AN ELEMENTARY PROOF OF FATOU'S LEMMA

IN FINITE DIMENSIONAL SPACES

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Abstract

We provide an elementary and very short proof of the Fatou Lemma in n-dimensions. In particular, we show that the latter result follows directly from Aumann's (1976) elementary proof of the fact that integration preserves upper-semicontinuity.

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1. INTRODUCTION

Aumann (1976) provided an elementary proof of the fact that integration preserves upper-semicontinuity, a result which is fundamental to prove the existence of an equilibrium in economies with a continuum of agents. Previous proofs of this result by Aumann (1965) and Schmeidler (1970) were far more complex than that of Aumann (1976). In particular, they were proved via Fatou's lemma in n-dimensions, an advanced mathematical result.

The purpose of this note (which is a "classroom note") is to provide an elementary and very short proof of the Fatou Lemma in n-dimensions. We will do so by showing that the latter result follows directly from Aumann's elementary proof of the fact that integration preserves upper-semicontinuity. Since Fatou's lemma in n-dimensions implies that integration preserves upper-semicontinuity (see for instance Aumann (1965, p.10)), our note enables us to conclude that these two remarkable results are equivalent.

Finally, since Fatou's Lemma in n-dimensions is one of the most advanced mathematical tools needed to prove the existence of a competitive equilibria in economies with a measure space of agents, we believe that our note will contribute to making the existence proof rather elementary and consequently, accessible to a broader class of audiences.

2. NOTATION DEFINITIONS AND THE RESULT

2.1 Notation

$\mathbb{R}^\ell$ denotes the ℓ-fold Cartesian product of the set of real numbers $\mathbb{R}$.

2^A denotes the set of all nonempty subsets of the set A.

ϕ denotes the empty set.

If A is a subset of $\mathbb{R}^\ell$, $\bar{A}$ denotes the closure of A.

Let X and Y be sets, the graph of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$.

2.2 Definitions

Let (T,τ,μ) be a complete atomless measure space and let $\phi : T \to 2^{\mathbb{R}^\ell}$ be a correspondence. The integral of ϕ is denoted by $\int_T \phi(t)d\mu(t)$ and it is equal to the set $\{\int_T x(t)d\mu(t) : x(t)\in\phi(t)\ \mu - \text{a.e.}\}$. The correspondence $\phi : T \to 2^{\mathbb{R}^\ell}$ is said to be integrably bounded if there exists a map $h\in L_1(\mu)$ such that $\sup\{\|x\| : x\in\phi(t)\} \leqslant h(t)\ \mu - \text{a.e.}$ Let P be a metric space. The correspondence $\phi : P \to 2^{\mathbb{R}^\ell}$ is said to be upper-semicontinuous (u.s.c.) if G_ϕ is closed in $P\times\mathbb{R}^\ell$. If A_n $(n=1,2,..)$ is a sequence of nonempty subsets of $\mathbb{R}^\ell$, we will denote by LsA_n the set of its limit superior points, i.e., $LsA_n = \{x\in\mathbb{R}^\ell : x = \lim_{k\to\infty} x_{n_k},\ x_{n_k} \in A_{n_k},\ k=1,2,..\}$. Recall that LsA_n is a closed set, indeed, $\overline{LsA_n} = Ls\bar{A}_n = LsA_n$.

2.3 The Result

The following two Lemmata are due to Aumann (1965, 1976).

Lemma 2.1 (Fatou's Lemma in ℓ-dimensions): Let $\phi_n : T \to 2^{\mathbb{R}^\ell}$, $(n=1,2..)$ be a sequence of integrably bounded correspondences.[1] Then

$$Ls \int_T \phi_n(t)d\mu(t) \subset \int_T Ls\phi_n(t)d\mu(t).$$

Lemma 2.2 (Integration preserves upper-semicontinuity): Let P be a metric space and $\psi : T \times P \to 2^{\mathbb{R}^\ell}$ be an integrably bounded correspondence such that for each fixed $t\in T$, $\psi(t,\cdot)$ is u.s.c. Then $\int_T \psi(t,\cdot)\, d\mu(t)$ is u.s.c.

Below we state our result:

Proposition 2.1: Lemma 2.1 is "equivalent" to Lemma 2.2, in the sense that each one can be directly derived from the other.

(i) Proof of Lemma 2.1 via Lemma 2.2

Let $P \equiv [0,1)$. Define the correspondence $\psi : T \times P \to 2^{\mathbb{R}^n}$ by

$$\psi(t,p) = \begin{cases} \bar{\phi}_n(t) & \text{if } p \in (\frac{1}{n+1}, \frac{1}{n}) \\ \bar{\phi}_n(t) \cup \bar{\phi}_{n+1}(t) & \text{if } p = \frac{1}{n+1} \\ \mathrm{Ls}\phi_n(t) & \text{if } p = 0. \end{cases}$$

It can be easily checked that for each fixed t, $\psi(t,\cdot)$ is u.s.c. and that ψ is integrably bounded. Therefore, ψ satisfies all the assumptions of Lemma 2.2 and we can conclude that $\int_T \psi(t,\cdot)d\mu(t)$ is u.s.c. Consequently, if $p_n = \frac{1}{n+\frac{1}{2}}$ (notice that $\lim_{n\to\infty} p_n = 0$) then,

$$(2.1) \qquad \mathrm{Ls}\int_T \psi(t,\frac{1}{n+\frac{1}{2}})\, d\mu(t) \subset \int_T \psi(t,0)d\mu(t) \equiv \int_T \mathrm{Ls}\, \phi_n(t)d\mu(t).$$

Observe that,

$$(2.2) \qquad \mathrm{Ls}\int_T \phi_n(t)d\mu(t) \subset \mathrm{Ls}\int_T \bar{\phi}_n(t)d\mu(t) \equiv \mathrm{Ls}\int_T \psi(t, \frac{1}{n+\frac{1}{2}})\, d\mu(t).$$

Combining now (2.1) and (2.2) we obtain that

$$\mathrm{Ls}\int_T \phi_n(t)d\mu(t) \subset \int_T \mathrm{Ls}\phi_n(t)d\mu(t).$$

This completes the proof of (i).

(ii) Proof of Lemma 2.2 via Lemma 2.1

The proof of this part is trivial but we give it for the sake of completeness. Let p_n, $(n=1,2..)$ be a sequence in P converging to $p \in P$. We must show that $\mathrm{Ls}\int_T \psi(t,p_n)d\mu(t) \subset \int_T \psi(t,p)d\mu(t)$. Since for each fixed $t \in T$, $\psi(t,\cdot)$ is u.s.c. we have that $\mathrm{Ls}\,\psi(t,p_n) \subset \psi(t,p)$ for all $t \in T$ and therefore

$$(2.3)\qquad \int_T \mathrm{Ls}\,\psi(t,p_n)d\mu(t) \subset \int_T \psi(t,p)d\mu(t).$$

It follows now from Lemma 2.1 that (2.3) can be written as:

$$\mathrm{Ls}\int_T \psi(t,p_n)d\mu(t) \subset \int_T \mathrm{Ls}\,\psi(t,p_n)d\mu(t) \subset \int_T \psi(t,p)d\mu(t).$$

This completes the proof of (ii).

We conclude by noting that a similar argument with that above can be carried through in an infinite dimensional setting in order to show that the approximate version of Fatou's Lemma implies that integration preserves upper-semicontinuity and vice versa.

FOOTNOTES

1. For the sequence ϕ_n: $T \to 2^{\mathbb{R}^\ell}$ to be integrably bounded we mean that there exists $g \in L_1(\mu)$ such that $\sup\{\|x\| : x \in \phi_n(t)\} \leqslant g(t)$ μ - a.e. for all n.

REFERENCES

R. J. Aumann (1965), "Integrals of Set-Valued Functions," _Journal of Mathematical Analysis and Applications_ 12, 1-12.

R. J. Aumann (1976), "An Elementary Proof that Integration Preserves Upper-Semicontinuity," _Journal of Mathematical Economics_ 3, 15-22.

D. Schmeidler (1970), "Fatou's Lemma in Several Dimenstions," _Proceedings of the American Mathematical Society_ 24, 300-306.