Implicit Functions and Diffeomorphisms without C^1

by

Leonid Hurwicz and Marcel K. Richter

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, MN 55455

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ABSTRACT. We prove implicit and inverse function theorems for non-C^1 functions, and characterize non-C^1 diffeomorphisms.

I Introduction

With K representing either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, suppose $f(\bar{x}, \bar{y}) = 0$ for some function $f : X \times Y \to K^k$, where $X \subseteqq K^n$ and $Y \subseteqq K^k$. We seek a solution, an "implicit function," ϕ on some neighborhood of $\bar{x}$ such that $f(x, \phi(x)) = 0$ holds locally.

Cauchy, working with complex scalars, originally assumed that f was analytic [**10**], and Dini's formulation for real scalars assumed that f was C^1 [**15**]. Similar smoothness assumptions have formed the backbone of most proofs since then. But what if f is only differentiable, rather than C^1 — and only at $(\bar{x}, \bar{y})$? We will show that a solution ϕ still exists.

We present two theorems that significantly weaken the C^1 assumption — to either differentiability at $(\bar{x}, \bar{y})$, or to partial differentiability with respect to y at $(\bar{x}, \bar{y})$, together with some continuity conditions on f. In the former case we obtain differentiability of all solutions at $\bar{x}$; in the latter case we obtain continuity of all solutions at $\bar{x}$. In neither case need there be a unique solution. However, when f is differentiable in a neighborhood of $(\bar{x}, \bar{y})$ and f_y is surjective there, then there is a unique solution and it is differentiable.

The Differentiable Implicit Function Theorem yields a General Inverse Function Theorem, which in turn characterizes diffeomorphisms without continuous differentiability.

Because implicit and inverse function theorems play a role in several parts of mathematics, there are many applications.

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II General Implicit Function Theorems

Let K represent either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We denote by $||\cdot||$ any norm on the finite dimensional Banach space K^q over K. For any $z \in K^q$ and real $\alpha \geqq 0$, we denote by $B_\alpha(z)$ the closed ball in K^q centered at z with $||\cdot||$-radius α.

We will need to bound the norm on K^{n+k} in terms of the norms on K^n and K^k. Since we have not specified any relationship between these norms, we note that the values $||(x,0)||$ of elements $(x,0) \in K^{n+k}$ define a norm on K^n. Since all norms on a finite dimensional real vector space are equivalent, there exists a real $\beta_1 > 1$ such that:

$$||(x,0)|| \leqq \beta_1 ||x|| \qquad \text{for all } x \in K^n. \tag{1a}$$

Analogously, there exists a real $\beta_2 > 1$ such that:

$$||(0,y)|| \leqq \beta_2 ||y|| \qquad \text{for all } y \in K^k. \tag{1b}$$

If X is a subset of K^q, we denote by id_X the identity function on X.

By differentiability we will always mean Fréchet differentiability. If $f : X \longrightarrow Y$ for some open set $X \subseteqq K^q$ and some set $Y \subseteqq K^q$, then we say that $f : X \longrightarrow f(X)$ is a diffeomorphism if f is injective and both f and its inverse $f^{-1} : f(X) \longrightarrow X$ are differentiable. And we say that f is a local diffeomorphism at $\bar{x} \in X$, if there is some nonempty open neighborhood $U \subseteqq X$ of $\bar{x}$ such that $f : U \longrightarrow f(U)$ is a diffeomorphism. If a local diffeomorphism and its inverse are r times differentiable, then we say it is a D^r diffeomorphism.

Where the classical proofs of implicit or inverse function theorems have often made use of contraction mapping theorems, we will use Brouwer's fixed point theorem for finite dimensional Banach spaces. (Although usually stated for the real case $K = \mathbb{R}$, it holds for $K = \mathbb{C}$ as well, since every finite dimensional complex Banach space is isomorphic (hence homeomorphic) to $\mathbb{C}^n$ for some n, which in turn is homeomorphic to $\mathbb{R}^{2n}$.)

We will also use the finite dimensional Banach space version of another theorem of Brouwer, which follows immediately from the result [6] for real scalars by the isomorphism and homeomorphism just mentioned:

> (Brouwer's Theorem on Invariance of Domain.) If U is an open subset of K^n and $g : U \longrightarrow K^n$ is a continuous injection, then $g(U)$ is an open subset of K^n, hence g is an open mapping, so g^{-1} is continuous and g is a homeomorphism of U onto $f(U)$.

Our first theorem assumes differentiability of f at a point, and yields differentiability of solutions at the point.

Theorem 1. Differentiable Implicit Function Theorem. With $K = \mathbb{R}$ or $K = \mathbb{C}$, let X be an open subset of K^n and let Y be an open subset of K^k. Suppose $(\bar{x}, \bar{y}) \in X \times Y$ and $f : X \times Y \to K^k$. Suppose that:

$$f(\bar{x}, \bar{y}) = 0; \tag{2a}$$

$$f(x, \cdot) \text{ is continuous on } Y, \text{ for all } x \in X; \tag{2b}$$

$$f(\cdot, \cdot) \text{ is differentiable at } (\bar{x}, \bar{y}); \tag{2c}$$

$$f_y(\bar{x}, \bar{y}) \text{ is surjective ; i.e.,} \tag{2d}$$

$$\det\left(\begin{bmatrix} \dfrac{\partial f^1(\bar{x}, \bar{y})}{\partial y_1} & \cdots & \dfrac{\partial f^1(\bar{x}, \bar{y})}{\partial y_k} \\ \vdots & & \vdots \\ \dfrac{\partial f^k(\bar{x}, \bar{y})}{\partial y_1} & \cdots & \dfrac{\partial f^k(\bar{z})}{\partial y_k} \end{bmatrix}\right) \neq 0.$$

Then:

a) There exists an open neighborhood $X_0 \times Y_0 \subseteqq K^n \times K^k$ of $(\bar{x}, \bar{y})$ and a function $\phi : X_0 \to Y_0$ such that:

$$f(x, \phi(x)) = 0 \qquad \text{for all } x \in X_0 \tag{3a}$$

$$\phi(\bar{x}) = \bar{y}. \tag{3b}$$

b) For all sets X_0, Y_0, and all functions $\phi : X_0 \to Y_0$ satisfying part (a), the function ϕ is differentiable at $\bar{x}$, with:

$$\phi'(\bar{x}) = -\left(f_y(\bar{x}, \bar{y})\right)^{-1} f_x(\bar{x}, \bar{y}). \tag{4}$$

c) If $(2b, c, d)$ are replaced by these stronger conditions:

$$f \text{ is differentiable on } X \times Y \tag{2e}$$

$$f_y \text{ is surjective on } X \times Y, \tag{2f}$$

then a neighborhood $X_0 \times Y_0$ of $(\bar{x}, \bar{y})$ as in part (a) can be chosen so that:

i) there is a unique function $\phi : X_0 \to Y_0$ satisfying part (a);

ii) ϕ is differentiable on X_0, with:

$$\phi'(x) = -(f_y(x, \phi(x)))^{-1} f_x(x, \phi(x)). \tag{5}$$

d) If f is C^r on $X \times Y$ for some $r \geqq 1$, then there is a neighborhood $X_1 \times Y_1$ of $(\bar{x}, \bar{y})$ and a unique function $\phi : X_1 \to Y_1$ satisfying $(3a, b)$, and ϕ is C^r on X_1.

Remark 1. We cannot totally eliminate the continuity hypothesis ($2b$). Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by :

$$f(x,y) = \begin{cases} x+y, & \text{if } x+y \neq 0 \\ x^2+y^2, & \text{otherwise.} \end{cases} \tag{6}$$

The local continuity hypothesis of Theorem 1 fails, since f is discontinuous at all $(x,y) \neq (0,0)$ with $x+y=0$. But all other hypotheses hold: $f(0,0)=0$, f is differentiable at $(0,0)$, and $f_y(0,0)=1$. Here one cannot solve for y as a function of $x \neq 0$, since no (x,y) in $\mathbb{R}^2$, other than $(0,0)$, satisfies $f(x,y)=0$.

Remark 2. Theorem 1 replaces the classical C^1 hypotheses of Cauchy's (complex) theorem and Dini's (real) theorem[(1)] by the weaker assumptions ($2b, c$). Its conclusions (a,b) are weaker than the classical versions, as the implicit functions ϕ are not necessarily unique or C^1. The conclusion (d) is part of the classical Implicit Function Theorems.

Proof of Theorem 1. (a) (Existence.) To prove the existence assertions (3), it suffices to prove:

$$\begin{aligned} &\exists\gamma_{\ \gamma>0}\ \exists\bar{\delta}_{\ \bar{\delta}>0}\ \forall\delta_{\ \bar{\delta}\geqq\delta\geqq 0}\ \forall x_{\ \|x\|\leqq\delta}\ \exists y_{\ \|y\|\leqq\gamma\delta} \\ &\qquad (\bar{x}+x, \bar{y}+y)\in X \ \ \&\ \ f(\bar{x}+x, \bar{y}+y)=0. \end{aligned} \tag{7}$$

For if we let $\gamma > 0$ and $\bar{\delta} > 0$ be as in (7), and if we define

$$X_0 = \text{Interior}(B_{\bar{\delta}}(\bar{x})) \tag{8}$$

then (7) guarantees existence of a point $y \in B_{\gamma\delta}(\bar{y})$ such that:

$$f(\bar{x}+x, \bar{y}+y) = 0 \tag{9a}$$

$$x = 0 \ \Rightarrow\ y = 0 \tag{9b}$$

$$\|y\| \leqq \gamma\|x\|. \tag{9c}$$

So we can use the Axiom of Choice to define a function ϕ on X with $\phi(x) = y$ and with the properties (3).[(2)]

We break the proof of (7) into parts corresponding to its quantifiers.

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(1) See the historical notes in Section IV.

(2) We could avoid the Axiom of Choice. For example, we could use the lexicographic ordering based on $y_1, \ldots, y_k$ to choose among multiple candidates y with minimum norm in each of the compact sets $(f(\bar{x}+x, \cdot))^{-1}(0) \cap B_{\gamma\delta}(\bar{y})$

i) To choose γ, first note that $(2d)$ implies that $f_y(\bar{x}, \bar{y})$ has an inverse $T = (f_y(\bar{x}, \bar{y}))^{-1}$. So we can define:

$$M = \max\{||Tf_x(\bar{x}, \bar{y}) \cdot x|| : ||x|| = 1\}, \tag{10}$$

and then choose any $\eta > 1$ and define:

$$\gamma = \eta \max\{M, 1\}, \tag{11}$$

so:

$$\gamma > 1. \tag{12}$$

ii) The choice of $\bar{\delta}$ is delicate, and will involve several steps. First note that, since f is differentiable at $(\bar{x}, \bar{y})$, we have: for all $(x, y) \in K^n \times K^k$ with $(\bar{x} + x, \bar{y} + y) \in X$,

$$f(\bar{x} + x, \bar{y} + y) = f(\bar{x}, \bar{y}) + f_x(\bar{x}, \bar{y}) \cdot x + f_y(\bar{x}, \bar{y}) \cdot y + r(x, y), \tag{13}$$

where $r(x, \cdot)$ is $o(||(x, y)||)$, i.e.,

$$\frac{||r(x, y)||}{||(x, y)||} \to 0 \quad \text{for all } (\bar{x} + x, \bar{y} + y) \in X \text{ with } 0 \neq ||(x, y)|| \to 0. \tag{14}$$

Note that the function $r(x, \cdot)$ defined in (13) is continuous for each x.

Now for any given x, the problem of finding y with $(\bar{x}+x, \bar{y}+y) \in X$ and with $f(\bar{x}+x, \bar{y}+y) = 0$ can be stated in each of these equivalent forms: $(\bar{x}+x, \bar{y}+y) \in X$ and

$$f(\bar{x} + x, \bar{y} + y) = 0 \tag{15a}$$

$$f_x(\bar{x}, \bar{y}) \cdot x + f_y(\bar{x}, \bar{y}) \cdot y + r(x, y) = 0 \quad \text{(by (13) and (2a))} \tag{15b}$$

$$r(x, y) = -f_x(\bar{x}, \bar{y}) \cdot x - f_y(\bar{x}, \bar{y}) \cdot y \quad \text{(rearranging)} \tag{15c}$$

$$Tr(x, y) = -Tf_x(\bar{x}, \bar{y}) \cdot x - Tf_y(\bar{x}, \bar{y}) \cdot y \quad \text{(applying } T\text{)} \tag{15d}$$

$$Tr(x, y) = -Tf_x(\bar{x}, \bar{y}) \cdot x - y \quad \text{(by definition of } T\text{)} \tag{15e}$$

$$y = -Tr(x, y) - Tf_x(\bar{x}, \bar{y}) \cdot x \quad \text{(rearranging).} \tag{15f}$$

In (iii) below, for every x near 0, we will apply a fixed point theorem to find a solution y of (15). To prepare for the fixed point theorem, we will bound the two terms on the right hand side of $(15f)$.

ii.1) To bound the first term on the right in $(15f)$, we note that, since $r(x, y)$ is $o(x, y)$ by (14), so is $Tr(x, y)$, and therefore:

$$\frac{||Tr(x, y)||}{||(x, y)||} \to 0 \quad \text{for all } (\bar{x} + x, \bar{y} + y) \in X \text{ with } 0 \neq ||(x, y)|| \to 0.$$

hence (16)

$$\forall \varepsilon_{\ \varepsilon>0}\ \exists \hat{\delta}_{\ \hat{\delta}>0}\ [B_{\hat{\delta}}(\bar{x},\bar{y}) \subseteqq X \\ \quad \&\ \forall (x,y)_{\ \|(x,y)\| \leqq \hat{\delta}}\ \|Tr(x,y)\| \leqq \varepsilon \|(x,y)\|].$$

Choosing $\varepsilon = 1 - 1/\eta$, we obtain from (16) a $\hat{\delta} > 0$ such that:

$$\begin{aligned} &B_{\hat{\delta}}(\bar{x},\bar{y}) \subseteqq X \\ &\qquad \text{and} \\ &\forall (x,y)_{\ \|(x,y)\| \leqq \hat{\delta}}\ \|Tr(x,y)\| \leqq (1 - \frac{1}{\eta})\|(x,y)\|. \end{aligned} \tag{17}$$

Using β_1 and β_2 from (1) we now define:

$$\bar{\delta} = \frac{1}{\beta_1 + \beta_2 \gamma} \hat{\delta}, \tag{18}$$

so:

$$\hat{\delta} > \bar{\delta} > 0 \tag{19a}$$

$$B_{(\beta_1+\beta_2\gamma)\bar{\delta}}(\bar{x},\bar{y}) \subseteqq X. \tag{19b}$$

Then:

$$\forall \delta_{\ \bar{\delta} \geqq \delta \geqq 0}\ \forall (x,y)_{\ \|(x,y)\| \leqq \delta}\ \|Tr(x,y)\| \leqq (1 - \frac{1}{\eta})\gamma\delta. \tag{20}$$

ii.2) To bound the second term in the right hand side of (15f), choose any δ with $\bar{\delta} \geqq \delta \geqq 0$. Then:

$$\begin{aligned} \forall x_{\ \|x\| \leqq \delta}\ \|Tf_x(\bar{x},\bar{y}) \cdot x\| &\leqq M\|x\| \qquad \text{(by (10))} \\ &\leqq M\delta \\ &\leqq \frac{1}{\eta}\gamma\delta \qquad \text{(by (11))}. \end{aligned} \tag{21}$$

iii) Now we complete the proof of (7) by picking any δ with $\bar{\delta} \geqq \delta \geqq 0$ and any $x \in K^n$ with $\|x\| \leqq \delta$; we must find a $y \in K^k$ with $\|y\| \leqq \gamma\delta$ such that $(\bar{x}+x, \bar{y}+y) \in X$ and $f(\bar{x}+x, \bar{y}+y) = 0$. With (15$f$) in mind, we define the function $F : K^k \to K^k$ by:

$$F(y) = -Tr(x,y) - Tf_x(\bar{x},\bar{y}) \cdot x, \tag{22}$$

which is continuous in y because $r(x, \cdot)$ is (by (13) and (2b)). Then for all y with $||y|| \leqq \gamma\delta$ we have:

$$\begin{aligned} ||(x,y)|| &= ||(x,0)+(0,y)|| \\ &\leqq ||(x,0)|| + ||(0,y)|| \\ &\leqq \beta_1||x|| + \beta_2||y|| \qquad \text{(by (1))} \\ &\leqq (\beta_1+\beta_2\gamma)\delta, \end{aligned} \tag{23}$$

so by (19b):

$$(\bar{x}+x, \bar{y}+y) \in B_{(\beta_1+\beta_2\gamma)\delta}(\bar{x},\bar{y}) \subseteqq X, \tag{24}$$

and

$$\begin{aligned} ||F(y)|| &= ||-Tr(x,y) - Tf_x(\bar{x},\bar{y})\cdot x|| \\ &\leqq ||Tr(x,y)|| + ||Tf_x(\bar{x},\bar{y})\cdot x|| \\ &\leqq (1-\frac{1}{\eta})\gamma\delta + \frac{1}{\eta}\gamma\delta \qquad \text{(by (20) and (21))} \\ &= \gamma\delta. \end{aligned} \tag{25}$$

Thus F is a continuous function from the ball $B_{\gamma\delta}(0)$ into itself. By Brouwer's Fixed Point Theorem, F must have a fixed point $F(y) = y \in B_{\gamma\delta}(0)$, and by (24) and (15$f$) this is equivalent to $||y|| \leqq \gamma\delta$ and $f(\bar{x}+x, \bar{y}+y) = 0$.

(b) (Differentiability at $\bar{x}$.) To prove differentiability of ϕ at $\bar{x}$, and the formula (4), consider any function ϕ as in (3). Since f is differentiable at $(\bar{x},\bar{y})$, we can suppose it has some Lipschitz constant $\tilde{\gamma} > 0$ at $(\bar{x},\bar{y})$. Without loss of generality, suppose that:

$$\bar{x} = 0 \qquad \text{and} \qquad \bar{y} = 0, \tag{26a}$$

hence:

$$\begin{aligned} &f(0,0) = 0 \qquad \text{(by (2a))} \\ &\phi(0) = 0 \qquad \text{(by (3b))}. \end{aligned} \tag{26b}$$

We must show that:

$$\frac{||\phi(x) - \phi(0) + (f_y(0,0))^{-1}f_x(0,0)x||}{||x||} \xrightarrow[||x||\to 0]{} 0. \tag{27}$$

Since f is differentiable at $(0,0)$,

$$f(x,\phi(x)) = f(0,0) + f_x(0,0)x + f_y(0,0)\phi(x) + o(||(x,\phi(x))||), \tag{28}$$

so by (26) and ($3a$) we have:

$$\frac{||f_x(0,0)x + f_y(0,0)\phi(x)||}{||(x,\phi(x))||} \xrightarrow[||(x,\phi(x))||\to 0]{} 0. \tag{29}$$

With β_1 and β_2 as in (1), we have:

$$\begin{aligned} ||(x,\phi(x))|| &\leqq ||(x,0)|| + ||(0,\phi(x))|| \\ &\leqq \beta_1||x|| + \beta_2||\phi(x)|| \\ &\leqq (\beta_1 + \beta_2\tilde{\gamma})||x||. \end{aligned} \tag{30}$$

So (29) implies:

$$\frac{||f_x(0,0)x + f_y(0,0)\phi(x)||}{||x||} \xrightarrow[||x||\to 0]{} 0. \tag{31}$$

Then because $(f_y(0,0))^{-1}$ is linear,

$$\frac{||(f_y(0,0))^{-1}f_x(0,0)x + \phi(x)||}{||x||} \xrightarrow[||x||\to 0]{} 0, \tag{32}$$

and this reformulation of (27) verifies the existence and the claimed value of $\phi'(0)$.

(c.ii) (Differentiability on X_0.) Let $\phi : X_0 \to Y_0$ be any implicit function whose existence was established in part (a). With the stronger hypotheses ($2e, f$), differentiability of ϕ on all of X_0 can be established by a proof analogous to that just given in (b) for differentiability at $\bar{x}$. Thus let ϕ be any function satisfying ($3a, b$) as in in part (a). Let $\tilde{x} \in X_0$ and $\tilde{y} = \phi(\tilde{x})$. Then by ($2e, f$) f satisfies all the conditions ($2a, b, c, d$) with $(\bar{x}, \bar{y})$ replaced by $(\tilde{x}, \tilde{y})$, so, as in part (b), ϕ is differentiable at $\tilde{x}$ and $\phi'(\tilde{x}) = -(f_y(\tilde{x},\tilde{y}))^{-1}f_x(\tilde{x},\tilde{y})$.

(c.i) (Uniqueness.) To prove the uniqueness of ϕ, define $G : X \times Y \to X \times K^k$ by:

$$G(x,y) = (x, f(x,y)). \tag{33}$$

By hypothesis ($2e$), G is differentiable on $X \times Y$, and from hypothesis ($2f$) we see that the derivative $G'(x,y)$ is surjective at all $(x,y) \in X \times Y$. So by part (c.ii) of Theorem 4 (the General Inverse Function Theorem)[3] applied to G, in some open neighborhood $X_1 \times Y_1 \subseteqq X \times K^k$ of $G(\bar{x},\bar{y}) = (\bar{x}, 0)$ there is a unique inverse function $H : X_1 \times Y_1 \to X \times Y$, and H is a local diffeomorphism at $(\bar{x}, 0)$. In particular for each $x \in X_1$ there is a unique $y \in Y_1$ satisfying $G(x,y) = (x,0)$, i.e.,

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[3] **There is no circularity in our reasoning here. Our proof later of part (c) of Theorem 4 will depend only on parts (a) and (b) (but not (c)) of the present theorem.**

satisfying $f(x, y) = 0$. Since $f(x, \phi(x)) = 0$ for the function ϕ established in part (a), this proves the uniqueness of ϕ on X_1.

(d) (C^r properties.) The uniqueness and C^r assertions are part of the classical Implicit Function Theorem. ∎

Even as simple a function as this:

$$f(x, y) = y - (x)^{\frac{1}{3}} \tag{34}$$

fails to satisfy the differentiability hypothesis of the preceding theorem. And, while it clearly admits the unique implicit function $\phi(x) = (x)^{\frac{1}{3}}$, that solution is not differentiable at 0. Indeed, the same would be true of any function

$$f(x, y) = y - \psi(x), \tag{35}$$

with ψ nondifferentiable. Such examples motivate our next implicit function theorem, which in the spirit of Goursat [**19**] has a weaker differentiability hypothesis (no differentiability with respect to x) but a stronger continuity hypothesis (joint in x and y). The conclusion of our second theorem will be weaker than our first, asserting merely continuity of solutions rather than differentiability at the initial point.

Theorem 2. Continuous Implicit Function Theorem. With $K = \mathbb{R}$ or $K = \mathbb{C}$, let X be an open subset of K^n and let Y be an open subset of K^k. Suppose $(\bar{x}, \bar{y}) \in X \times Y$ and $f : X \times Y \to K^k$. Suppose that:

$$f(\bar{x}, \bar{y}) = 0; \tag{36a}$$

$$f(\cdot\,, \cdot) \text{ is continuous on } X \times Y; \tag{36b}$$

$$f(\bar{x}, \cdot) \text{ is differentiable at } \bar{y}; \tag{36c}$$

$$f_y(\bar{x}, \bar{y}) \text{ is surjective ; i.e.,} \tag{36d}$$

$$\det\left(\begin{bmatrix} \dfrac{\partial f^1(\bar{x}, \bar{y})}{\partial y_1} & \cdots & \dfrac{\partial f^1(\bar{x}, \bar{y})}{\partial y_k} \\ \vdots & & \vdots \\ \dfrac{\partial f^k(\bar{x}, \bar{y})}{\partial y_1} & \cdots & \dfrac{\partial f^k(\bar{x}, \bar{y})}{\partial y_k} \end{bmatrix}\right) \neq 0.$$

Then:

a) There exists an open neighborhood $X_0 \times Y_0 \subseteqq K^n \times K^k$ of $(\bar{x}, \bar{y})$ and a function $\phi : X_0 \to Y_0$ such that:

$$f(x, \phi(x)) = 0 \qquad \text{for all } x \in X_0 \tag{37a}$$

$$\phi(\bar{x}) = \bar{y}. \tag{37b}$$

b) The neighborhood $X_0 \times Y_0$ in (a) can be chosen so that there exists at least one function $\phi : X_0 \to Y_0$ satisfying (37), and so that every such function is continuous at $\bar{x}$.

c) (Goursat.) If assumption ($36c$) is replaced by this stronger condition:

$$f(x, \cdot) \text{ is } C^1 \text{ on } Y \qquad \text{for all } x \in X, \tag{36e}$$

then there is a neighborhood $X_1 \times Y_1$ of $(\bar{x}, \bar{y})$ and a unique function $\phi : X_1 \to Y_1$ satisfying ($37a, b$), and ϕ is continuous on X_1.

Remark 3. As W. H. Young showed [**35**], the existence result (a) for the case $k = 1$ only requires that f be continuous separately in x and y, rather than jointly. See the historical remarks below.

Remark 4. Theorem 2(a,b) replaces the C^1 hypothesis of Goursat's result [**19**][(4)] by the weaker assumptions ($36b, c$). Its conclusions in (a) and (b) are weaker than in Goursat [**19**], as the implicit functions ϕ are not necessarily unique or continuous. Here is an example for $K = \mathbb{R}$ in which the uniqueness and continuity conclusions of part (c) cannot be obtained under the non-C^1 hypotheses of parts (a) and (b): Let $f(x, y) = x - g(y)$, where

$$g(y) = \begin{cases} (y/2) + y^2 \sin(1/y), & \text{if } y \neq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

Proof of Theorem 2. Without loss of generality, we assume that:

$$\bar{x} = 0 \qquad \text{and} \qquad \bar{y} = 0, \tag{39}$$

and we choose $\bar{\varepsilon} > 0$ such that:

$$B_{\bar{\varepsilon}}(0) \subseteqq X. \tag{40}$$

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(4) See the historical notes in Section IV, and the comparisons with results of W. H. Young and of Bliss.

(a) (Existence.) Without loss of generality, we assume that:[5]

$$f_y(0,0) = \mathrm{id}_{K^k}. \tag{41}$$

This is justified since $f_y(0,0)$ is invertible ($36d$), so we could define the function $g(x,y) = f(x,(f_y(0,0))^{-1}y)$, which would have all the properties (36) with $g_y(0,0)$ the identity on K^k, and g would admit an implicit function ϕ if and only if f admits an implicit function $\tilde{\phi} = (f_y(0,0))^{-1}\phi$:

$$g(x,\phi(x)) = 0 \;\Leftrightarrow\; f(x,\tilde{\phi}(x)) = 0. \tag{42}$$

Now define $F : X \times Y \to K^k$ by:

$$F(x,y) = y - f(x,y). \tag{43}$$

By hypothesis ($36c$) and by (39) and (41),

$$\begin{aligned} f(0,y) &= f(0,0) + f_y(0,0)y + R(y) \\ &= y + R(y) \end{aligned} \tag{44}$$

for some function $R(y)$ of y that is $o(||y||)$:

$$\frac{||R(y)||}{||y||} \xrightarrow[0\neq||y||\to 0]{} 0. \tag{45}$$

Then:

$$\begin{aligned} \frac{||F(0,y)||}{||y||} &= \frac{||y - f(0,y)||}{||y||} \qquad \text{(by (43) and (44))} \\ &= \frac{||R(y)||}{||y||} \xrightarrow[0\neq||y||\to 0]{} 0 \qquad \text{(by (45))}, \end{aligned} \tag{46}$$

so there exists a $\tilde{\gamma} > 0$ such that:[6]

$$\forall\gamma_{\;0<\gamma\leqq\tilde{\gamma}}\;\forall y_{\;||y||\leqq\gamma}\;||F(0,y)|| < \gamma. \tag{47}$$

We next show that:

$$\forall\gamma_{\;0<\gamma\leqq\tilde{\gamma}}\;\exists\varepsilon_{\gamma\;0<\varepsilon_\gamma\leqq\bar{\varepsilon}}\;\forall\varepsilon_{\;0\leqq\varepsilon\leqq\varepsilon_\gamma}\;\forall x_{\;||x||\leqq\varepsilon}\;\forall y_{\;||y||\leqq\gamma}\;||F(x,y)|| < \gamma. \tag{48}$$

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(5) Recall that id_{K^k} is the identity function on K^k. Cf. p. 2.

(6) Our proof of (47) is based on the differentiability hypothesis ($36c$). Alternatively, we could obtain (47) by simply assuming that $F(\bar{x},\cdot)$ has a Lipschitz constant less than 1 at $y = 0$. That property is a weakened version of the hypotheses of Goursat's lemma [19] (p. 185, §1, for $k = 1, n = 1$, and p. 188, §3, for arbitrary k), which sets the stage for his application of what is now known as the Contraction Mapping Theorem. But our continuity proof in part (b) will still use the differentiability hypothesis ($36c$).

First define the correspondence $C : \{\varepsilon \in K : 0 \leqq \varepsilon \leqq \bar{\varepsilon}\} \twoheadrightarrow X \times Y$:

$$C(\varepsilon) = \{(x, y) : \|x\| \leqq \varepsilon \ \ \& \ \ \|y\| \leqq \gamma\}, \tag{49}$$

and the corresponding maximum function:

$$M(\varepsilon) = \max\{\|F(x, y)\| : (x, y) \in C(\varepsilon)\}. \tag{50}$$

Since F is continuous by hypothesis (36*b*), and since $C(\varepsilon)$ is clearly a continuous correspondence,(7) it follows from the Maximum Theorem(8) that $M(\cdot)$ is a continuous function. So (48) follows from (47).

Thus for every $\gamma \leqq \tilde{\gamma}$ and every x with $\|x\| \leqq \varepsilon_\gamma$, the function $F(x, \cdot)$ carries $B_\gamma(0)$ into $B_\gamma(0)$. It is also continuous by (36*b*), so by Brouwer's Fixed Point Theorem there exists a $y \in B_\gamma(0)$ such that

$$F(x, y) = y, \tag{51}$$

i.e.,

$$f(x, y) = 0. \tag{52}$$

If $x \neq 0$, then we define $\phi(x) = y$, choosing any such y;(9) if $x = 0$ we define $\phi(0) = 0$. Then (37) holds, with $X_0 = B_\varepsilon(0)$ and $Y_0 = B_\gamma(0)$.

We have thus shown:

> For every $\gamma > 0$ with $\gamma \leqq \tilde{\gamma}$, and every $\varepsilon \geqq 0$ with $\varepsilon \leqq \varepsilon_\gamma$, there exists a function $\phi : X_\gamma \to Y_\gamma$ such that (37) holds with $X_0 = B_\varepsilon(0)$ and $Y_0 = B_\gamma(0)$. (53)

(b) (Continuity.) We now prove that a neighborhood $X_0 \times Y_0$ of $(\bar{x}, \bar{y})$ can be chosen so that not only does there exist a function $\phi : X_0 \to \tilde{Y}_0$ satisfying (37), but every such function is continuous at $x = 0$.

We can choose $\delta > 0$ with $\delta \leqq \tilde{\gamma}$ such that:

$$\|R(y)\| < \|y\| \qquad \text{for all } y \in B_\delta(0) \qquad \text{(using (45))}, \tag{54a}$$

and by (44) we can also ensure that:

$$y \neq 0 \ \Rightarrow \ f(0, y) \neq 0 \qquad \text{for all } y \in B_\delta(0). \tag{54b}$$

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(7) I.e., both upper and lower hemicontinuous. Cf. Berge [1], [2], where "semicontinuity" is used for what we are calling "hemicontinuity."

(8) Cf. [12, p. 889, Remark], [13, p. 19, 1.8(4)], [1, p. 122], [2, p. 116].

(9) As in footnote 2, the Axiom of Choice is not needed.

Defining $\gamma = \delta$, $X_0 = B_{\varepsilon_\gamma}(0)$ and $Y_0 = B_\gamma(0)$, it follows from (53) that there exists a function $\phi : X_0 \to Y_0$ satisfying (37).

Now let ϕ be any function from X_0 into Y_0 satisfying (37). Suppose that ϕ is not continuous at 0. Then there is a sequence $x_1, x_2, \ldots$ converging to 0, but with $\phi(x_i)$ not converging to $\phi(0) = 0$. So there is a subsequence, whose elements we again denote by x_i, for which

$$\phi(x_i) \xrightarrow[i\to\infty]{} \tilde{y} \neq 0. \tag{55}$$

By (54b) that implies:

$$f(0, \tilde{y}) \neq 0, \tag{56}$$

so by the continuity hypothesis (36b) we have $f(x,y) \neq 0$ for all (x,y) in some neighborhood of $(0,\tilde{y})$. But defining $y_j = \phi(x_j)$ we also have $f(x_j, y_j) = 0$ by the solution property (37a), and $(x_j, y_j) \xrightarrow[i\to\infty]{} (0,\tilde{y})$ by (55). This contradiction completes the proof of continuity.

(c) (Uniqueness and continuity.) If $f(x, \cdot)$ is also C^1 on Y for all $x \in X$, then the local uniqueness and continuity of the implicit function are part of Goursat's result [**19**]. ∎

Remark 5. While some of the hypotheses of the Continuous and the Differentiable Implicit Function Theorems are the same, in total they are noncomparable. The differentiable version has stronger differentiability hypotheses, but weaker continuity properties. While the function f in (34) satisfies the hypotheses of the Continuous Theorem but not those of the Differentiable Theorem, the reverse is true for the function

$$f(x,y) = \begin{cases} y + x^2 + y^2, & \text{if } x \text{ is rational} \\ y, & \text{otherwise.} \end{cases} \tag{57}$$

Here $f(x,y)$ is not continuous in x at $(0,y)$ for $y \neq 0$, so it violates (36b) for $(\bar{x},\bar{y}) = (0,0)$; but it is differentiable at $(0,0)$, and satisfies the other hypotheses of the Differentiable Theorem.

We have obtained implicit functions under two different hypotheses about the independent variable x. In our first theorem $f(x,y)$ was differentiable jointly in x and y at $(0,0)$; and we concluded that the implicit function $\phi(x)$ was differentiable at $x = 0$. In our second theorem $f(x,y)$ was only differentiable with respect to y at $(0,0)$, and $f(x,y)$ was continuous with respect to x at $x = 0$; and we concluded that the implicit function $\phi(x)$ was continuous at $x = 0$. Now we show that these two results can be combined, allowing two types of independent variables — those satisfying differentiability hypotheses yield differentiability conclusions for the implicit function, and those satisfying continuity hypothesis yield continuity conclusions. We are interested in solving $f(v, w, y) = 0$ for y as a function ϕ of (v, w): $f(v, w, \phi(v, w)) = 0$. The role played by x in Theorem 1 is played here by v, and the role of x in Theorem 2 is played here by w.

Theorem 3. General Implicit Function Theorem. With $K = \mathbb{R}$ or $K = \mathbb{C}$, let V, W, and Y be open subsets of K^m, K^p, and K^k, respectively, where $m + p > 0$; we allow $m = 0$ or $p = 0$, in which case V or W, respectively, is empty. Suppose $(\bar{v}, \bar{w}, \bar{y}) \in V \times W \times Y$ and $f : V \times W \times Y \to K^k$. Suppose that:

$$f(\bar{v}, \bar{w}, \bar{y}) = 0; \tag{57a}$$

$$f(v, \cdot, \cdot) \text{ is continuous on } W \times Y, \text{ for all } v \in V; \tag{57b}$$

$$f(\cdot, \bar{w}, \cdot) \text{ is differentiable at } (\bar{v}, \bar{y}); \tag{57c}$$

$$f_y(\bar{v}, \bar{w}, \bar{y}) \text{ is surjective ; i.e.,} \tag{57d}$$

$$\det \left(\begin{bmatrix} \dfrac{\partial f^1(\bar{v}, \bar{w}, \bar{y})}{\partial y_1} & \cdots & \dfrac{\partial f^1(\bar{v}, \bar{w}, \bar{y})}{\partial y_k} \\ \vdots & & \vdots \\ \dfrac{\partial f^k(\bar{v}, \bar{w}, \bar{y})}{\partial y_1} & \cdots & \dfrac{\partial f^k(\bar{v}, \bar{w}, \bar{y})}{\partial y_k} \end{bmatrix} \right) \neq 0.$$

Then:

a) There exist open neighborhoods V_0 of $\bar{v}$ and Y_0 of $\bar{y}$ such that, for every $v \in V_0$ there exists an open neighborhood W_v of $\bar{w}$ and a function $\phi(v, \cdot) : W_v \times Y_0$ such that:

$$f(v, w, \phi(v, w)) = 0 \qquad \text{for all } w \in W_v \tag{58a}$$

$$\phi(\bar{v}, \bar{w}) = \bar{y}. \tag{58b}$$

b) The neighborhood Y_0 in (a) can be chosen so that every function ϕ satisfying (58) also satisfies:

$$\phi(\bar{v}, \cdot) \text{ is continuous at } \bar{w}, \text{ for all } v \in V_0 \tag{59a}$$

$$\phi(\cdot, \bar{w}) \text{ is differentiable at } \bar{v}, \text{ and:} \tag{59b}$$

$$\phi_v(\bar{v}, \bar{w}) = -(f_y(\bar{v}, \bar{w}, \bar{y}))^{-1} f_v(\bar{v}, \bar{w}, \bar{y}).$$

c) If $f(\cdot, \cdot, \cdot)$ is continuous on $V \times W \times Y$, then there is a neighborhood $V_0 \times W_0 \times Y_0$ and a function ϕ satisfying (58) that is continuous at $(\bar{v}, \bar{w}, \bar{y})$.

d) (Goursat [19], [20].) If $f(\cdot,\cdot,\cdot)$ is continuous on $V \times W \times Y$, and if $f(v,w,\cdot)$ is C^1 on Y for all $(v,w) \in V \times W$, then there is a neighborhood $V_1 \times W_1 \times Y_1$ of $(\bar{v},\bar{w},\bar{y})$ and a unique function $\phi : V_1 \times W_1 \to Y_1$ satisfying $(58a,b)$, and ϕ is continuous on $V_1 \times W_1$.

Remark 6. This theorem is a generalization of the Differentiability and the Continuous Implicit Function Theorems: when w is absent, the hypotheses and conclusions are those of the Differentiable Implicit Function Theorem, and when v is absent, they are those of the Continuous Implicit Function Theorem.

Proof of Theorem 3. Without loss of generality, we simplify by assuming that:

$$\bar{v} = 0 \quad \text{and} \quad \bar{w} = 0 \quad \text{and} \quad \bar{y} = 0. \tag{60}$$

Also without loss of generality, we assume that:

$$f_y(0,0,0) = \mathrm{id}_{K^k}. \tag{61}$$

This is justified since $f_y(0,0,0)$ is invertible $(57d)$, so we could define the function $g(v,w,y) = f(v,w,(f_y(0,0,0))^{-1}y)$, which would have all the properties (57) with $g_y(0,0,0)$ the identity on K^k, and g would admit an implicit function ϕ if and only if f admits an implicit function $\tilde{\phi} = (f_y(0,0,0))^{-1}\phi$:

$$g(v,w,\phi(v,w)) = 0 \Leftrightarrow f(v,w,\tilde{\phi}(v,w)) = 0. \tag{62}$$

(a) (Existence.) Define $F : V \times W \times Y$ by:

$$F(v,w,y) = y - f(v,w,y). \tag{63}$$

By hypothesis $(57c)$,

$$\begin{aligned} f(v,0,y) &= f(0,0,0) + f_v(0,0,0)v + f_y(0,0,0)y + R(v,y) \\ &= y + f_v(0,0,0)v + R(v,y) \qquad \text{(by (61), (57}a\text{) and (60))} \end{aligned} \tag{64}$$

for some function $R(v,y)$ of (v,y) that is $o(\|(v,y)\|)$:

$$\frac{R(v,y)}{\|(v,y)\|} \xrightarrow[0 \neq \|(v,y)\| \to 0]{} 0. \tag{65}$$

By (63) and (64) we have:

$$\begin{aligned} \|F(v,0,y)\| &= \|y - f(v,0,y)\| \\ &= \|R(v,y) + f_v(0,0,0)v\| \\ &\leq \frac{\|R(v,y)\|}{\|(v,y)\|}\|(v,y)\| + \|f_v(0,0,0)\|\,\|v\|, \end{aligned} \tag{66}$$

so by (65) there exists a $\gamma > 0$ such that, for all $||(v, y)|| < \gamma$ we have

$$\forall (v, y)_{\;||(v,y)|| \leqq \gamma} \; ||F(v, 0, y)|| < \frac{1}{2}\gamma + ||f_v(0, 0, 0)|| \; ||v||. \tag{67}$$

Now define:

$$\varepsilon_0 = \begin{cases} \dfrac{\gamma}{2||f_v(0,0,0)||}, & \text{if } ||f_v(0,0,0)|| \neq 0 \\ \\ \frac{1}{2}\gamma, & \text{otherwise,} \end{cases} \tag{68}$$

so $\varepsilon_0 > 0$. Now let

$$\varepsilon = \min\{\varepsilon_0, \gamma\}. \tag{69}$$

Then:

$$\forall v_{\;||v|| < \varepsilon} \; \forall y_{\;||(v,y)|| \leqq \gamma} \; ||F(v, 0, y)|| < \frac{1}{2}\gamma + \frac{1}{2}\gamma = \gamma. \tag{70}$$

Without loss of generality, we may suppose that the norm on $V \times Y$ is the maximum norm, and then (70) and (69) imply :

$$\forall v_{\;||v|| < \varepsilon} \; \forall y_{\;||y|| \leqq \gamma} \; ||F(v, 0, y)|| < \gamma. \tag{71}$$

Next we will show that this inequality holds also for all w near 0. Since $F(v, 0, y)$ is defined for $||v|| < \varepsilon$ & $||y|| \leqq \gamma$, openness of W ensures there is some $\bar{\delta} > 0$ for which $F(v, w, y)$ is defined for all $||x|| < \varepsilon$ & $||w|| \leqq \bar{\delta}$ & $||y|| \leqq \gamma$. Then paralleling the proof of Theorem 2,[10] we define:

$$C(\delta) = \{(w, y) : ||w|| \leqq \delta \; \& \; ||y|| \leqq \gamma\}, \tag{72}$$

for all δ with $0 \leqq \delta \leqq \bar{\delta}$. Clearly $C(\cdot)$ is a continuous correspondence, so by the continuity hypothesis (57b) and the Maximum Theorem, for every v with $||v|| < \varepsilon$, the function

$$M_v(\delta) = \max\{||F(v, w, y)|| : ||w|| \leqq \delta \; \& \; ||y|| \leqq \gamma\} \tag{73}$$

is continuous for $0 \leqq \delta \leqq \bar{\delta}$. So by (71) for every v with $||v|| < \varepsilon$, there exists a $\delta_v > 0$ with $\delta_v \leqq \bar{\delta}$ such that:

$$\forall v_{\;||v|| < \varepsilon} \; \forall w_{\;||w|| \leqq \delta_v} \; \forall y_{\;||y|| < \gamma} \; ||F(v, w, y)|| \leqq \gamma. \tag{74}$$

Thus for every v with $||v|| < \varepsilon$ and every w with $||w|| \leqq \delta_v$, the function $F(v, w, \cdot)$ carries $B_\gamma(0)$ into $B_\gamma(0)$. It is also continuous by (57b), so by Brouwer's Fixed Point Theorem there exists a $y \in B_\gamma(0)$ such that

$$F(v, w, y) = y, \tag{75}$$

[10] Cf. (48), (49), and (50).

i.e.,

$$f(v,w,y)=0. \tag{76}$$

If $w \neq 0$, then we define $\phi(v,w)=y$, for any such y;[11] if $w=\bar{x}=0$ we define $\phi(0)=0$. Then (58) holds with $V_0=B_\varepsilon(0)$, $W_v=B_{\delta_v}(0)$, and $Y_v=B_\gamma(0)$.

(b) (Continuity with respect to w.) To prove part (59a), we note that the function defined by $\tilde{f}(w,y)=f(0,w,y)$ satisfies the hypotheses of the Continuous Implicit Function Theorem, and so by part (b) of that theorem we immediately obtain property (59a).

(Differentiability with respect to v.) To prove (59b), we note that the function $\hat{f}$ defined by $\hat{f}(v,y)=f(v,0,y)$ satisfies the hypotheses of the Differentiable Implicit Function theorem, and so by part (b) of that theorem we immediately obtain property (59b).

(c) (Continuity with respect to (v,w).) Part (c) follows immediately from part (b) of the Continuous Implicit Function Theorem, since if we define $x=(v,w)$, then all the hypotheses of that theorem apply.

(d) (Uniqueness and local continuity.) See part (c) of Theorem 2. ∎

III A General Inverse Function Theorem

As usual, it is straightforward to prove an inverse function theorem from an implicit function.

Theorem 4. General Inverse Function Theorem. Let $K=\mathbb{R}$ or $K=\mathbb{C}$, and let X be an open subset of K^n, and let $g: X \rightarrow K^n$ be continuous on X and differentiable at $\bar{x} \in X$. Let $\bar{y}=g(\bar{x})$. Suppose $g'(\bar{x})$ is surjective, i.e., $\operatorname{rank}(g'(\bar{x}))=n$. Then:

a) There exists an open neighborhood Y of $\bar{y}$ and an injection $h: Y \rightarrow X$ such that:

$$g \circ h = \mathrm{id}_Y \tag{77a}$$

$$h(\bar{y})=\bar{x}. \tag{77b}$$

[11] As in footnote 2, the Axiom of Choice is not needed.

b) Every function h satisfying (77) is differentiable at $g(\bar{x})$, with:

$$h'(g(\bar{x})) = (g'(\bar{x}))^{-1}. \tag{78}$$

c) If the differentiability of g at $\bar{x}$ is strengthened to differentiability on X, and surjectivity of $g'(\bar{x})$ is strengthened to surjectivity of g' on X, then:

i) the neighborhood Y in part (a) can be chosen so that h is also differentiable and surjective on Y;

ii) the neighborhood X_0 can also be chosen so that $g\restriction_{X_0}: X_0 \to g(X_0)$ is a diffeomorphism (in particular, the inverse function h is unique) and:

$$h'(g(x)) = (g'(x))^{-1} \qquad \text{for all } x \in X_0. \tag{79}$$

d) If g is C^r on X for some $r \geqq 1$, then there is an open neighborhood Y of $\bar{y}$, and there is a unique function $h : Y \to X$ mapping Y onto an open set, and satisfying the conditions $(77a, b)$, and it is C^r.

Proof. We will prove the General Inverse Function Theorem from the Differentiable Implicit Function Theorem, reversing the roles of x and y. Define

$$f(x, y) = g(x) - y \tag{80}$$

for all $x \in X$, so f is differentiable at $(\bar{x}, \bar{y})$.

(a) (Existence.) To prove part (a), note that because $f_x(\bar{x}, \bar{y}) = g'(\bar{x})$ is surjective, part (a) of the Differentiable Implicit Function Theorem implies that there exists an open neighborhood Y of $g(\bar{x})$ and a function $\psi : Y \to X$ such that:

$$f(\psi(y), y) = 0 \qquad \text{for all } y \in Y \tag{81a}$$

$$\psi(\bar{y}) = \bar{x}. \tag{81b}$$

So in view of (80), $h = \psi$ satisfies (77). And h is an injection, since if $h(y) = h(\tilde{y})$ then $y = g(h(y)) = g(h(\tilde{y})) = \tilde{y}$ by $(77a)$.

(b) (Differentiability at $\bar{x}$.) To prove part (b), we note that part (b) of the Differentiable Implicit Function Theorem implies that any ψ satisfying (81) is differentiable at $\bar{y} = g(\bar{x})$. Then by (77) the Chain Rule implies that $g'(h(\bar{y}))h'(\bar{y}) =$ the identity on Y, so $h = \psi$ also satisfies (78).

(c.i) (Differentiability on Y.) Assume now that $g'(x)$ exists and is surjective for all $x \in X$. The same argument used in (b) to prove differentiability of any right inverse of g at $g(\bar{x})$ now shows that h is differentiable at all $y \in Y$, hence continuous on Y. To see that, let $h : Y \to X$ be as in part (a), and let $\tilde{x} \in X$ and $\tilde{y} = g(\tilde{x})$. Since g is differentiable on X, it is continuous on X; and since $g'(\tilde{x})$ is surjective, part (a) applies with $(\tilde{x}, \tilde{y})$ replacing $(\bar{x}, \bar{y})$ so, as in part (b), h is differentiable at $\tilde{y}$ and $h'(\tilde{y}) = (g'(\tilde{x}))^{-1}$.

(c.ii) (Uniqueness and diffeomorphism.) Since any right inverse h of g is also injective on Y (by part (a)) and continuous (by part (c.i)), it follows from Brouwer's theorem on Invariance of Domain that h is a homeomorphism from Y to the open neighborhood $h(Y)$ of $\bar{x}$. So $X_0 = h(Y)$ is an open neighborhood of $\bar{x}$, and $g{\restriction}_{X_0}: X_0 \to g(X_0)$ is a homeomorphism.

It follows that $g{\restriction}_{X_0}$ has a unique inverse. Indeed, for any $y \in g(X_0)$ we have $g(h(y)) = y$ because h is a right inverse of g. Applying h to each side yields $h(g(h(y))) = h(y)$. Thus, since h is a homeomorphism onto the open neighborhood X_0 of $\bar{x}$, we have $h(g(x)) = x$ for all $x \in X_0$, so $h \circ g = id_{X_0}$. Then h is a left inverse as well as a right inverse of g.

By (c.i) the inverse h of g is differentiable, so $g{\restriction}_{X_0}$ is a diffeomorphism, and (79) holds.

(d) (C^r properties.) The uniqueness and C^r assertions are part of the classical Inverse Function Theorem. ∎

Diffeomorphism Corollary. Let $K = \mathbb{R}$ or $K = \mathbb{C}$, let X be an open subset of K^n, let $g : X \to K^n$, and let the integer $r \geqq 1$. Then g is a local D^r diffeomorphism[12] at $\bar{x} \in X$ if and only if there is some neighborhood of $\bar{x}$ on which g is D^r and g' is surjective.

Proof. The necessity of local D^r differentiability and surjectivity is obvious. The sufficiency is contained in part (c.ii) of the General Inverse Function Theorem when $r = 1$. For $r > 1$ the proof follows by induction, paralleling the usual proofs of the classical Inverse Function Theorem.[13] ∎

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(12) Cf. page 2.

(13) Cf. [**14**, p. 272, (10.2.3)].

Remarks 7. a) There are examples[14] in which no inverse h in part (a) of the General Inverse Function Theorem is unique or continuous. In such examples, with f as defined by (80), none of the functions ϕ in Theorems 1(a,b), 2(a,b), and 3(a,b) can be unique or continuous.

b) The continuity hypothesis on h cannot be dropped altogether, as shown by instances based on example (6) above.

c) The uniqueness and differentiability properties that follow from the additional hypotheses in part (c) are new results in the real case $K = \mathbb{R}$. When $K = \mathbb{C}$, however, the local differentiability assumption in (c) is only apparently weaker than the classical assumptions when $K = \mathbb{R}$, it is only apparently weaker when $K = \mathbb{C}$, since then the local differentiability hypothesis of (c)[15] implies, by Goursat's Theorem [**11**, p. 100] on analytic functions, that g is actually analytic — in which case the classical hypotheses and conclusions of part (d) hold.

d) As an example of a diffeomorphism that is not C^1, yet satisfies our hypotheses in part (c.ii) of Theorem 4 and the Corollary, consider:

$$g(x) = \begin{cases} x + x^2 \sin(1/x), & \text{for } -1 < x < 1 \\ 0, & \text{for } x = 0. \end{cases} \tag{82}$$

e) The theorems have useful applications, since the classical Implicit Function Theorem is imbedded in many parts of mathematics, including differentiable manifolds and optimization theory. The weaker hypotheses of the theorems above admit the possibility of extending existing results to these and other areas.

For example, replacing the classical Implicit Function Theorem by the general version (Theorem 1) allows a proof of the Lagrange Multiplier Theorem that weakens the classical C^1 hypothesis to just continuity of the constraint functions and their differentiability at the extremum [**23**]. As another example, comparative statics questions for equilibrium systems[16] can now be addressed with weaker assumptions.

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(14) E.g., (38) or [**14**, p. 273, Problem 2].

(15) But not those of parts (a) and (b).

(16) Cf. [**33**, Part I].

IV Some Historical Background and Comparisons

We originally developed the general theorems to solve an applied problem [**23**]. However, attempts to ascertain their novelty led to a historical study, and we present a few of the highlights that may be of interest to some readers.

We sketch the development of the implicit function theorem, with primary emphasis on the smoothness assumptions underlying the proofs. For the early period we have been guided by the information in Osgood [**30**], with Cauchy the earliest mentioned. We have not tried to examine the history prior to Cauchy's work. We use the notation of the previous sections.

The are several dimensions we could use for making historical comparisons. We could discuss assumptions: real scalars or complex scalars; the number of x and y variables; the smoothness assumptions on f with respect to x or y or (x, y); smoothness assumptions at $(\bar{x}, \bar{y})$ or in a neighborhood of the point. We could discuss conclusions: existence of implicit function, uniqueness of implicit functions, and smoothness of implicit functions. We could discuss methods of proof: Cauchy's calculus of residues, Cauchy's *Calcul des limites*, differential equations, or fixed point theorems.

We will give a brief chronological history, highlighting major differences among the various contributions, comparing our three theorems above with the major earlier results.

1831. Cauchy. Cauchy's results [**10**] are presumably the earliest rigorous existence proofs of the Implicit Function Theorem. Here the underlying scalar field K is that of the complex numbers. He mentions the general case in the introduction, and gives a detailed analysis of the special case $k = 1, n = 1$. The function f is assumed to be represented by a power series — i.e., to be analytic. A unique implicit function ϕ is obtained, and it is shown to be analytic.

Cauchy's tools were his calculus of residues and his Calculus of Limits. Their flavor may be experienced by considering the simple case where $k = 1$ and $n = 1$, so that both x and y are one-dimensional. Then a presentation of Cauchy's approach, in modern language looks roughly like the following.

Assuming f is analytic and not identically zero, for any given x, the generalized Argument Principle implies:[17]

$$\frac{1}{2\pi i}\int_C y\frac{f_y(x,y)}{f(x,y)}dy = \sum_{r=1}^{p} \alpha_r a_r, \tag{83}$$

[17] Cf. [**11, p. 124, Theorem 3.6**]

where the a_r are zeros of $f(x,\,\cdot\,)$ and the α_r are their multiplicities, and where C is a closed rectifiable curve not passing through any root a_r. Since the roots of an analytic function that is not identically zero are isolated, one can pick a small enough circle C in the complex plane about the root a_1 of $f(x,\,\cdot\,)$ so that there are no other zeros in the circle. In particular, Cauchy considers the case that the root a_1 has multiplicity 1.[18] Then denoting a_1 by $\phi(x)$, (83) becomes:[19]

$$\frac{1}{2\pi i}\int_C y\frac{f_y(x,y)}{f(x,y)}dy = \phi(x). \tag{84}$$

In view of the continuity of f and its derivatives, Rouché's Theorem[20] implies that for all nearby x the number of zero's (counting multiplicities) of the function $f(x,\,\cdot\,)$ remains the same; so there is a unique local solution function ϕ, and it can be calculated by the integral formula (84).

Cauchy further verifies that ϕ is (locally) an analytic function — that it is represented by its Maclaurin series. First he applies the calculus of residues to calculate derivatives for the Maclaurin series. Without loss of generality assuming $x = 0$ and $\phi(x) = 0$:[21]

$$\begin{aligned}\phi^{(m)}(0) &= \frac{1}{2\pi i}\int_C y\left[\frac{\partial^m}{\partial x^m}\left(\frac{f_y(x,y)}{f(x,y)}\right)_{x=0}\right]dy\\ &= \frac{1}{2\pi i}\int_C yA_m(y)dy \quad \text{(defining } A_m)\\ &= \frac{1}{2\pi i}(2\pi i\ \mathrm{Res}(yA_m(y);0)) \quad \text{(by the Residue Theorem)}\\ &= \frac{1}{(m-1)!}\lim_{y\to 0}\frac{d^{m-1}}{dy^{m-1}}((y^m)yA_m(y)),\end{aligned} \tag{85}$$

where the last equality follows [22] since $yA_m(y)$ has a pole of order m. Then he shows that the series converges to ϕ.

Of course this is an anachronistic account of his methods: the generalized Argument Principle, Rouché's Theorem, etc., had perhaps not yet crystallized

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[18] When the usual assumption is made that $f_y(\bar{x},\bar{y})$ is surjective, it follows that the unique root within the circle is a simple root.

[19] Cf. [**10**, p. 76 (52)]. Cauchy expresses y in polar coordinates, so his integrand has additional factor y.

[20] Cf. [**11**, p. 125, Theorem 3.8].

[21] Cf. [**10**, p. 83, equation (97)].

[22] Cf. [**11**, p. 113, Proposition 2.4].

as named theorems, and Cauchy established them in the context of his proof.[(23)]

1852. Cauchy. In his memoir [9], Cauchy used an alternative approach for proving the Implicit Function Theorem, namely existence theorems for differential equations. Again the underlying scalar field is that of the complex numbers, and the function f is analytic ("synectic"). The implicit function ϕ is unique and analytic. It is explicitly assumed that $f_y(\bar{x}, \bar{y})$ is nonsingular.

Although he mentions more general cases, formal proofs are given only for the case where x is one-dimensional but y can be multi-dimensional (i.e., $n = 1$ and $k \geqq 1$). In this situation Cauchy is able to apply existence theorems for ordinary differential equations to establish existence, uniqueness, and differentiability of the implicit function ϕ.

We can illustrate the logic of his approach again for the simple case $f(x, y) = 0$ with both x and y one-dimensional. Assume[(24)] that f is C^1 and that$f_y(\bar{x}, \bar{y}) \neq 0$. In that case, the right hand side of the ordinary differential equation

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)} \tag{86}$$

with the initial condition $f(\bar{x}, \bar{y}) = 0$ is well defined and has a unique local solution, say $y = \phi(x)$, satisfying $\bar{y} = \phi(\bar{x})$. It can be shown that the function ϕ is the unique function locally satisfying $f(x, \phi(x)) = 0$ with $\bar{y} = \phi(\bar{x})$; i.e., it is the implicit function for the given function f. Furthermore, its derivative satisfies the relation

$$\phi'(x) = \frac{f_x(x, \phi(x))}{f_y(x, \phi(x))}. \tag{87}$$

Thus we have obtained the standard Implicit Function Theorem (for this simple case), but at the cost again of assuming the given function f to be C^1. Hence it could have been used to obtain Dini's result for a system involving a single independent variable. Perhaps an analogous approach using existence theorems for partial differential equations could be used when there are several independent variables, i.e., when $n \geqq 1$.

Theorem 1 above weakens the assumptions of both Cauchy papers by reducing the smoothness requirements on f (from analytic in a neighborhood to differentiable at a point). Although Theorem 1 obtains existence of an implicit

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(23) A lucid exposition of the methods he used in [10] for establishing (84) is contained in [27].

(24) This logic applies both to the real scalars and also to the complex scalars (in which case C^1 is equivalent to analytic).

function and asserts that all implicit functions are differentiable, it cannot claim uniqueness of the implicit function,

1875. Briot and Bouquet. (25) Briot and Bouquet in their [**5**] are clearly aware of and following Cauchy's work in the theory of complex variables. The scalars are complex, and f is analytic. In Section 211, p. 336, Theorem III, they state and prove the theorem for the special case $k = 1, n = 1$; in Section 212, p. 337, Theorem IV, they state and sketch an argument for the more general case $k \geqq 1$, but still with $n = 1$.

They use solution methods for ordinary differential equations to establish existence of an analytic implicit function. The uniqueness of the implicit function is implied by their results on ordinary differential equations (an explicit uniqueness argument is given in Section 209, p. 332, for the case of a single ordinary differential equation).

1877-78. Dini. In his lecture notes [**15**], Dini states the Implicit Function Theorem in the form found in most textbooks today. The scalars are real, and f is C^1 (or C^r for some $r \geqq 1$). He solves for a unique implicit function ϕ, which he proves is C^1 (respectively, C^r).

At that time, of course, Brouwer's Fixed Point Theorem [**7**] was not available.(26) Instead, Dini used the Intermediate Value Theorem together with a C^1 assumption. To indicate the essence of his approach, consider the simple case where both x and y are one-dimensional and there is only a single given function f and a single equation $f(x, y) = 0$.

In this context, Dini assumes that, in some neighborhood of $(\bar{x}, \bar{y})$, the function f is C^1 and furthermore, that the partial derivative $f_y(\bar{x}, \bar{y}) \neq 0$. His proof of the existence of an implicit function $y = \phi(x)$ then uses two propositions of differential calculus: the Intermediate Value Theorem and Taylor's Formula with Remainder (Extended Theorem of the Mean) in the form $f(\bar{x} + h, \bar{y} + k) = f(x, y) + hf_x(\bar{x} + \theta h, \bar{y} + \theta k) + kf_y(\bar{x} + \theta h, \bar{y} + \theta k)$, with $0 < \theta < 1$.

Thus even to establish existence, this proof uses the existence of the derivatives of f with respect to x and y in a neighborhood of $(\bar{x}, \bar{y})$. Furthermore it uses the continuity of the partial f_y to show that, in a sufficiently small neighborhood, it retains the same sign as at $(\bar{x}, \bar{y})$, and the continuity of f_x is also used.

(25) (We have not had access to Briot and Bouquet's first edition of 1859. (Osgood [**30**] mistakenly dates the 2nd edition as 1873).)

(26) The paper itself is dated July, 1910.

The first printed version of Dini's implicit function theorem is in Genocchi's [**17**], as edited by Peano. (The German translation is [**18**].) A clear formulation of Dini's theorem and a proof is also found in [**31**], with a helpful diagram.

Theorem 1 weakens Dini's assumptions by allowing both real and complex scalars, and by weakening the smoothness requirements on f. Although Theorem 1 asserts that all the implicit functions are differentiable at $(\bar{x}, \bar{y})$, it cannot claim uniqueness of the implicit functions.

1893. Jordan. In Sections 91-95 of [**24**] the scalars are real, and the results do not seem to go beyond those of Dini. The general case $k \geqq 1, n \geqq 1$ is considered, f is C^1, and a unique implicit function ϕ is obtained. Jordan notes that its partial derivatives exist, and his discussion makes clear that they are continuous.

In Section 191, page 178, the scalars are complex. The case $k = 1, n \geqq 1$ is considered in detail; the possibility of an extension to $k \geqq 1$ is mentioned. The function f is analytic ("synectic"), and a unique implicit function ϕ is obtained, and it is shown to be analytic, with the usual formula for its partial derivatives.

A clear proof along the same lines, for $k = 1, n = 1$ is also to be found in [**31**, pp. 345-3467].

Our Theorem 1 weakens the analytic hypothesis on f to differentiability at $(\bar{x}, \bar{y})$, while retaining existence of an implicit function and differentiability of every implicit function at $(\bar{x}, \bar{y})$. It cannot, however, guarantee uniqueness or analyticity of the implicit function.

1899. Lindelöf. In [**26**] the scalars are complex. The general case $k \geqq 1, n \geqq 1$ is considered. The function f is assumed analytic, an implicit function ϕ is obtained, and it is shown to be analytic.

The proof is based on power series expansions of f and the implicit function ϕ, using Cauchy's *Calcul des limites*, in contrast to Jordan (1893), which reduced the complex case to the real case.

Our Theorem 1 weakens Lindelöf's assumption that f is analytic to differentiability at $(\bar{x}, \bar{y})$ and local continuity, while retaining existence of an implicit function. While it obtains differentiability of every implicit function at $(\bar{x}, \bar{y})$, it cannot claim uniqueness.

1901. Osgood. This contains a clear statement of an implicit function theorem for complex scalars, with $k \geqq 1, n \geqq 1$, and with an explicit statement of uniqueness of the implicit function. We have also found this to be a very useful source for historical information; see especially pp. 19 (footnote 30) and 103 (footnote 247).

1903. Goursat. In [**19**] the scalars are real, and $n \geqq 1$ and $k \geqq 1$. Goursat makes no differentiability assumption with respect to x. He assumes that f is continuous in a neighborhood of $(\bar{x}, \bar{y})$, and he retains the assumptions that $f(x, \cdot)$ is C^1 for x near 0. He obtains a unique implicit function, and shows that it is continuous.

Goursat's proof used a fixed point theorem, what we would now call a contraction mapping theorem, which he proved using Picard's method of successive approximations. Defining

$$g(x, y) = y - \bar{y} - (f_y(\bar{x}, \bar{y}))^{-1} f(x, y) \tag{88}$$

he shows that $g(x, y) = y$ has a unique solution for $y = \phi(x)$ that is a continuous function of x.(27)

Theorem 2 weakens Goursat's assumptions by allowing both real and complex scalars, and by weakening the smoothness requirements on f. Although Theorem 2 obtains existence of an implicit function and asserts that all implicit functions are continuous, it cannot claim uniqueness of the implicit functions.

The Goursat result (dropping differentiability of f with respect to x, and assuming continuity of f near $(\bar{x}, \bar{y})$) is also found in Bliss's published lecture [**3**, pp. 8-9] on the "Fundamental Existence Theorems." Although Bliss cites Goursat's paper, his method of proof is based on Taylor's Theorem.

In [**22**] Graves notes that existence, uniqueness, and continuity can be obtained without assuming any differentiability of f with respect to the variable x. His remark (p. 139) is in the midst of his proof of his Theorem 2 (p. 138).

1909. W. H. Young. There are several theorems to consider in Young's [**35**]. In all of them, the scalars are the reals.

Young's Theorem 9 modifies Dini's result, replacing the C^r differentiability assumption on f with an assumption that f is differentiable of order r for some $r \geqq 2$. It asserts the existence of a unique implicit function ϕ, which is also differentiable of order r. Neither our Theorem 1 nor Dini's theorem require second order differentiability for existence of an implicit function.

Our Theorem 1 uses weaker assumptions than Young's Theorem 9 in allowing both real and complex scalars, and it uses weaker smoothness requirements on f (differentiability at $(\bar{x}, \bar{y})$, instead of twice differentiable at $(\bar{x}, \bar{y})$, hence C^1 in a neighborhood). Although our Theorem 1 obtains existence of an implicit

(27) He notes that the solution is unique provided that g is Lipschitz continuous with constant $K < 1$, observing that this automatically holds when f_y is continuous in a neighborhood of $(\bar{x}, \bar{y})$. (See also footnote 6, p. 11 above.)

function and asserts that all implicit functions are differentiable at $\bar{x}$, it cannot claim uniqueness of the implicit functions.

Young's Theorem 10 has a weaker assumption on f than Dini's theorem, since it requires a lower order differentiability of f with respect to y (differentiability in a neighborhood of $(\bar{x}, \bar{y})$, rather than C^1). However it strengthen's Dini's assumption that $f_y(\bar{x}, \bar{y})$ is nonsingular (hence nonsingular in some neighborhood, since f is C^1 for Dini), by imposing nonsingularity conditions on certain principal minors (in some neighborhood). Young then concludes that there exists a unique implicit function ϕ, and it is differentiable at $(\bar{x}, \bar{y})$.

Young remarks that only the uniqueness of ϕ is affected if one weakens the assumptions of his Theorem 10 by only requiring that the Jacobian and principal minor conditions at $(\bar{x}, \bar{y})$, rather than throughout some neighborhood.[28]

Our Theorem 2 uses weaker assumptions than in Young's Theorem 10 in allowing both real and complex scalars, in only requiring continuity of f locally and differentiability at $(\bar{x}, \bar{y})$ (rather than differentiability locally), in not requiring Young's conditions on principal minors of the Jacobian in a neighborhood (it only requires that the Jacobian itself be nonsingular at $(\bar{x}, \bar{y})$). Our Theorem 2, however, cannot claim uniqueness of the implicit functions.

Young claims (in his Corollary 3 of Theorem 10) that Dini's result is a corollary of his Theorem 10, although that is not clear to us.

In neither Theorem 9 or Theorem 10 does Young consider a situation in which f is not differentiable with respect to x, in contrast to Goursat's result and our Theorem 2. However, his Theorem 5, as extended in his section 10, contains a special case that does not require differentiability. For the special case $k = 1, n \geqq 1$, Assuming only that $f_y(\bar{x}, \bar{y})$ is nonzero, and that f is continuous separately in each argument, he obtains in parts (1) and (2) the existence of an implicit function ϕ. (While there need not be a unique implicit function, he notes that there exists an upper semicontinuous one and a lower semicontinuous one.)

The basic idea of Young's proof is similar in spirit to what others have done. Because it yields a better result for the special case $k = 1, n \geqq 1$ than either his general Theorems 9 or 10, or our Theorem 2, we present here a proof sketch. (We weaken his continuity assumption with respect to x by only requiring continuity at $\bar{x}$.)

Because $f(\bar{x}, \bar{y}) = 0$ and $f_y(\bar{x}, \bar{y}) \neq 0$, there exists a $\hat{y}$ such that $f(\bar{x}, \hat{y}) > 0$ and $f(\bar{x}, -\hat{y}) < 0$. Because $f(\,\cdot\,, \hat{y})$ is continuous at $\bar{x}$, then

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(28) Page 420, paragraph 16.

for all small enough x we have both $f(\bar{x}+x, \hat{y}) > 0$ and $f(\bar{x}+x, -\hat{y}) < 0$. Then continuity of $f(\bar{x}+x, \cdot)$ and the Intermediate Value Theorem imply $f(\bar{x}+x, y^*) = 0$ for some y^* in the interval $[-\hat{y}, \hat{y}]$. By continuity of $f(\bar{x}+x, \cdot)$, the set of such y^* is closed, so one way to define the value of an implicit function at x is to chose the maximum such y^*.

Other. Without aiming at completeness, we mention a few other treatments of the implicit function theorem. Careful statements of the classical (Dini) version are found in Bolza [**4**] and Carathéodory [**8**].

In the complex domain, Goursat [**20**, pp. 399 ff.], [**21**, pp. 233 ff.] and Osgood [**32**, §§45, 105], [**32**, pp. 86-86], and Markushevich [**28**, p. 109] all utilize the Weierstrass Preparation Theorem [**34**, pp. 107-114].(29) This theorem deals with cases where the equation $f(x, y) = 0$ has roots $y(x)$ whose multiplicity m may equal or exceed 1, and is applicable to cases where $f_y(\bar{x}, \bar{y}) = 0$, and in this respect has broader applicability than the results established in the present paper. When $m = 1$, the Preparation Theorem can be used to prove the Implicit Function Theorem for the case where $f_y(\bar{x}, \bar{y}) \neq 0$ and f is analytic. This is shown most explicitly by Markushevich [**28**, p. 109, Theorem 3.11].

Evgrafov's proof [**16**] uses series expansions and Cauchy's integral formulae, as well as the Fréchet property of the differential. In addition to proving existence, uniqueness, and continuity, it also derives the formula $\phi'(x) = -f_y(x, y)f(x, y)$.

The paper by Hildebrand and Graves [**22**] was perhaps the first to use Banach space techniques to establish the Implicit Function Theorem through methods applying to both the real and complex domains.

Nijenhuis [**29**], building on a notion of Leach [**25**], obtained inverse and implicit functions in the real case, under an assumption ("strong differentiability") weaker than C^1, but stronger than Fréchet differentiability.

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(29) Published in 1886, but presented in lectures since 1860.

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