ON NASH IMPLEMENTATION OF THE WALRASIAN
OR LINDAHL CORRESPONDENCE
IN THE TWO-AGENT ECONOMY

by

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September 1987

Abstract: Various possibility and impossibility theorems are obtained concerning Nash implementation of the Walrasian or Lindahl correspondence in two-agent economies.   The result is drastically different from the case with more than two agents.   There is neither a continuous and balanced, nor a smooth and weakly balanced mechanism which implements either of these two correspondences.   For both the Walrasian and Lindahl cases, however, there are mechanisms which implement the correspondence with properties which are either continuous and weakly balanced, or smooth but not weakly balanced.

1. Introduction

This paper investigates the existence and non-existence of mechanisms which yield Walrasian or Lindahl allocations at Nash equilibrium points for a two-agent economy in a classical environment.

In the literature, some regularity conditions are presupposed on the outcome functions. These regularity conditions include continuity or smoothness, (weak) balancedness, and individual feasibility. Individual feasibility means that all possible outcomes are in the consumption set for each agent. (Weak) balancedness together with individual rationality require that the game should remain solvent at all times. While it is certainly desirable to have (weak) balancedness and individual feasibility, the desire for continuity or smoothness may need some justification. Postlewaite (1985) argued that since arriving at a Nash equilibrium presupposes mutually compatible expectations among all agents, it would be informationally less "bothersome" for the agents if the outcome function is continuous or smooth. The reason is that in using Nash strategies, an agent needs information about other agents' optimal strategies in order to compute his own. If the outcome function is assumed to be continuous or smooth, a small error in prediction will only result in a small change in the outcome, and hence, in strategies. Another conceivable reason why it is less informationally "bothersome" for agents is that if the outcome function is smooth the agents can approximate prediction error linearly to determine their optimal amount of information to be collected. Another advantage of having a continuous or smooth outcome function comes from practical considerations. Suppose that the (true) characteristics of some agents have changed a little bit but the new characteristics are still in the class of the admissible environment. In this case, the form of the mechanism may remain the same but resources need to be reallocated. If resource reallocation is costly, it may be more desirable not to reallocate even if the original allocation is no longer optimal. The loss of efficiency will be small if the outcome function is smooth. Moreover, even if resource reallocation is costless, arriving at the new equilibrium may need a dynamic adjustment process. Presumably, an outcome function with regularity will require fewer iterations in computation than an irregular one.

In the many-agent case[1], mechanisms proposed by Schmeidler (1976), Groves and Ledyard (1977), Hurwicz (1979a, c), Walker (1981), and Hurwicz, Maskin and Postlewaite (1984) met some of the requirements mentioned above, while at the same time Nash-implementing a Walrasian or Lindahl correspondences on classical environments. Recently Tian (1985) has successfully designed a relevant mechanism with all the properties required except smoothness.

The picture is less rosy in the two-agent case. Mechanisms by Hurwicz (1979c) and Miura (1982) have balanced but discontinuous outcome functions. Reichelstein (1984) proved that there is no smooth balanced mechanism which Nash-implements the Walrasian correspondence. Subsequently, Vega-Redondo (1985) obtained the corresponding impossibility result for the Lindahl case. Furthermore, Hurwicz and Weinberger (1984) showed that the Pareto correspondence cannot be implemented by a smooth and strictly concave mechanism. Nakamura (1987) constructed a mechanism which is continuous, individually feasible, weakly balanced, and Nash implements the Walrasian/Lindahl correspondence.

But what about the smooth and weakly balanced or continuous and balanced mechanisms? We will answer this question in the negative. We prove that there is no smooth and weakly balanced or continuous and balanced mechanism that could Nash-implement the Walrasian/Lindahl correspondence on a classical environment for the two-agent cases. Moreover, these impossibility theorems are independent of individual feasibility and informational efficiency requirements. For purpose of completeness and comparison, the results in Nakamura (1987) are also reported without proof.

The above discussion is summarized in the following table.

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[1] In this paper, by "many-agent" we mean at least three but a finite number of agents.

	balanced	weakly balanced	not necessarily weakly balanced
smooth	Non-Existence ⇐ Reichelstein (1984) Vega-Redondo (1985) Hurwicz & Weinberger (1984) ⇑	Non-Existence Theorems 2.3 3.3	Existence Theorems 2.4 3.4 ⇓
continuous	Non-Existence Theorem 2.2 3.2	Existence ⇒ Nakamura (1987)	Existence ⇓
discontinuous	Existence ⇒ Hurwicz (1979c) Miura (1982)	Existence ⇒	Existence

⇒ denotes logical implication.

As can be seen from the table, a tradeoff must be made between continuity (*resp.* smoothness) and balancedness (*resp.* weak balancedness). This is in sharp contrast to the many agents case, where all entries of the table would be read "existence" . In this case, the Walker (1981) mechanism occupies the top left corner which, as a logical consequence, implies existence for all other situations. Finally we note that both Reichelstein (1984) and Vega-Redondo (1985) are special cases of our impossibility theorems.

This paper is organized as follows: The Walrasian (*resp.* Lindahl) case is examined in Section 2 (*resp.* 3). Notations and definitions are introduced in Section 2.1 (3.1), existence and non-existence theorems can be found in Section 2.2 (3.2). All proofs are relegated to Appendices.

2. Walrasian Case

2.1. Notations and Definitions

Consider a two-consumer pure exchange economy with $\ell + 1$ goods, which are denoted by $(x, y) \in \Re_+ \times \Re_+^\ell$, where x is chosen to be the numeraire. The characteristics of each agent i consist of his consumption set $\Re_+^{\ell+1}$, his initial endowment $(\omega_{ix}, \omega_{iy}) \in \Re_{++}^{\ell+1}$ both of which are fixed in this paper, and his preference relation.

Let $\mathcal{E}$ be the set of all admissible preference profiles. It is a subset of $\wp[(\Re_+^{\ell+1})^2] \times \wp[(\Re_+^{\ell+1})^2]$ where $\wp(X)$ means the set of all subsets of X, $i.e.$, the power set of X. The representative element $(R_1, R_2) \in \mathcal{E}$ is interpreted as for all i, R_i the i-th consumer's weak preference relation.

For all $R \in \mathcal{E}$, the set of Walrasian allocations is denoted by $W(R)$. W is said to be the Walrasian correspondence.

The mechanism discussed here is a vector $(M_1 \times M_2, h)$ where M_i is a non-empty set which is called a message space of agent i and h is a function from $M_1 \times M_2$ into $(\Re^{\ell+1})^2$ which is said to be an outcome function. More explicitly, we use the following notation:

$$h(m) \equiv (X_1(m), Y_1(m), X_2(m), Y_2(m)) \qquad \forall m \in M \equiv M_1 \times M_2$$

. The set of all Nash equilibria is denoted by $\nu(R)$. The set of the corresponding allocations $h(\nu(R))$ is denoted by $N(R)$. N is said to be a Nash correspondence.

Let us consider the following conditions about the mechanism.

DEFINITION 2.1: A mechanism (M, h) is said to *implement the Walras correspondence* if

$$W(R) = N(R) \qquad \forall R \in \mathcal{E}.$$

DEFINITION 2.2: A mechanism (M, h) is said to be *weakly balanced* if

$$X_1(m) + X_2(m) \leq \omega_{1x} + \omega_{2x}$$

$$Y_1(m) + Y_2(m) \leq \omega_{1y} + \omega_{2y}$$

$\forall m \in M$. It is said to be *balanced* if inequality $\leq$ is replaced by equality $=$.

DEFINITION 2.3: A mechanism (M, h) is said to be *individually feasible* if for each i,

$$(X_i(m), Y_i(m)) \in \Re_+^{\ell+1} \qquad \forall m \in M.$$

2.2. Main Results

In this section, we investigate the existence and non-existence of mechanisms which implement the Walrasian mechanism which is continuous, weakly balanced, and individually feasible.

THEOREM 2.1: *(Nakamura (1987)):   Assume $\mathcal{E}$ satisfies the following conditions: For each $R \in \mathcal{E}$ and for each i,*
(1) R_i is transitive, reflexive, and total on $\Re_+^{\ell+1}$,
(2) R_i is strictly monotone on $\Re_{++}^{\ell+1}$,
(3) R_i is convex in the sense that for all (x, y) and $(x', y') \in \Re_+^{\ell+1}$,

$$(x, y)P_i(x', y') \quad \Longrightarrow \quad t(x, y) + (1 - t)(x', y')P_i(x', y')$$

for all $t \in (0, 1)$, where P_i represents the asymmetric part of R_i.
(4) For all $(x, y) \in \Re_{++}^{\ell+1}$, and $(x', y') \in \partial\Re_+^{\ell+1}$, $(x, y)P_i(x', y')$.
Then there exists a continuous, weakly balanced, and individually feasible mechanism which implements the Walrasian correspondence.

For the proof see Nakamura (1987). Notice that the preferences represented by the familiar Cobb-Douglas utility functions belong to the class $\mathcal{E}$. It is impossible to strengthen the conditions of weak balancedness or continuity stronger to balancedness or smoothness. The following two theorems state these impossibility results when the environments are large enough.

THEOREM 2.2: *Assume that $\ell = 1$ and for all $\alpha \in (0,1)$, there exists $R^\alpha = (R_1^\alpha, R_2^\alpha) \in \mathcal{E}$ such that for each i, R_i^α is represented by the utility function u_i of the form*

$$u_i(x,y) = x + \alpha y \qquad \forall (x,y) \in \Re_+^2.$$

Then there exists no mechanism (M,h) which satisfies the following conditions:
(1) (M,h) implements the Walras correspondence,
(2) For each i, M_i is a second countable topological space and h is continuous,
(3) (M,h) is balanced.

THEOREM 2.3: *Assume that $\ell = 1$ and for all $\alpha \equiv (\alpha_1, \alpha_2) \in (0,1)^2$, there exists $(R_1^\alpha, R_2^\alpha) \in \mathcal{E}$ such that for each i, R_i^α is represented by the Cobb-Douglas utility function u_i of the form:*

$$u_i(x,y) = x^{\alpha_i} y^{1-\alpha_i} \qquad \forall (x,y) \in \Re_+^2.$$

Then there exists no mechanism (M,h) which satisfies the following conditions:
(1) (M,h) implements the Walrasian correspondence
(2) For each i, M_i is a k_i-dimensional differential manifold and h is $(k_1 + k_2)$-times continuously differentiable,
(3) (M,h) is weakly balanced.

Hurwicz (1979b) showed that if a mechanism implements a correspondence which is Pareto optimal and individually rational, then it essentially implements the Walrasian correspondence, when the environments are sufficiently rich. Hence Theorems 2.2 and 2.3 imply there is no "well-behaved" mechanism which attains Pareto optimality and individual rationality for two consumers when the environments are large enough.

Can we weaken balancedness (weak balancedness) or continuity (smoothness) so as to obtain a stronger result? The answer is no. We seek a mechanism that is:

(1) balanced, but not continuous; or

(2) smooth, but not weakly balanced.

The answer to (1) is known from Hurwicz (1979c), who constructed a mechanism which implements the Walrasian correspondence, and is balanced but not continuous. The following theorem treats the second case.

THEOREM 2.4: *Assume that for all $R \in \mathcal{E}$ and for each i, R_i satisfies the following conditions:*

(1) R_i is transitive, reflexive, and total,

(2) R_i is strictly monotone on $R_+^{\ell+1}$.

Define a mechanism (M, h) as follows.

$$M_i = \Re_{++}^{\ell} \times \Re^{\ell} \qquad \text{for all } i.$$

For all $m \equiv (p_1, y_1; p_2, y_2) \in M$, define

$$Y_i(m) = y_i - y_{-i} + \omega_{iy}$$

$$X_i(m) = -p_{-i}(Y_i(m) - \omega_{iy}) - (p_1 - p_2)^2 + \omega_{ix}$$

for each i. Then this mechanism implements the Walrasian correspondence, and is smooth but not weakly balanced.

The fact that this mechanism really works is readily verified. Note that in this mechanism, the "central planning bureau" need not to know the initial endowments if we use net trade in all definitions and modify the mechanism accordingly.

3. Lindahl Case

3.1. Notations and Definitions

Consider a two-consumer economy with one private good and one public good, both of which are connected by a linear production function:

$$\alpha x + \beta y \leq 0, \qquad \alpha > 0, \quad \beta > 0,$$

where $-x$ represents the net input of the private good and y represents the net output of the public good. Without loss of generality, we can assume $\alpha = \beta = 1$. Each agent i has his initial endowment $\omega_i \in \Re_{++}$ of the private good. The characteristics of each agent i consist of his consumption set $\Re_+^2$, his initial endowment ω_i, both of which are fixed in this paper, and his preference relation. Let $\mathcal{E}$ be a subset of $\wp[(\Re_+^2)^2] \times \wp[(\Re_+^2)^2]$. The representative element $(R_1, R_2) \in \mathcal{E}$ is interpreted as for all i, R_i the i-th consumer's weak preference relation.

For all $R \in \mathcal{E}$ the set of Lindahl allocations is denoted by $L(R)$. L is said to be a Lindahl correspondence.

The mechanism discussed here is a vector $(M_1 \times M_2, h)$ where M_i is non-empty set which is called a message space and h is a function from $M_1 \times M_2$ into $\Re^3$ which is said to be an outcome function. More explicitly, we use the following notation:

$$h(m) \equiv (X_1(m), X_2(m), Y(m)) \qquad \forall m \in M \equiv M_1 \times M_2.$$

The set of all Nash equilibria is denoted by $\nu(R)$. The set of the corresponding allocations $h(\nu(R))$ is denoted by $N(R)$. N is said to be a Nash correspondence.

Let us consider the following conditions about the mechanism.

DEFINITION 3.1: A mechanism (M, h) is said to *implement the Lindahl correspondence* if

$$L(R) = N(R) \qquad \forall R \in \mathcal{E}.$$

DEFINITION 3.2: A mechanism (M, h) is said to be *weakly balanced* if

$$X_1(m) + X_2(m) + Y(m) \leq \omega_1 + \omega_2 \qquad \forall m \in M.$$

It is said to be *balanced* if inequality $\leq$ is replaced by equality $=$.

DEFINITION 3.3: A mechanism (M, h) is said to be *individually feasible* if for each i,

$$(X_i(m), Y(m)) \in \Re^2_+ \qquad \forall m \in M.$$

3.2. Main Results

In this section we give the parallel results for Lindahl case.

THEOREM 3.1: *(Nakamura (1987)):* *Assume $\mathcal{E}$ satisfies the following conditions: For each $R \in \mathcal{E}$ and for each i,*
(1) R_i is transitive, reflexive, and total on $\Re^2_+$,
(2) R_i is strictly monotone on $\Re^2_{++}$,
(3) R_i is convex in the sense that for all (x, y) and $(x', y') \in \Re^2_+$,

$$(x, y)P_i(x', y') \quad \Longrightarrow \quad t(x, y) + (1 - t)(x', y')P_i(x', y')$$

for all $t \in (0, 1)$, where P_i represents the asymmetric part of R_i.
(4) For all $(x, y) \in \Re^2_{++}$, and $(x', y') \in \partial\Re^2_+$, $(x, y)P_i(x', y')$.
Then there exists a continuous, weakly balanced, and individually feasible mechanism which implements the Lindahl correspondence.

For the proof see Nakamura (1987).

THEOREM 3.2: *Assume that for all $\alpha \in (0, 1)$, there exists $R^\alpha \in \mathcal{E}$ such that R_i^α is represented by the utility function of the form:*

$$u_1(x_1, y) = x_1 + \alpha y \qquad \forall(x_1, y) \in \Re^2_+$$

and R_2 is represented by the utility function u_2 of the form:

$$u_2(x_2, y) = x_2 + (1 - \alpha)y \qquad \forall(x_2, y) \in \Re_+^2$$

Then there exists no mechanism (M, h) which satisfies the following conditions:

(1) (M, h) implements the Lindahl correspondence,

(2) For each i, M_i is a second countable topological space and h is continuous,

(3) (M, h) is balanced.

THEOREM 3.3: *Assume for all $\alpha \equiv (\alpha_1, \alpha_2) \in (0, 1)^2$, there exists $R^\alpha \in \mathcal{E}$ such that for each i, R_i^α is represented by the utility function u_i of the form*

$$u_i(x_i, y) = x_i^{\alpha_i} y_i^{1-\alpha_i} \qquad \forall(x_i, y) \in R_+^2.$$

Then there exists no mechanism (M, h) which satisfies the following conditions:

(1) (M, h) implements the Lindahl correspondence,

(2) For each i, M_i is a k_i-dimensional differentiable manifold and h is $(k_1 + k_2)$-times continuously differentiable,

(3) (M, h) is weakly balanced.

The assumptions of balancedness (weak balancedness) or continuity (smoothness) in Theorem 3.1 (3.2) cannot be weakened anymore. In fact, there exists mechanism that is

(1) balanced, but not continuous; or

(2) smooth, but not weakly balanced.

In case (1), Hurwicz (1979c) and Miura (1982) constructed such a mechanism which implements the Lindahl correspondence. The following theorem treats the second case.

THEOREM 3.4: *Assume that for all $R \in \mathcal{E}$ and for each i, R_i satisfies the following conditions:*

(1) R_i is transitive, reflexive, and total,

(2) R_i is strictly monotone on $\Re_+^2$.

Define a mechanism (M, h) as follows:

$$M_i = \Re_{++} \times \Re \qquad \text{for all } i.$$

For all $m \equiv (p_1, y_1; p_2, y_2) \in M$, define

$$Y(m) = y_1 + y_2$$

$$X_i(m) = -p_{-i} Y(m) - (1 - p_1 - p_2)^2 + \omega_i \qquad \text{for each } i.$$

Then this mechanism implements the Lindahl correspondence, and is smooth but not weakly balanced.

Appendix A: Proofs of Theorems 2.2 and 2.3

PROOF OF THEOREM 2.2: For all $m \in M$, let the vector R^α be the preference profile appeared in the statement of Theorem 2.2. The set of all such preference profiles are denoted by $\mathcal{E}_L \subset \mathcal{E}$.

Suppose that there exists a continuous and balanced mechanism $(M, h) \equiv (M_1 \times M_2; X_1, Y_1; X_2, Y_2)$ which implements Walrasian correspondence. Define

$$B^\circ \equiv \{(x_1, y_1, x_2, y_2) \in \Re^4_{++} | \quad x_1 + x_2 = \omega_{1x} + \omega_{2x}, \quad y_1 + y_2 = \omega_{1y} + \omega_{2y}$$

$$x_1 < \omega_{1x}, \quad y_1 > \omega_{1y}, \quad x_1 + y_1 > \omega_{1x} + \omega_{1y}\}.$$

LEMMA 4.1: *For all $\alpha \in (0, 1)$, $B^\circ \cap W(R^\alpha)) \neq \emptyset$.*

PROOF: Fix $\alpha \in (0, 1)$. Choose y_1 such that $\omega_{1y} < y_1$ and sufficiently close to ω_{1y}. Define

$$y_2 = \omega_{1y} + \omega_{2y} - y_1, \qquad x_1 = \omega_{1x} + \alpha(\omega_{1y} - y_1), \qquad x_2 = \omega_{1x} + \omega_{2x} - x_1.$$

Then $(x_1, y_1, x_2, y_2) \in B^\circ \cap W(R^\alpha)$.

<div align="right">

Q.E.D.

</div>

For all $m \in h^{-1}(B^\circ) \equiv V$, define

$$p(m) \equiv \frac{\omega_{1x} - X_1(m)}{Y_1(m) - \omega_{1y}} = \frac{\omega_{2x} - X_2(m)}{Y_2(m) - \omega_{2y}}.$$

Let $\{O_i\}$ be a countable basis of M_i, and U be a subset of $\{O_1\} \times \{O_2\}$ such that $O_1 \times O_2 \subset V$. Since balancedness implies

$$V = h^{-1}\left[\{(x_1, y_1, x_2, y_2) \in \Re^4_{++} | \quad x_1 < \omega_{1x}, \quad y_1 > \omega_{1y}, \quad x_1 + y_1 > \omega_{1x} + \omega_{1y}\}\right],$$

which is open, U is an open covering of V.

LEMMA 4.2: *For all $O_1 \times O_2 \in U$, and for all $m^* \in (O_1 \times O_2) \cap \nu(\mathcal{E}_L)$, we have the following "saddle point" properties:*

(a)
$$p(m^*) \leq p(m_1, m_2^*) \qquad \forall m_1 \in O_1$$

(b)
$$p(m^*) \geq p(m_1^*, m_2) \qquad \forall m_2 \in O_2.$$

PROOF: Fix $m^* \in (O_1 \times O_2) \cap \nu(\mathcal{E}_L)$. Then for some α, $m^* \in \nu(R^\alpha)$ and $h(m^*) \in B^o$. By the definition of p, we know that

$$X_1(m^* + p(m^*)Y_1(m^*) = \omega_{1x} + p(m^*)\omega_{1y}. \tag{1}$$

Fix $m_1 \in O_1$. Then since $m^* \in \nu(R^\alpha)$, it follows that

$$X_1(m^*) + \alpha Y_1(m^*) \geq X_1(m_1, m_2^*) + \alpha Y_1(m_1, m_2^*) \tag{2}$$

By Nash implementation, we know that $\alpha = p(m^*)$. Hence by (1) and (2) we can get

$$\omega_{1x} + p(m^*)\omega_{1y} \geq X_1(m_1, m_2^*) + p(m^*)Y_1(m_1, m_2^*) \tag{3}$$

On the other hand, by the definition of p,

$$\omega_{1x} + p(m_1, m_2^*)\omega_{1y} = X_1(m_1, m_2^*) + p(m^*)Y_1(m_1, m_2^*). \tag{4}$$

Since $(m_1, m_2^*) \in O_1 \times O_2 \subset h^{-1}(B^o)$ implies $Y_1(m_1, m_2^*) > \omega_{1y}$, it follows that $p(m*) \leq p(m_1, m_2^*)$.

Analogously, one can show the second statement.

$Q.E.D.$

LEMMA 4.3: *For each $O_1 \times O_2 \in U$, p is constant on $(O_1 \times O_2) \cap \nu(\mathcal{E}_L)$.*

PROOF: Suppose that there exist m and $n \in (O_1 \times O_2) \cap \nu(\mathcal{E}_L)$ such that $p(n) < p(m)$. Since $n_1 \in O_1$, $p(m) \leq p(n_1, m_2)$ by part (a) of Lemma 4.2. Thus $p(n) < p(n_1, m_2)$.

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This contradicts part (b) of Lemma 4.2.

<div align="right">*Q.E.D.*</div>

Lemma 4.3 implies the range of p is countable on $V \cap \nu(\mathcal{E}_L)$. But by Nash implementation and Lemma 4.1, the range of p of this set is equal to $(0,1)$, which is a contradiction. This completes the proof of Theorem 2.2.

<div align="right">*Q.E.D.*</div>

PROOF OF THEOREM 2.3: For all $(\alpha_1, \alpha_2) \in (0,1)^2$, let R^α be the corresponding preference profile. Then one can show that $(x_1^*, y_1^*, x_2^*, y_2^*) \in W(R^\alpha)$ if and only if

$$x_1^* = \alpha_1(\omega_{1x} + p^*\omega_{1y}), \quad y_1^* = \frac{(1-\alpha_1)(\omega_{1x} + p^*\omega_{1y})}{p^*},$$

$$x_2^* = \alpha_2(\omega_{2x} + p^*\omega_{2y}), \quad y_2^* = \frac{(1-\alpha_2)(\omega_{2x} + p^*\omega_{2y})}{p^*},$$

$$\text{where } p^* = \frac{(1-\alpha_1)\omega_{1x} + (1-\alpha_2)\omega_{2x}}{\alpha_1\omega_{1y} + \alpha_2\omega_{2y}}.$$

Hence for sufficiently small $\epsilon > 0$, we can get

$$\alpha \in (0,\epsilon) \times (1-\epsilon, 1) \quad \Longrightarrow \quad x_1^* < \omega_{1x} \quad \text{and} \quad y_1^* > \omega_{1y}.$$

The set of all associated preference profiles parameterized by $\alpha \in (0, \epsilon) \times (1 - \epsilon, 1)$ is denoted by $\mathcal{E}_C \subset \mathcal{E}$. Define

$$B^o \equiv \{(x_1, y_1, x_2, y_2) \in R^4_{++} | \quad x_1 < \omega_{1x}, \quad y_1 > \omega_{1y}, \quad x_2 > \omega_{2x}, \quad y_2 < \omega_{2y}\}.$$

Note that $W(\mathcal{E}_C) \subset B^o$.

Suppose that there exits a $(k_1 + k_2)$-times continuously differentiable and weakly balanced mechanism $(M, h) = (M_1 \times M_2; X_1, Y_1; X_2, Y_2)$ which implements Walrasian correspondence. For each i $(i = 1, 2)$, define

$$p^i(m) \equiv \frac{\omega_{ix} - X_i(m)}{Y_i(m) - \omega_{iy}} \qquad \forall m \in h^{-1}(B^o).$$

Note that $\nu(\mathcal{E}_C) \subset h^{-1}h(\nu(\mathcal{E}_C)) = h^{-1}(W(\mathcal{E}_C)) \subset h^{-1}(B^o)$.

LEMMA 4.4: *For all* $m \in \nu(\mathcal{E}_C)$, $D_i p^i(m) = 0$ *for each* i, *where* D_i *indicates the partial derivatives with respect to* m_i.

PROOF: Fix $m \in \nu(\mathcal{E}_L)$. Then there exists α such that $m \in \nu(R^\alpha)$. Let U_1 be the associated Cobb-Douglas utility function parameterized by α_1. Then since m is a Nash equilibrium,

$$D_x U_1(X_1(m), Y_1(m)) D_1 X_1(m) + D_y U_1(X_1(m), Y_1(m)) D_1 Y_1(m) = 0. \qquad (1)$$

By the definition of p^1,

$$p^1(m)\big[Y_1(m) - \omega_{1y}\big] = \omega_{1x} - X_1(m) \qquad \forall m \in h^{-1}(B^o).$$

By differentiating with respect to m_1, we can get

$$D_1 p^1(m)\big[Y_1(m) - \omega_{1y}\big] + p^1(m) D_1 Y_1(m) = -D_1 X_1(m). \qquad (2)$$

By Nash implementation, we know that h(m) is the Walrasian allocation. Hence

$$p^1(m) = \frac{D_y U_1(X_1(m), Y_1(m))}{D_x U_1(X_1(m), Y_1(m))}. \qquad (3)$$

From (1), (2) and (3), we can get $D_1 p^1(m)[Y_1(m) - \omega_{1y}] = 0$. Since $h(m) \in B^o$, $D_1 p^1(m) = 0$.

Analogously, it can be shown that $D_2 p^2(m) = 0$.

Q.E.D.

LEMMA 4.5: *For all* $m \in h^{-1}(B^o)$, $p^2(m) \leq p^1(m)$.

PROOF: Fix m. By the definition of p^i's, we can get

$$p^1(m)\big[Y_1(m) - \omega_{1y}\big] = \omega_{1x} - X_1(m), \qquad (4)$$

$$p^2(m)\big[Y_2(m) - \omega_{2y}\big] = \omega_{2x} - X_2(m). \qquad (5)$$

By weak balancedness,

$$X_1(m) + X_2(m) \le \omega_{1x} + \omega_{2x}, \tag{6}$$

$$Y_1(m) + Y_2(m) \le \omega_{1y} + \omega_{2y}. \tag{7}$$

From (4), (5), (6) and (7),

$$p^2(m)[Y_2(m) - \omega_{2y}] \ge p^1(m)[Y_2(m) - \omega_{2y}].$$

Since $Y_2(m) < \omega_{2y}$, $p^2(m) \le p^1(m)$.

<div align="right">*Q.E.D.*</div>

LEMMA 4.6: *For all $m \in \nu(\mathcal{E}_C)$, $D_i p^j(m) = 0$ for each i and j.*

PROOF: If $i = j$, it is proven in Lemma 4.4. Fix $m \in \nu(\mathcal{E}_C)$. By Nash implementation, we know that $p^1(m) = p^2(m)$. Hence by Lemma 4.5, we know that m is a minimizer of $p^1(\cdot) - p^2(\cdot)$ over an open set $h^{-1}(B^o)$. Hence by the first order necessary conditions, we know that $D_i p^1(m) - D_i p^2(m) = 0$ for each i. Hence by Lemma 4.4, $D_i p^j(m) = 0$[1].

<div align="right">*Q.E.D.*</div>

Thus we know that for each $m \in \nu(\mathcal{E}_C)$, $Dp^1(m) \equiv (D_1 p^1(m), D_2 p^1(m)) = 0$. By Sard's Theorem, $p^1(\nu(\mathcal{E}_C))$ is of measure zero. But by Nash implementation, $p^1(\nu(\mathcal{E}_C))$ must have a positive measure, which is a contradiction. This completes the proof of Theorem 2.3.

<div align="right">*Q.E.D.*</div>

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[1] Reichelstein (1984) assumed that the set of Nash equilibria is a manifold. Under this assumption, however, we should have proven $D_i p^j(m) = 0$ without assuming weak balancedness. Thus we can still obtain the impossibility theorem. But from Theorem 2.4, which assures the existence in such a case, this assumption is very strong.

Appendix B: Sketch of Proofs of Theorems 3.2 and 3.3

The proofs of Theorems 3.2 and 3.3 are similar to the proofs of Theorems 2.2 and 2.3. So we will sketch the proof.

PROOF OF THEOREM 3.2: For all $\alpha \in (0,1)$, let R^{α} be the corresponding preference profile. The set of all such preference profiles are denoted by $\mathcal{E}_L \subset \mathcal{E}$.

Suppose that there exists a continuous and balanced mechanism $(M, h) = (M_1 \times M_2; X_1, X_2, Y)$ which implements Lindahl correspondence. Define

$$B^{\circ} \equiv \{(x_1, x_2, y) \in \Re^3_{++} | \quad x_1 + x_2 + y = \omega_1 + \omega_2, \quad x_1 < \omega_1, \quad x_2 < \omega_2\}.$$

LEMMA 5.1: *For all* $\alpha \in (0,1)$, $B^{\circ} \cap L(R^{\alpha}) \neq \emptyset$.

PROOF: Fix α. Choose y such that $0 < y < \min\{\omega_1/\alpha, \omega_2/(1 - \alpha)\}$. Define $x_1 = \omega_1 - \alpha y$ and $x_2 = \omega_2 - (1 - \alpha)y$.

$$Q.E.D.$$

For all $m \in h^{-1}(B^0) \equiv V$, define

$$q^i(m) = \frac{\omega_i - X_i(m)}{Y(m)}.$$

Let $\{O_i\}$ be a countable basis of M_i, and U be the subset of $\{O_1\} \times \{O_2\}$ such that $O_1 \times O_2 \subset V$. Since V is open, U is an open covering of V.

LEMMA 5.2: *For all* $O_1 \times O_2 \in U$, *and for all* $m^* \in (O_1 \times O_2) \cap \nu(\mathcal{E}_L)$,

$$q^1(m^*) \leq q^1(m_1, m_2^*) \qquad \forall m_1 \in O_1,$$

$$q^2(m^*) \leq q^2(m_1^*, m_2) \qquad \forall m_2 \in O_2.$$

The proof is analogous to the proof of Lemma 4.2.

LEMMA 5.3: *For each $O_1 \times O_2 \in U$, q^i is constant on $(O_1 \times O_2) \cap \nu(\mathcal{E}_L)$.*

The proof is similar to that of Lemma 4.3, if we use the fact that $q^1(m)+q^2(m) \equiv 1$.

Lemma 5.3 implies the range of q^i is countable on $V \cap \nu(\mathcal{E}_L)$. But by Nash implementation, the range of q^i is equal to $(0,1)$, which is a contradiction.

$Q.E.D.$

PROOF OF THEOREM 3.3: For all $\alpha \in (0,1)^2$, let R^α be the corresponding preference profile. The set of all such preference profiles are denoted by $\mathcal{E}_C \subset \mathcal{E}$. Suppose that there exists a $(k_1 + k_2)$-times continuously differentiable and weakly balanced mechanism $(M,h) = (M_1 \times M_2; X_1, X_2, Y)$ which implements Lindahl correspondence. Let $B^o \equiv \Re^3_{++}$. For all $m \in h^{-1}(B^o) \equiv V$, define

$$q^i(m) \equiv \frac{\omega_i - X_i(m)}{Y(m)}.$$

LEMMA 5.4: *For all $m \in \nu(\mathcal{E}_C)$, $D_i p^i(m) = 0$ for each i.*

The proof is analogous to that of Lemma 4.4.

LEMMA 5.5: *For all $m \in h^{-1}(B^o)$, $q^1(m) + q^2(m) \geq 1$.*

This assertion is straightforward from weak balancedness of the outcome function.

LEMMA 5.6: *For all $m \in \nu(\mathcal{E}_C)$, $D_i q^j(m) = 0$ for each i and j.*

This assertion is obvious.

Thus we know that for each $m \in \nu(\mathcal{E}_C)$, $Dq^1(m) \equiv (D_1 q^1(m), D_2 q^1(m)) = 0$. By Sard's Theorem, $q^1(\nu(\mathcal{E}_C))$ is of measure zero. But by Nash implementation, the range of q^1 must have a positive measure, which is a contradiction.

$Q.E.D.$

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