APPROXIMATION OF CONTRACTIBLE VALUED

CORRESPONDENCES BY FUNCTIONS

by

Andrew McLennan

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, MN 55455

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Andrew McLennan*
Department of Economics
University of Minnesota
1035 Management and Economics
271 19th Avenue South
Minneapolis, MN 55455

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Abstract

Let X and Y be absolute neighborhood retracts (this is a large class of spaces) with X compact, and let $F : X \to Y$ be an upper hemicontinuous correspondence whose values are compact and contractible. It is shown that any neighborhood of the graph of F contains the graph of a continuous function $f : X \to Y$. The relevance of this result to fixed point theory is indicated. It is also shown that if X is "locally infinite," then F can be approximated in the stronger sense of the graph of f being close to the graph of F and every point in the graph of F being close to some point in the graph of f. A conjectured generalization of the main result is stated.

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1. Introduction

The purpose of this note is to put on record Theorem 1 below, which is a generalization of a Theorem of Mas-Colell (1974) (Proposition 2 below). After stating this result this section describes its application to fixed point theory and explains the generality of the class of spaces it considers. The proof is in §2. In §3 we show that approximations in a stronger sense exist provided the domain of the correspondence is "locally infinite." A conjectured generalization of Theorem 1 is stated in §4.

The terms occurring in the statement of Theorem 1 have the following definitions. If X is a topological space and $A \subset X$, a continuous function $r : X \to A$ with $r(a) = a$ for all $a \in A$ is a *retraction*, and if such a function exists we say that A is a *retract* of X. A metric space X is an *absolute neighborhood retract* (ANR) if $i(X)$ is a retract of a neighborhood of itself whenever $i : X \to Y$ is an embedding (homeomorphism onto its image) of X in a metric space Y. (As we explain below, "most" well behaved spaces are ANR's.) A *correspondence* $F : X \to Y$ is a nonempty valued set valued mapping. The *graph* of F is $Gr(F) = \{(x, y) \in X \times Y | y \in F(x)\}$. For topological spaces X and Y a correspondence $F : X \to Y$ is *upper hemicontinuous* if for every $x \in X$ and every neighborhood V of $F(x)$ there is a neighborhood U of x such that $F(x') \subset V$ for all $x' \in U$. A topological space X is *contractible* if there is a continuous function $c : X \times [0, 1] \to X$ with $c(x, 0) = x$ for all $x \in X$ and $c(\cdot, 1)$ a constant map.

Theorem 1: Let X and Y be ANR's with X compact. Suppose $D \subset U \subset X$ with D compact and U open. If $F : U \to Y$ is an upper hemicontinuous correspondence whose values $F(x)$ are compact and contractible, and $W \subset U \times Y$ is a neighborhood of $Gr(F)$, then there is a continuous function $f : D \to Y$ with $Gr(f) \subset W$.

Of particular interest is the case $D = U = X$, in which case we find that any neighborhood of the graph of a upper hemicontinuous compact contractible valued (henceforth u.h.c.c.c.v.) correspondence from X to Y contains the graph of a continuous function from X to Y. This allows fixed point theorems to be extended to the case of contractible valued correspondences.

Definition: A topological space X has *the fixed point property* if every continuous function $f : X \to X$ has a fixed point.

Corollary: If a compact ANR X has the fixed point property, then every u.h.c.c.c.v. correspondence from X to itself has a fixed point.

Proof: Suppose that $Gr(F)$ does not intersect the diagonal in $X \times X$. Then $Gr(F)$ has a neighborhood that does not intersect the diagonal, and this neighborhood contains the graph of a continuous function whose graph cannot intersect the diagonal, a contradiction of the hypothesis. ∎

The theory of the Lefschetz fixed point index can be used to show that any compact contractible ANR has the fixed point property (c.f. McLennan (1989c)), and in conjunction with the Corollary this implies that any u.h.c.c.c.v. correspondence from a compact contractible ANR to itself has a fixed point. From the point of economics the Eilenberg Montgomery theorem - any upper hemicontinuous acyclic valued correspondence from a compact acyclic ANR to itself has a fixed point - is only slightly more general. In McLennan (1989a) it is shown that the set of sequential equilibria (Kreps and Wilson (1982)) of an extensive game is the set of fixed points of an u.h.c.c.c.v. correspondence from a space homeomorphic to a Euclidean ball to itself.

The class of compact ANR's is very large and includes virtually all compact spaces that arise in mathematical economics. The following characterization gives some indication of this.

Proposition 1: A metric space X is an ANR if it is homeomorphic to a retract of an open subset of a convex set in a locally convex linear space, and an ANR is necessarily homeomorphic to a retract of an open subset of a convex set in a normed linear space.

Proof: E.g. Borsuk (1967, IV.3.1). ∎

Combining the Whitney embedding theorem (e.g. Hirsch (1976, Th. 3.5, p. 24, and remarks on p. 27)), the tubular neighborhood theorem (Hirsch (1976, Th. 5.1, p. 109)), and the collaring theorem (Hirsch (1976, Th. 6.1, p. 113)), one can show that any (paracompact Hausdorff) manifold with boundary is a retract of an open subset of some $\mathbb{R}^k$, hence an ANR. A polyhedron (as defined below) is a retract of an open subset of some $\mathbb{R}^k$ by virtue of Whitehead's regular neighborhood theorem (e.g. Hudson (1969, Th. 2.11, p. 55)). The space of probability measures on a compact space, with the weak* topology, is an ANR, as is the unit ball, with the weak* topology, in the dual of a Banach space, by Proposition 1. The Banach-Alaoglu Theorem asserts that the unit ball in the dual of a Banach space is weak* compact (e.g. Dunford and Schwartz (1958, §V.4)), and the space of probability measures on a compact space is a closed, hence compact, subset of such a ball. These examples suggest that the restriction of our results to compact ANR's is quite mild from the point of view of application in mathematical economics.

2. The Proof of Theorem 1

In addition to Proposition 1, the proof of Theorem 1 depends on two known results, Propositions 2 and 3 below. The statement of each is preceded by relevant definitions.

An *n-simplex* σ in $\mathbb{R}^k$ is the convex hull of a set $\{v_0, v_1, \ldots, v_n\} \subset \mathbb{R}^k$ of *vertices* that are affinely independent: the affine hull $\{\Sigma_i \alpha_i v_i | \alpha_0, \alpha_1, \ldots, \alpha_n \in I\!R, \Sigma_i \alpha_i = 1\}$ of $\{v_0, v_1, \ldots, v_n\}$ is an n-plane. The *faces* of σ are the convex hulls of nonempty subsets of $\{v_0, v_1, \ldots, v_n\}$. A *polyhedron* (or a *simplicial complex*) is (any space homeomorphic to) a finite union of simplices in some $I\!R^k$, of various dimensions, such that the intersection of any two simplices is either empty or a face of both. A representation of a polyhedron K as a finite union of this sort is called a *triangulation* of K. A *subcomplex* of K is a union of the simplices in some subset of the set of simplices in some triangulation. A *subdivision* of a triangulation is another triangulation each of whose simplices is contained in a simplex of the first triangulation. The *mesh* of a triangulation is the maximum diameter of its simplices. Any triangulation of a polyhedron has subdivisions of arbitrarily small mesh (e.g. Dold (1980, §III.6)).

Proposition 2: Let $K \subset I\!R^k$ be a polyhedron, let Z be a convex subset of a normed linear space, and let $F : K \to Z$ be an u.h.c.c.c.v. correspondence. Then any neighborhood of $Gr(F)$ contains the graph of a continuous $f : K \to Z$.

Proof: The Lemma of Mas-Colell (1974) asserts this for Z a convex subset of a Euclidean space and F defined on a neighborhood of K. A careful examination of the proof shows that MasColell's argument does not depend on the additional strength of his hypotheses. ∎

For metric spaces X and Y, two maps $f, g \in C(X, Y)$ are *ε-homotopic*, $\varepsilon > 0$, if there is a map $h : X \times [0, 1] \to Y$ with $h_0 = f$, $h_1 = g$, and $\mathrm{diam}(h(\{x\} \times [0, 1])) < \varepsilon$ for all x. A space Y *ε-dominates* a space X if there exist maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi \circ \varphi$ is ε-homotopic to Id_X.

Proposition 3: Given X, a compact ANR, and $\varepsilon > 0$, there exists a polyhedron K that ε-dominates X.

Proof: Brown (1971, pp. 40-41).

The proof of Theorem 1 now has three steps.

Lemma 1: Suppose $F : S \to T$ is an upper hemicontinuous compact valued correspondence where S is a compact metric space (with metric d_S) and T is a topological space. Let W be an open neighborhood of $Gr(F)$. Then there is $\varepsilon > 0$ and a neighborhood $W' \subset W$ of $Gr(F)$ such that $(s', t) \in W$ whenever $(s, t) \in W'$ and $d_S(s', s) < \varepsilon$.

Proof: The definition of the product topology implies that for every $(s, t) \in Gr(F)$ we can find $\delta_{(s,t)} > 0$ and $V_{(s,t)} \subset T$, an open neighborhood of y, such that $\mathbf{B}(s;\, \delta_{(s,t)}) \times V_{(s,t)} \subset W$. (In general we implicitly endow any metric space A with a metric, and we let $\mathbf{B}(a;\, \varepsilon)$ (resp $\mathbf{B}(E;\, \varepsilon)$) be the open ε-ball around $a \in A$ (resp. $E \subset A$).) For any $s \in S$ the compactness of $F(s)$ implies the existence of $t_1, \ldots, t_K$ such that $V_{(s,t_1)} \cup \ldots \cup V_{(s,t_K)} \supset F(s)$. Letting $\delta_s = \min\{\delta_{(s,t_k)}\}$ and $V_s = \cup_k V_{(s,t_k)}$, we have shown the existence of $\delta_s > 0$ and V_s, an open subset of T, such that $\{s\} \times F(s) \subset \mathbf{B}(s;\, \delta_s) \times V_s \subset W$. Since F is upper semicontinuous, we may assume without loss of generality that $F(s') \subset V_s$ for all $s' \in \mathbf{B}(s;\, \delta_s)$ by replacing δ_s with a smaller number if need be. Choose $x_1, \ldots, x_M$ such that $\cup_m \mathbf{B}(s_m;\, \delta_{s_m}/2) = X$, let $W' = \cup_m \mathbf{B}(s_m;\, \delta_{s_m}/2) \times V_{s_m}$, and let $\varepsilon = \min\{\delta_{s_m}/2\}$. ∎

Lemma 2: Let X be a compact ANR, and let Z be an open subset of a convex subset of a locally convex space. Suppose $D \subset U \subset X$ where D is compact and U is open. If $F : U \to Z$ is an u.h.c.c.c.v. correspondence and $W \subset U \times Z$ is a neighborhood of $Gr(F)$, then there is $f \in C(D, Z)$ with $Gr(f) \subset W$.

Proof: Since U may be replaced with a smaller neighborhood of D whose closure is contained in U, no generality is lost in assuming that $W \subset \bar{U} \times Z$ is a neighborhood of $Gr(F)$ in $\bar{U} \times Z$ where $F : \bar{U} \to Z$ is an u.h.c.c.c.v. correspondence. We may now choose ε_1 and $W' \subset W$ as per Lemma 1. Let ε_2 be small enough that the open ε_2 ball around D is contained in U. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Let K be a polyhedron that ε-dominates X by means of the maps $\varphi : X \to K$ and $\psi : K \to X$. Since $d_X(x, \psi(\varphi(x))) < \varepsilon$ for all x, we have $\varphi(D) \subset \psi^{-1}(U)$. Since $\varphi(D)$ is compact and $\psi^{-1}(U)$ is open, after suitably fine subdivision we can find a subcomplex $J \subset K$ with $\varphi(D) \subset J \subset \psi^{-1}(U)$. Proposition 1 implies the existence of $g \in C(J, Z)$ with $Gr(g) \subset (\psi \times Id_Z)^{-1}(W')$. Let $f = g \circ \varphi|_D$. For $(x, f(x)) \in Gr(f)$ we have $(\psi(\varphi(x)), f(x)) \in W'$ and $d_X(x, \psi(\varphi(x))) < \varepsilon$, so $(x, f(x)) \in W$. ∎

Proof of Theorem 1: As per Proposition 1, fix a retraction $r : Z \to Y$ with inclusion $i : Y \to Z$ where Z is a relatively open subset of a convex subset of a locally convex space. The $(Id_U \times r)^{-1}(W)$ is a neighborhood of $Gr(i \circ F)$ in $U \times Z$, so Lemma 2 implies the existence of a continuous $g : U \to Z$ with $Gr(f) \subset (Id_U \times r)^{-1}(W)$. Letting $f = r \circ g$, we clearly have $Gr(f) \subset W$. ∎

3. Approximation in a Stronger Sense

Though of no apparent importance for the theory of fixed points, it is interesting to consider whether an u.h.c.c.c.v. correspondence $F : X \to Y$ can be approximated by a continuous function $f : X \to Y$ in the following strong sense: $Gr(f)$ is close to $Gr(F)$, as above, and every point in $Gr(F)$ is near some point in $Gr(f)$. This is not true without an additional assumption since, for instance, X could be a singleton, but with this sort of example ruled out such approximations exist.

Theorem 2: Let X and Y be ANR's with X compact. Suppose $D \subset U \subset X$ with D compact and U open, and every neighborhood of every point in D contains infinitely many elements of D. If $F : U \to Y$ is an u.h.c.c.c.v. correspondence, $W \subset U \times Y$ is a neighborhood of $Gr(F)$, and $\varepsilon > 0$, then there is a continuous function $f : D \to Y$ with $Gr(f) \subset W$ such that for every $(x, y) \in Gr(F|_D)$ there is $(x', y') \in Gr(f) \subset \mathbf{B}((x, y); \varepsilon)$.

Proof: As per Proposition 1, fix a retraction $r : Z \to Y$ with inclusion $i : Y \to Z$ where Z is a relatively open subset of a convex subset of a normed space. Replacing Z with a smaller set if need be, we may assume that $\|z - ir(z))\| < \varepsilon/4$ for all $z \in Z$. Let $W' = (Id_U \times r)^{-1}(W)$.

Replacing U by a smaller neighborhood of D, we may assume that F is actually defined on all of $\bar{U}$. By a construction similar to the proof of Lemma 1 we may choose $0 < \rho \leq \varepsilon/4$ and (pairwise distinct) $x_1, \ldots, x_L \in \bar{U}$ such that $\mathbf{B}(x_k;\, \rho) \times \mathbf{B}(iF(x_k);\, \rho) \subset W'$ for each k and $Gr(iF) \subset W''$ where $W'' = \cup_{k=1,\ldots,L}\ \mathbf{B}(x_k;\, \rho) \times \mathbf{B}(iF(x_k);\, \rho)$. With a certain amount of care in the construction we can insure that there is $K \leq L$ such that $x_1, \ldots, x_K \in D$ and $Gr(iF|_D) \subset \cup_{k=1,\ldots,K}\ \mathbf{B}(x_k;\, \rho) \times \mathbf{B}(iF(x_k);\, \rho)$. Each x_k has a neighborhood U_k with $iF(x') \subset \mathbf{B}(iF(x_k);\, \rho)$ for all $x' \in U_k$, so, applying this style of construction one more time, we can construct a neighborhood $W''' \subset W''$ of $Gr(iF)$ such that for each k there is a neighborhood $U_k \subset \mathbf{B}(x_k;\, \rho)$ such that $W''' \cap (U_k \times Z) \subset \mathbf{B}(x_k;\, \rho) \times \mathbf{B}(iF(x_k);\, \rho)$. Without loss of generality we may assume that the sets $U_1, \ldots, U_K$ are pairwise disjoint.

Now let $g' : D \to Z$ be a continuous function with $Gr(g') \subset W'''$, as per Lemma 2. A new function $g : D \to Z$ is defined by the following construction. For each k choose $w_{k1}, \ldots, w_{kQ(k)} \in iF(x_k)$ such that $iF(x_k) \subset \cup_{q=1,\ldots,Q(k)} \mathbf{B}(y_{kq};\, \varepsilon/4)$, choose distinct $x_{k1}, \ldots, x_{kQ(k)} \in U_k$, and for each x_{kq} choose $z_{kq} \in F(x_k)$ with $\|g'(x_{kq}) - z_{kq}\| < \rho$ (this is possible since $(x_{kq},\, g'(x_{kq})) \in W'''$). Choose numbers $\eta_{kq} > 0$ small enough that for each k the balls $\mathbf{B}(x_{kq};\, \eta_{kq})$ are pairwise disjoint and contained in U_k with $\|g'(x) - z_{kq}\| < \rho$ for all $x \in \mathbf{B}(x_{kq};\, \eta_{kq})$.

For points x not in any $\mathbf{B}(x_{kq};\, \eta_{kq})$ set $g(x) = g'(x)$. For each k let $c_k : iF(x_k) \times [0,\, 1] \to iF(x_k)$ be a contraction. For $x \in \mathbf{B}(x_{kq};\, \eta_{kq})$ let $\delta_{kq}(x) = \frac{\|x_{kq} - x\|}{\eta_{kq}}$, and set

$$g(x) = \begin{cases} (1 - 3\delta_{kq}(x))g'(x) + 3\delta_{kq}(x)z_{kq}, & 0 \leq \delta_{kq}(x) \leq \frac{1}{3}, \\ c_k(z_{kq},\, 3\delta_{kq}(x) - 1), & \frac{1}{3} < \delta_{kq}(x) \leq \frac{2}{3}, \\ c_k(w_{kq},\, 3 - 3\delta_{kq}(x)), & \frac{2}{3} < \delta_{kq}(x) \leq 1. \end{cases}$$

Clearly g is continuous. Let $f = r \circ g$. Our construction implies that $Gr(g) \subset W'' \subset W'$, so $Gr(f) \subset W$.

Consider $(x,\, y) \in Gr(F)$. There is some k such that $x \in \mathbf{B}(x_k;\, \rho)$ and $y \in \mathbf{B}(iF(x_k);\, \rho)$. Choose $y \in F(x_k)$ with $\|y - y'\| < \rho$, and choose q such that $\|y' - w_{kq}\| < \varepsilon/4$. Since $f(x_{kq}) = r(g(x_{kq})) = r(w_{kq}) = w_{kq}$, and $\rho < \varepsilon/4$, we have

$$\|x - x_k\| + \|y - f(x_{kq})\| \leq \|x - x_k\| + \|x_k - x_{kq}\| + \|y - y'\| + \|y' - w_{kq}\| \leq 4(\varepsilon/4). \blacksquare$$

4. A Conjecture

We conclude with a conjecture which, if confirmed, would allow the extension of the theory of the Lefschetz fixed point index and the theory of essential sets of fixed points to be extended to u.h.c.c.c.v. correspondences $F : X \to X$, X a compact ANR, along the lines laid out in McLennan (1989c).

Conjecture: Let X and Y be ANR's with X compact, and let $F : X \to Y$ be an u.h.c.c.c.v. correspondence. Suppose $D \subset U \subset X$ where D is compact and U is open. Then for any neighborhood W of $Gr(F)$ there is a neighborhood W' of $Gr(F|_U)$ such that for any continuous function $f' : U \to Y$ with $Gr(f') \subset W'$ and any neighborhood W'' of $Gr(f'|_D)$ there is a continuous function $f : X \to Y$ with $Gr(f) \subset W$ and $Gr(f|_D) \subset W''$.

Proposition C.8 of McLennan (1989c) is a version of this result with X a compact metric space, Y a compact ANR that is a convex subset of a locally convex linear space, and F an upper hemicontinuous compact convex valued correspondence. In McLennan (1989b) the following result is established.

Proposition 4: Let K be a polyhedron, and let J be a subcomplex. Let Z be a convex subset of a normed linear space. Let $F : K \to Z$ be an u.h.c.c.c.v. correspondence. Then for every neighborhood W of $Gr(F)$ there is a neighborhood W' of $Gr(F|_D)$ such that for every continuous $f' : J \to Z$ with $Gr(f') \subset W'$ there exists a continuous $f : K \to Z$ with $f|_J = f'$ and $Gr(f) \subset W$.

(The statement in McLennan (1989b) requires Z to be a convex subset of a Euclidean space, but the proof is a straightforward extension of Mas-Colell's (1974) argument and does not depend on Y being finite dimensional.)

A natural idea is to attempt to prove the Conjecture by combining Proposition 4 with the techniques used to prove Theorem 1, but this fails for a subtle reason. Let X be a compact ANR, let Z be a convex subset of a normed linear space, and let $F : X \to Z$ be an u.h.c.c.c.v. correspondence. Let W be a neighborhood of $Gr(F)$. Let K be a polyhedron that ε-dominates X by means of the maps $\varphi : X \to K$ and $\psi : K \to X$. If ε is small then there will be a subcomplex $J \subset K$ with $\varphi(D) \subset J \subset \psi^{-1}(U)$, as in Lemma 2. If the graph of $f' : U \to Y$ is close $Gr(F|_U)$ then $Gr(f' \circ \psi|_J)$ will be close to $Gr(F \circ \psi)$. Since $(\psi^{-1} \circ Id_Z)(W)$ is a neighborhood of $Gr(F \circ \psi)$, for sufficiently close f' Proposition 4 implies the existence of $g : K \to Z$ with $Gr(g) \subset (\psi^{-1} \circ Id_Z)(W)$ and $g|_J = f' \circ \psi|_J$. With ε small the graph of $g \circ \varphi$ will be contained in W and $g \circ \varphi|_D = f' \circ \psi \circ \varphi|_D$ will be close to f'. The difficulty is that the relevant neighborhood of $Gr(F \circ \varphi|_J)$ must be chosen after ε, K, φ, and ψ are chosen, so this method does not guarantee that the distance between $f'|_D$ and $f' \circ \psi \circ \varphi|_D$ can be made small in comparison with the distance between $Gr(f')$ and $Gr(F|_D)$.

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