A COUNTEREXAMPLE AND AN EXACT VERSION OF FATOU'S LEMMA IN INFINITE DIMENSION

by

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Abstract

An example is presented to show that approximate versions of Fatou's Lemma in infinite dimension cannot be improved, and a sufficient condition for an exact version is provided.

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1. INTRODUCTION

Versions of the well known Fatou's Lemma for real valued, non-negative measurable functions ($\int \liminf f_n \leqslant \liminf \int f_n$) have been obtained for functions, or correspondences, with values in n-dimensional spaces (see Aumann [2], Schmeidler [10], Hildenbrand and Mertens [7], Artstein [1]).

The further step to infinite dimensional spaces (Banach spaces) is the object of current research. One of the motivations for this has been the problem, arising in mathematical economics, on the existence of an equilibrium in economies with an infinite dimensional commodity space and a measure space of agents (see e.g. Yannelis [11], and in particular remark 6.5). In this contest an essential ingredient is an infinite dimensional version of Fatou's Lemma.

A first major step in this direction has been the recent important work of Khan and Majumdar [9], where an approximate version has been proved for functions with values in a Banach space. This result has been extended to correspondences by Yannelis [12]. Here approximate means that in the inclusion $w - \limsup \int F_n \subset s\text{-cl} \int w\text{-}\limsup F_n$ the s-closure is required.

The question arises if the strong closure can be removed, thus providing an exact version of the lemma. The primary purpose of this work is to provide a simple example to the contrary, under the assumptions adopted in [9] and [12]. At the same time, analogous results true in the finite dimensional case are shown to be false in the infinite dimensional setting. In particular, it is shown that an important result of Khan [8] (which extends a theorem of Datko [4])is the best one can do in infinite dimensional spaces.

Moreover, we provide a sufficient condition for the "exact" version (i.e. without the strong closure) to hold. An exact version has already been proved, under different assumptions by Yannelis [12].

2. NOTATION AND DEFINITIONS.

2.1 Notation

X is a separable Banach space, with norm $\|\cdot\|$.

X^* is its topological dual, its norm is also denoted $\|\cdot\|$.

For $x \in X$, $p \in X^*$ the inner product is $p \cdot x$.

$\text{w-Ls}F_n = \{x \in X : x = \text{w-lim}\, x_k,\ x_k \in F_{n_k} \quad k=1,2\ldots\}$ (weak limit superior).

$\text{s-Ls}F_n = \{x \in X : x = \text{s-lim}\, x_k, x_k \in F_{n_k} \quad k=1,2\ldots\}$ (strong limit superior).

For K non empty subset of X, $\sigma_K(p) = \sup\limits_{x \in K} p \cdot x$ is the support function of K.

$\overline{\text{con}}K$ is the closed convex hull of K.

(T,τ,μ) a complete, finite, measure space

$L_1(\mu,X)$ is the space of equivalence classes of X-valued Bochner integrable functions.

For any $A \in 2^X \setminus \{\phi\}$, $|A| = \sup\limits_{a \in A} \|a\|_X$

$d(\cdot,A)$ the distance function

$$d(x,A) \equiv \inf_{a \in A} \|x-a\|_X$$

$e : 2^X \times 2^X \to \mathbb{R}_+ \cup \{+\infty\}$ excess distance

$$e(A,B) = \sup_{x \in A} d(x,B)$$

$h : 2^X \times 2^X \to \mathbb{R}_+ \cup \{+\infty\}$

$$h(A,B) = \max\{e(A,B),\ e(B,A)\}.$$

the Hausdorff distance.

2.2 Definitions

The sequence of correspondences $\{F_n(\cdot)\}$ is Cauchy in measure if

$\forall \varepsilon > 0\ \forall \delta > 0\ \exists N(\varepsilon,\delta) : m,n \geq N(\varepsilon) \Rightarrow \mu(\omega : h(F_m(t),F_n(t)) > \varepsilon) < \delta.$

(or $\forall \varepsilon > 0 \lim\limits_{m,n\to+\infty} \mu\,(\omega : h(F_m(t),\ F_n(t)) > \varepsilon) = 0$).

$P_f(X)$ [resp. $P_b(X)$, $P_c(X)$, and so on] is the set of all closed [resp. bounded, convex, and so on] subsets of X.

A correspondence $F : T \to P_f(X)$ is said to be measurable if for all $x \in X$, $d(x, F(t))$ is a measurable function.

$\mathcal{L}_F$ is the set of all selectors of $F(\cdot)$ in $L_1(\mu, X)$, i.e., $\mathcal{L}_F \equiv \{f \in L_1(\mu, X) : f(t) \in F(t) \mu\text{-a.e.}\}$.

We also define the integral of the correspondence F as follows: $\int F(t) d\mu(t) \equiv \{\int f(t) d\mu(t) : f \in \mathcal{L}_F\}$. A correspondence $F : T \to 2^X$ is said to be integrably bounded if there exists a map $g \in L_1(\mu)$ such that for μ-a.e. t in T, $|\phi(t)| \leqslant g(t)$.

3. A COUNTEREXAMPLE TO FATOU'S LEMMA IN INFINITE DIMENSIONAL SPACES

3.1 The Example

In this example the sequence $\phi_n : T \to X = l_2$, n=1,2,..., is a sequence of functions from (T,τ,μ) where $T = [0,2\pi]$, τ the Borel sets, μ the Lebesgue measure. We shall see that $\phi_n(t) \in K \equiv \{x \in l_2 : \|x\| \leqslant 4\pi\}$; a.e. t; K is a weakly compact, convex set. Also $Ls\phi_n(\cdot)$ is lower measurable (this follows immediately from Lemma 4.1 in Dsosu and Shaozhong [6]). The assumptions of the main theorems in Khan-Majumdar [9] and Yannelis [12] are therefore satisfied; but under these conditions the inclusion $s\text{-}Ls \int_T \phi_n(t)d\mu(t) \subset \int w\text{-}Ls\phi_n(t)d\mu(t)$ does not hold. Since it is always the case that $s\text{-}Ls\int\phi_n(t)d\mu(t) \subset w\text{-}Ls \int_T \phi_n(t)d\mu(t)$, we can conclude that the inclusion $w\text{-}Ls\int_T\phi_n(t)d\mu(t) \subset \int_T w\text{-}Ls\phi_n(t)d\mu(t)$ does not hold in general.

The example is a modification of the well known Liapunoff example on the range of a vector valued measure (see Diestel and Uhl [5, p. 262]).

We recall here the content of the example for the reader's convenience. Select a complete orthogonal system $(w_n)_{n=0}^{\infty}$ in $L_2(\mu)$ such that each w_n assumes only the values ± 1 and $w_0 = \chi_{[0,2\pi]}$, while $\int_0^{2\pi} w_n d\mu = 0$ $(n \geqslant 1)$. Then for each n define λ_n on τ by

$$\lambda_n(E) = 2^{-n} \int_E [1+w_n(t)/2]d\mu(t) \qquad (E \in \tau)$$

and finally let $G : t \to l_2$ be defined by

$$G(E) = (\lambda_0(E), \lambda_1(E), \ldots, \lambda_n(E), \ldots).$$

Then $\|G(E)\|_{l_2} \leqslant 2\mu(E)$ $(E \in \tau)$, from which it follows that G is of bounded variation and μ-continuous. Clearly 0 and G(T) are included in $G(\tau)$, so that G(T)/2 is in the convex hull of $G(\tau)$. It is proved in Diestel-Uhl [5, p. 262], that there is no $E \in \tau$ with $G(E) = G(T)/2$. From here we proceed with our example.

From the Radon-Nikodym theorem (see e.g. Diestel-Uhl [5, p. 59]) there exists a function $g \in L_1(\mu, l_2)$ such that:

$$G(E) = \int_E g(s)\,d\mu(s) = \int_T \chi_E(s)g(s)\,d\mu(s) \quad (E \in \tau).$$

From Uhl's theorem (Diestel-Uhl [5, Theorem 10, p. 266]) the norm closure of the range of G is closed, and convex; then G(T)/2 is in the closure. Hence, we conclude that there exists a sequence of sets $\{E_n\}_{n=1}^{\infty}$, $E_n \in \tau$, such that

$$\text{s-}\lim_{n\to\infty} G(E_n) = G(T)/2.$$

Let now $X : T \to 2^{l_2}$ be defined as $X(t) = \{x : \|x\|_{l_2} \leqslant 4\pi\}$; clearly X is an integrably bounded correspondence, with weakly compact and non-empty, convex values. Define the sequence $\phi_n : T \to l_2$ by $\phi_n(t) = \chi_{E_n}(t)g(t)$ n=1,2....

Clearly $\phi_n(t) \in \{0, g(t)\}$ for μ-a.e. $t \in T$, for every n; and therefore $\text{w-Ls}\,\phi_n(t) \subset \{0, g(t)\}$ (if non-empty). It follows that for any function $\phi \in \mathcal{L}_{\text{w-Ls}\phi_n}$ we have $\phi = \chi_E g$ for some set $E \in \tau$. But for the inclusion $\text{s-Ls} \int_T \phi_n(t)\,d\mu(t) \subset \int_T \text{w-Ls}\,\phi_n(t)\,d\mu(t)$ to hold we must in particular have $G(T)/2 \in \int_T \text{w-Ls}\,\phi_n(t)\,d\mu(t)$, that is $G(T)/2 = \int_E g(t)\,d\mu(t) = G(E)$; and that no such set E exists is proved in the quoted Liapunoff example.

3.2 Remarks:

(a) A classical result for the finite dimensional case (see Aumann [2, Theorem 4]) is that if F is closed valued and integrably bounded, then $\int F$ is compact. For the infinite dimensional case, under assumptions (that are easily checked to be satisfied in our case), Khan [8] proves that $\int \overline{\text{con}}F = \text{s-cl} \int F$. So if, in addition, F is convex, weakly closed, then $\int F$ is weakly closed, convex. The above example shows that the convex cannot be removed. Let in fact $F : T \to 2^{l_2}$ be defined as

$$F(t) = \{o, g(t)\}.$$

Then, from what we have seen above

$\int F$ is not s-closed.

The above example also shows that the convexity assumption of Lemma 3.1 in Yannelis [12] cannot be removed.

(b) Another classical result for the finite dimensional case is that if F is non negative and Borel measurable, then $\int \text{co}F = \int F$. (See Aumann [2, Theorem 3]). The correspondence F defined above proves that this is false already for l_2, because clearly $G(T)/2 = \int_T 1/2g(t)d\mu(t) \in \int \text{con}F$, and $G(T)/2 \notin \int F$.

(Notice that the above correspondence assumes values in the positive cone of l_2.)

(c) The correspondences $w - Ls\phi_n$ takes values (when non-empty) in the set $\{0, g(t)\}$, and is therefore compact valued. However it is not convex valued; it should be noted that with the assumption of convexity Yannelis [12] has proved an exact version of Fatou's Lemma.

(d) We finally observe that ϕ_n is not a Cauchy sequence in measure. In fact it is easy to see, under the assumptions of, say, the Main Theorem of Yannelis [12], and the additional condition that ϕ_n is a Cauchy sequence of <u>functions</u>, in measure, that:

$$s\text{-}Ls\int_T \phi_n(t)d\mu(t) = s\text{-}\lim\int_T \phi_n(t)d\ (t) = w\text{-}Ls\int_T \phi_n(t)d\mu(t)$$

$$\subset \int_T w\text{-}Ls\phi_n(t)d\mu(t).$$

(Note that $w\text{-}Ls\phi_n(t)$ may be a non convex set).

Indeed X(t) has only to be non-empty, integrably bounded, and not necessarily weakly compact and convex. The equalities follow simply from dominated convergence, and the existence of a limit function (so that $w\text{-}Ls\phi_n \neq \phi$) follows from the Cauchy condition. This will be examined in detail for the case of correspondences in the sequel.

4. A SUFFICIENT CONDITION FOR AN EXACT VERSION OF THE FATOU LEMMA

We shall need these simple lemmata.

Lemma 1: Let $F_n : T \to P_f(X)$, $n \geq 1$, be a sequence of measurable correspondences, such that $\{F_n(\cdot)\}_{n\geq 1}$ is uniformly integrable, and Cauchy in measure. Then there exists an $F : T \to P_f(X)$, measurable correspondence, such that $F_n \overset{h}{\to} F$ in measure, the family $((F_n)_{n>1}, F)$ is uniformly integrable. Also if $F_n : T \to P_{fc}(X)$, then $F : T \to P_{fc}(X)$.

Proof: Since $|F_n(t)| \leq g(t)$ μ-a.e. t, $g \in L_1(\mu)$, then clearly for every $n \geq 1$, $F_n(t) \in P_{fb}(X)$ μ-a.e.t. Choose now N_j, $j=1,2,\ldots$, so that if $k, l > N_j$ then $\mu(t : h(F_k(t), F_l(t)) > 2^{-j}) < 2^{-j}$. We may assume N_j is increasing. Then $h(F_{N_{j+1}}(t), F_{N_j}(t)) \leq 2^{-j}$ except for a set E_j, with $\mu(E_j) < 2^{-j}$. Let $H_i = \bigcup_{j=i}^{\infty} E_j$. Then $h(F_{N_{j+1}}(t), F_{N_j}(t)) \leq 2^{-j}$ for $t \notin H_i$, $\{F_{N_j}(t)\}_{j>i}$ is a Cauchy sequence in $(P_{cb}(X), h)$. Since such space is complete (Castaing-Valadier [3, Theorem II-14, p. 47]) there exists a set $F^i(t) \in P_{cb}(X)$ such that $F_{N_j}(t) \overset{h}{\to} F^i(t)$. Note that $F^{i+k}(t) = F^i(t)$, $k \geq 1$, $t \in T \setminus H^i$. Define the correspondence F as the limit on $T \setminus \bigcap_{i=1}^{\infty} H_i$ of the F^i correspondences. It is standard to check that $F(\cdot)$ satisfies the properties stated. The proof of the second part is similar.

Lemma 2: Let $F_n : T \to P_{fc}(X)$, $n > 1$, be a sequence of measurable correspondences. Let $F_n(t) \subset K$ μ-a.e.t, K a weakly compact convex subset of X). Let F_n be a Cauchy sequence in measure, and $F : T \to P_{fc}(X)$ as in Lemma 1. Then w-Ls $\int_T F(t)d\mu(t) \subset \int_T F(t)\, d\mu(t)$.

Proof: Let the sequence $x_{n_k} = \int_T f_{n_k}(t)d\mu(t)$, $f_{n_k} \in \mathcal{L}_{F_{n_k}}$ converge weakly to x_0. We claim that $x_0 \in \int_T F(t)d\mu(t)$, so arguing by contradiction we assume the opposite. We notice that $\int_T F(t)d\mu(t)$ is closed, bounded and convex. This follows from Lemma 3.1 in Yannelis [12], which gives s-cl $\int_T F(t)d\mu(t) = \int_T F(t)d\mu(t)$.

Therefore there exists a $\lambda \in X^*$, and $a_1, a_2 \in \mathbb{R}$ such that:

$$\lambda \cdot x_0 > a_1 > a_2 > \lambda \cdot x \ (x \in \int_T F(t)d\mu(t).$$

Let $\varepsilon = a_2 - \lambda \cdot x_0$; then since x_{n_k} converges weakly to x_0, there exists a $k(\varepsilon)$ such that $\lambda \cdot x_{n_k} + \varepsilon/3 \geqslant \lambda \cdot x_0$ $(k \geqslant k(\varepsilon))$, and therefore $\lambda \cdot x_{n_k} \geqslant a_2 + \varepsilon/3$. It is easy to show that $\mathcal{L}_{F_n} \overset{h}{\to} \mathcal{L}_F$, and so for every $\varepsilon' > 0$ there exists and $n(\varepsilon')$ such that $\mathcal{L}_{F_n} \subset \mathcal{L}_{F,\varepsilon} = \{f \in L_1(\mu,X) : d(f, \mathcal{L}_F) < \varepsilon\}$. Set now $\varepsilon' = \varepsilon/3\|x\|$. Then, if B_ε is the ball in $L_1(\mu,X)$ with radius ε,

$$\sigma_{\int \mathcal{L}_{F,\varepsilon}}(\lambda) = \sigma_{\int \mathcal{L}_F + B\varepsilon}(\lambda) = \sigma_{\int \mathcal{L}_F}(\lambda) + \sigma_{\int B_\varepsilon}(\lambda)$$

$$\leqslant a_2 + \sigma_{\int B_\varepsilon}(\lambda) \leqslant a_2 + \|\lambda\|\varepsilon' < a_2 + \varepsilon/3.$$

Since $\sigma_{\int \mathcal{L}_{F_n}}(\lambda) \leqslant \sigma_{\int \mathcal{L}_F}(\lambda)$, then $\sigma_{\int \mathcal{L}_{F_n}}(\lambda) \leqslant a_2 + \varepsilon/3$, a contradiction.

Lemma 3: Let $F_n : T \to P_f(X)$ be a sequence of measurable correspondences, $F_n \overset{h}{\to} F$ in measure; then $\int_T F(t)d\mu(t) \subset \int_T \text{w-Ls}F_n(t)d\mu(t)$.

Proof: We prove that in fact $\mathcal{L}_F \subset \text{s-Ls}F_n$. Let $f \in \mathcal{L}_F$. We have $h(F(t),F_n(t)) > d(f(t),F_n(t))$, so that $\{t : h(F_n(t),F(t)) > \varepsilon\} \subset \{t : d(f(t),F_n(t)) > \varepsilon\}$. Define now the sequence of real valued functions

g_n by $g_n(t) = d(f(t), F_n(t))$; each g_n is measurable because f, F_n are. From the assumption $F_n \xrightarrow{h} F$ in measure we have now that g_n converges in measure to zero. Hence on a subsequence $\{n_k\}$ we have $g_{n_k} \xrightarrow{\mu\text{-a.e.}} 0$; let Z be the exceptional set. Then $d(f(t), F_{n_k}(t)) \to 0$ $(t \in T \setminus Z)$, and so $f(t) \in s\text{-}\mathrm{Ls}F_n(t)$ $(t \in T \setminus Z)$, $\mu(Z) = 0$.

From Lemma 1, 2, 3 above then follows:

Theorem 1: Let $F_n : T \to P_{fc}(X)$, $n \geqslant 1$, be a sequence of measurable correspondences, h-Cauchy in measure. Let $F_n(t) \subset K$, μ-a.e.$t \in T$, for every n, where K is weakly compact and convex; then

$$w\text{-}\mathrm{Ls} \int_T F_n(t)\,d\mu(t) \subset \int_T w\text{-}\mathrm{Ls}F_n(t)\,d\mu(t).$$

Remark: The exact version in Yannelis [12, Theorem 3.1] does not require the Cauchy condition as well as the convex valuedness of the correspondences F_n; but the $w\text{-}\mathrm{Ls}F_n$ is assumed to be convex. Of course the convergence in measure does not imply this last condition (which may fail even when the F_n are functions) consequently, neither result implies the other.

Remark: The assumption in Lemma 2 and in the Theorem that $F_n(t) \subset K$, where K is a weakly compact convex subset of X, can be replaced by $(F_n)_{n \geqslant 1}$ is uniformly integrable and X is reflexive.

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