FEASIBLE NASH IMPLEMENTATION OF COMPETITIVE EQUILIBRIA IN AN ECONOMY WITH EXTERNALITIES

by

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Abstract: This paper proves feasible Nash implementability of the Pigouvian competitive equilibria in an economy with consumption externalities, as well as proofs of the existence of Pigouvian equilibria and the fundamental theorems of welfare economics with regard to the Pigouvian equilibria.

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1. Introduction

In this paper, I will consider the possibility of Nash implementation of competitive equilibria in an economy with externalities. This kind of incentive problem has been investigated for a long time, originally in the context of a public goods economy. As Samuelson [16] pointed out, Lindahl equilibrium has the "free rider problem," so that we must not expect that Lindahl equilibrium will be achieved if each agent is rational and behaves strategically.

In 1972, Hurwicz [3] pointed out that this kind of problem could occur even in the standard Walrasian economy. Subsequently, there have been many contributions concerning Nash implementation of both Walras and Lindahl equilibria, including those of Schmeidler [17], Hurwicz [5] and Walker [19]. These papers propose mechanisms which do implement either Walrasian or Lindahl equilibrium, and also satisfy the condition guaranteeing the balance of demand and supply even outside the equilibrium. Jordan [7] found a general way to modify these mechanisms so that they can treat production. Recently, Hurwicz, Maskin and Postlewaite [6] and Postlewaite and Wettstein [14] constructed mechanisms satisfying, in addition to the balance condition, the individual feasibility condition, namely that the outcome belongs to each individual's consumption set even outside equilibrium.

As far as stability is concerned, Jordan [8] proves the non-existence of mechanisms which implement Walrasian equilibrium and are dynamically stable among the classical environments. Kim [9] proves the same impossibility theorem in an economy with public goods. He also constructs a mechanism which implements Lindahl equilibrium and is dynamically stable if we restrict the environments within the quasi-linear utility functions.

The main concern of this paper is economies which allows consumption externalities. This type of economy is a generalization of the standard public goods economy and the "free rider problem" is still valid in this economy. Moreover, in this economy, all prices must be privatized with some tax-subsidy system if we want to attain Pareto optimality with the market system in an informationally decentralized way. Hence, the incentive problem is much more serious than in a public goods economy.

In an economy with externalities, Aoki [1] discusses the relation between competitive equilibria and Pareto optimal allocations. But he restricts himself to an economy of very special type in which there is only a single consumer and externalities exist only within each industry. In a more general framework, Osana [12] shows the existence of equilibria and proves that every Pareto optimal allocation is a competitive equilibrium if some suitable tax-subsidy system is adopted. Otani and Sicilian [13] use a stronger equilibrium concept and consider the fundamental theorems of welafare economics. But they do not prove the existence of such equilibria. Furthermore all these arguments do not consider implementability of the competitive equilibrium.

On the other hand, Hurwicz and Schmeidler [4] construct some mechanisms guaranteeing the existence of Nash equilibrium and the Pareto optimality of the equilibrium for every admissible profile of preferences, when the set of alternatives is finite. Saijo [15] proves Maskin's theorem [11] which states necessary conditions and sufficient conditions for the mechanisms to implement any given target correspondence in a general social choice framework. But in their mechanisms, each agent must know the social attainable set and his individual message space has infinite dimensions, so that practically, it is very difficult to exchange messages between agents.

In section 2 of this paper, the basic framework of economies with externalities will be discussed. In section 3, I will define "Pigouvian" competitive equilibrium, which is close to that of Otani and Sicilian [13], in a pure exchange economy with consumption externalities, and will prove existence of the equilibria and the fundamental theorems of welfare economics, namely: every Pigouvian competitive equilibrium is Pareto optimal and every Pareto optimal allocation is attainable via a Pigouvian competitive equilibrium provided that the initial endowments are suitably redistributed. But as stated above, this equilibrium has a serious incentive compatibility problem when we insist on informational decentralization. This is the main issue of section 4 of this paper. I will construct a continuous and feasible mechanism which implements the Pigouvian competitive equilibrium even when the mechanism designer knows neither individuals' preferences nor initial endowments, and each individual only knows his own preferences and his own initial endowments.

In section 5, I will consider an economy with a linear production function, and assert the same statements as in section 3 and 4. Note, however, that this economy does not include the pure exchange economy discussed in section 2.

2. Environments

Consider a pure exchange economy(1) with n consumers and $l+1$ commodities. A commodity bundle is denoted by (x, y), where $x \in \Re_+$ (numéraire without externalities) and $y \in \Re^l_+$ (with externalities). The i-th consumer's preference relation is denoted by $\succeq_i$ which is a binary relation(2) on the set $\Re_+ \times \Re^l_+ \times \Re^{l(n-1)}_+$. Let $P_i : \Re_+ \times \Re^l_+ \times \Re^{l(n-1)}_+ \twoheadrightarrow \Re_+ \times \Re^l_+ \times \Re^{l(n-1)}_+$ be the strict upper-contour correspondence. I will assume the following monotonicity assumption of the preferences:

ASSUMPTION 2.1: (Monotonicity) For all $(x_i, y_i; y_{-i}) \in \Re_+ \times \Re^l_+ \times \Re^{l(n-1)}_+$ and for all $\epsilon \in \Re_{++}$

$$(x_i + \epsilon, y_i; y_{-i}) \in P_i(x_i, y_i; y_{-i}).$$

His initial endowment is given by $(\omega^x_i,\ \omega^y_i) \in \Re_+ \times \Re^l_+$.

Note that I assume implicitly that "the consumption set" is equal to the non-negative orthant, although I do not assume the completeness of the preferences.

The attainable set of this economy is denoted by the following set.

$$A \equiv \left\{ (x, y) \in \Re^n_+ \times \Re^{ln}_+ : \quad \sum_i (x_i - \omega^x_i) = 0 \qquad \text{and} \qquad \sum_i (y_i - \omega^y_i) = 0 \right\}.$$

Pareto optimality and individual rationality are defined as follows:

DEFINITION 2.1: $(x^*,\ y^*) \in A$ is *Pareto optimal* if there is no $(x,\ y) \in A$ such that for all i,

$$(x_i,\ y_i;\ y_{-i}) \in P_i((x^*_i,\ y^*_i;\ y^*_{-i})).$$

DEFINITION 2.2: $(x, y) \in A$ is *individually rational* if for all i,

$$(\omega^x_i,\ \omega^y_i;\ \omega^y_{-i}) \notin P_i(x_i,\ y_i;\ y_{-i}).$$

(1) For more general case, see Osana [12].

(2) Note that we do not assume either transitivity or completeness of the preferences at this stage.

3. Competitive Equilibria

3.1. Definitions

In this section, we will define a Pigouvian competitive equilibrium. The first definition is a transfer system which takes a role to equate the private marginal cost with the social marginal cost.

Consider two distinct consumers i and j $(i \neq j)$. Let t_{ij} be a transfer rate from i to j. Then if consumer j consumes y_j unit of commodity y, then consumer i pays $t_{ij}(y_j - \omega_j^y)$ for j's consumption. Similarly, consumer j pays $t_{ji}(y_i - \omega_i^y)$ for i's consumption of commodity y of y_i unit. Thus i's net transfer to consumer j is

$$t_{ij}(y_j - \omega_j^y) - t_{ji}(y_i - \omega_i^y).$$

Hence the sum of transfers paid by i is equal to

$$\sum_{j \neq i} \Big(t_{ij}(y_j - \omega_j^y) - t_{ji}(y_i - \omega_i^y) \Big) = \sum_{j \neq i} t_{ij}(y_j - \omega_j^y) + (-\sum_{j \neq i} t_{ji})(y_i - \omega_i^y).$$

Thus if we write $-\sum_{j \neq i} t_{ji}$ as t_{ii}, then the total transfer from i can be written as

$$\sum_j t_{ij}(y_j - \omega_j^y),$$

which is very simple.

Formally, the transfer system is defined as follows.

DEFINITION 3.1: $t \in \Re^{ln^2}$ is called a *transfer system* if for all j,

$$\sum_{i \neq j} t_{ij} = -t_{jj}.$$

t_{ij} $(i \neq j)$ is interpreted as the transfer rate from consumer i to consumer j. t_{jj} is the tax rate of consumer j. So the above condition means that the subsidy $(= -t_{jj})$ of the j-th consumer is equal to the sum of the transfer to consumer j.

REMARK 3.1: *$t \in \Re^{ln^2}$ is a transfer system if and only if*

$$\sum_i \sum_j t_{ij}(y_j - \omega_j^y) = 0 \qquad \textit{for every } y \in \Re_+^{ln}.$$

$t_{ij}(y_j - \omega_j^y)$ is an amount of transfer from consumer i to consumer j. Hence this remark means that total transfer is always equal to zero so that the budget constraint of the government is always satisfied, which guarantees the Walras law.

Secondly, we will define the budget set of consumer i in this economy.

DEFINITION 3.2: For each price $p \in \Re^l$ and a transfer system $t \in \Re^{ln^2}$, let

$$B_i(p,t) \equiv \Bigg\{ (x_i,\ y_i;\ y_{-i}) \in \Re_+ \times \Re_+^{ln} : \\ x_i + py_i \leq \omega_i^x + p\omega_i^y - \sum_j t_{ij}(y_j - \omega_j^y) \quad \text{and} \quad \sum_{\mathrm{j}} \mathrm{y_j} = \sum_{\mathrm{j}} \omega_{\mathrm{j}}^{\mathrm{y}} \Bigg\}.$$

The second condition in the budget constraint is the balanced condition of commodity y. Since consumer i specifies his desired consumption of the others', it is natural for him to consider the balance of demand of commodity y. Note that he must know others' initial endowments in order to know his budget constraint. This creates another informational problem concerning to the following Pigouvian competitive equilibrium.

DEFINITION 3.3: $(p^*,\ t^*,\ x^*,\ y^*) \in \Re^l \times \Re^{ln^2} \times \Re_+^n \times \Re_+^{ln}$ is called a *Pigouvian competitive equilibrium* if

(1) t^* is a transfer system,

(2) $(x_i^*,\ y_i^*;\ y_{-i}^*)$ is a maximal element of $\succeq_i$ in $B_i(p^*,\ t^*)$, namely,

(2.1) $(x_i^*, y^*) \in B_i(p^*, t^*)$

(2.2) $P_i(x_i^*, y^*) \cap B_i(p^*, t^*) = \emptyset$

(3) $(x^*,\ y^*) \in A$.

The corresponding allocation (x^*, y^*) is called a *Pigouvian competitive allocation.*

Conditions (1) and (3) are clear. Condition (2) means that consumers maximize their preferences given the equilibrium prices and transfers. Namely condition (2) means that the allocation y^* is optimal for each consumer under his own budget set. It should be noted that the Pigouvian equilibrium in this context is different from the equilibrium introduced by Osana [12] in which y_i^* is optimal given not only prices and transfers but also others' consumption $(y_1^*, \ldots, y_{i-1}^*; y_{i+1}^*, \ldots, y_n^*)$. In fact, our definition of the Pigouvian equilibrium in this context is stronger than that of the equilibrium in Osana [12], so that we can assure non-wastefulness and individual rationality.

3.2. Theorems

Now we can state the following theorems. Note that in most of the following analysis, the convexity of preferences will be assumed. This assumption is a necessary evil, since Calsamiglia [2] proved the impossibility of realization of Pareto optimal correspondence with finite dimensional message spaces in non-convex environments.

THEOREM 3.1: (Existence) *For all i, assume the following:*

(1) *$P_i(\cdot)$ has an open graph.* (Continuity)

(2) *For all $(x_i, y) \in \Re_+ \times \Re_+^l \times \Re_+^{l(n-1)}$ $\quad (x_i, y) \notin \mathrm{conv}\ P_i(x_i, y)$.*[(3)] (Convexity)

(3) $\omega_i^x \in \Re_{++}$.

Then there exists a Pigouvian competitive equilibrium.

(3) For a given set X, conv X denotes the convex hull of X.

PROOF: Let us introduce the following notations:

$$X_i \equiv \Re_+ \times \{(0,\ldots,0)\} \times \Re_+^{ln} \times \{(0,\ldots,0)\} \subset \Re_+ \times \Re_+^{ln(i-1)} \times \Re_+^{ln} \times \Re_+^{ln(n-i)}.$$

$$Y \equiv \Big\{(x, y_1, \ldots, y_n) \in \Re \times \Re^{ln^2} :$$

$$y_1 = \cdots = y_n, \qquad x = 0 \qquad \text{and} \qquad \sum_j y_{ij} = 0 \quad \text{for all } i\Big\}.$$

$$\tilde{\omega}_i \equiv (\omega_i^x, 0, \ldots, 0, \omega^y, 0, \ldots, 0) \in X_i.$$

We will extend P_i on X_i in the natural way, that is

$$(x_i, 0, \ldots, 0, y, 0, \ldots, 0) \in P_i(x_i', 0, \ldots, 0, y', 0, \ldots, 0) \qquad \Longleftrightarrow \qquad (x_i, y) \in P_i(x', y').$$

Denote

$$\tilde{A} = \left\{(u, v) \in \prod_i X_i \times Y : \quad \sum_i u_i = \tilde{\omega}_i + v\right\}.$$

Then $\tilde{A}$ is compact since X_i's are lower bounded and closed. Hence there is a convex and compact set $K \subset \Re \times \Re^{ln^2}$ such that

$$\text{proj}_{X_i} \; \tilde{A} \subset \text{int } K \qquad \text{and} \qquad \text{proj}_Y \; \tilde{A} \subset \text{int } K.$$

Let $\tilde{X}_i \equiv X_i \cap K$ and $\tilde{Y} \cap K$.

Fix any natural number ν. Consider the following disk as the set of combination of prices and transfers:

$$D^\nu \equiv \left\{q \in \Re \times \Re^{ln^2} : \quad \|q\| \le \nu\right\}.$$

For all $q \in D^\nu$, define the budget set of consumer i as

$$C_i(q) \equiv \left\{u_i \in \tilde{X}_i : \quad (1, q)u_i \le (1, q)\tilde{\omega}_i\right\}.$$

Since $\omega_i^x \in \Re_{++}$, C_i is a continuous correspondence from D^ν into $\tilde{X}_i$.

Now consider the following $n+2$ players' abstract economy:

First n players: Consumers

Strategy Set: $\tilde{X}_i$

Preference Correspondence: $P_i(\cdot)$

Constraint Correspondence: $C_i(\cdot)$

$(n+1)$-st player: Firm

Strategy Set: $\tilde{Y}$

Preference Correspondence: $P_f(\cdot)$ which is defined by $\forall q \in D^\nu$ and $\forall v \in \tilde{Y}$

$$P_f(q,v) \equiv \left\{ v' \in \tilde{Y} : \quad (1,q)v' > (1,q)v \right\}$$

Constraint Correspondence: $\tilde{Y}$

$(n+2)$-nd player: Auctioneer

Strategy Set: D^ν

Preference Correspondence: $P_a(\cdot)$ which is defined by

$$\forall q \in D^\nu,\ \forall (u_1, \cdots, u_n) \in \prod_{i=1}^{n} \tilde{X}_i \text{ and } v \in \tilde{Y}$$

$$P_a(q,u,v) \equiv \left\{ q' \in D^\nu : \qquad (1,q') \left(\sum_{i=1}^{n} (u_i - \tilde{\omega}_i) - v \right) > (1,q) \left(\sum_{i=1}^{n} (u_i - \tilde{\omega}_i) - v \right) \right\}$$

Constraint Correspondence: D^ν

Then by Shafer and Sonnenschein [18], there is a generalized Nash equilibrium $(q^\nu, u^\nu, v^\nu) \in D^\nu \times \prod_{i=1}^{n} \tilde{X}_i \times \tilde{Y}$, which satisfies the following conditions:

$$(1,q^\nu)u_i^\nu \leq (1,q^\nu)\tilde{\omega}_i \tag{1}$$

$$\text{conv } P_i(u^\nu) \cap \left\{ u_i \in \tilde{X}_i : \quad (1,q^\nu)u_i \leq (1,q^\nu)\tilde{\omega}_i \right\} = \emptyset \tag{2}$$

$$(1,q^\nu)v^\nu \geq (1,q^\nu)v \qquad \forall v \in \tilde{Y} \tag{3}$$

$$(1,q^\nu)\left(\sum_{i=1}^{n}(u_i^\nu-\tilde{\omega}_i)-v^\nu\right)\geq(1,q)\left(\sum_{i=1}^{n}(u_i^\nu-\tilde{\omega}_i)-v^\nu\right)\qquad\forall q\in D^\nu. \tag{4}$$

Using standard argument one can prove $\|q^\nu\|\not\to\infty$ by the monotonicity of the preferences. Moreover, since $\tilde{X}_i$ and $\tilde{Y}$ are compact, we may assume, without loss of generality, that

$$q^\nu\longrightarrow q^*,\qquad u^\nu\longrightarrow u^*,\qquad\text{and}\qquad v^\nu\longrightarrow v^*\qquad\qquad\text{as }\nu\longrightarrow\infty. \tag{5}$$

Hence using equations (1), (2), (3), and (4), one can assert

$$(1,q^*)u_i^*\leq(1,q^*)\tilde{\omega}_i \tag{6}$$

$$\text{conv }P_i(u^*)\cap\left\{u_i\in\tilde{X}_i:\quad(1,q^*)u_i\leq(1,q^*)\tilde{\omega}_i\right\}=\emptyset \tag{7}$$

$$(1,q^*)v^*\geq(1,q^*)v\qquad\forall v\in\tilde{Y} \tag{8}$$

$$(1,q^*)\left(\sum_{i=1}^{n}(u_i^*-\tilde{\omega}_i)-v^*\right)\geq(1,q)\left(\sum_{i=1}^{n}(u_i^*-\tilde{\omega}_i)-v^*\right)\qquad\forall q\in\Re^{ln^2}. \tag{9}$$

Hence if we write:

$$u_i^*\equiv(x_i^*;0,\cdots,0,y_i^*,0,\cdots,0)\qquad\text{and}\qquad v^*\equiv(\bar{x},\bar{y},\cdots,\bar{y}),$$

then using (6) and (9), one can prove the following equations:

$$\sum_{i=1}^{n}x_i^*\leq\bar{x}+\sum_{i}^{n}\omega_i^x$$
$$y_i^*=\bar{y}+\omega^y\equiv y^*\qquad\text{for all }i.$$

But since $(\bar{x},\bar{y},\cdots,\bar{y})\in Y$, it follows that

$$\bar{x}=0\qquad\text{and}\qquad\sum_{i=1}^{n}\bar{y}_i=0$$

Moreover, using (8), one can prove

$$\sum_{i=1}^{n}q_{ij}^*=\sum_{i=1}^{n}q_{ik}^*\equiv p^*\qquad\text{for all }j\text{ and }k.$$

$$(1, q^*)v^* = 0$$

Hence

$$(x_1^*, \cdots, x_n^*; y^*) \in A.$$

In order to get the equilibrium transfer system, define:

$$t_{ij}^* \equiv q_{ij}^* \qquad \text{if } i \neq j$$

$$t_{ii}^* \equiv q_{ii}^* - p^*$$

Q.E.D.

THEOREM 3.2: (Non-Wastefulness) *Every Pigouvian competitive allocation is Pareto optimal.*

PROOF: The proof is straightforward using the standard argument.

Q.E.D.

THEOREM 3.3: (Unbiasedness) *For all i, assume the following:*

(1) $P_i(\cdot)$ *is open-valued.* (Continuity)

(2) $P_i(\cdot)$ *is convex-valued.* (Convexity)

Then every Pareto optimal allocation $(x^*,\ y^*) \in \Re^n_{++} \times \Re^{ln}_{++}$ can be attained as a Pigouvian competitive allocation provided that the initial endowments are suitably redistributed.

PROOF: Let

$$D \equiv \left\{ (x; y_1, \cdots, y_n) \in \Re \times \Re^{ln^2} : \qquad \text{There exists } (x_1, \cdots, x_n) \in \Re^n \text{ such that} \right.$$
$$\left. x = \sum_i x_i \quad \text{and} \quad (x_i + x_i^*, y_i + y^*) \in P_i(x_i^*, y^*) \quad \forall i \right\}.$$

and

$$F \equiv \left\{ (x; y_1, \cdots, y_n) \in \Re \times \Re^{ln^2} : y_1 = \cdots = y_n, \qquad x \leq 0, \qquad \text{and} \qquad \sum_j y_{ij} = 0 \quad \forall i \right\}.$$

Then D and F are convex and $D \cap F = \emptyset$ since (x^*, y^*) is Pareto optimal. Hence there is a hyperplane $(q^x; q_1^y, \cdots, q_n^y) \in \Re \times \Re^{ln^2} \setminus \{0\}$ and $r \in \Re$ such that

$$q^x x + \sum_i q_i^y y \leq r \qquad \forall (x; y, \cdots, y) \in F \tag{1}$$

$$q^x x + \sum_i q_i^y y \geq r \qquad \forall (x; y_1, \cdots, y_n) \in D. \tag{2}$$

By monotonicity, $q^x \geq 0$.

Since $(x^*; \overbrace{y^*, \cdots, y^*}^{n \text{ times}}) \in F$ and $(x^* + (\overbrace{\epsilon, \cdots, \epsilon}^{n \text{ times}}); \overbrace{y^*, \cdots, y^*}^{n \text{ times}}) \in D \quad \forall \epsilon \in \Re_{++}$, it follows that

$$q^x x^* + \sum_i q_i^y y^* = r.$$

Hence by (1), for all $(x, y) \in \Re^{ln^2}$ with $x \leq 0$ and $\sum_j y_j = 0$,

$$q^x x^* + \sum_i q_i^y y^* \geq q^x x + \sum_i q_i^y y.$$

So we can show that

$$\sum_i q_{ij}^y = \sum_i q_{ik}^y \equiv p^* \qquad \text{for all } j \text{ and } k.$$

Let i be such that $(q^x, q_i^y) \neq 0$. Without loss of generality, assume $i = 1$. Then for all $(x_1, y) \in P_1(x_1^*, y^*)$,

$$q^x x_1 + q_1^y y \geq q^x x_1^* + q_1^y y^*.$$

Namely

$$P_1(x_1^*, y^*) \cap \left\{ (x_1, y) \in \Re_+ \times \Re_+^{ln} : \quad q^x x_1 + q_1^y y < q^x x_1^* + q_1^y y^* \right\} = \emptyset.$$

Since $P_1(x_1^*, y^*)$ is open,

$$P_1(x_1^*, y^*) \cap \left\{ (x_1, y) \in \Re_+ \times \Re_+^{ln} : \quad q^x x_1 + q_1^y y \leq q^x x_1^* + q_1^y y^* \right\} = \emptyset.$$

Hence by monotonicity, $q^x > 0$. Hence we may assume that

$$q^x = 1 \qquad \text{and} \qquad \sum_i q_i^y = (\overbrace{p^*, \cdots, p^*}^{n \text{ times}}).$$

so that $r = 0$. Define

$$\begin{aligned} t_{ij}^* &\equiv q_{ij}^y \qquad \text{if } i \neq j \\ t_{ii}^* &\equiv q_{ii}^y - p^*, \qquad \text{and} \\ (\omega_i^x, \omega_i^y) &\equiv (x_i^*, y_i^*) \end{aligned}$$

Then t is a transfer system. Hence one can prove that for all i and for all $(x_i, y) \in P_i(x_i^*, y^*)$,

$$x_i + q_i^y y \geq x_i^* + q_i^y y^*.$$

Namely

$$P_i(x_i^*, y^*) \cap \{(x_i, y) \in \Re_+ \times \Re_+^{ln} : \quad x_i + q_i^y y < x_i^* + q_i^y y^*\} = \emptyset.$$

Since $P_i(x_i^*, y^*)$ is open,

$$P_i(x_i^*, y^*) \cap \{(x_i, y) \in \Re_+ \times \Re_+^{ln} : \quad x_i + q_i^y y \leq x_i^* + q_i^y y^*\} = \emptyset.$$

Namely

$$P_i(x_i^*, y^*) \cap \left\{ (x_i, y) \in \Re_+ \times \Re_+^{ln} : \quad x_i + p^* y_i \leq \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^* (y_j - \omega_j^y) \right\} = \emptyset.$$

Q.E.D.

THEOREM 3.4: (Individual Rationality) *Every Pigouvian competitive allocations is individually rational.*

PROOF: Obvious.

Q.E.D.

4. Feasible Nash Implementation

In this section, I will prove the possibility of feasible Nash implementation of Pigouvian competitive equilibrium when the mechanism designer does not know either the individual preferences or the individual initial endowments. I also assume that each individual only knows his own preferences and his own initial endowment and does not know the others'. Again, a convexity assumption on preferences plays an important role.

4.1. Mechanism

From now on, we will write the i-th consumer's true initial endowments as $\mathring{\omega}_i \equiv (\mathring{\omega}_i^x,\ \mathring{\omega}_i^y)$. Let us make the following assumptions:

ASSUMPTION 4.1: $n \geq 3$.[4]

ASSUMPTION 4.2: $\mathring{\omega}_i \gg 0$.

ASSUMPTION 4.3: $\succeq_i$ is complete, transitive, and convex.

ASSUMPTION 4.4: (Boundary Condition)

$\forall (x_i, y_i) \in \text{int } \Re_+^{l+1}$,[5] $\forall (x_i', y_i') \in \partial\, \Re_+^{l+1}$,[6] and $\forall y_{-i} \in \Re_+^{l(n-1)}$,

$$(x_i,\ y_i;\ y_{-i}) \succ_i (x_i',\ y_i';\ y_{-i}).$$

(4) When we have only two consumers, some difficulties arise. In particular, one can get some impossibility results concerning Nash implementation of even Walrasian or Lindahl equilibria. For details, see Kwan and Nakamura [10].

(5) For any set X, int X denotes the topological interior of X.

(6) For any set X, ∂X denotes the topological boundary of X.

Let us consider the following mechanism.

DEFINITION 4.1: (Message Space) For all i, let

$$M^i \equiv \Re^l \times \Re^{ln} \times (0,\ \mathring{\omega}_i] \qquad \text{and} \qquad M \equiv \prod_i M^i.$$

The representative strategy of consumer i, is denoted by $m_i \equiv (p_i,\ (y_{ij})_j,\ \omega_i)$, which can be interpreted in a following way:

(1) p_i: proposed price.

(2) y_{ij}: proposed total consumption of consumer j.

(3) ω_i: reported initial endowments.

For a given $m \equiv (m_i)_i \equiv (p_i,\ (y_{ij})_j,\ \omega_i)_i \in M$, define the following mechanism:

DEFINITION 4.2:

$$\alpha_i(m) \equiv \sum_{k,\ k' \neq i} (p_k - p_{k'})^2$$

$$\alpha(m) \equiv \sum_i \alpha_i(m)$$

$$\beta_i(m) \equiv \begin{cases} \alpha_i(m)/\alpha(m), & \text{if } \alpha(m) > 0; \\ 1/n & \text{otherwise.} \end{cases}$$

$$p(m) \equiv \sum_i \beta_i(m) p_i$$

Namely, the actual price $p(m)$ is a weighted average of the proposed price p_i by each consumer i with a coefficient $\beta_i(m)$ which is also affected by individuals' strategies. Note that this $p(\cdot)$ is continuous even though $\beta_i(\cdot)$ is discontinuous. The transfer system will be defined in the following way:

DEFINITION 4.3:

$$t_{ij}(m) \equiv y_{i+1,j} - y_{i+2,j}.$$

Note that this $(t_{ij}(m))_{ij}$ is actually a transfer system.

DEFINITION 4.4:

$$D(m) \equiv \Big\{ (y_1, \cdots, y_n) \in \Re_+^{ln} :$$

$$\forall\, i \quad p(m)y_i + \sum_j t_{ij}(m)(y_j - \omega_j^y) \leq \omega_i^x + p(m)\omega_i^y \quad \text{and} \quad \sum_i y_i = \sum_i \omega_i^y \Big\}.$$

Note that the above $D(\cdot)$ is convex-valued and continuous (*i.e.*, both upper semi- and lower semi-continuous.)

DEFINITION 4.5:

$$y_j(m) \equiv \sum_i y_{ij} - \sum_i y_{i,j+1} + \omega_j^y.$$

DEFINITION 4.6:

$$(Y_1, \ldots, Y_n)(m) \equiv \text{argmin}\, \{\|y - y(m)\| : \quad y \in D(m)\}.$$

and

$$X_i(m) \equiv p(m)\Big(\omega_i^y - Y_i(m)\Big) + \sum_j t_{ij}(m)\Big(\omega_j^y - Y_j(m)\Big) + \omega_i^x.$$

4.2. Theorems

THEOREM 4.1: *This mechanism is continuous and feasible.*

PROOF: Obvious.

Q.E.D.

THEOREM 4.2: *The set of Nash allocations coincides with the set of Pigouvian competitive allocations.*

PROOF: Let (p^*, t^*, x^*, y^*) be a Pigouvian competitive equilibrium. We will define a strategy profile $m^* = (p_i^*, (y_{ij}^*)_j, \omega_i^*)_i$ in the following way. Let $p_i^* = p^*$ and $\omega_i^* = \mathring{\omega}_i$. In order to define y_{ij}^*, we will consider the following devices: Let $(\tilde{y}_1^*, \ldots, \tilde{y}_n^*)$ be a solution of the following:

$$\tilde{y}_{jh}^* - \tilde{y}_{j+1,h}^* = y_{jh}^* - \mathring{\omega}_{jh}^y \qquad \forall j = 1, \ldots, n \quad \forall h = 1, \ldots, l.$$

For a fixed $j = 1, \ldots n$ and a fixed $h = 1, \ldots, l$, consider the following linear equation system:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ & & \cdots & & \\ & & \cdots & & \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1^{jh} \\ y_2^{jh} \\ \cdot \\ \cdot \\ y_n^{jh} \end{pmatrix} = \begin{pmatrix} \tilde{y}_{jh}^* \\ t_{njh}^* \\ t_{1jh}^* \\ \cdot \\ t_{n-2,jh}^* \end{pmatrix}$$

According to Walker [19], the above system of equations has the unique solution $(y_i^{jh})_i$. Define $y_i^* = (y_i^{jh})_{jh}$. Then it is easy easy to see this s^* attains the Pigouvian competitive allocation. Moreover, since for all m_i, $(X_i(m_i, m_{-i}^*),\ Y(m_i, m_{-i}^*))$ satisfies his budget constraint by the definition of mechanism, one can show

$$(X_i(m^*),\ Y(m^*)) \succeq_i (X_i(m_i, m_{-i}^*),\ Y(m_i, m_{-i}^*)).$$

Hence m^* is a Nash equilibrium.

Conversely, let $m^* = (p_i^*, (y_{ij}^*)_j, r_i^*, \omega_i^*)$ be a Nash equilibrium. Suppose that $\omega_i^* \neq \mathring{\omega}_i$. Then by increasing his reported initial endowments, he can attain a larger y_i, which is the contradiction by monotonicity. Hence

$$\omega_i^* = \mathring{\omega}_i.$$

We will prove that for any i, $(X_i(m^*), Y(m^*))$ maximizes his preference relation subject to his budget constraint determined by $p(m^*)$ and $t(m^*)$.

It is straightforward to show this $(X_i(m^*), Y(m^*))$ does satisfy his budget constraint. In order to prove the preference maximization, suppose, on the contrary, that there exist i and (x_i, y) such that

$$(x_i,\ y) \in B_i(p(m^*),\ t(m^*)) \tag{1}$$

$$(x_i,\ y) \succ_i (X_i(m^*),\ Y(m^*)). \tag{2}$$

By monotonicity, we can assume, without loss of generality, that the budget constraint is satisfied with equality. By the boundary condition, one can prove that

$$0 \ll X_k(m^*) \ll \sum_{j=1}^{n} \mathring{\omega}_j^x \qquad \forall k.$$

Hence by the convexity of preferences, taking the convex combination if necessary, we can assume without loss of generality that y is sufficiently close to $Y(m^*)$, so that (x_i, y) is attainable for him, which contradicts the fact that m^* is a Nash equilibrium.

Q.E.D.

5. Linear Production Function and Pigouvian Equilibria

In this section, I will consider an economy with a linear production function. Consider the same economy as the previous sections except that we have a linear production function of the form:

$$x + \rho y = 0, \qquad \text{where } \rho \in \Re^l_+ \setminus \{0\}.$$

The attainable set of this economy is defined as

$$A \equiv \left\{ (x, y) \in \Re^n_+ \times \Re^{ln}_+ : \quad \sum_i (x_i - \omega^x_i) + \rho \sum_i (y_i - \omega^y_i) = 0 \right\}.$$

The Pigouvian equilibrium is defined in the same manner, namely:

DEFINITION 5.1: $(p^*,\ t^*,\ x^*,\ y^*) \in \Re^l \times \Re^{ln^2} \times \Re^n_+ \times \Re^{ln}_+$ is called a *Pigouvian competitive equilibrium* if

(1) t^* is a transfer system,

(2) $(x^*_i,\ y^*_i;\ y^*_{-i})$ is a maximal element of $\succ_i$ in $B_i(p^*,\ t^*)$, namely,

(2.1) $(x^*_i, y^*) \in B_i(p^*, t^*)$

(2.2) $P_i(x^*_i, y^*) \cap B_i(p^*, t^*) = \emptyset$

where

$$B_i(p, t) \equiv \left\{ (x_i,\ y_i;\ y_{-i}) \in \Re_+ \times \Re^{ln}_+ : \quad x_i + py_i \leq \omega^x_i + p\omega^y_i - \textstyle\sum_j t_{ij}(y_j - \omega^y_j) \right\}$$

(3) $p^* = \rho$

(4) $(x^*,\ y^*) \in A$.

The corresponding allocation allocation.

Now using proofs similar to those appeared in section 3, one can prove the following theorems:

THEOREM 5.1: (Existence) *For all i, assume the following:*

(1) $P_i(\cdot)$ *has an open graph.* (Continuity)

(2) For all $(x_i, y) \in \Re_+ \times \Re^l_+ \times \Re^{l(n-1)}_+$ $\quad (x_i, y) \notin \text{conv } P_i(x_i, y)$. (Convexity)

(3) $\omega^x_i \in \Re_{++}$.

Then there exists a Pigouvian competitive equilibrium.

THEOREM 5.2: (Non-Wastefulness) *Every Pigouvian competitive allocation is Pareto optimal.*

THEOREM 5.3: (Unbiasedness) *For all i, assume the following:*

(1) $P_i(\cdot)$ *is open-valued.* (Continuity)

(2) $P_i(\cdot)$ *is convex-valued.* (Convexity)

Then every Pareto optimal allocation $(x^*,\ y^*) \in \Re^n_{++} \times \Re^{ln}_{++}$ *can be attained as a Pigouvian competitive allocation provided that the initial endowments are suitably redistributed.*

THEOREM 5.4: (Individual Rationality) *Every Pigouvian competitive allocations is individually rational.*

Now let us consider th feasible Nash implementation. We will modify the previous mechanism and will get the following mechanism:

DEFINITION 5.2: (Message Space) For all i, let

$$M^i \equiv \Re^{ln} \times (0,\ \mathring{\omega}_i] \qquad \text{and} \qquad M \equiv \prod_i M^i.$$

The representative strategy of consumer i, is denoted by $m_i \equiv ((y_{ij})_j,\ \omega_i)$.

For a given $m \equiv (m_i)_i \equiv ((y_{ij})_j,\ \omega_i)_i \in M$, define the following mechanism:

DEFINITION 5.3:

$$t_{ij}(m) \equiv y_{i+1,j} - y_{i+2,j}.$$

DEFINITION 5.4:

$$D(m) \equiv \left\{ (y_1, \cdots, y_n) \in \Re_+^{ln} : \quad \rho y_i + \sum_j t_{ij}(m)(y_j - \omega_j^y) \leq \omega_i^x + \rho\omega_i^y \quad \forall\ i \right\}.$$

DEFINITION 5.5:

$$y_j(m) \equiv \sum_i y_{ij} - \sum_i y_{i,j+1} + \omega_j^y.$$

DEFINITION 5.6:

$$(Y_1,\ \ldots,\ Y_n)(m) \equiv \textbf{argmin}\left\{ \|y - y(m)\| : \quad y \in D(m) \right\}.$$

and

$$X_i(m) \equiv \rho\Big(\omega_i^y - Y_i(m)\Big) + \sum_j t_{ij}(m)\Big(\omega_j^y - Y_j(m)\Big) + \omega_i^x.$$

Then one can assert the following theorems assuming Assumptions 4.1 - 4.4:

THEOREM 5.5: *This mechanism is continuous and feasible.*

THEOREM 5.6: *The set of Nash allocations coincides with the set of Pigouvian competitive allocations.*

REFERENCES

1. M. Aoki, Marshallian external economies and optimal tax-subsidy structure, *Econometrica* 39 (1971), 35-53.
2. X. Calsamiglia, Decentralized resource allocation and increasing returns, *Journal of Economic Theory* 14 (1977) 262-273.
3. L. Hurwicz, On informationally decentralized systems, in *Decision and Organization* eds. R. Radner and C. B. McGuire, Amsterdam, North-Holland (1972), 297-336.
4. L. Hurwicz and D. Schmeidler, Construction of outcome functions guaranteeing existence and Pareto optimality of Nash equilibria, *Econometrica* 46 (1978), 1447-1474.
5. L. Hurwicz, Outcome functions yielding Walrasian and Lindahl allocations at Nash equilibrium points, *Review of Economic Studies* 46 (1979), 217-225.
6. L. Hurwicz, E. Maskin and A. Postlewaite, Feasible implementation of social choice correspondence by Nash equilibria, *mimeographed.* (1984).
7. J. S. Jordan, A noncooperative game form implementing Walrasian equilibria for production environments, *mimeographed.* (1982).
8. J. S. Jordan, Instability in the implementation of Walrasian allocations, *Journal of Economic Theory* 39, (1986), 301-328.
9. T. Kim, Stability problems in the implementation of Lindahl allocations, ph.D. Thesis in University of Minnesota, (1987).
10. Y. Kwan and S. Nakamura, On Nash implementation of the Walrasian or Lindahl correspondence in the two-agent economy, Discussion Paper no. 243, University of Minnesota (1987).
11. E. Maskin, Nash equilibrium and welfare optimality, *mimeographed.* (1977).
12. H. Osana, Optimal tax-subsidy system for an economy with Marshallian external-

ities, *Econometrica* 45, (1977), 329-340.

13. Y. Otani and J. Sicilian, Externalities and problems of nonconvexity and overhead costs in welfare economics, *Journal of Economic Theory* 14, (1977) 239-251.
14. A. Postlewaite and D. Wettstein, Implementing constrained Walrasian equilibria continuously, CARESS Working paper #83-24, University of Pennsiylvania, (1983).
15. T. Saijo, Strategy space reduction in Maskin's theorem: sufficient conditions fo Nash implementation, *Econometrica* 56, (1988), 693-700.
16. P. Samuelson, The pure theory of public expenditure, *Review of Economic Studies* 36, (1954), 387-389.
17. D. Schmidler, Walrasian analysis via strategic outcome functions, *Econometrica* 48, (1980), 1585-1594.
18. W. J. Shafer and H. Sonnenshein, Equilibrium in abstract economies without ordered preferences, *Journal of Mathematical Economics* 2, (1975), 345-348.
19. M. Walker, A simple incentive compatible scheme for attaining Lindahl allocations, *Econometrica* 49, (1981), 65-71.