COMPETITIVE MATCHING EQUILIBRIUM AND MULTIPLE PRINCIPAL-AGENT MODELS

by

Shuhe Li

Discussion Paper No. 267, January 1993

Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, MN 55455

COMPETITIVE MATCHING EQUILIBRIUM AND MULTIPLE PRINCIPAL-AGENT MODELS

Shuhe Li

Department of Economics

University of Minnesota

Minneapolis, MN 55455

E-mail shuhe@atlas.socsci.umn.edu

May 1992

Revised January 27, 1992

This paper was presented at the 1992 Midwest Mathematical Economics Conference at Pittsburg, and at the 1992 Mathematical Economics Workshop at University of Minnesota. I greatly appreciate the guidance and encouragement of Leonid Hurwicz. I would like to thank Beth Allen, James Jordan, Andrew McLennan and Marcel Richter for many good suggestions. In particular, Edward Green's help and insights cannot be over emphasized. Alvin Roth kindly provided two references. Finally, I am grateful for Neelam Jain's helpful editorial suggestions. All errors are the sole responsibility of the author.

ABSTRACT

This paper studies one-to-one two-sided matchings with externalities and explores its application to *multiple* principal-agent models with principal-agent assignment being endogenously determined. Each individual has to make two strategic decisions: to choose a partner and to sign a contract with his partner. Each individual may care about not only to whom he matches and what contract he signs but also other people's matchings and contracts, that is, there may exist *externalities*. Equilibrium is defined *inductively*. A blocking pair takes all the equilibria in the residual market into account. To reflect the asymmetric distribution of power between principals and agents, the concept *principal-equilibrium* is introduced, which refines the equilibrium.

I show that under certain conditions (i) there always exists a Pareto optimal equilibrium, (ii) there always exists a Pareto optimal principal-equilibrium and (iii) the set of equilibria coincides with the core. In general, the set of equilibria could be empty and an equilibrium could be non-Pareto optimal.

This paper provides a unified framework for one-to-one two-sided matching models such as job matchings, housing markets, auctions and marriages. A generalization of this framework to a *social partitioning game* is provided.

1. INTRODUCTION

Consider an arrangement of close ties between banks and industrial firms such as the *keiretsu* in Japan: banks hold equity stakes in firms to which they lend and banks have memberships on the boards of directors that manage these firms. Suppose there are four banks and four industrial firms where one bank matches to one firm and vice versa. For each bank-firm pair, there is a set of conceivable contracts to be signed. A bank (firm) may care about not only to which firm (bank) it matches and what contract it signs, but also the matchings and contracts between other banks and other firms, that is, there may exist *externalities*. Who will match to whom and with what contract? Is there a stable arrangement? Is a stable arrangement efficient? This paper constructs a framework based on two-sided matching models to answer these questions.

In the initial marriage model by Gale and Shapley (1962), one person's preference order for the other side is fixed. Sasaki and Toda (1986) introduced externalities into the Gale-Shapley marriage model. Shapley and Shubik (1972) introduced a TU assignment game, where one's preference order for the other side depends on the sharing of each pair's total utility. Since the TU assignment game is very restrictive, Kaneko (1982) and Moldovanu (1990) generalized it to an NTU assignment game. The NTU assignment game is still very restrictive. It allows only cardinal preferences and is in characteristic function form in which there is no room for externalities.

Based on the job matching model of Kelso and Crawford (1982), Roth (1984, 1985) provided an abstract framework for two-sided matching markets. Although Roth's framework is quite general (it incorporates one-one, many-one and many-many matchings), it does not incorporate externalities that could be an important feature in multiple principal-agent models (see Myerson (1982)). Externalities involve the situation where a person cares about not only with whom he matches and what contract he signs but also other persons' matchings and contracts. Externalities exist in marriage markets if there is jealousy. They may also exist in job matchings, international partnerships of firms such as airlines, law firm-client (industrial firm) matchings and bank-industrial firm matchings if firms are linked by markets and/or government regulations or if there is invention or pollution.

Two-sided matching models can incorporate the feature that the participants in a market have bargaining power. Many markets such as job markets, housing markets and auctions have this feature. The classical general equilibrium framework assumes that all participants are price-takers (impersonal), hence it can not be applied to these markets.

Two-sided matching models with externalities could potentially be used to analyze multiple principal-agent models. Whenever there exist asymmetric power and/or asymmetric information across different persons (e.g., some persons (principals) have less information but more power——the right to design a contract, while others (agents) have more information——hidden types and/or hidden actions but less power), the principal-agent problem arises.(1)

Principal-agent models play an important role in informational economics and contract theory. Numerous applications have been found in accounting, labor, credit, tax and sharecropping issues. Yet for about the last two decades, the research has been focusing on partial equilibrium analysis. In one principal-one agent models, one principal-many agents models and one agent-many principals models, the reservation utility levels of the agent(s) are exogenously given. Myerson (1982) considered a multiple principal-agents model, but he fixed the principal-agents assignment exogenously.

In the modern world, however, typically there are two or more principal-agent pairs and people are linked by market. Consequently, competition arises and thus the principal-agent assignment and the reservation utility levels are determined endogenously.

In this paper, the equilibrium concept is defined in the spirit of *stable matching*—no blocking pair exists. Because of externalities, however, when a pair considers whether to block or not, it needs to take the reactions of the residual coalition into account. A consistent rational hypothesis is that a blocking pair takes all the equilibria in the residual market into account. That is, it neither conjectures that the outsiders will not change their strategies, nor conjectures that the outsiders will do everything, but it conjectures that the outsiders will play equilibrium strategies among themselves. Thus I assume that players are neither myopic (or careless) as in Nash equilibrium, nor prudent as in α-core. An equilibrium is defined inductively.

To reflect the asymmetric distribution of power between principals and agents, the concept *principal-equilibrium* is introduced, which refines the equilibrium.

For one-to-one two sided matchings, it turns out that (i) there always exists an equilibrium for any finite market with weak externalities, where weak externalities mean that each individual may care about other people's partner assignments and contracts, but the most important thing for him is "whom his partner is and what contract he signs"; (ii) there always exists a principal-equilibrium for any finite market with *weak externalities*; (iii) the set of equilibria coincides with the core for any market with weak externalities and there is no logical relationship in general; (iv) there exists a one-side-optimal equilibrium (and polarization: the one-side-optimal equilibrium is another-side-worst equilibrium) for any finite market without externalities, but there is no one-side-optimal equilibrium (and polarization) in general; and (v) a principal-equilibrium (hence an equilibrium) is not efficient in general, but the set of equilibria (if not empty) contains at least one Pareto optimal equilibrium, and so is the set of principal-equilibria.

This suggests that once externalities are present in a matching market, existence, core equivalence, polarization and efficiency of equilibrium no longer hold (in general). It also suggests that government intervention may be necessary if there exist nonweak externalities. In contrast, in conventional principal-agent models, there is no room for externalities, hence, in general, (constrained) efficiency could be attained.

Of course, one-to-one two-sided matching models are very restrictive. In reality, we observe various coalition forms such as political parties and labor unions; it is not equally easy to form every coalition and only certain coalitions are effective. It turns out that we could significantly generalize this framework to a broad model where, (1) an arbitrary permissible coalition structure is given, and (2) the feasibility of each coalition's joint strategy is contingent on the joint strategy of the residual coalition. I call this broad model a *social partitioning game*, which is a generalization of the standard (noncooperative) normal form game, the *partitioning game* of Kaneko and Wooders (1983) and the *social coalitional game* of Ichiishi (1981). The solution concept (*inductive* equilibrium) here, however, is different from those (Nash, core-like or strong equilibrium-like).

The solution concepts are defined in Section 2. Existence theorems are proved in Section 3. The properties of equilibrium are discussed in Section 4. In Section 5, I give an example of an application to sharecropping. Finally, concluding remarks are made in Section 6; in particular, a social partitioning game and an inductive solution concept are defined.

2. THE BASIC FRAMEWORK AND THE CLASSES OF MARKETS

A. An Illustration

I will use three examples to illustrate the general approach in this paper. The first example is about pair figure skating, which shows how the framework is motivated. The second example is a modified version of Myerson's example (1982, Proposition 3), which shows: (1) how to avoid nonexistence of principal-equilibrium by introducing endogenous assignment between principals and agents to internalize externalities in some cases, (2) how to transform a basic multiple principal-agent setting into the standard structure of this paper. The third example shows how the solution concepts are defined.

Example 1 (Figure Skating) [2] Suppose in a pair figure skating competition, there are three female candidates w_1, w_2 and w_3, three male candidates m_1, m_2 and m_3; w_i (also m_i) is the *i*th rated female (male) individual skater. There are six partner assignments and the corresponding competition results (awards):

(1)	w_1m_1	w_2m_2	w_3m_3	(2)	w_1m_1	w_2m_3	w_3m_2
	A	B	C		A	B	C
(3)	w_1m_2	w_2m_1	w_3m_3	(4)	w_1m_2	w_2m_3	w_3m_1
	A	B	C		A	C	B
(5)	w_1m_3	w_2m_2	w_3m_1	(6)	w_1m_3	w_2m_1	w_3m_2
	B	A	C		B	A	C

where A (first place) is worth \$500, B (second place) is worth \$300, C (third place) is worth \$100.

For each woman-man pair, suppose there are three ways to share the money: 1:1, 2:1 and 1:2. Each skater prefers more money to less, and with a given amount of money she (or he) prefers a higher rated partner.

An *arrangement* specifies who is matched to whom and with what contract. Each pair determines what contract to be signed. For example, the following is an arrangement: w_1 is matched to m_1 with sharing 1:1, w_2 is matched to m_2 with 1:2, and w_3 and m_3 with 1:1.

An *equilibrium* is defined in the spirit of *stable matching*—no blocking pair exists. Because of externalities, when a pair considers whether to block or not, it needs to take the reactions of the residual coalition into account. A consistent rational hypothesis is that a blocking pair takes all the equilibria in the residual coalition into account. An equilibrium is defined inductively. A pair will block if by signing a new contract between themselves they could be better off for every equilibrium in the residual market.

For instance, suppose the arrangement of the above example is the status quo. Suppose that w_2 and m_1 were to jointly consider whether to block or not. I will argue that they would decide not to block. There are two possible partner assignments in the residual coalition: (i) $(w_1m_2,\ w_3m_3)$, (ii) $(w_1m_3,\ w_3m_2)$. If (i) happens, w_2 and m_1 will be in the second place and cannot be better off; if (ii) happens, w_2 and m_1 will be in the first place and can be better off. Clearly (ii) is not sustainable since w_1 and m_2 have incentive to block, while (i) is sustainable and is in fact the equilibrium partner assignment in the residual market. Anticipating this, w_2 and m_1 would not block.

It turns out that there are 162 (6×3^3) arrangementss and 15 equilibria as follows:

Type (1):	w_1m_1	w_2m_2	w_3m_3
	(250, 250)	(100, 200)	(x, y)
	(1000/3, 500/3)	(150, 150)	(x, y)
	(1000/3, 500/3)	(200, 100)	(x, y),

Type (2): w_1m_1 w_2m_3 w_3m_2

(1000/3, 500/3) (200, 100) (x, y),

Type (3): w_1m_2 w_2m_1 w_3m_3

(1000/3, 500/3) (100, 200) (x, y),

with $x, y \in \{100/3, 50, 200/3\}$ and $x + y = 100$.

As an example, let us check that Type (1) (w_1m_1, w_2m_2, w_3m_3) with sharing {(250, 250), (100, 200), (50, 50)} is an equilibrium. First, w_1 is only willing to join m_2 with sharing 2:1. But m_2 will reject since he can get 500/3 that is less than 200. Second, if w_2 joins m_1, then w_1 will join m_2 with sharing 2:1 (and w_3 matches to m_3) (this is the equilibrium partner assignment w.r.t. w_2m_1), w_2m_1 will be in the second place and get 300 that is less than 350 (the total of their current awards). If w_2 joins m_3, then (w_1m_2, w_3m_1) will be the equilibrium partner assignment w. r. t. w_2m_3, w_2m_3 will be in the third place and get 100 that is less than their current total 150. Finally, if w_3 joins m_1, they can get at most 300 hence m_1 can get at most 200 that is less than his current share 250. Clearly m_2 is not willing to join w_3. This exhausts all the cases.

If we allow each pair to share their total award arbitrarily, the following arrangement is an equilibrium:

Type (1): w_1m_1 w_2m_2 w_3m_3

(350, 150) (150, 150) (50, 50).

First w_1 is only willing to join m_2 with a share greater than 350, but m_2 will not accept. Second, w_2 is only willing to try to join m_1, but the equilibrium partner assignment in the residual market will be (w_1m_2, w_3m_3) that makes (w_2, m_1) in the second place. Finally, w_3 can only try to join m_1, but the equilibrium partner assignment in the residual market will be $(w_1m_3,\ w_2m_2)$ that makes $(w_3,\ m_1)$ in the third place. (Notice that $(w_1m_2\ \ w_2m_3)$ is not an equilibrium partner assignment w.r.t. w_3m_1. Suppose that w_1 and m_2 (w_2 and m_3) are matched and x is w_1's share and y is m_2's share. If $x < 200$, w_1 would join m_3 to block. If $x \geq 200$, then $y < 300$, m_2 would join w_2 to block.)

Example 2 (Myerson) Suppose there are two principals p_1 and p_2, two agents a_1 and a_2, each agent has two (private) types—α or β with equal probabilities, and the two agents' types are independent. The payoff matrix is as follows (determined by nature): for $i,\ j \in \{1,\ 2\}$

		a_j's type	
		α	β
	a	(6, 1)	(0, z^j)
p_i's action	b	(0, z^j)	(6, 1)
	c	(5, 0)	(5, 0)

where the first (second) term in the parenthesis is p_i's (a_j's) payoff, and

$$\begin{aligned} z^1 &= 2 \quad \text{if } p_2 \text{ chooses } a \text{ or } b, \\ &= 1 \quad \text{if } i = j \text{ and } p_2 \text{ chooses } c, \\ z^2 &= 2 \quad \text{if } i = j \text{ and } p_1 \text{ chooses } c, \\ &= 1 \quad \text{if } p_1 \text{ chooses } a \text{ or } b. \end{aligned}$$

Assume one principal hires one agent. Each agent's strategy is to report what type he is. Based on an agent's reporting, a principal's strategy is to choose a probability measure on $\{a,\ b,\ c\}$. Notice that if $z^j = 1$ for sure then the optimal incentive compatible contract for p_i is a if a_j says his type is α and b if

β. On the other hand, if there is any positive probability that $z^j = 2$, then a_j will want to steer p_i to the wrong corner (b if α, a if β), and the best incentive compatible contract is c for sure. Therefore, we may confine our attention to the contracts f_k: $\{\alpha, \beta\} \longrightarrow \{a, b, c\}$ for $k = 1, 2$, where $f_1(\alpha) = a$, $f_1(\beta) = b$, $f_2(\alpha) = f_2(\beta) = c$.

Myerson assumed that the principal-agent assignment is fixed—a_1 is p_1's agent, a_2 is p_2's agent. Then he demonstrated that there is no principal-equilibrium. (For the fixed assignment case, a principal-equilibrium is a Bayesian Nash equilibrium of the game played by the principals with incentive compatible strategies). If p_2 chooses a or b with positive probability then p_1 will choose c, if p_1 chooses c then p_2 will choose c, if p_2 chooses c then p_1 will choose a or b, and if p_1 chooses a or b then p_2 will choose a or b.

But if we allow the principal-agent assignment to be determined endogenously, there exists a principal-equilibrium. In this case, besides having the strategies mentioned above, each principal (agent) has an additional choice: to which agent (principal) he matches. An *arrangement* is said to be a *principal-equilibrium* if it is stable in the sense that there is no blocking pair (a principal with an agent) and no blocking principal (see more discussion in the next example).

It turns out that the following arrangement is a principal-equilibrium: a_1 is matched to p_2 with contract f_2 (p_2 chooses c regardless of a_1's reporting) and a_2 is matched to p_1 with contract f_1 (p_1 chooses a if a_2 reports α and b if a_2 reports β). There is no blocking pair since p_1 has already achieved his best and has no incentive to change, and a_2 cannot be made better off if p_1 keeps f_1. There is no blocking principal since the only incentive compatible contract for p_2 is f_2 (given that a_2 is matched to p_1 with contract f_1) and p_1 has already achieved his best.

Now I can transform the basic setting in the example into a *competitive matching model* $(P, A, M, \bar{\mathcal{R}})$, where $P = \{p_1, p_2\}$ is the set of principals, $A = \{a_1, a_2\}$ is the set of agents, $M = \{(a_i, p_j, f_k; a_r, p_s, f_t): i, j, k, r, s, t \in \{1, 2\}, i \neq r, j \neq s\}$, where (a_i, p_j, f_k) means a_i is assigned to p_j with contract f_k, is the set of all conceivable arrangements, and $\bar{\mathcal{R}}$ is the set of all conceivable preference profiles $\bar{R} = (\bar{R}^1, \bar{R}^2, \bar{R}_1, \bar{R}_2)$, where $\bar{R}^i$ ($\bar{R}_j$) is p_i's (a_j's, given his type) preference order on M induced by the expected utility levels. I will use C^i_j to denote the set of all conceivable contracts between p_i and a_j. In the example, $C^i_j = \{f_1, f_2\}$ for $i, j \in \{1, 2\}$.

Example 3 Let $P = \{p_1, p_2\}$, $A = \{a_1, a_2\}$, $C^1_1 = \{a, b\}$, $C^1_2 = \{d\}$, $C^2_1 = \{c\}$, $C^2_2 = \{e, f\}$, and $M = \{ae, af, be, bf, cd\}$, where ae means that a_1 is matched to p_1 with contract a, and a_2 is matched to p_2 with contract e, and so on. The preference profile $\bar{R}$ is as follows: ($\bar{R}^1$: $x \quad y \quad z$ means that $x \bar{R}^1 y \bar{R}^1 z$, and so on.)

$\bar{R}_1$: *ae af bf cd be*,

$\bar{R}_2$: *af bf cd be ae*,

$\bar{R}^1$: *be bf cd ae af*,

$\bar{R}^2$: *bf cd ae af be*.

It turns out that there are two equilibria in this example: *af* and *bf*. Notice that *ae* is blocked by a_2 and p_1 with $(c)d$, *be* is blocked by a_1 and p_2 with $c(d)$, and *cd* is blocked by a_1 and p_1 with $b(f)$. Since both p_2 and a_2 prefer *bf* to *be*, the equilibrium between p_2 and a_2 is f provided that p_1 is matched to a_1 with contract b. Although p_1 and a_1 can only control b, they anticipate that p_2 and a_2 will choose f.

A Principal-equilibrium is defined to reflect the fact that principals posses more power than agents do. In the example, there is one principal-equilibrium: *bf*. Since p_1 prefers *bf* to *af* and he knows that *bf* is "stable" (*bf* is an equilibrium), he will choose b.

B. The Basic Framework

Let $P = \{p_1, \cdots, p_m\}$ and $A = \{a_1, \cdots, a_n)$ be two disjoint sets of persons. Let $C = \prod_{i=0}^{m} \prod_{j=0}^{n} C^i_j$, where C^i_j is the set of feasible contracts between p_i and a_j with the convention: C^i_0 (a singleton) means that p_i matches to himself if $i \neq 0$, C^0_j (a singleton) means that a_j matches to himself if $j \neq 0$, and C^0_0 (a singleton) means that everyone matches to himself. Let $C^i = \bigcup_{j=0}^{n} C^i_j$, $C_j = \bigcup_{i=0}^{m} C^i_j$.

An *arrangement* specifies who matches to whom and with what contract. Formally, an *arrangement* is a pair (μ, c), where μ is a *partner assignment* that is a mapping $\mu: P \cup A \longrightarrow P \cup A$ with (i) $\mu^2(x) = x$ for all $x \in P \cup A$, and (ii) $\mu(p) \in A$ if $\mu(p) \neq p$, $\mu(a) \in P$ if $\mu(a) \neq a$; and c is a *contract assignment* with c^i_μ (c^μ_j) denoting the contract assigned between p_i and $\mu(p_i)$ (between a_j and $\mu(a_j)$) under the arrangement (μ, c).

Let $\mathcal{M}$ be the set of all conceivable arrangements. A strict preference order is a binary relation on $\mathcal{M}$ that is total, asymmetric and transitive.[3] Let $\bar{\mathcal{R}}$ be the set of all conceivable strict preference profiles. A generic element of $\bar{\mathcal{R}}$ is $\bar{R} = (\bar{R}^1, \cdots, \bar{R}^m, \bar{R}_1, \cdots, \bar{R}_n)$, where $\bar{R}^i$ is p_i's preference order on $\mathcal{M}$, $\bar{R}_j$ is a_j's preference order on $\mathcal{M}$. We will use $\bar{R}^i$: x y z to mean x $\bar{R}^i$ y $\bar{R}^i$ z, and so on.

A competitive matching model (with externalities) is an ordered tuple $(P, A, \mathcal{M}, \bar{\mathcal{R}})$. A *competitive matching market* (or for short, *market*) is an ordered tuple $(P, A, \mathcal{M}, \bar{R})$.

A strict *direct* preference order of p_i (a_j) is a binary relation R^i (R_j) on C^i (on C_j) that is total, asymmetric and transitive. Let $\mathcal{R}$ be the set of all conceivable strict direct preference profiles.

A market is said to be with *weak externalities* if each individual may care about other people's arrangements, but the most important thing for him is "whom his partner is and what contract he has". Formally, a market $(P, A, \mathcal{M}, \bar{R})$ is said to be with *weak externalities* if $\exists$ R such that $\forall$ (μ, c), $(\lambda, b) \in \mathcal{M}$, c^i_μ R^i b^i_λ implies (μ, c) $\bar{R}^i$ (λ, b) for all i, and c^μ_j R_j b^λ_j implies (μ, c) $\bar{R}_j$ (λ, b) for all j; R is called the *associated* direct preference profile of $\bar{R}$.

A market is said to be with *no externality* if each person cares only about whom his partner is and what contract he has. Formally, a market with *no externality* is an ordered tuple (P, A, C, R). Clearly a market with no externality is a 'special case' of markets with weak externalities. All the following analyses for markets with weak externalities are also valid for markets with no externality unless mentioned otherwise.

In a market with weak externalities $(P, A, M, \bar{R})$, an arrangement (μ, c) is said to be *individually rational* if there is no individual (say p_i) who prefers being unmatched to (μ, c) (say $c_0^i\ R^i\ c_\mu^i$, where R^i is the associated direct preference of $\bar{R}^i$).

Given a market, for a given coalition and a given arrangement within this coalition, we call the new market formed by excluding the given coalition from the original market (with the naturally induced preference orders for the members of the residual coalition) the *residual market*. Choose an equilibrium (principal-equilibrium) (defined below) in the residual market, and construct a new arrangement for the original market such that the new arrangement agrees with the given arrangement within the given coalition and agrees with the chosen equilibrium (principal-equilibrium) in the residual market. We call this new arrangement *reactive* (*principal-reactive*) *arrangement w. r. t. the given arrangement within the given coalition.*

The following definition is a generalization of *stable matching*. The equilibrium is defined inductively. Because of externalities, when a pair (or an individual) considers whether to block or not, it needs to take the reactions of the residual market into account. A consistent rational hypothesis is that a blocking pair (or an individual) takes all the reactive arrangements (and nothing else) into account.

DEFINITION 2.1 A *competitive matching equilibrium* (or for short, *equilibrium)* is defined as follows:

(i) For the case $|P| = |A| = 1$, an equilibrium is an individually rational and Pareto optimal arrangement.

(ii) Suppose an equilibrium has been defined for each of the cases with $|P| < m$ and $|A| = 1$. For the case with $|P| = m$ and $|A| = 1$, an arrangement (μ, c) is an equilibrium if (a) there is no *blocking individual*, that is, there is no p_i (or a_j) such that p_i (or a_j) prefers every reactive arrangement w. r. t. c_0^i (or c_j^0) to (μ, c); and (b) there is no *blocking pair*, that is, there are no $p_i \in P$, $a_j \in A$ and $b_j^i \in C_j^i$ such that both p_i and a_j prefer every reactive arrangement w. r. t. b_j^i to (μ, c).

(iii) Similarly define an equilibrium for each of the cases with $|P| = 1$ and $|A| \leq n$. Next define an equilibrium for the case with $|P| = 2$ and $|A| = 2$. Then define an equilibrium for each of the cases with $|P| \leq m$ and $|A| = 2$, and similarly for all the cases with $|P| = 2$ and $|A| \leq n$. Next define an equilibrium for the case with $|P| = |A| = 3$, and so on. Finally define an equilibrium for the case with $|P| = m$ and $|A| = n$.

The following equilibrium concept reflects the asymmetric distribution of power between principals and agents: each principal squeezes his agent to the extent such that if he further squeezes, his agent will "divorce" him.

Let us call a contract c_j^i *achievable* to p_i if there is an equilibrium in which p_i is assigned to a_j with contract c_j^i.

DEFINITION 2.2 A *principal competitive matching equilibrium* (or for short, *principal-equilibrium)* is defined as follows:

(i) For the case $|P| = |A| = 1$, a principal-equilibrium is the principal's most favorable arrangement among all the individually rational arrangements.

(ii) Suppose a principal-equilibrium has been defined for each of the cases with $|P| < m$ and $|A| = 1$. For the case with $|P| = m$ and $|A| = 1$, an arrangement (μ, c) is a principal-equilibrium if (μ, c) is an equilibrium and there is no $p_i \in P$, and $b_\mu^i \in C_\mu^i$ such that b_μ^i is achievable to p_i and p_i prefers every principal-reactive arrangement w. r. t. b_μ^i to (μ, c).

(iii) Similarly define a principal-equilibrium for each of the cases with $|P| = 1$ and $|A| \leq n$. Next define a principal-equilibrium for the case with $|P| = 2$ and $|A| = 2$. Then define a principal-equilibrium for each of the cases with $|P| \leq m$ and $|A| = 2$, and similarly for all the cases with $|P| = 2$ and $|A| \leq n$. Next define a principal-equilibrium for the case with $|P| = |A| = 3$, and so on. Finally define a principal-equilibrium for the case with $|P| = m$ and $|A| = n$.

The following observation is straightforward.

PROPOSITION 2.3 *(i) Consider a market with weak externalities $(P, A, M, \bar{R})$. Let R be the associated direct preference profile of $\bar{R}$. Then an arrangement (μ, c) is an equilibrium if and only if it is individually rational, and there is no blocking pair, i.e., there are no p_i, a_j and b_j^i such that $b_j^i \; R^i \; c_\mu^i$, and $b_j^i \; R_j \; c_j^\mu$.*

A principal-equilibrium is an equilibrium (μ, c) such that there is no p_i and b_μ^i such that b_μ^i is achievable to p_i and $b_\mu^i \; R^i \; c_\mu^i$.

(ii) For a market with no externality, a principal-equilibrium

is an equilibrium such that every p_i (principal) likes this equilibrium at least as well as any other equilibrium that has the same partner assignment as that in this equilibrium.

C. The Classes of Markets

Marriage Model Let P be the set of men (written M), A be the set of women (written W), and C_j^i be a singleton for all i and j. Then $(P, A, C, \mathcal{R})$ reduces to $(M, W, \mathcal{R})$, the classical marriage model of Gale and Shapley (1962). Notice that competitive matching equilibrium reduces to *stable matching* in this model.

NTU Assignment Game For $i, j \neq 0$, let $C_j^i \subset \mathbb{R}_+^2$, and a generic element of C_j^i be $c_j^i = (u_i, v_j)$. Let R^i be represented by the utility function $U^i(c_j^i) = u_i$ for $j \neq 0$, and $U^i(c_0^i) = 0$; R_j by $U_j(c_j^i) = v_j$ for $i \neq 0$, and $U_j(c_j^0) = 0$. (Any possible indifference could be ruled out by introducing a tie-breaking rule.) Then $(P, A, C, \mathcal{R})$ reduces to (P, A, C), the NTU assignment game of Moldovanu (1990).

A special case of NTU assignment game is TU assignment game:

TU Assignment Game For $i, j \neq 0$, let $C_j^i = \{c_j^i = (u_i, v_j) \in \mathbb{R}_+^2: u_i + v_j = \alpha_{ij}$ for some nonnegative fixed number $\alpha_{ij}\}$. Let R^i be represented by the utility function $U^i(c_j^i) = u_i$ for $j \neq 0$, and $U^i(c_0^i) = 0$; R_j by $U_j(c_j^i) = v_j$ for $i \neq 0$, and $U_j(c_j^0) = 0$. Then $(P, A, C, \mathcal{R})$ reduces to (P, A, α) $(\alpha = (\alpha_{ij}))$, the TU assignment game of Shapley and Shubik (1972).

A special case of TU assignment game is auction:

Auction Let $m = 1$ and P be a seller, A be the set of bidders. Let r_0 be the seller's reservation price, r_j be bidder j's reservation price. Let $C_j = \{c_j = (u, v_j) \in \mathbb{R}^2_+: u + v_j = max\ \{0, r_j - r_0\}\}$. If the object is sold to bidder j at price p, then the seller gets utility $u = p - r_0$, bidder j gets utility $v_j = r_j - p$. If the object is not sold, everyone gets zero utility. Then $(P, A, C, \mathcal{R})$ reduces to (P, A, r) $(r = (r_0, r_1, \cdots, r_n)$, the auction model of Vickrey (1961).

Principal-agent model Let $m = n = 1$, P be a principal, and A be an agent. Let C be the set of incentive-compatible and individually rational contracts. Let the preference orders be represented by the indirect expected utility functions defined on the set of contracts. Then $(P, A, C, \mathcal{R})$ reduces to the conventional principal-agent model of Ross (1973) and others.

Multiple principal-agent model Let P be the set of principals, A be the set of agents, and C^i_j be the set of all conceivable (e.g., incentive-compatible) contracts between p_i and a_j. (If there exist externalities, then whether a contract between a pair (a principal and an agent) is incentive compatible depends on other people's contracts. By the revelation principle, we may postulate that each pair only uses incentive compatible contracts (contingent on other people's contracts).) Let the preference orders be represented by the indirect expected utility functions defined on $\mathcal{M}$. Then $(P, A, \mathcal{M}, \bar{R})$ becomes a multiple principal-agent model.

A special case of the multiple principal-agent model is the one-to-one matching version of the *job matching model* of Kelso and Crawford (1982), where P is a set of firms, A is a set of workers, and C_j^i is the set of possible job descriptions (e.g., working hours and payments) between firm i and worker j.

3. EXISTENCE OF EQUILIBRIUM

A. The Case with A Finite Set of Arrangements

I believe that most realistic economic problems are discrete and finite. This is because practically most commodities and even money are indivisible, and people do not care about 1/2 gain of rice or 1/10 penny due to transaction cost.

Let $(P, A, \mathcal{M}, \bar{R})$ be a market with weak externalities and R be the associated direct preference profile of $\bar{R}$. We will use "R-" to mean "in terms of direct preference", and call a contract *acceptable* to a person if he R-prefers this contract to being unmatched and *unacceptable* otherwise.

THEOREM 3.1 *In any market with weak externalities and with a finite set of arrangements, there always exists an equilibrium.*

PROOF: Let $(P, A, \mathcal{M}, \bar{R})$ be a market with weak externalities and with a finite set of arrangements. Let R be the associated direct preference profile of $\bar{R}$. A revised version of the (Gale-Shapley) deferred acceptance algorithm will be used here to construct an equilibrium.

Let us call the persons from P principals and the persons from A agents. (This will be justified by the proof of Theorem 3.4.) The algorithm goes as follows.

At step one, each principal proposes his R-top contract to the relevant agent. Each agent rejects any unacceptable contract, and each agent who receives more than one proposal rejects all but his R-top one among these. Any principal whose proposed contract is not rejected at this point is kept "assigned" to the relevant agent with the proposed contract.

At any step, any principal who was rejected at the previous step proposes his next R-preferred contract (i.e., his R-top contract among those that have not yet been rejected), as long as there remains an acceptable contract that he has not yet proposed (otherwise he issues no further proposals). Each agent receiving proposals rejects any unacceptable contract, and also rejects all but his R-top contract among the set consisting of new proposals together with any contract he may have kept assigned from the previous step.

The procedure stops after any step in which no principal is rejected. At this point, match each principal to the relevant agent with the contract to which they kept assigned. Agents who did not receive any acceptable proposal, and principals whose all proposals were rejected will not be assigned.

The algorithm must eventually stop because there are only finite principals and finite contracts (the set of arrangements is finite), and no principal proposes any contract more than once. Clearly it produces an individually rational arrangement (μ, c).

It remains to show that (μ, c) is not blocked by any pair. By contradiction, suppose there exist p_i, a_j, and $\bar{c}^i_j$ such that $\bar{c}^i_j$ R^i c^i_μ, and $\bar{c}^i_j$ R_j c^μ_j. Then $\bar{c}^i_j$ is acceptable to p_i, and p_i must have proposed $\bar{c}^i_j$ to a_j before proposing c^i_μ to $\mu(p_i)$. Since p_i is not assigned to a_j with $\bar{c}^i_j$ when the algorithm stopped, a_j must have rejected $\bar{c}^i_j$ in favor of c^μ_j. Thus c^μ_j R_j $\bar{c}^i_j$, a contradiction.

Q.E.D.

COROLLARY 3.2 *(Gale and Shapley) There exists a stable matching for every marriage market.*

LEMMA 3.3 *In a market with weak externalities, no principal's achievable contract is ever rejected in the revised version of the deferred acceptance algorithm.*

PROOF: Let $(P, A, M, \bar{R})$ be a market with weak externalities and R be the associated direct preference profile of $\bar{R}$. Consider the revised version of the deferred acceptance algorithm in Theorem 3.1. By induction, assume that up to a given step in the procedure no achievable contract has yet been rejected. If there will be no rejection of any achievable contract from this point on, then we are done. Suppose, at this step, some a_j rejects some c^i_j. If c^i_j is not acceptable to a_j, then c^i_j is not achievable and we are done. Suppose a_j rejects c^i_j in favor of some $\bar{c}^k_j$. We want to show that c^i_j is not achievable. By contradiction, suppose p_i is assigned to a_j with c^i_j under some equilibrium (μ, c). Since no achievable contract proposed by p_k has been rejected at this step and c^k_μ is achievable, p_k has not proposed c^k_μ yet at this step (otherwise p_k is kept "assigned" with c^k_μ and can not propose any other proposal like $\bar{c}^k_j$ at this step), so $\bar{c}^k_j\ R^k\ c^k_\mu$. Also $\bar{c}^k_j\ R_j\ c^i_j$. Thus $\bar{c}^k_j$ blocks c^i_j, a contradiction. Q.E.D.

THEOREM 3.4 *Consider any market with weak externalities. If the set of arrangements is finite, then there always exists a principal-equilibrium.*

PROOF: We show that the equilibrium (μ, c) obtained by the revised version of the deferred acceptance algorithm in Theorem 3.1 is a principal-equilibrium. By contradiction, suppose there exist p_i and b^i_μ such that b^i_μ is achievable to p_i and p_i prefers b^i_μ to c^i_μ. Then p_i must have proposed b^i_μ before proposing c^i_μ to $\mu(p_i)$. Since p_i is not assigned to $\mu(p_i)$ with b^i_μ when the algorithm stopped, $\mu(p_i)$ must have rejected b^i_μ, a contradiction to Lemma 3.3 since b^i_μ is achievable. Q.E.D.

PROPOSITION 3.5 *There does not always exist a principal-equilibrium.*

PROOF: The proof is done by one example. Let $P = \{p_1,\ p_2\}$, $A = \{a_1,\ a_2\}$, $C^1_1 = \{a,\ b\}$, $C^1_2 = \{d\}$, $C^2_1 = \{c\}$, $C^2_2 = \{e,\ f\}$, and $M = \{ae,\ af,\ be,\ bf,\ cd\}$, where ae means that a_1 is matched to p_1 with contract a, and a_2 is matched to p_2 with contract e, and so on. The preference profile $\bar{R}$ is as follows:

$\overline{R}^1$: *af be ae bf cd*,

$\overline{R}^2$: *be af ae bf cd*,

$\overline{R}_1$: *be af ae bf cd*,

$\overline{R}_2$: *af be ae bf cd*.

There are two equilibria: *be* and *af*. There is no principal -equilibrium. Notice that *be* is blocked by p_1 with *a* since the principal-reactive arrangement w. r. t. *a* is *af*. Also *af* is blocked by p_2 with *e* since the principal-reactive arrangement w. r. t. *e* is *be*. Q.E.D.

PROPOSITION 3.6 *There does not always exist an equilibrium.*

PROOF: The proof is done by one example. The example is the same as the one in Proposition 3.5 except that the preference profile $\overline{R}$ is as follows:

$\overline{R}^1 = \overline{R}_1$: *be af ae bf cd*,

$\overline{R}^2 = \overline{R}_2$: *ae af be bf cd*.

Notice that *ae* is blocked by p_1 and a_1 with *b* since the reactive arrangement w. r. t. *b* is *be*, *be* is blocked by p_2 and a_2 with *f* since the reactive arrangement w. r. t. *f* is *af*, *af* is blocked by p_1 and a_1 with *b* since the reactive arrangement w. r. t. *b* is *be*. Clearly *cd* and *bf* are blocked by p_2 and a_2 with *e*. So there is no equilibrium. Q.E.D.

B. The Case with An Infinite Set of Arrangements

When there is an infinite set of arrangements, we can not guarantee existence of equilibrium any more even for the case without externality. But under certain regularity conditions, the set of equilibria for NTU assignment game is not empty (see Kaneko (1985) and Moldovanu (1990)). In particular, there always exists an equilibrium for any TU assignment game (Shapley and Shubik (1972)).

PROPOSITION 3.7 *There does not always exist an equilibrium if the set of arrangements is infinite (even for the case without externality).*

PROOF: The proof is done by an example. Let $P = \{p_1, p_2\}$, $A = \{a_1, a_2\}$, $C_1^1 = \{c_1^1\}$, $C_2^1 = \{c_2^1(k): k=1, 2, \cdots\}$, $C_1^2 = \{c_1^2\}$, $C_2^2 = \{c_2^2(k): k=1, 2, \cdots\}$, and

$$c_2^1(k) \quad R^1 \quad c_1^1 \quad R^1 \quad p_1 \qquad \text{for all } k,$$
$$c_2^2(k) \quad R^2 \quad c_1^2 \quad R^2 \quad p_2 \qquad \text{for all } k,$$
$$c_1^1 \quad R_1 \quad c_1^2 \quad R_1 \quad a_1,$$
$$c_2^1(k+1) \quad R_2 \quad c_2^2(k) \quad R_2 \quad c_2^1(k) \quad R_2 \quad a_2 \qquad \text{for } k = 1, 2, \cdots.$$

Clearly the two principals will compete for agent two in the see-saw way for infinite rounds. There is no equilibrium.

Q.E.D.

For any TU assignment game, however, there always exists an equilibrium.

THEOREM 3.8 *(Shapley and Shubik) There always exists an equilibrium for any TU assignment game.*

PROOF: See Shapley and Shubik (1972). Q.E.D.

Since auction is a special TU assignment game, we have the following corollary:

COROLLARY 3.9 *For any auction, there always exists an equilibrium.*

4. THE PROPERTIES OF EQUILIBRIUM

In this section, we discuss the relationships between equilibrium (also principal-equilibrium) and the notions of *principal-optimality* and *core* (defined below) as well as Pareto optimality.

Let CME be the set of competitive matching equilibria, and PCME be the set of principal competitive matching equilibria.

A. Principal-Optimality and Polarization

For a given market $(P, A, M, \bar{R})$, an equilibrium (μ, c) is said to be *principal-optimal* if every p_i likes (μ, c) at least as well as any other equilibrium. If the principal-optimal equilibrium is the worst equilibrium for every a_j among all equilibria, then it is said to be *polarized*. Clearly every principal-optimal equilibrium is a principal-equilibrium.

PROPOSITION 4.1 *(i) There does not always exist a* *principal-optimal equilibrium (even for the case with weak externalities and with a finite set of arrangements).*

(ii) Polarization does not hold in general.

PROOF: (i) The proof is done by the following example: $P = \{p_1, p_2\}$, $A = \{a_1, a_2\}$, $C_1^1 = \{a\}$, $C_2^1 = \{c\}$, $C_1^2 = \{b\}$, $C_2^2 = \{d, e\}$, everyone prefers *ad*, *ae*, *bc* to all other arrangements, and

$\bar{R}^1$: *ae ad bc*,

$\bar{R}^2$: *ad ae bc*,

$\bar{R}_1$: *ae ad bc*,

$\bar{R}_2$: *ae ad bc*.

Notice that this is the case with weak externalities. It is easy to check that CME = {*ae*, *ad*}, PCME = {*ad*} and there is no principal-optimal equilibrium.

(ii) The proof is done by the following example: $P = \{p_1, p_2\}$, $A = \{a_1, a_2\}$, $C_1^1 = \{a, b, h\}$, $C_2^1 = \{d\}$, $C_1^2 = \{c\}$, $C_2^2 = \{e, f, g\}$, everyone prefers *ae*, *af*, *ag*, *be*, *bf*, *bg*, *he*, *hf*, *hg* to all other arrangements, and

$\bar{R}^1 = \bar{R}_1$: *hf hg bf ae be bg he af ag*,

$\bar{R}^2 = \bar{R}_2$: *he bg bf ae af ag hf hg be*.

It is easy to check that CME = {*ae*, *bf*}, and *bf* is both principal-optimal and agent-optimal. Q.E.D.

However, the following is true:

THEOREM 4.2 *In any market with no externality and with a finite set of arrangements, there always exists a* *principal-optimal equilibrium.*

PROOF: By Lemma 3.3, it is easy to check that the equilibrium obtained by the revised version of the deferred acceptance algorithm in Theorem 3.1 is principal-optimal. Q.E.D.

Even in a market without externality, a principal-equilibrium may be not principal-optimal as the following example shows. It follows that the principal-optimal equilibrium is unique, while principal-equilibrium is not unique in general.

Example Let $P = \{p_1, p_2\}$, $A = \{a_1, a_2\}$, $C_1^1 = \{a\}$, $C_2^1 = \{c\}$, $C_1^2 = \{b\}$, $C_2^2 = \{d\}$, and

R^1: a c p_1,

R^2: d b p_2,

R_1: b a a_1,

R_2: c d a_2.

It is easy to check that POCME = $\{ad\}$, while PCME = $\{ad, bc\}$.

B. Coalition-Stability: Core

Given a market $(P, A, M, \bar{R})$, an arrangement (μ, c) is said to be in the *core* if there is no coalition T and an arrangement $(\lambda, b)_T$ within T such that every member of T prefers every arrangement, that agrees with $(\lambda, b)_T$ within T and agrees with an arrangement within the residual market, to (μ, c).

PROPOSITION 4.3 *In general, PCME (hence CME) is not contained* *in the Core.*

PROOF: It is easy to check that in the example in the proof of part (ii) of Proposition 4.1, $ae \in$ PCME, but $ae \notin$ Core. Q.E.D.

However, the following is true:

THEOREM 4.4 *For any market with weak externalities, the set of equilibria equals the core, and the set of principal-equilibria is contained in but not necessarily equal to the core.*

PROOF: Let $(P, A, M, \bar{R})$ be a market with weak externalities and R be the associated direct preference profile of $\bar{R}$. Clearly the core is contained in the set of equilibria. Suppose (μ, c) is an equilibrium but not in the core. Then there exists $(\bar{\mu}, \bar{c})$ and a coalition T such that each member of T prefers $(\bar{\mu}, \bar{c})$ to (μ, c). Then there must exist a pair (p_i, a_j) in T such that $\bar{\mu}(p_i) = a_j$, $\bar{c}^i_j \; R^i \; c^i_\mu$, and $\bar{c}^i_j \; R_j \; c^\mu_j$ since (μ, c) is individually rational. So (p_i, a_j) with $\bar{c}^i_j$ blocks (μ, c) and (μ, c) is not an equilibrium, a contradiction.

The second part of the theorem follows from the definition of principal-equilibrium and the last example in this section. Q.E.D.

C. Efficiency: Pareto Optimality

PROPOSITION 4.5 *A principal-equilibrium (hence an equilibrium)* *is not necessarily Pareto optimal in general.*

PROOF: In the example in the proof of part (ii) of Proposition 4.1, *ae* is a principal-equilibrium, but it is not Pareto optimal since it is Pareto dominated by *bf*. Q.E.D.

However, the following is true:

THEOREM 4.6 *(i) If CME is nonempty, then CME always contains a Pareto optimal arrangement.*

(ii) If PCME is nonempty, then PCME always contains a Pareto optimal arrangement.

PROOF: (i) The result follows from the finiteness of the market and the fact that if $(\mu, c) \in$ CME and (λ, b) Pareto dominates (μ, c), then $(\lambda, b) \in$ CME.

(ii) The proof is the same as that in (i). Q.E.D.

In the example in the proof of part (ii) of Proposition 4.1, *ae* is a principal-equilibrium and *bf* Pareto dominates *ae*. It follows that *bf* is a principal-equilibrium. Furthermore, *bf* is Pareto optimal.

In particular, from Theorem 4.4, we have the following corollary:

COROLLARY 4.7 *For any market with weak externalities, every equilibrium (hence every principal-equilibrium) is Pareto optimal.*

D. *Summary: Logical Relationships*

Let POCME be the set of principal-optimal competitive matching equilibria, PO be the set of Pareto optimal arrangements.

PROPOSITION 4.8 *(i) POCME $\neq \phi$ if the set of arrangements is finite and there is no externality.*

(ii) PCME $\cap$ PO $\neq \phi$ if the set of arrangements is finite and there are weak externalities (only) or no externality.

(iii) CME $\cap$ PO $\neq \phi$ if the set of arrangements is finite and there are weak externalities (only) or no externality.

(iv) CME = Core if there are weak externalities (only) or no externality.

(v) POCME $\subseteq$ PCME $\subseteq$ CME always hold.

(vi) In general, POCME, PCME and CME could be empty, and an equilibrium may not be Pareto optimal.

The following example shows that an equilibrium is not necessarily a principal-equilibrium (even if there is no externality). It also shows the difference between conventional (single) principal-agent models and multiple principal-agent models in terms of allocations and welfare.

Example Let $P = \{p_1, p_2\}$, $A = \{a_1, a_2\}$, $C_1^1 = \{c_1^1(1), c_1^1(2)\}$, $C_2^1 = \{c_2^1(1), c_2^1(2)\}$, $C_1^2 = \{c_1^2\}$, $C_2^2 = \{c_2^2(1), c_2^2(2)\}$, and

R^1: $c_2^1(1)$ $c_2^1(2)$ $c_1^1(1)$ $c_1^1(2)$ p_1,

R^2: $c_2^2(1)$ $c_2^2(2)$ c_1^2 p_2,

R_1: c_1^2 $c_1^1(2)$ $c_1^1(1)$ a_1,

R_2: $c_2^2(2)$ $c_2^1(2)$ $c_2^2(1)$ $c_2^1(1)$ a_2.

There are two equilibria: $(c_1^1(1),\ c_2^2(2))$ and $(c_1^1(2),\ c_2^2(2))$. Clearly $(c_1^1(2),\ c_2^2(2))$ is not a principal-equilibrium. In fact, it is easy to check that $(c_1^1(1),\ c_2^2(2))$ is the only principal -equilibrium. If $(p_2,\ a_2)$ is isolated, then $c_2^2(1)$ will be the only principal-equilibrium in the (single) principal-agent setting. Clearly p_2 is strictly worse off because of the competition from p_1.

5. AN EXAMPLE OF APPLICATION: COMPETITIVE SHARECROPPING

In this section, we take sharecropping as an example to illustrate how to set a multiple principle-agent model into the general framework of this paper. The limitations of conventional (single) principal-agent models are addressed. Notice that a joint stock cooperation could be seen as a form of sharecropping. Also a society consisting of a government financed by taxes and the public is in the form of sharecropping.

Consistent and strong evidence has shown that in backward economies, the dominant form of contracts between a landowner and a tenant is sharecropping, and the dominant sharing rule is 50%-50%.[(4)] Hurwicz and Shapiro (1978) considered the following problem: Suppose outputs are publicly observable, but labor inputs are not verifiable. Knowing only the form of production function and the tenant's (agent's) utility function—not the exact values of their parameters, the landowner (principal) seeks to choose a contract which maximizes the minimum (over all possible parameter values) of a quantity related to his residual gain. They showed that in a broad class of cases the only such contract is the 50%-50% sharing rule.

Although they considered a deterministic principal-agent model, their work could be generalized into a stochastic setting. To see how it works, consider the following simple stochastic linear-quadratic model. There is a risk-neutral landowner and a tenant with utility function $u(c, \ell) = c - \gamma\ell^2/2$, where c is his consumption, ℓ is his labor input. They both have the same prior about a constant return to scale production: $y = k_i\ell$ with probability π_i, $i = 1, 2$, where k_1 and k_2 are two positive constants, and y is output. Assume that the realizations of shocks are publicly observable.(5) Let α_i be the tenant's contingent share of output if event i happens.

Given a contract (α_1, α_2), the tenant chooses a contingent labor input plan (ℓ_1, ℓ_2) to maximize

$$Eu = \pi_1(\alpha_1 k_1 \ell_1 - \gamma\ell_1^2/2) + \pi_2(\alpha_2 k_2 \ell_2 - \gamma\ell_2^2/2).$$

The optimal solution is $\ell_i = \alpha_i k_i/\gamma$, $i = 1, 2$. Then the landowner chooses (α_1, α_2) to maximize the expected residual $\sum_{i=1}^{2}\pi_i(1 - \alpha_i)k_i\alpha_i k_i/\gamma$. The optimal sharing rule is $\alpha_i = 50\%$ for i = 1, 2.

In reality there exist non-50%-50% sharing rules and the share of the tenant working in good land tends to be higher than that in bad land in some economies (see Bardhan (1981)). In equilibrium, the following multiple principal-agent model generates a 50% or more share for tenants, and the share of the tenant working in good land is greater than or equal to that in bad land.

Suppose there are two landowners p_1 and p_2, two tenants a_1 and a_2. Landowner p_i's technology is $y = k''\ell$ with probability π_i, $y = k'\ell$ with probability $1 - \pi_i$, where $k'' > k' > 0$, and $1 > \pi_1 > \pi_2 > 0$. Tenant a_i has utility function $u_i(c, \ell) = c - \gamma_i \ell^2/2$, and $\gamma_2 > \gamma_1 > 0$. Let $k_i = \pi_i k'' + (1 - \pi_i)k'$. Then $k_1 > k_2$. We may say that landowner 1 (2) has good (bad) land, tenant 1 (2) is diligent (lazy).

Suppose technologies are common knowledge but shocks are not observable (hence labor inputs can not be inferred from the observations of outputs), tenants' preferences are common knowledge but labor inputs are not verifiable. This is the case without adverse selection but with moral hazard. By the certainty equivalence principle, we can consider only the equivalent deterministic case. Also due to the empirical evidence, we consider only the contracts in the form of sharecropping.[6] Assume that one landowner hires one tenant, and an unmatched person gets zero utility.

Consider a pair consisting of a landowner and a tenant. If the pair is isolated, then the optimal sharing rule is 50%-50%. On the other hand, if all the four players are linked by market, then the following proposition shows that in equilibrium, the diligent tenant works in the good land and has a 50% or more share of output, the lazy tenant works in the bad land and shares 50% of output. For such a class of economies, we cannot expect to observe a tenant's share being less than 50%; instead we will observe tenant's shares being greater than 50% on average.

PROPOSITION 5.1 *In this competitive sharecropping setting, there exists a unique principal-equilibrium:*

$$\mu(p_1) = a_1,$$
$$\mu(p_2) = a_2,$$
$$\alpha_2^* = 1/2,$$

and

$$\alpha_1^* = \begin{cases} 1/2 & \text{if } \bar{\alpha} < 1/2 \\ \bar{\alpha} & \text{if } \bar{\alpha} \geq 1/2 \end{cases}$$

where $\bar{\alpha} = (1 + (1 - \gamma_1/\gamma_2)^{1/2})k_2/(2k_1)$.

PROOF: It is easy to check that if a_j works for p_i with a share (of output) α, then the optimal labor input is $\ell = \alpha k_i/\gamma_j$; hence the expected utility of a_j is $\alpha^2 k_i^2/(2\gamma_j)$, and the expected residual of p_i is $(1 - \alpha)\alpha k_i^2/\gamma_j$.

Clearly the arrangement stated in the proposition is individually rational since $0 < \bar{\alpha} < 1$. It remains to check pairwise blocking and principal's blocking. If p_2 hires a_2, he can get maximum residual $k_2^2/(4\gamma_2)$. So he is willing to offer a_1 a share at most $\hat{\alpha}$, where $\hat{\alpha}$ is the largest number satisfying $(1 - \alpha)\alpha k_2^2/\gamma_1 \geq k_2^2/(4\gamma_2)$, or $(1 - \alpha)\alpha \geq \gamma_1/(4\gamma_2)$. It turns out that $\hat{\alpha} = (1 + (1 - \gamma_1/\gamma_2)^{1/2})/2$. By sharing $\hat{\alpha}$ with p_2, a_1 can get $\hat{\alpha}^2 k_2^2/(2\gamma_1)$. Thus to attract a_1 to work for him, p_1 must choose α_1 such that $\alpha_1^2 k_1^2/(2\gamma_1) \geq \hat{\alpha}^2 k_2^2/(2\gamma_1)$, or $\alpha_1 \geq \hat{\alpha}k_2/k_1$. Notice $1/2 \leq \alpha_1^* \leq \hat{\alpha}$ since $k_1 > k_2$ and $\gamma_1 < \gamma_2$; hence $(1 - \alpha_1^*)\alpha_1^* \geq \gamma_1/(4\gamma_2)$, which implies that p_1 prefers sharing $1 - \alpha_1^*$ with a_1 to being assigned to a_2 or he is indifferent. Clearly, α_1^* is the only optimal choice for p_1 among all the stable arrangements (equilibria), so is α_2^*

for p_2. Therefore, the arrangement stated in the proposition is the unique principal-equilibrium. Q.E.D.

A numerical example Let $k_1 = 3/2$, $k_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 10$. Then $\alpha_1 = 65\%$ and $\alpha_2 = 50\%$.

Notice that in Proposition 5.1, $\bar{\alpha}$ is positively correlated with both k_2/k_1 and γ_2/γ_1. This implies that the more relatively productive the second landowner's land is, and/or the more relatively diligent the first tenant is, the higher share the first tenant may have. This reward scheme is a 'tournament' among all principals and agents. In contrast, in a one principal-many agents model, a contest reward scheme is a tournament among all agents only (see Green and Stokey (1983)).

6. CONCLUDING REMARKS: GENERALIZATIONS

(i) Although all the analyses have been made for the case with variable partner assignment, all the theorems and propositions stated in this paper are also valid for the case with an (exogenously) fixed partner assignment (i.e., $\mathcal{M} = \{(\mu, c): \mu = \mu^*$ for a fixed $\mu^*\}$). An exception is that in some cases (e.g., the Myerson example) externalities exist, hence existence of principal-equilibrium may fail for a fixed partner assignment. Externalities may be internalized by choosing a partner assignment endogenously and thus existence of principal-equilibrium may be restored.

(ii) The framework here could be generalized to incorporate one-to-many or (even) many-to-many two-sided matchings (e.g., one firm may hire more than one worker, and one worker may work for more than one firm). Furthermore, two-sided matching models are very restrictive. It turns out that we could significantly generalize this framework to a broad model where, (1) an arbitrary permissible coalition structure is given, and (2) the feasibility of each coalition's joint strategy is contingent on the joint strategy of the residual coalition. I call this broad model *social partitioning game*, which is a generalization of the standard normal form game, the *partitioning game* of Kaneko and Wooders (1983) and the *social coalitional game* of Ichiishi (1981). The solution concept here, however, is different from those. In Ichiishi's model, coalitional structure generates no externality, and "strong Nash equilibrium" is used as solution concept that assumes players are extremely myopic or careless. The Kaneko-Wooders model is in characteristic function form and hence there is no room for externalities, and core is used as solution concept that assumes players are extremely prudent. The solution concept here assumes that players are rational in the sense when they form a coalition and take a joint action, they neither conjecture that the outsiders will not change their strategies, nor conjecture that the outsiders will do everything, but they conjecture that the outsiders will play equilibrium strategies within themselves.

Social Partitioning Game Let I be a finite set of players. A nonempty subset of I is called a coalition. For each coalition S, let Π^S be a fixed collection of partitions of S, X^S be the strategy space of S, and f^S: $X^{I\setminus S} \twoheadrightarrow X^S$ be a *feasibility correspondence*. For $\pi \in \Pi^I$, let $X^\pi = \times_{S\in\pi} X^S$. An *arrangement* is a pair (π, x), where $\pi \in \Pi^I$, $x \in X^\pi$, and $x^S \in f^S(x)$ for all $S \in \pi$. Let Γ be the set of all arrangements. For each $i \in I$, his preference order is a binary relation R^i on Γ. Let P^i be the asymmetric part of R^i. Let $\Pi = (\Pi^S)_{S\subseteq I}$, $X = (X^S)_{S\subseteq I}$, $f = (f^S)_{S\subseteq I}$, $R = (R^i)_{i\in I}$.

A *social partitioning game* is a tuple (I, Π, X, f, R). We call the number of players in a game $\#I$ the *size* of the game.

Compare this game with the standard normal form game, there are two new elements here: "coalition structure" Π and "strategy feasibility" f, and the latter is a special case of the former in which each permissible coalition is a single person and each person's strategy space is independent of other people's strategies.

Given a game (I, Π, X, f, R). For a coalition S and an arrangement within the coalition $(\pi^{*S}, x^{*S}) \in \Pi^S \times X^S$, let $(I, \Pi, X, f, R)|(\pi^{*S}, x^{*S}) \stackrel{\text{def}}{=} (\bar{I}, \bar{\Pi}, \bar{X}, \bar{f}, \bar{R})$ be the subgame, where $\bar{I} = I\setminus S$, $\bar{\Pi} = (\Pi^T)_{T\subseteq\bar{I}}$, $\bar{X} = (X^T)_{T\subseteq\bar{I}}$, $\bar{f} = (\bar{f}^T)_{T\subseteq\bar{I}}$ with $\bar{f}^T(\cdot)$: $X^{\bar{I}\setminus T} \twoheadrightarrow X^T$ and $\bar{f}^T(\cdot) = f^T(\cdot, x^{*S})$, and $\bar{R} = (\bar{R}^i)_{i\in\bar{I}}$ with $(\pi^{I\setminus S}, x^{I\setminus S})$ $\bar{R}^i$ $(\pi'^{I\setminus S}, x'^{I\setminus S})$ iff $(\pi^{I\setminus S} \cup \pi^{*S}, (x^{I\setminus S}, x^{*S}))$ R^i $(\pi'^{I\setminus S} \cup \pi^{*S}, (x'^{I\setminus S}, x^{*S}))$ for all arrangements $(\pi^{I\setminus S}, x^{I\setminus S})$, $(\pi'^{I\setminus S}, x'^{I\setminus S})$ in the subgame.

We define equilibrium inductively:

(i) For any subgame with minimum size, an *equilibrium* is a Pareto optimal arrangement within this subgame.

(ii) Suppose an equilibrium has been defined for all subgames with size smaller than n. For a game (I, Π, X, f, R) with n players, an arrangement (π, x) is an *equilibrium* if there are no $\hat{\pi} \in \Pi^I$, $S \in \hat{\pi}$ and $(\pi*^S, x*^S) \in \Pi^S \times X^S$ with $x*^S \in f^S(x^{I\setminus S})$, such that for all equilibrium $(\pi'^{I\setminus S}, x'^{I\setminus S})$ in the subgame $(I, \Pi, X, f, R)|(\pi*^S, x*^S)$, $(\pi'^{I\setminus S} \cup \pi*^S, (x'^{I\setminus S}, x*^S))\ P^i\ (\pi, x)$ for all $i \in S$.

(iii) The framework here is at a level of abstraction in which the informational/contractual structure used in most of the literature on principal-agent problems is not explicit. Nevertheless, the informational/contractual structure couldbe incorporated into this framework as the competitive sharecropping example in the last section has shown.

NOTES

(1) In a repeated game setting, we typically assume that there exist unobservable uncertainties or unknown parameters in the production function so that the private information cannot be perfectly revealed by a stage-play. In a one-shot (turnpike) game setting, however, this may not be necessary.

(2) This example was suggested by Edward Green.

(3) The "strict preference order" assumption could be relaxed by introducing a tie-breaking rule. By defining preference order this way, we allow externalities to exist.

(4) See the excellent survey in India in Bardhan (1981).

(5) Since the landowner does not know the parameters in the production function, he can not infer the type or the action of the tenant from the observation of output even though the shocks are publicly observable.

(6) This could also be justified partially by limited liability.

REFERENCES

Bardhan, Pranab K. (1981): *Land, Labor, and Rural Poverty: Essays in Development Economics*. Oxford, London: Oxford University Press, 112-126.

Bernheim, Douglas B. and Michael D. Whinston (1986): "Common Agency," *Econometrica*, 54, 923-942.

Gale, David and Lloyd Shapley (1962): "College Admission and the Stability of Marriage," *American Mathematical Monthly*, 69, 9-15.

Green, Jerry and Nancy Stokey (1983): "A Comparison of Tournaments and Contracts," *Journal of Political Economy*, 91, 349-364.

Hurwicz, Leonid, and Leonard Shapiro (1978): "Incentive Structure Maximizing Residual Gain under Incomplete Information," *Bell Journal of Economics*, 9, 180-191.

Ichiishi, Tatsuro (1981): "A Social Coalitional Equilibrium Existence Lemma," *Econometrica*, 49, 369-377.

Kaneko, Mamoru (1985): "The Central Assignment Game and the Assignment Markets," *Journal of Mathematical Economics*, 10, 205-32.

Kaneko, Mamoru, and Myrna H. Wooders (1983): "Cores of Partitioning Games," Cowles Foundation Paper No. 566.

Kelso, A., and Crawford, V. (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica,* 50, 1483-1504.

Moldovanu, B. (1990): "Stable Bargaining Equilibrium for Assignment Games without Side Payments," *International Journal of Game Theory*, 19, 177-90.

Myerson, Roger B. (1982): "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," *Journal of Mathematical Economics*, 10, 67-81.

Page, F. H. Jr. (1991): "Optimal Contract Mechanisms for Principal-Agent Problems with Moral Hazard and Adverse Selection," *Economic Theory,* 1, 323-338.

Ross, S. (1973): "The Economic Theory of Agency: the Principal's Problem," *American Economic Review*, 63, 134-139.

Roth, Alvin E. (1984): "Stability and Polarization of Interests in Job Matching," *Econometrica*, 52, 47-57.

Roth, Alvin E. (1985): "Conflict and Coincidence of Interest in Job Matching," *Mathematical Operation Research,* 10, 379-389.

Roth, Alvin E. and M. O. Sotomayor (1990): *Two Sided Matching—A Study in Game-Theoretic Modeling and Analysis*. Cambridge: Cambridge University Press.

Sasaki, Hiroo and Manabu Toda (1986): "Marriage Problem Reconsidered: Externality and Stability," *Mimeo,* Department of Economics, University of Rochester.

Shapley, Lloyd S., and Martin Shubik (1972): "The Assignment Game I: the Core," *International Journal of Game Theory*, 1, 111-30.

Vickrey, W. (1981): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8-37.