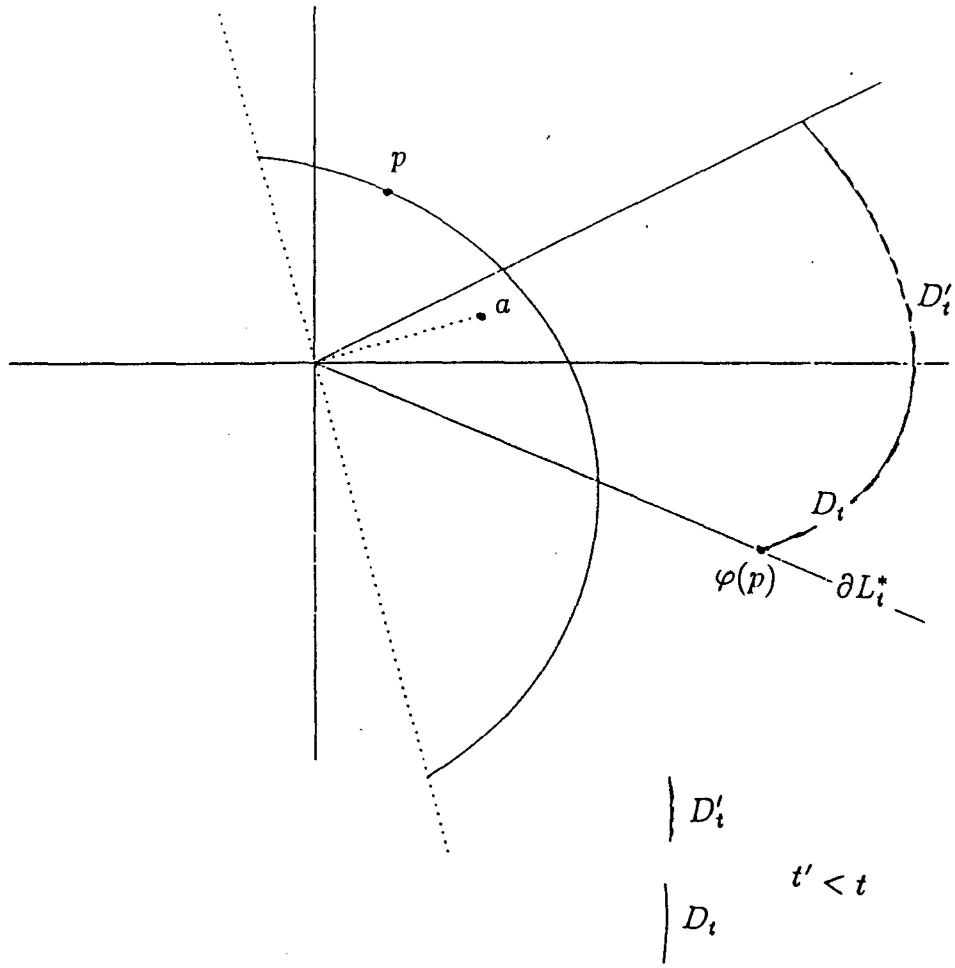

Figure 3: Direct Stage.

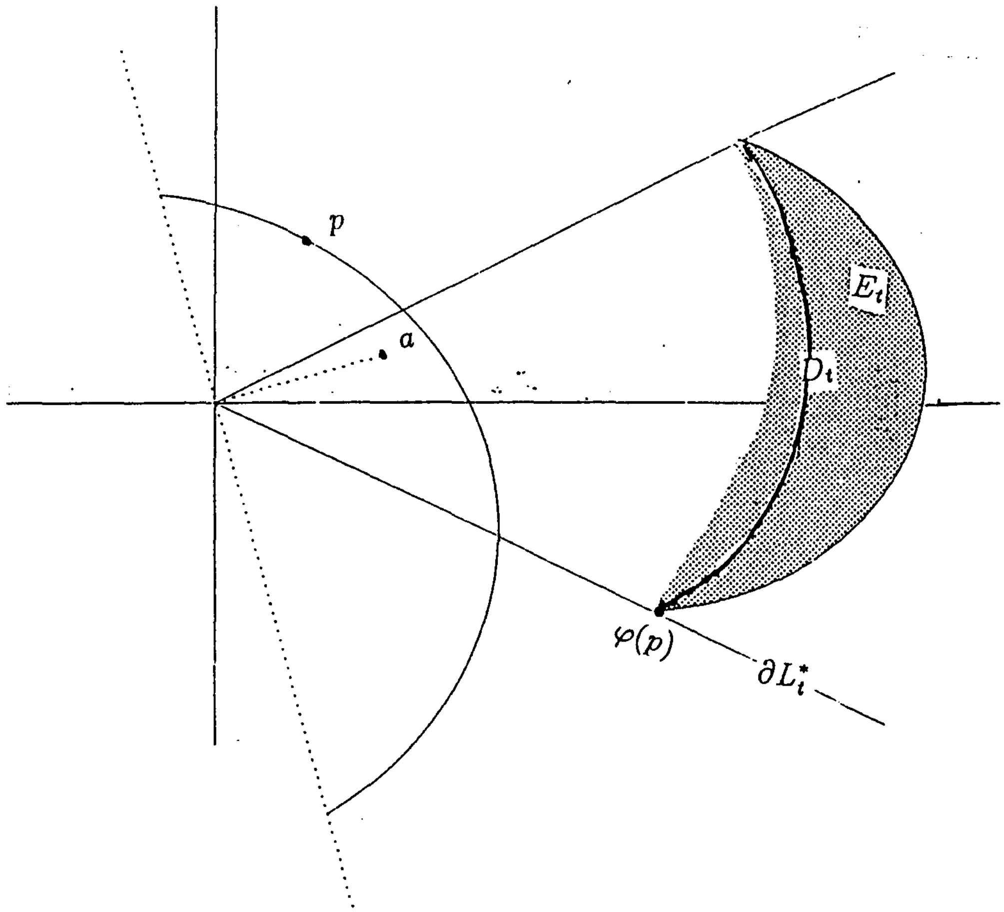

Figure 4: Open Hereditary Stage.

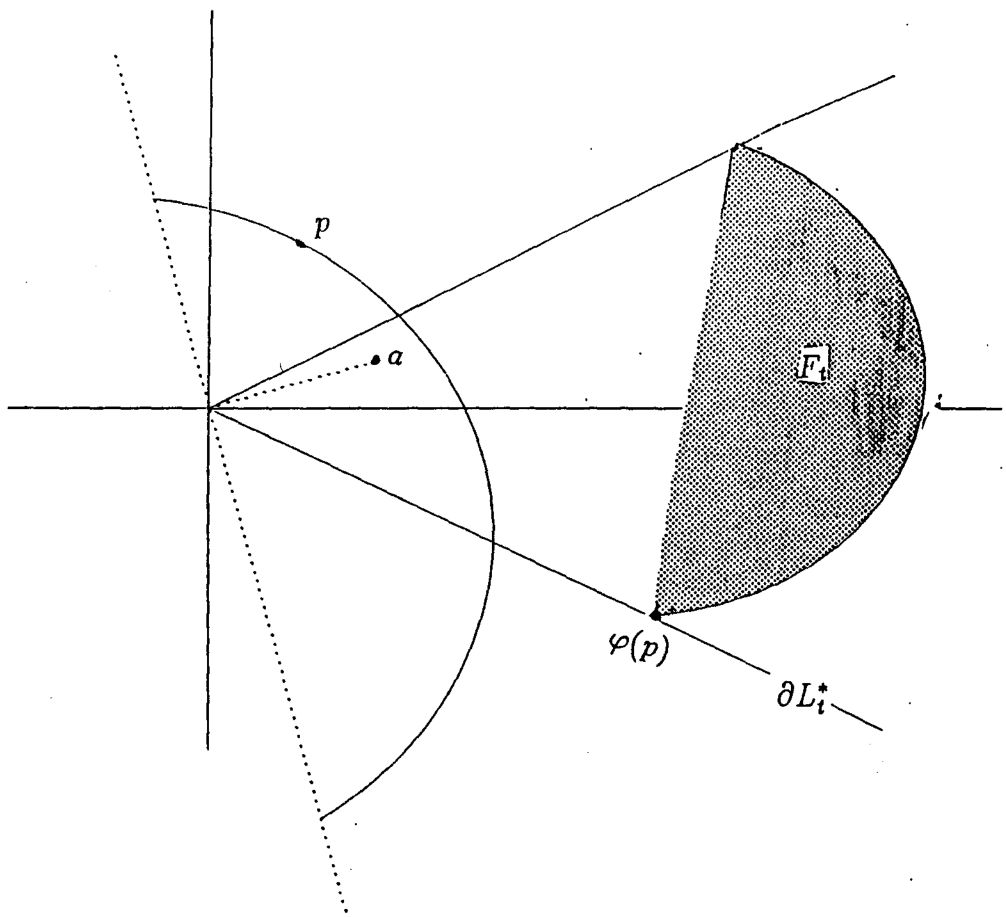

Figure 5: Convexity Stage.

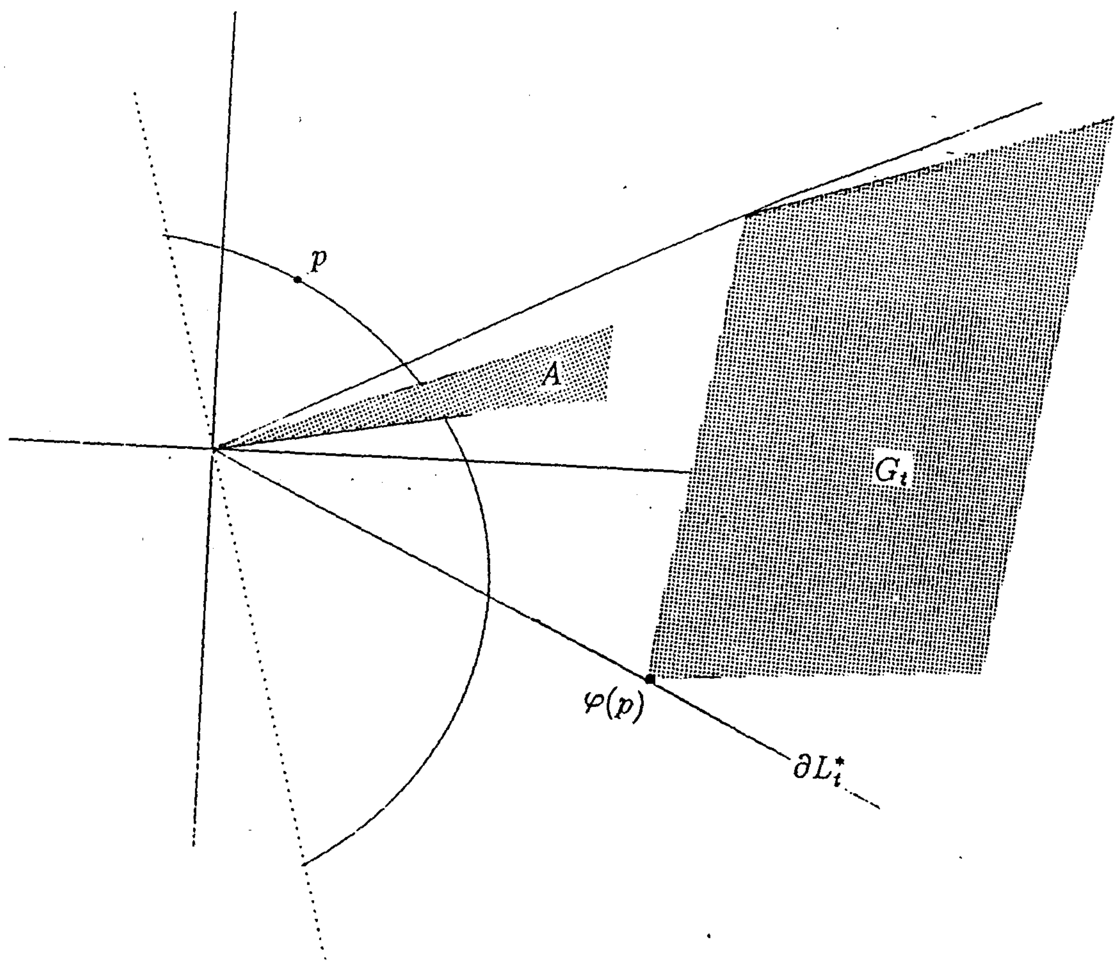

Figure 6: Insatiability Stage.

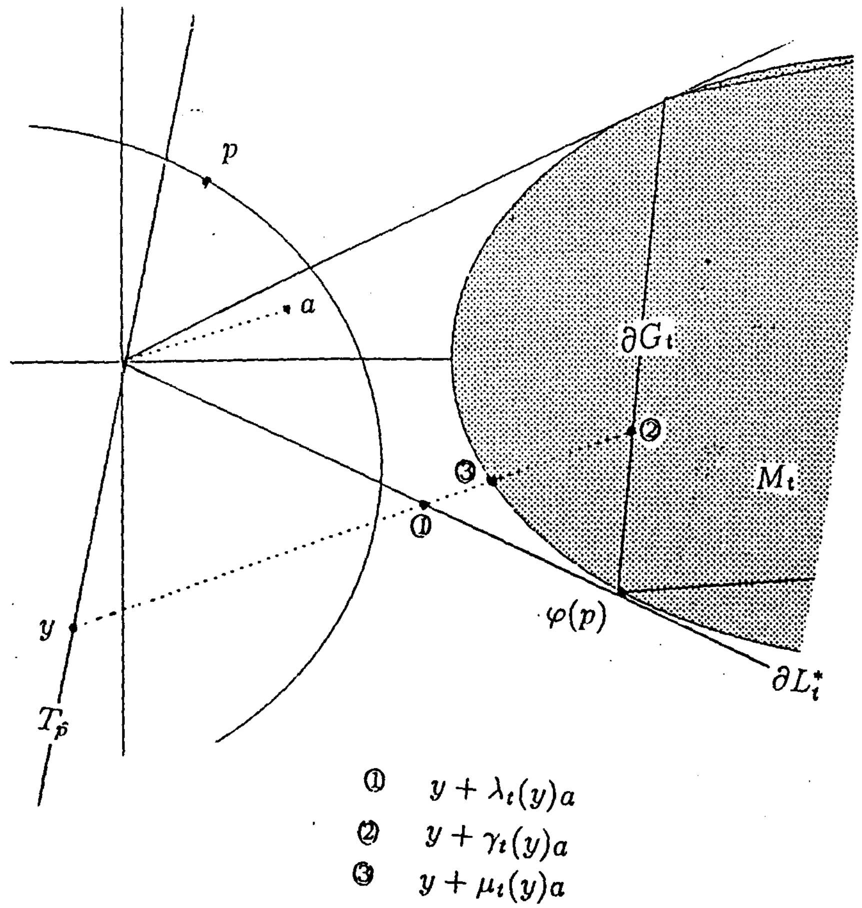

Figure 7: Strict Convexity Stage.