CORE-WALRAS EQUIVALENCE IN ECONOMIES WITH A CONTINUUM OF AGENTS AND COMMODITIES†

by

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Abstract - This paper contains the following results for economies with infinite dimensional commodity spaces. (i) We establish a core-Walras equivalence theorem for economies with an atomless measure space of agents and with an ordered separable Banach commodity space whose positive cone has a non-empty norm interior. This result includes as a special case the Aumann (1964) and Hildenbrand (1974) finite dimensional theorems. (ii) We provide a counterexample which shows that the above result fails in ordered Banach spaces whose positive cone has an empty interior even if preferences are strictly convex, monotone and weak* continuous and initial endowments are strictly positive. (iii) After introducing a new assumption on preferences called "commodity pair desirability," (which is automatically satisfied whenever preferences are monotone and the positive cone of the commodity space has a non-empty interior), we establish core-Walras equivalence in any arbitrary separable Banach lattice whose positive cone may have an empty (norm) interior. (iv) We provide a proof that in some concrete spaces whose positive cone may have an empty interior, the assumption of an extremely desirable commodity or uniform properness suffices for core-Walras equivalence. Finally, (v) we indicate how our methods can be used to obtain core-Walras equivalence results for the space $M(\Omega)$ of measures on a compact metric space.

Key Words: Core allocations, Walrasian allocations, extremely desirable commodity, commodity pair desirability, Bochner integral, Banach lattice.

I. INTRODUCTION

Two of the most widely used solution concepts in economic theory are the competitive equilibrium and the core. The first concept is usually associated with Walras, and refers to the noncooperative allocation of resources via a price system. The essential idea behind this concept is that when agents are assumed to know only the price system (which they treat parametrically) and their own preferences and endowments, then are allowed to trade freely in a decentralized market, this process results in allocations which maximize agents' utilities (subject to their budgets) and equate supply with demand. The second concept is usually associated with Edgeworth, and refers to the allocation of resources via a pure quantity bargaining process. The essential idea behind this concept is that when agents are allowed to bargain freely (either multi- or bilateraly), this process leads to an allocation of resources where it is not possible for any coalition of agents to redistribute their initial endowments among themselves in a way that makes each member of the coalition better off. Thus, in contrast to the competitive equilibrium, the core allows for the possibility of cooperation among agents in the economy.

A classical conjecture about the relationship between these two concepts, attributed to Edgeworth, is that the core shrinks to the competitive equilibrium as the number of agents in the economy becomes large. This conjecture, often called the Edgeworth conjecture, and indeed the notion of the core (although not by this name) were first discussed by Edgeworth in 1881. However, the core was not the subject of modern research until it was formally introduced in the general (mathematical) theory of games by Gilles in

1953. Aumann (1964), in a pathbreaking paper, reformulated rigorously the Edgeworth conjecture in economics by showing that in perfectly competitive economies (i.e., economies with an atomless measure space of agents) with finitely many commodities, the core coincides with the competitive (or Walrasian) equilibrium. Hence, in perfectly competitive economies, core allocations completely characterize competitive equilibrium allocations.

The formal proof of this coincidence result has come to be known as the core-Walras equivalence theorem. In the past two decades, many researchers have studied this problem extensively in economies with finitely many commodities. This research has led to very general core-Walras equivalence results and approximate core-Walras equivalence results in economies with finitely many commodities. However, since our goal in this paper is to study core-Walras equivalence results in economies with infinitely many commodities, we will not elaborate further on these finite dimensional results, except where they have particular bearing on our work. However, we do refer the reader to Anderson (1986) for an excellent survey of this interesting literature.

Before proceeding to a discussion of the main results of our paper, it may be useful to discuss the general importance of infinite dimensional commodity spaces in economics. As others have observed (e.g., Court (1941), Debreu (1954), Gabszewicz (1967), Bewley (1970) and Peleg-Yaari (1970)), infinite dimensional commodity spaces arise very naturally in economics. In particular, an infinite dimensional commodity space may be desirable in problems involving an infinite time horizon, uncertainty about the possibly infinite number of states of nature of the world, or infinite varieties of commodity characteristics. For instance, the Lebesgue space

L_∞ of bounded measurable functions on a measure space considered by Bewley (1970), Gabszewicz (1967) and Mertens (1970) is useful in modeling uncertainty or an infinite time horizon. The space L_2 of square-integrable functions on a measure space considered by Gabszewicz (1968) and Duffie-Huang (1985), is useful in modeling the trading of long-lived securities over time. Finally, the space $M(\Omega)$ of measures on a compact metric space considered by Mas-Colell (1975), Hart (1979) and Jones (1984) is useful in modeling differentiated commodities.

In this paper, we study core-Walras equivalence results for perfectly competitive economies with an infinite dimensional commodity space which is general enough to include all of the spaces that have been found most useful in equilibrium analysis. In particular, we cover all the Lebesgue spaces L_p, $(1 \leqslant p \leqslant \infty)$, the space of measures, $M(\Omega)$ and the space of continuous functions on a compact metric space $C(X)$.[1] The results that we obtain in this context are four-fold.

Firstly, we prove core-Walras equivalence results for perfectly competitive economies with an infinite dimensional commodity space whose positive cone has a non-empty (norm) interior. Parts of this problem have been addressed by other researchers (i.e., Gabszewicz (1968), Mertens (1970) and Bewley (1973) for the space L_∞).[2] However, since our assumptions are less restrictive than those adopted in these previous papers, we obtain as corollaries of our results the finite dimensional theorems of Aumann (1964) and Hildenbrand (1974, Theorem 1, p. 133). The proof of this result is similar in spirit to that of Hildenbrand, except that owing to the infinite dimensional setting, we appeal to results on the integration of correspondences having values in a Banach space. The work of Khan (1985) is especially helpful in this regard.

Secondly, in infinite dimensional commodity spaces whose positive cone has an empty (norm) interior, we show that even under quite strong assumptions on preferences and endowments, core-Walras equivalence fails. In particular, we show that even when preferences are strictly convex, monotone, and weak* continuous and initial endowments are strictly positive, core-Walras equivalence fails to hold. It is interesting to note that this failure results despite the fact that these assumptions are much stronger than the standard assumptions which guarantee equivalence in either Aumann and Hildenbrand or our first theorem.

Thirdly, we obtain core-Walras equivalence for infinite dimensional commodity spaces (in particular, Banach lattices) whose positive cone may have an empty (norm) interior and are general enough to cover the spaces L_p $(1 \leqslant p < \infty)$ and $M(\Omega)$. In view of the above counterexample to core-Walras equivalence in spaces whose positive cone has an empty interior, we introduce a new condition on preferences called commodity pair desirability. In essence, this assumption is a strengthening of the assumption of an extremely desirable commodity used in Yannelis-Zame (1986), which in turn is related to the condition of uniform properness in Mas-Colell (1986).[3] All of these assumptions are essentially bounds on the marginal rate of substitution, and in practice turn out to be quite weak. For example, all three of these assumptions are automatically satisfied whenever preferences are monotone and the positive cone of the commodity space has a non-empty (norm) interior. Hence this assumption is implicit in the infinite dimensional work of Gabszewicz (1968), Mertens (1970), and Bewley (1973), and is automatically satisfied in the finite dimensional work of Aumann (1964) and Hildenbrand (1974). We also wish to note that in addition to the commodity pair desirability assumption, the lattice structure of the commodity space will play a crucial role in our analysis. However, since this role is rather technical in nature we will defer discussion of this point to Sections 7 and 9.

Finally, in spaces whose positive cone has an empty interior, we wish to determine whether extreme desirability (or uniform properness) is sufficient to ensure core-Walras equivalence. In particular, we introduce a linearity assumption (which is automatically satisfied in the Lebesgue space L_1) and show that the assumption of an extremely desirable commodity suffices for core-Walras equivalence in infinite dimensional spaces whose positive cone may have an empty (norm) interior.

The remainder of the paper is organized as follows: Section 2 contains notation, definitions and some results on Banach lattices and the integration of correspondences. The economic model is outlined in Section 3. In Section 4 we state and prove a core-Walras equivalence theorem for an ordered separable Banach space of commodities, whose positive cone has a non-empty (norm) interior. The failure of this result for spaces whose positive cone has an empty interior is established in Section 5. The central assumption of the paper, commodity pair desirability, is introduced in Section 6. In Section 7, we prove a core-Walras equivalence result for a commodity space which can be any arbitrary separable Banach lattice, provided that this assumption holds. In Section 8 we show that the extremely desirable commodity assumption suffices in some concrete spaces for core-Walras equivalence. Finally, some concluding remarks are given in Section 9.

2. PRELIMINARIES

2.1 Notation

$\mathbb{R}^\ell$ denotes the ℓ-fold Cartesian product of the set of real numbers $\mathbb{R}$.

intA denotes the interior of the set A.

2^A denotes the set of all nonempty subsets of the set A.

ϕ denotes the empty set.

/ denotes the set theoretic subtraction.

dist denotes distance.

If $A \subset X$ where X is a Banach space, $c\ell A$ denotes the norm closure of A.

If X is a Banach space its dual is the space X^* of all continuous linear functionals on X.

If $q \in X^*$ and $y \in X$ the value of q at y is denoted by $q \cdot y$.

2.2 Definitions

We begin by collecting some useful notions on Banach lattices (a more detailed exposition may be found in Aliprantis-Burkinshaw (1978, 1985) or Schaefer (1974)), which will be needed in the sequel.

A normed vector space is a real vector space E equipped with a norm $\|\cdot\|: E \to [0,\infty)$ satisfying:

(i) $\|x\| \geqslant 0$ for all x in E, and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all x in E and all α in $\mathbb{R}$;

(iii) $\|x+y\| \leqslant \|x\| + \|y\|$ for all x,y in E.

A Banach space is a normed vector space for which the metric induced by the norm is complete.

If E is a Banach space, then its dual space E^* is the set of continuous linear functionals on E. The dual space E^* is itself a Banach space, when equipped with the norm

$$\|\phi\| = \sup\{|\phi(x)| : x \in E, \|x\| \leq 1\}.$$

A Banach lattice is a Banach space L endowed with a partial order $\leq$ (i.e., $\leq$ is a reflexive, antisymmetric, transitive relation) satisfying:

1) $x \leq y$ implies $x + z \leq y + z$ (for all $x,y,z \in L$);
2) $x \leq y$ implies $tx \leq ty$ (for all $x,y \in L$, all real numbers $t \geq 0$);
3) every pair of elements $x,y \in L$ has a supremum (least upper bound) $x \vee y$ and an infimum (greatest lower bound) $x \wedge y$;
4) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ (for all $x,y \in L$).

Here we have written, as $|x| = x^+ + x^-$ where $x^+ = x \vee 0$, $x^- = (-x) \wedge 0$; we call x^+, x^- the positive and negative parts of x respectively and $|x|$ the absolute value of x. Note that $x = x^+ - x^-$, and that $x^+ \wedge x^- = 0$. We say that $x \in L$ is positive if $x \geq 0$; we write L_+ (or L^+) for the set of all positive elements of L and refer to L_+ (or L^+) as the positive cone of L.

We will say that an element x of L is strictly positive (and write $x >> 0$) if $\phi(x) > 0$ whenever ϕ is a positive non-zero element of L_+. Strictly positive elements are usually called quasi-interior to L_+. Note that if the positive cone L_+ of L has a non-empty (norm) interior, then the set of strictly positive elements coincides with the interior of L_+. However, many Banach lattices contain strictly positive elements even though the positive cone L_+ has an empty interior (see Aliprantis-Burkinshaw (1985, p. 259)). We will now give basic examples of separable Banach lattices.

(i) the Euclidean space $\mathbb{R}^N$;

(ii) the space ℓ_p $(1 \leq p < \infty)$ of real sequences (a_n) for which the norm $\|(a_n)\|_p = (\Sigma\ |a_n|^p)^{1/p}$ is finite;

(iii) the space $L_p(\Omega,\mathcal{R},\mu)$ $(1 \leq p < \infty)$ of equivalence classes of measurable functions f on the separable measure space $(\Omega,\mathcal{R},\mu)$ for which the norm $\|f\|_p = (\int_\Omega |f|^p d\mu)^{1/p}$ is finite;

(iv) the space $C(\Omega)$ of continuous, real-valued functions on the compact Hausdorff space Ω (with the supremum norm);

A basic property of Banach lattices which will play a crucial role in the sequel, is the Riesz Decomposition Property.

Riesz Decomposition Property: Let L be a Banach lattice and let x, $y_1,\ldots,y_n$ be positive elements of L such that $0 \leq x \leq \sum_{i=1}^{n} y_i$. Then there are positive elements $x_1,\ldots,x_n$ in L such that $\sum_{i=1}^{n} x_i = x$ and $0 \leq x_i \leq y_i$ for each i.

We now define some measure theoretic notions as well as the concepts of a Bochner integrable function and the integral of a correspondence.

Let X, Y be sets. The graph of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. Let (T,τ,μ) be a finite measure space, (i.e., μ is a real-valued, non-negative countably additive measure defined on a σ-field τ of subsets of T such that $\mu(T) < \infty$), and X be a Banach space. The correspondence $\phi : T \to 2^X$ is said to have a measurable graph if $G_\phi \in \tau \otimes \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-field. A function $f : T \to X$ is called simple if there exist $x_1, x_2,\ldots,x_n$ in X and $a_1, a_2,\ldots a_n$ in τ such that

$f = \sum_{i=1}^{n} x_i \chi_{a_i}$ where $\chi_{a_i}(t) = 1$ if $t \in a_i$ and $\chi_{a_i}(t) = 0$ if $t \notin a_i$. A function $f : T \to X$ is said to be μ-measurable if there exists a sequence of simple functions $f_n : T \to X$ such that $\lim_{n\to\infty} \| f_n(t) - f(t)\| = 0$ μ - a.e. A μ-measurable function $f : T \to X$ is said to be Bochner integrable if there exists a sequence of simple functions $\{f_n : n = 1,2,\ldots\}$ such that

$$\lim_{n\to\infty} \int_T \| f_n(t) - f(t)\| d\mu(t) = 0.$$

In this case we define for each $E \in \tau$ the integral to be $\int_E f(t)d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)d\mu(t)$. It can be easily shown (see Diestel-Uhl (1977, p. 45)) that if $f : T \to X$ is a μ-measurable function then f is Bochner integrable if and only if $\int_T \|f(t)\| d\mu(t) < \infty$. We denote by $L_1(\mu,X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : T \to X$ normed by $\|x\| = \int_T \|x(t)\| d\mu(t)$. Moreover, we denote by $\mathcal{L}_\phi$ the set of all X-valued Bochner integrable selections from the correspondence $\phi : T \to 2^X$, i.e.

$$\mathcal{L}_\phi = \{x \in L_1(\mu,X) : x(t) \in \phi(t)\ \mu \text{ - a.e.}\}.$$

The integral of the correspondence $\phi : T \to 2^X$ is defined as:

$$\int_T \phi(t)d\mu(t) = \{\int_T x(t)d\mu(t) : x \in \mathcal{L}_\phi\}.$$

In the sequel we will denote the above integral by

$$\int \phi \text{ or } \int_T \phi.$$

2.3 Lemmata

If (T,τ,μ) is atomless and $X = \mathbb{R}^{\ell}$, it follows from Lyapunov's Theorem that the integral of the correspondence $\phi : T \to 2^X$, i.e., $\int\phi$, is convex. However, this result is false in infinite dimensional spaces (see for instance Diestel-Uhl (1977, p.262)). Nevertheless, it can be easily deduced (see for instance Datko (1970) or Hiai-Umegaki (1977) or Khan (1985)) from the approximate version of Lyapunov's Theorem in infinite dimensional spaces that the norm closure of $\int\phi$, i.e., $c\ell\int\phi$, is convex. More formally the following Lemma is true.

Lemma 2.1: Let (T,τ,μ) be a finite atomless measure space, X be a Banach space and $\phi : T \to 2^X$ be a correspondence. Then $c\ell\int\phi$ is convex.

We will also need the following result whose proof follows from the measurable selection theorem and can be found in Hiai-Umegaki (1977, Theorem 2.2, p. 156).

Lemma 2.2: Let (T,τ,μ) be a finite measure space, X be a separable Banach space, and $\phi : T \to 2^X$ be a correspondence having a measurable graph. Suppose that $\int \phi \neq \emptyset$. Then for every $p \in X^*$ we have that

$$\inf_{z \in \int\phi} p \cdot z = \int \inf_{y \in \phi(\cdot)} p \cdot y.$$

It should be noted that Lemma 2.2 has been proved in Hildenbrand (1974, Proposition 6, p. 63) for $X = \mathbb{R}^{\ell}$. However, by recalling that the Aumann measurable selection theorem holds in separable metric spaces, one can easily see that Hildenbrand's argument remains true in separable Banach spaces. In fact, it is even true in arbitrary Hausdorff separable and metrizable linear topological spaces.

With all these preliminaries out of the way, we can now turn to our model.

3. ECONOMY, CORE AND COMPETITIVE EQUILIBRIUM

Denote by E the commodity space. Throughout this section the commodity space E will be an ordered Banach space.

An <u>economy</u> $\mathcal{E}$ is a quadruple $[(T,\tau,\mu), X, \succ, e]$ where

(1) (T,τ,μ) is a <u>measure space of agents</u>,

(2) $X : T \to 2^E$ is the <u>consumption correspondence</u>,

(3) $\succ_t \subset X(t)\times X(t)$ is the <u>preference relation</u>[4] of agent t, and

(4) $e : T \to E$ is the <u>initial endowment</u>, where e is Bochner integrable and $e(t) \in X(t)$ for all $t \in T$.

An <u>allocation</u> for the economy $\mathcal{E}$ is a Bochner integrable function $x : T \to E_+$. An allocation x is said to be <u>feasible</u> if $\int_T x(t)d\mu(t) = \int_T e(t)d\mu(t)$. A <u>coalition</u> S is an element of τ such that $\mu(S) > 0$. The coalition S can <u>improve upon</u> the allocation x if there exists an allocation g such that

(i) $g(t) \succ_t x(t)$ μ - a.e. in S, and

(ii) $\int_S g(t)d\mu(t) = \int_S e(t)d\mu(t)$.

The set of all feasible allocations for the economy $\mathcal{E}$ that no coalition can improve upon is called the <u>core</u> of the economy $\mathcal{E}$ and it is denoted by $\mathcal{C}(\mathcal{E})$.

An allocation x and a price $p \in E^*_+ \setminus \{0\}$ are said to be a <u>competitive equilibrium</u> (or a <u>Walras equilibrium</u>) for the economy $\mathcal{E}$, if

(i) $x(t)$ is a maximal element for $\succ_t$ in the budget set $\{y \in X(t) : p \cdot y \leqslant p \cdot e(t)\}$ μ - a.e., and

(ii) $\int_T x(t)d\mu(t) = \int_T e(t)d\mu(t)$.

We denote by $W(\mathcal{E})$ the set of all competitive equilibria for the economy $\mathcal{E}$.

4. CORE-WALRAS EQUIVALENCE IN ORDERED BANACH SPACES WHOSE POSITIVE CONE HAS A NON-EMPTY NORM INTERIOR

We begin by stating some assumptions needed for the proof of our core-Walras equivalence result.

(a.0) E is an ordered separable Banach space whose positive cone E_+ has a non-empty norm interior, i.e., $\operatorname{int}E_+ \neq \phi$.

(a.1) (Perfect Competition): (T,τ,μ) is a finite atomless measure space.

(a.2) $X(t) = E_+$ for all $t \in T$.

(a.3) (Resource availability): The aggregate initial endowment $\int_T e(t)d\mu(t)$ is strictly positive, i.e., $\int e >> 0$.

(a.4) (Continuity): For each $x \in E_+$ the set $\{y \in E_+ : y \succ_t x\}$ is norm open in E_+ for all $t \in T$,

(a.5) $\succ_t$ is irreflexive and transitive for all $t \in T$.

(a.6) (Measurability): The set $\{(t,y) \in T \times E_+ : y \succ_t x\}$ belongs to $\tau \otimes \mathcal{B}(E_+)$.

(a.7) (Monotonicity): If $x \in E_+$ and $v \in E_+ \setminus \{0\}$, then $x + v \succ_t x$ for all $t \in T$.

We are now ready to state our first result.

Theorem 4.1: Under assumptions (a.0) - (a.7), $\mathcal{C}(\mathcal{E}) = W(\mathcal{E})$.

Remark 4.1: Note that the assumptions of the above theorem correspond to those in Aumann (1964) in the setting of an ordered separable Banach space E of commodities. It can be easily seen that for $E = \mathbb{R}^\ell$, Theorem 4.1 gives as a corollary Aumann's (1964) core equivalence result, (as well as Hildenbrand's (1974, Theorem 1, p. 133) core-Walras equivalence theorem). It may be instructive at this point to note that Bewley's (1973) infinite dimensional extension of Aumann's core equivalence theorem does not provide the above results as a corollary because it is based on quite stronger assumptions than those adopted by Aumann and Hildenbrand.

4.1 Proof of Theorem 4.1

The fact that $W(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$ is well known, and therefore its proof is not repeated here. We begin the proof by assuming that the allocation x is an element of the core of $\mathcal{E}$. We wish to show that for some price p, the pair (x,p) is a competitive equilibrium for $\mathcal{E}$.

To this end, define the correspondence $\phi : T \to 2^{E_+}$ by

$$(4.0) \quad \phi(t) = \{z \in E_+ : z \succ_t x(t)\} \cup \{e(t)\}.$$

We claim that:

$$(4.1) \quad c\ell\left(\int_T \phi - \int_T e\right) \cap \text{int}\, E_- = \emptyset,$$

or equivalently,[5]

$$(4.2) \quad \left(\int_T \phi - \int_T e\right) \cap \text{int}\, E_- = \emptyset.$$

Suppose otherwise, i.e.,

$$\left(\int_T \phi - \int_T e\right) \cap \text{int}\, E_- \neq \emptyset$$

then there exists $v \in \text{int}\, E_+$ such that

$$(4.3) \quad \int e - v \in \int \phi.$$

It follows from (4.3) that there exists a function $y : T \to E_+$ such that

$$(4.4) \quad \int_T y = \int_T e - v,$$

and $y(t) \in \phi(t) \quad \mu$ - a.e.

Let

$$S = \{t : y(t) \succ_t x(t)\}, \text{ and}$$

$$S' = \{t : y(t) = e(t)\}.$$

Since $\int y \neq \int e$ we have that $\mu(S) > 0$. Define $\tilde{y} : S \to E_+$ by $\tilde{y}(t) = y(t) + \frac{v}{\mu(S)}$ for all $t \in S$. By monotonicity (assumption (a.7)) $\tilde{y}(t) \succ_t y(t)$. Since $y(t) \succ_t x(t)$ for all $t \in S$, by transitivity (assumption (a.5)) $\tilde{y}(t) \succ_t x(t)$ for all $t \in S$. Moreover, it can be easily seen that $\tilde{y}(\cdot)$ is feasible for the coalition S, i.e.,

$$\int_S \tilde{y} = \int_S y + v = \int_T y - \int_{S'} e + v$$

$$= \int_T e - \int_{S'} e = \int_S e, \text{ (recall (4.4))}.$$

Therefore, we have found an allocation $\tilde{y}(\cdot)$ which is feasible for the coalition S and is also preferred to the allocation x, which in turn was assumed to be in the core of $\mathcal{E}$, a blatant contradiction. The above contradiction establishes the validity of (4.1).

We are now in a position to separate the set $c\ell(\int\phi - \int e) = c\ell\int\phi - \int e$ from int E_-. Clearly the set int E_- is convex and non-empty. We want to show that $c\ell\int\phi - \int e$ is convex and non-empty as well. Observe first that by the definition of $\phi(\cdot)$, 0 is an element of $\int\phi - \int e$ and this shows that $c\ell\int\phi - \int e$ is non-empty. Since, (T,τ,μ) is atomless (assumption (a.1)) by Lemma 2.1 $c\ell\int\phi$ is convex. Thus, by Theorem 9.10 in Aliprantis-Burkinshaw (1985, p. 136) there exists a continuous linear functional $p \in E^* \setminus \{0\}$, $p \geq 0$ such that

(4.5) $p \cdot y \geqslant p \cdot \int e$ for all $y \in \int \phi$.

Since by assumption (a.6), $\succ_t$ has a measurable graph, so does $\phi(\cdot)$, i.e., $G_\phi \in \tau \otimes \mathcal{B}(E_+)$. Therefore, it follows from Lemma 2.2 that

(4.6) $$\inf_{y \in \int \phi} p \cdot y = \int \inf_{z \in \phi(\cdot)} p \cdot z \geqslant \int p \cdot e.$$

It follows from (4.6) that

(4.7) μ - a.e. $p \cdot z \geqslant p \cdot e(t)$ for all $z \succ_t x(t)$.

To see this, suppose that for $z \in \phi(\cdot)$, $p \cdot z < p \cdot e(t)$ for all $t \in S$, $\mu(S) > 0$.

Define the function $\tilde{z} : T \to E_+$ by

$$\tilde{z}(t) = \begin{cases} z(t) & \text{if } t \in S \\ e(t) & \text{if } t \notin S. \end{cases}$$

Obviously, $\tilde{z} \in \phi(\cdot)$. Moreover,

$$\int_T p\tilde{z} = \int_S p \cdot z + \int_{T \setminus S} p \cdot e$$

$$< \int_S p \cdot e + \int_{T \setminus S} p \cdot e = \int p \cdot e,$$

a contradiction to (4.6).

We now show that μ - a.e. $p \cdot x(t) = p \cdot e(t)$. First note that it follows directly from (4.7) that $p \cdot x(t) \geqslant p \cdot e(t)$ μ - a.e. If now $p \cdot x(t) > p \cdot e(t)$ for all $t \in S$, $\mu(S) > 0$ then,

$$p \cdot \int_T x = p \cdot \int_{T \setminus S} x + p \cdot \int_S x$$

$$> p \cdot \int_{T \setminus S} e + p \cdot \int_S e = p \cdot \int_T e,$$

contradicting $\int_T x = \int_T e$, since $p \geq 0$, $p \neq 0$.

To complete the proof we must show that $x(t)$ is maximal in the budget set $\{z \in E_+ : p \cdot z \leq p \cdot p(t)\}$ μ - a.e. The argument is now routine. Since $\int_T e$ is strictly positive (assumption (a.3)) it follows that $\mu(\{t: p \cdot e(t)\}) > 0$, for if $p \cdot e(t) = 0$ μ - a.e. then $p \cdot \int_T e = 0$ contradicting the fact that $\int_T e$ is strictly positive since $p \geq 0$, $p \neq 0$.

Thus, we can safely pick an agent t with positive income, i.e., $p \cdot e(t) > 0$. Since $p \cdot e(t) > 0$ there exists an allocation x' such that $p \cdot x' < p \cdot e(t)$. Let y be such that $p \cdot y \leq p \cdot e(t)$ and let $y(\lambda) = \lambda x' + (1-\lambda)y$ for $\lambda \in (0,1)$. Then for any $\lambda \in (0,1)$, $p \cdot y(\lambda) < p \cdot e(t)$ and by (4.7) $y(\lambda) \not\succ_t x(t)$. It follows from the norm continuity of $\succ_t$ (assumption (a.4)) that $y \not\succ_t x(t)$. This proves that $x(t)$ is maximal in the budget set of agent t, i.e., $\{w : p \cdot w \leq p \cdot e(t)\}$. This, together with the monotonicity of preferences (assumption (a.7)) implies that prices are strictly positive, i.e, $p >> 0$. Indeed, if there exists $v \in E_+ \setminus \{0\}$ such that $p \cdot v = 0$ then $p \cdot (x(t) + v) = p \cdot x(t) = p \cdot e(t)$ and by monotonicity $x(t) + v \succ_t x(t)$ contradicting the maximality of $x(t)$ in the budget set.

Thus $p >> 0$ and $x(t)$ is maximal in the budget set whenever $p \cdot e(t) > 0$. Consider now an agent t with zero income, i.e., $p \cdot e(t) = 0$. Since $p >> 0$ his/her budget set $\{z : p \cdot z = 0\}$ consists of zero only, and moreover, $p \cdot x(t) = p \cdot e(t) = 0$. Hence, $x(t) = 0$ for almost all $t \in T$, with $p \cdot e(t) = 0$; i.e., zero in this case is the maximal element in the budget set. Consequently, (p,x) is a competitive equilibrium for $\mathcal{E}$, and this completes the proof of Theorem 4.1.

5. THE FAILURE OF THE CORE-WALRAS EQUIVALENCE IN COMMODITY SPACES WHOSE POSITIVE CONE HAS AN EMPTY INTERIOR

In the previous section we showed that if the commodity space is an ordered separable Banach space E whose positive cone has a non-empty norm interior (i.e., $\mathrm{int}E_+ \neq \phi$), then the standard assumptions (i.e., the assumptions of Theorem 4.1) guarantee the core-Walras equivalence. We now show that if one drops the assumption that the positive cone of the space E has a non-empty norm interior, then Theorem 4.1 fails. The following example will illustrate this.

Example 5.1: Consider the economy $\mathcal{E} = [(T,\tau,\mu), X, \succ, e]$ where,

(1) The space of agents is $T = [0,1]$, τ = Lebesgue measurable sets, μ = Lebesgue measure,

(2) The consumption set of each agent is, $X(t) = \ell_2^+$ for all $t \in T$, where ℓ_2 is the space of real sequences (a_n) for which the norm $\|a_n\| = (\Sigma|a_n|^2)^{1/2}$ is finite,

(3) The preference relation of each agent $\succ_t$, is represented by a strictly concave, monotone weakly continuous utility function, i.e., $u_t(x) = \sum_{i=1}^{\infty} i^{-2}(1-\exp(-i^2 x_i))$ for all $t \in T$, and

(4) The initial endowment of each agent is $e(t) = e = (\frac{1}{i^2})_{i=1}^{\infty}$ for all $t \in T$.

We will show that for the above economy, $\mathcal{C}(\mathcal{E}) \neq \phi$ and $W(\mathcal{E}) = \phi$. In particular, we will show that the core of $\mathcal{E}$, is unique and consists of the initial endowments e, i.e., $\mathcal{C}(\mathcal{E}) = \{e\}$ and $W(\mathcal{E}) = \phi$. The latter (i.e., $W(\mathcal{E}) = \phi$) will easily follow from the fact that $\mathcal{C}(\mathcal{E}) = \{e\}$. Indeed, since $W(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$, $W(\mathcal{E}) \subset \{\{e\}, \phi\}$, but the only candidate as a supporting price p for the allocation e are multiples of $p = (1,1,\ldots.)$ which are not in the dual of ℓ_2. Hence, all we need to show is that $\mathcal{C}(\mathcal{E}) = \{e\}$.

To prove that $\mathcal{C}(\mathcal{E}) = \{e\}$ we will first need to show that e is Pareto optimal, i.e., there does not exist a feasible allocation x such that $u(x(t)) \geq u(e)$ for all $t \in T$ and $u(x(t)) > u(e)$ for all $t \in S$, $S \subset T$, $\mu(S) > 0$ (note that the subscript t on u is dropped). To this end suppose by way of contradiction that there exists an allocation x such that $\int_T x = \int_T e \equiv e$, $u(x(t)) \geq u(e)$ for all $t \in T$ and $u(x(t)) > u(e)$ for all $t \in S$, $\mu(S) > 0$. Without loss of generality we may assume that there exist positive real numbers ε, δ, with $u(x(t)) \geq u(e) + \varepsilon$, $t \in S$, $\mu(S) = \delta$. Extend x to $\tilde{x}(t) = x(t-[t])$, ([t] = the integer part of t), and let $x^k(t) = \sum_{i=1}^{k-1} \frac{\tilde{x}(t+\frac{i}{k})}{k}$.

Then

$$\int_0^1 u(x^k(t))d\mu(t) = \int_0^1 u\left(\sum_{i=1}^{k-1} \frac{\tilde{x}(t+\frac{i}{k})}{k}\right) d\mu(t)$$

$$\geq \int_0^1 \sum_{i=1}^{k-1} \frac{1}{k} u(\tilde{x}(t+\frac{i}{k}))d\mu(t)$$

$$= \int_0^1 u(x(t))d\mu(t) \geq u(e) + \varepsilon\delta. \tag{5.1}$$

Notice that each component of the $x^k(\cdot)$, denoted by $x_i^k(\cdot)$ (an L_1 function), converges to e_i μ - a.e., i.e., $x_i^k(t) = \sum_{i=1}^{k-1} \frac{\tilde{x}(t+\frac{i}{k})}{k} \underset{k\to\infty}{\to} \int_0^1 x_i(s)d\mu(s) = e_i$ for almost all t in T; so $(x_i^i(t))_{i=1}^{\infty}$ converges weakly in ℓ_2 to $(e_i)_{i=1}^{\infty}$. Since u is weakly continuous it follows that $u(x^k(t)) \to u(e)$ μ - a.e. Notice that by definition u is bounded, in particular, $u(x) < \pi^2/6$ for every $x \in \ell_2^+$ (recall the definition of $u(\cdot)$ in (3)) and therefore by the Lebesgue dominated convergence theorem $\lim_{k\to\infty} \int_0^1 u(x^k(t))d\mu(t) = \int_0^1 \lim_{k\to\infty} u(x^k(t))d\mu(t) = u(e) = u(\int e)$, a contradiction to (5.1). Thus, e is Pareto optimal.

We are now ready to complete the proof of the fact that $\mathcal{C}(\mathcal{E}) = \{e\}$. To this end we first show that:

(5.2) $\mathcal{C}(\mathcal{E}) \subset \{e\}$.

Suppose that (5.2) is false, then there exists an allocation $x \in \mathcal{C}(\mathcal{E})$ such that $x(t) \neq e$ for all $t \in S$, $\mu(S) > 0$. Let $\tilde{x} = \frac{x+e}{2}$. Then $\tilde{x}$ is feasible and for all $t \in T$,

$$u(\tilde{x}(t)) > \frac{1}{2} u(x(t)) + \frac{1}{2} u(e)$$

$$\succcurlyeq u(e) \quad \text{(recall that } u(x(t)) \succcurlyeq u(e) \text{ for all } t \in T \text{ since } x \in \mathcal{C}(\mathcal{E})).$$

Moreover, by strict concavity of $u(\cdot)$ we have that

$$u(\tilde{x}(t)) > u(e) \text{ for all } t \in S,$$

a contradiction to the fact that e is Pareto optimal.

We now show that:

(5.3) $\{e\} \subset \mathcal{C}(\mathcal{E})$.

Suppose that (5.3) is false, then there exists a coalition S and an allocation x such that $\int_S x = \int_S e$ and $u(x(t)) > u(e)$ for all $t \in S$. Define the allocation $\tilde{x}(\cdot)$ as follows:

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } t \in S \\ e(t) & \text{if } t \notin S. \end{cases}$$

Then $u(\tilde{x}(t)) \succcurlyeq u(e)$ for all $t \in T$ and $u(\bar{x}(t)) > u(e)$ for all $t \in S$, contradicting the fact that e is Pareto optimal.

It follows from (5.2) and (5.3) that $\mathcal{C}(\mathcal{E}) = \{e\}$ and this completes the proof of the fact that $\mathcal{C}(\mathcal{E}) \neq \phi$ and $W(\mathcal{E}) = \phi$.

Notice that since example 5.1 satisfies all the conditions of Theorem 4.1 except assumption (a.0), (note that int$\ell_2^+ = \phi$) we can conclude that if positive results are to be obtained in spaces whose positive cone have an empty norm interior, some additional assumption needs to be imposed. The additional assumption is that of an extremely desirable commodity introduced in Yannelis-Zame (1986), (which is related to the assumption of proper preferences introduced by Mas-Colell (1986)), provided that the commodity space is the Lebesgue space L_1. However, if the commodity space is any arbitrary separable Banach lattice, a slightly stronger form of the assumption of an extremely desirable commodity will be needed, called commodity pair desirability. The next section is devoted to the nature of this assumption.

6. EXTREMELY DESIRABLE COMMODITIES AND COMMODITY PAIR DESIRABILITY

For notational convenience, below we drop the subscript t on the preference relation $\succ$. We begin by defining the notion of an extremely desirable commodity. Let E be a Banach lattice and denote its' positive cone (which may have an empty norm interior) by E_+. Let $v \in E_+$, $v \neq 0$ and U be an open neighborhood. We say that $v \in E_+$ is an extremely desirable commodity if there exists U such that

/ for each $x \in E_+$ we have that $x + \alpha v - z \succ x$ whenever $\alpha > 0$, $z \leq x + \alpha v$ and $z \in \alpha U$. In other words, v is extremely desirable if an agent would prefer to trade any commodity bundle z for an additional increment of the commodity bundle v, provided that the size of z is sufficiently small compared to the increment of v. The above notion has a natural geometric interpretation. In particular, let $v \in E_+$, $v \neq 0$, U be an open neighborhood and define the open cone C as follows:

$$C = \{\alpha\, v - z : \alpha > 0,\ z \in E,\ z \in \alpha U\}.$$

The bundle v is said to be an extremely desirable commodity, if for each $x \in E_+$, we have $y \succ x$ whenever y is an element of $(C + x) \cap E_+$. Note this implies that v is an extremely desirable commodity if for each $x \in E_+$ we have that $((-C + x) \cap E_+) \cap \{y : y \succ x\} = \phi$, or equivalently $-C \cap \{y - x \in E_+ : y \succ x\} = \phi$. The latter property is precisely the assumption we need if we consider L_1 as a commodity space (see Section 8).

Recall that if the preference relation $\succ$ is monotone and $\text{int}E_+ \neq \phi$, then the assumption of an extremely desirable commodity is automatically satisfied (see for instance Yannelis-Zame (1986)).

We now turn to a strengthening of the above assumption.

A pair $(x,y) \in E_+ \times E_+$ is said to be a desirable commodity pair if for every $z \in E_+$ we have $z + x - y \succ_t z$ whenever $y \lesssim x + z$ for each $t \in T$. The pair $(x,y) \in E \times E$ is said to have the splitting property if for any m-tuple $(s_1,\dots,s_m) \in E \times \dots \times E$ such that $\sum_{i=1}^{m} s_i = (x-y)^-$ there exists an m-tuple $(d_1,\dots,d_m) \in E \times \dots \times E$ such that $\sum_{i=1}^{m} d_i = (x-y)^+$ and the pair (d_i,s_i) is a desirable commodity pair.

We are now ready to define our key notion.

Definition 6.1 (Commodity pair desirability): There exist $v \in E_+$, $v \neq 0$ and a neighborhood U such that any commodity pair (u,w) of the form $w = \alpha v$, $\alpha > 0$ and $u \in \alpha U$ has the splitting property.

Obviously the above concept is a substitutability condition which roughly speaking says that an agent would accept a sufficiently small amount of the commodity bundle s_i if he/she would be compensated by consuming more of the desirable commodity bundle d_i.[6]

A couple of comments are in order. First notice that for $m = 1$ in the above definition we have that for any $z \in E_+$, $z + (w-u)^+ - (w-u)^- = z + w-u \succ z$ whenever $u \lesssim z + w$, $w = \alpha v$, $\alpha > 0$, and $u \in \alpha U$, i.e., for $m = 1$ we are reduced to the assumption of an extremely desirable commodity.

Moreover, it is easy to show that if $\text{int}E_+ \neq \phi$ and the preference relation $\succ$ is monotone then the condition of commodity pair desirability is automatically satisfied. Specifically, let $v \in \text{int}E_+$ and U be such that $v + U \subset E_+$, then for any pair (w,u) with $w = \alpha v$, $\alpha > 0$, $u \in \alpha U$ we have $(w-u) \in E_+$ so $(w-u)^- = 0$ and therefore by monotonicity for any $z \in E_+$ $z + w-u \succ z$.

7. CORE-WALRAS EQUIVALENCE IN SEPARABLE BANACH LATTICES WHOSE POSITIVE CONE HAS AN EMPTY INTERIOR

In this section we state and prove a core-Walras equivalence theorem for a commodity space which can be any arbitrary separable Banach Lattice whose positive cone may have an empty norm interior. We begin by stating the following assumptions:

(a.0') E is any separable Banach Lattice.

(a.8) (Commodity pair desirability): There exists $v \in E_+ \setminus \{0\}$ and an open neighborhood U such that any commodity pair (u,w) of the form $w = \alpha v$, $\alpha > 0$ and $u \in \alpha U$, has the splitting property.

We are now ready to state and prove the following result:

Theorem 7.1: Under assumptions (a.0'), (a.1)-(a.8), $\mathcal{C}(\mathcal{E}) = W(\mathcal{E})$.

Proof: It can be easily shown that $W(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$. Hence, we will show that if $x \in \mathcal{C}(\mathcal{E})$ then for some price p, the pair (x,p) is a competitive equilibrium for $\mathcal{E}$. Define the correspondence $\phi : T \to 2^{E_+}$ by

(7.1) $\phi(t) = \{z \in E_+ : z \succ_t x(t)\} \cup \{e(t)\}$.

Let C be the open cone spanned by the set v + U given by assumption (a.8), i.e., $C = \text{span}\,\{0, v + U\} \equiv \bigcup_{\alpha>0} \alpha(v + U)$. We claim that:

(7.2) $c\ell\,(\int \phi - \int e) \cap - C = \emptyset$

or equivalently

(7.3) $(\int \phi - \int e) \cap - C = \emptyset$.

Since -C is open it suffices to show that for any $y \in \mathcal{L}_\phi$ there exists a sequence $\{(\bar{y}^k, \bar{e}^k) : k=1,2,..\}$ such that $\bar{y}^k$ converges in the $L_1(\mu,E)$ norm to y, $\int \bar{e}^k = \int e$, and

(7.4) $\int_T \bar{y}^k - \int_T \bar{e}^k \notin -C$.

Let $S = \{t : y(t) \succ_t x(t)\}$, $S' = T \setminus S$. Without loss of generality we may assume that $\mu(S) > 0$, (for if $\mu(S) = 0$ then $y(t) = e(t)$ μ - a.e. which implies that $\int y - \int e = 0 \notin -C$. Consequently (7.3) holds). In the argument below y and e are restricted to S. Moreover, denote by μ_S the restriction of μ to S.

Since, $y : S \to E_+$ is Bochner integrable and $\succ_t$ is norm continuous (assumption (a.4)) there exist $y_1^k,\ldots,y_{m_k}^k$ in E_+ and $T_1^k, T_2^k,\ldots,T_{m_k}^k$ in τ such that y^k converges in the $L_1(\mu_S,E)$ norm to y, and

(7.5) $y^k = \sum_{i=1}^{m_k} y_i^k \chi_{T_i^k}$

(7.6) $y_i^k \succ_t x(t)$ for all $t \in T_i^k$ and all i, $i=1,\ldots,m_k$, and

(7.7) $\mu_S(T_i^k) = \xi$, $i=1,\ldots,m_k$.

Let $e^k = \sum_{i=1}^{m_k} (\int_{T_i^k} e(t)d\mu(t))\, \chi_{T_i^k}$.

In order to establish (7.4) we first show that

(7.8) $\int_S y^k - \int_S e^k \notin -C$.

Suppose that (7.8) is false, then

$\sum_{i=1}^{m_k} y_i^k \xi - \sum_{i=1}^{m_k} e_i^k \xi \in -\alpha(v + U)$, and therefore

$$(7.9)\quad \sum_{i=1}^{m_k} y_i^k + w - u = \sum_{i=1}^{m_k} e_i^k,$$

where $w = \frac{\alpha}{\xi} v$, $u \in \frac{\alpha}{\xi} U$.

Since $\sum_{i=1}^{m_k} e_i^k \geqslant 0$, it follows from (7.9) that

$$(7.10)\quad (w - u)^- \leqslant \sum_{i=1}^{m_k} y_i^k.$$

Applying the Riesz Decomposition Property to (7.10) we can find $(s_1,\ldots,s_{m_k}) \in E_+ \times \ldots \times E_+$ such that

$$(7.11)\quad \sum_{i=1}^{m_k} s_i = (w - u)^- \quad 0 \leqslant s_i \leqslant y_i^k \quad \text{for all } i.$$

It follows from the assumption of commodity pair desirability that there exists an m_k-tuple $(d_1,\ldots,d_{m_k}) \in E_+ \times \ldots . \times E_+$, such that $\sum_{i=1}^{m_k} d_i = (w - u)^+$ and

$$(7.12)\quad \tilde{y}_i^k = y_i^k + d_i - s_i \succ_t y_i^k \quad \text{for all } t \in T_i^k \text{ and for all } i.$$

Note that since $y_i^k \geqslant s_i$ it follows that $\tilde{y}_i^k \in E^+$. Moreover, since $y_i^k \succ_t x(t)$ for all $t \in T_i^k$ and all i, and $\tilde{y}_i^k \succ_t y_i^k$ for all $t \in T_i^k$ and all i, by transitivity of $\succ_t$ we have that $\tilde{y}_i^k \succ_t x(t)$ for all $t \in T_i^k$ and all i. Also,

$$\sum_{i=1}^{m_k} \tilde{y}_i^k \xi = \sum_{i=1}^{m_k} e_i^k \xi = \int e,$$

Define $\tilde{y}^k = \sum_{i=1}^{m_k} \tilde{y}_i^k \chi_{T_i^k}$. Notice that $\int_S \tilde{y}^k = \int_S e$. Therefore, we have found an allocation $\tilde{y}^k(\cdot)$ feasible for the coalition S and preferred to $x(\cdot)$ which in turn was assumed to be in the core of $\mathcal{E}$, a contradiction. Hence, (7.8) holds.

We are now ready to construct the sequence $\{(\bar{y}^k, \bar{e}^k) : k=1,2,\ldots\}$. In particular, define $\bar{y}^k : T \to E_+$ by

$$\bar{y}^k(t) = \begin{cases} y^k(t) & \text{if } t \in S \\ y(t) & \text{if } t \notin S. \end{cases}$$

Similarly define $e^k : T \to E_+$ by

$$\bar{e}^k(t) = \begin{cases} e^k(t) & \text{if } t \in S \\ e(t) & \text{if } t \notin S. \end{cases}$$

Note that $\int_T \bar{y}^k - \int_T \bar{e}^k \notin -C$ and therefore (7.4) holds.
We can now separate the convex nonempty set $c\ell \int \phi - \int e$ from the convex nonempty set $-C$. Proceeding as in the proof of Theorem 4.1 one can now complete the proof.

Remark 7.1: Since in separable Banach Lattices whose positive cone has a nonempty (norm) interior and preferences are monotone, assumption (a.8) is automatically satisfied, it follows that in such spaces Theorem 4.1 becomes a corollary of Theorem 7.1. However, since in Theorem 4.1 the commodity space is any arbitrary ordered separable Banach space (i.e. no lattice structure is required) we cannot derive Theorem 4.1 as a corollary of Theorem 7.1.

8. IS THE ASSUMPTION OF AN EXTREMELY DESIRABLE COMMODITY SUFFICIENT FOR THE CORE-WALRAS EQUIVALENCE IN COMMODITY SPACES WHOSE POSITIVE CONE HAS AN EMPTY INTERIOR?

This section is devoted to the problem of whether the assumption of an extremely desirable commodity is sufficient to prove core-Walras equivalence in commodity spaces which are separable Banach Lattices whose positive cone may have an empty (norm) interior. To put the problem differently; is it possible to restrict ourselves to some commodity spaces whose positive cone may have an empty interior and still ensure core-Walras equivalence provided that we only assume that the condition of an extremely desirable commodity is satisfied? (Of course we assume that (a.2)-(a.7) hold as well). The answer to the above question is affirmative. In particular, if the commodity space is the Lebesgue space L_1, the assumption of an extremely desirable commodity will suffice. However, before we state and prove our next result, we will need the following assumptions:

(a.9) Let E be any separable Banach lattice. If $x_i \in E_+$, $x_i \notin B(0,\delta_i)$ $i=1,2,\ldots n$ then $\sum_{i=1}^{m} x_i \notin B(0, \sum_{i=1}^{m} \delta_i)$. (Here $B(0,\delta)$ denotes the open ball centered at 0 of radius δ).

Note that if $E = L_1$, (a.9) is automatically satisfied.

(a.10) (Extremely desirable commodity): Let $v \in E_+/\{0\}$ and U be an open neighborhood. Let C be the cone spanned by v + U. The bundle v is said to be an extremely desirable commodity, if for each $x \in E_+$ and each $t \in T$, we have $y \succ_t x$ whenever y is an element of $(C + x) \cap E_+$.

(a.10') For each $x \in E_+$ the sets $\{y \in E_+ : y \succ_t x\}$ and $\{y \in E_+ : x \succ_t y\}$ are norm open in E_+ for all $t \in T$.

We now have the following result:

Theorem 8.1: Under assumptions (a.0'), (a.2) - (a.7), (a.9) and (a.10), $\mathcal{C}(\mathcal{E}) = W(\mathcal{E})$.

Proof: Adopting an identical argument with that used in the proof of Theorem 7.1, we end up with equation (7.9) which is renumbered here as

$$(8.1) \quad \sum_{i=1}^{m_k} y_i^k + w - u = \sum_{i=1}^{m_k} e_i^k, \; w = \frac{\alpha}{\xi} v, \; u \in \frac{\alpha}{\xi} U.$$

(Recall that we want to derive a contradiction to the fact that the allocation $x(\cdot)$ is in the core of $\mathcal{E}$). For notational convenience we drop the superscript k in (8.1). Moreover, note that without loss of generality we may assume that $u \geq 0$, (otherwise since $u = u^+ - u^-$, we may define $\hat{y}_i = y_i + \frac{u^+}{m}$. Then $\hat{y}_i \geq 0$ and $u^- \in \alpha U$ (recall that U can be assumed to be solid), $\hat{y}_i \succ_t x(t)$ for all $t \in T_i^k$ and all i and one can now proceed by substituting y_i for $\hat{y}_i$).

It follows from (8.1) that for any m-tuple $(\theta_1,\ldots,\theta_m)$, $\theta_i \geq 0$ (i=1, ...,n), $\sum_{i=1}^{m} \theta_i = 1$ we have

that $\sum_{i=1}^{m} (y_i + \theta_i w) - u = \sum_{i=1}^{m} e_i \geq 0$, and therefore

$$(8.2) \quad u \leq \sum_{i=1}^{m} (y_i + \theta_i w).$$

Applying the Riesz Decomposition Property in (8.2) we obtain $u_1,\ldots,u_m$ in E_+ such that

(8.3) $\sum_{i=1}^{m} u_i = u$, $u_i \leqslant y_i + \theta_i w$ for all i.

For each i, define $\tilde{y}_i : [0,1] \to E_+$ by $y_i(\theta) = y_i + \theta_i w - u_i$. Moreover, for each i set $F_i(\tilde{y}_i(\theta)) = \text{dist}\,(\tilde{y}_i(\theta),\, C + y_i) = \delta_i(\theta)$. Let $\Delta = \{q \in \mathbb{R}^m_+ : \sum_{i=1}^{m} q_i = 1\}$. Define the continuous mapping $f : \Delta \to \Delta$ by $f(\theta) = \left(\frac{(\theta_i + \delta_i)}{1 + \sum_{j=1}^{m} \delta_j}\right)_{i=1}^{m}$. By Brouwer's fixed point theorem, there exists $\theta^* \in \Delta$ such that $\theta^* = f(\theta^*)$, i.e.,

(8.4) $\delta_i = \theta_i^* \sum_{j=1}^{m} \delta_j \quad i=1,\ldots,m.$

If we show that $\sum_{j=1}^{m} \delta_j = 0$ then by the definition of F_i, $\tilde{y}_i(\theta^*) \in c\ell(C + y_i) \cap E_+$ and by continuity (assumption (a.4')) and the assumption of an extremely desirable commodity (assumption (a.11)) $\tilde{y}_i(\theta^*) \succ_t y_i$ for all $t \in T_t$, and all i. One can now proceed as in the proof of Theorem 7.1 to complete the proof of the theorem. (Since the argument is identical with that in the proof of Theorem 7.1 we do not repeat it here). Hence, all which remains to be shown is that $\sum_{j=1}^{m} \delta_j = 0$. To this end suppose that $\sum_{j=1}^{m} \delta_j > 0$. Notice that by (8.4) we have that $\theta_i^* = 0$ if and only if $\delta_i = 0$. Define $J, K \subset I = \{1,2,\ldots,m\}$ as follows:

$J = \{i \in I : \delta_i = 0\}$, $K = I \setminus J$. Note that $J = \{i \in I : \theta_i^* = 0\}$. Consider any $i \in J$; then by the definition of $\tilde{y}_i(\cdot)$ we have that $\tilde{y}_i(\theta^*) = y_i - u_i$. Now if $u_i \neq 0$ by monotonicity $y_i \succ_t \tilde{y}_i(\theta^*)$ and by virtue of continuity and extreme desirability we can conclude that $\tilde{y}_i(\theta^*) \notin c\ell(C + y_i)$. By the definition of F_i, $\delta_i > 0$, a contradiction to the fact that $i \in J$. Hence, $u_k = 0$ for $i \in J$ and so

(8.5) $\sum_{i \in I} u_i = \sum_{i \in K} u_i = u.$

Consider any $i \in K$, i.e., $\delta_i > 0$, then by the definition of F_i, it follows that $\tilde{y}_i(\theta^*) = y_i + \theta_i^* w - u_i \notin C + y_i$ for every $i \in K$, and therefore, $\theta_i^* w - u_i \notin C$ for all $i \in K$ which in turn implies that $u_i \notin \theta_i^* \frac{\alpha}{\xi} U$ for all $i \in K$. It follows from (8.5), (8.3), the fact that $u_i \notin \theta_i^* \frac{\alpha}{\xi} U$ for all $i \in K$, $\sum_{i=1}^{m} \theta_i^* = 1$ and assumption (a.9) that $\sum_{i \in I} u_i = \sum_{i \in K} u_i = u \notin \frac{\alpha}{\xi} U$, which contradicts (8.1), (i.e., $u \in \frac{\alpha}{\xi} U$). The above contradiction establishes that $\sum_{i=1}^{m} \delta_i = 0$ and this completes the proof of the theorem.

As we remarked earlier, the linearity assumption (a.9) is automatically satisfied if $E = L_1$. Therefore, in L_1 the assumption of an extremely desirable commodity suffices to obtain core-Walras equivalence. Note that a remark of the same nature was made in Yannelis-Zame (1986, p. 96) for the problem of the existence of an equilibrium with a countable number of agents.

9. CONCLUDING REMARKS

Remark 9.1: The separability of our commodity space E was used in the proof of our theorems at one point only (recall (4.6)). Specifically, it was used to make Lemma 2.2 applicable (recall that Lemma 2.2 is a consequence of the measurable selection theorem and the latter requires separability). We do not know if Lemma 2.2 can be proved in a different way without the separability assumption.

Remark 9.2: Bewley (1973) has proved a core equivalence result for a non-separable commodity space which is $L_\infty(T,\tau,\mu)$, i.e., the space of essentially bounded measurable functions on a measure space (T,τ,μ) (with the essential sumpremum norm), but his assumptions are than those adopted in the proof of our Theorem 4.1. Nevertheless, we would like to indicate how Theorem 4.1 can be modified to cover $L_\infty(T,\tau,\mu)$. Suppose now that $E_+ = L_\infty^+$ and that the integral of the correspondence $\phi : T \to 2^{E_+}$ defined in Section 4 by (4.0), is contained in K, where K is a weakly compact subset of L_∞^+. Moreover, suppose that assumptions (a.0) - (a.7) still hold. Then, Theorem 4.1 remains true. This is due to the fact that weakly compact subsets of L_∞ are norm separable (see Diestel-Uhl (1977, Theorem 13, p. 252)) and consequently $\int\phi$ takes values in a separable subset of a Banach space. Therefore, Lemma 2.2 applies and the proof of Theorem 4.1 remains unchanged.

Remark 9.3: By replacing assumption (a.2) by:

(a.2') $X(t) = K$ for all $t \in T$, where K is a non-empty convex subset of E_+, the reader can easily check that the proof of Theorem 4.1 goes through. Moreover, if in (a.2') above we add that K is weakly compact, then for $E = L_\infty(T,\tau,\mu)$ Theorem 4.1 still holds since $\int\phi$ takes values in a separable subset of a Banach space and therefore, Lemma 2.2 applies (recall the previous remark).[7]

Remark 9.4: It is of interest to know whether or not under the assumption of Theorem 4.1 we have core-Walras existence as well. It is known, for instance, that this is not true if $X = \mathbb{R}^n$, (see Aumann (1966, Section 8, p.17)); this is also the case here. For instance, under (a.0) - (a.7) a competitive equilibrium may not exist. However, by replacing (a.2) by

(a.2') $X(t) = K$ for all $t \in T$, where K is a weakly compact convex non-empty subset of E_+,

and adding the assumption:

(a.11) for all $t \in T$, and for all $x(t) \in X(t)$ the set $\{y : y \succ_t x(t)\}$ is convex and the set $\{y : x(t) \succ_t y\}$ is norm open,

one can conclude, by virtue of the main theorem in Khan-Yannelis (1986) that both sets, i.e., $\mathcal{C}(\&)$ and $W(\&)$ are non-empty. Moreover, under the assumptions of Theorem 7.1 for $E_+ = L_p^+$, $(1 \leq p < \infty)$ and (a.11) it follows from the main result in Rustichini-Yannelis (1986) that the sets $\mathcal{C}(\&)$ and $W(\&)$ are non-empty.[8]

Remark 9.5: Vind (1964) introduced the coalitional preference framework and obtained core-Walras equivalence. His approach has been extended in several directions by Richter (1971) and Armstrong-Richter

(1985). It is of interest to know whether or not one can obtain core-Walras equivalence with an infinite dimensional commodity space in the very general setting of Armstrong-Richter (1985). This seems to be an interesting question.

Remark 9.6: Notice that in Theorem 7.8 and 8.1 the commodity space was assumed to be a Banach lattice. However, in Theorem 4.1 we only needed the commodity space to be an ordered Banach space, i.e., no lattice structure was required. The reason we needed the lattice structure in Theorems 7.1 and 8.1 was to apply the Riesz Decomposition Property which in turn has a natural economic meaning as was indicated in Yannelis-Zame (1986, p. 89). However, we do not know whether or not one can dispense with the lattice structure in Theorems 7.1 and 8.1.

Remark 9.7: Using the notion of commodity pair desirability one can easily prove the second welfare economics theorem for economies with an atomless measure space of agents and with a commodity space which can be any arbitrary separable Banach lattice. We hope to take up the details in a subsequent paper.

Remark 9.8: We now indicate how our methods can cover the space $M(\Omega)$, used by Mas-Colell (1975). Specifically, Mas-Colell considers as commodity spaces the set of bounded, signed (Borel) measures on Ω, denoted by $m(\Omega)$. He endows $m(\Omega)$ with the weak* topology. Note the weak* topology on norm bounded subsets of $m(\Omega)$ is separable and metrizable. Since preferences are also endowed with the weak* topology in order to obtain the counterpart of Theorem 7.1, one needs to work with allocations which are Gelfand integrable functions (see Khan (1985, 1986) for a definition). The argument used to prove Theorem 7.1 remains unchanged, provided that one uses the fact that the weak* closure of the Gelfand integral of correspondence (7.1) is convex (see Khan (1985, Claim 3, p. 265)), and by noting that since we are in a setting of a locally convex, separable and metrizable linear topological space, the measurable selection theorem is applicable and therefore the counterpart of Lemma 2.2 for the Gelfand integral holds as well. However, since the space $M(\Omega)$ has no strictly positive elements (unless Ω is countable) we show only that core allocations are quasi equilibria and not competitive equilibria.

FOOTNOTES

1. Recently, substantial progress has been made in establishing existence results for the competitive equilibrium in exchange economies with finitely many agents and with a commodity space which is general enough to encompass all of the spaces mentioned above, (see for instance Mas-Colell (1986) or Yannelis-Zame (1986) among others). Moreover, some progress has been made in obtaining equilibrium existence results for perfectly competitive economies, i.e., economies with an atomless measure space of agents a la Aumann (1966) with an infinite dimensional commodity space, (see for instance, Khan-Yannelis (1986), Rustichini-Yannelis (1986), Yannelis (1987) and Zame (1986)). However, the core has received significantly less attention in infinite dimensional settings.

2. Gabszewicz (1967) has also proved a core-Walras equivalence result for a commodity space which is L_2. Notice that the positive cone of L_2 has an empty (norm) interior. However, throughout the paper Gabszewicz treats the positive cone of L_2 as having a non-empty norm interior.

3. It should be noted that the precursors of the assumption of uniform properness are Chichilnisky-Kalman (1980). In particular, in order to apply Hahn-Banach-type separation theorems in spaces whose positive cone has an empty interior, they introduced a related assumption with that of uniform properness used by Mas-Colell.

4. $\succ$ is defined to be the asymmetric part of the weak preference relation $\succsim$, i.e., we say that $x \succ y$ if and only if $x \succsim y$ and not $y \succsim x$. This is not needed for Theorems 4.1 and 7.1. However, it is used in the proof of Theorem 8.1.

5. This is so since intE_- is an open set. In particular if A and B are subsets of any topological space and B is open, then it can be easily seen that $A \cap B = \phi$ if and only if $c\ell A \cap B = \phi$.

6. Note that since we have allowed splitting for $(w - u)^-$ and $(w - u)^+$, we may think of i, (i=1, 2, ..., m) as agents and the splitting property as a kind of redistribution. Hence, the notion of commodity pair desirability is a "coalitional type" of uniform properness.

7. It should be noted that if we endow L_∞ with the Mackey topology (as Bewley does) and restrict results to bounded consumption sets, Theorem 4.1 is still true. In particular, on bounded sets the Mackey topology is metrizable and separable. Hence, Lemma 2.2 holds. Furthermore, the Mackey closure of the integral (4.0) is still convex (see Khan (1985, Claim 3, p. 265)). Hence the proof of Theorem 4.1 remains unchanged.

8. Note that in the above papers it is assumed that preferences are convex, an assumption which is dispensible in the finite dimensional case. In fact as Aumann (1964) showed, the Lyapunov convexity theorem convexifies the aggregate demand set. However, in infinite dimensional spaces, as was remarked earlier, the Lyapunov convexity result fails. It is also worth mentioning that in addition to the failure of Lyapunov's theorem, Fatou's Lemma fails in infinite dimensional spaces as well (see Rustichini (1986)). Hence, there is no exact analogue to Schmeidler's (1970) version of Fatou's Lemma in infinite dimensions. However, approximate versions of Fatou's Lemma have been obtained by Khan-Majumdar (1986) and Yannelis (1986). Moreover, with additional assumptions exact versions of Fatou's Lemma in infinite dimensional spaces can be obtained as well (see Rustichini (1986) and Yannelis (1986)).

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