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Abstract: We provide random equilibrium existence theorems for games with a countable number of players. Our results give as corollaries random versions of the ordinary equilibrium existence Theorems of Nash [10], Fan [5] and Browder [1].

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# 1. INTRODUCTION

A finite game consists of a set of players $I = \{1,2,..,n\}$ each of whom is characterized by a strategy set $X_i$ and a payoff function $u_i : \prod_{j \in I} X_j \to \mathbb{R}$. An equilibrium for this game is a strategy vector such that no player can increase his/her payoff by deviating from his/her equilibrium strategy, given that the other players use their equilibrium strategies, i.e., $x^* \in \prod_{i \in I} X_i$ is an equilibrium if $u_i(x^*) = \max_{y_i \in X_i} u_i(x_i^*,..,x_{i-1}^*,y_i,x_{i+1}^*,..,x_n^*)$ for all $i \in I$. The above game form and the notion of equilibrium were both introduced in a seminal paper by Nash [10].[1] In that same paper Nash proved by means of the Brouwer fixed point theorem, the existence of an equilibrium for the above game, where strategy sets were subsets of $\mathbb{R}^\ell$, i.e., the $\ell$-fold product of the real numbers. The work of Nash has found very interesting applications in Game Theory and Mathematical Economics. Generalization of Nash's equilibrium existence theorem to games where strategy sets were subsets of arbitrary Hausdorff linear topological spaces, were obtained by Fan [5] and Browder [1] among others. The results of Fan and Browder were proved by means of infinite dimensional fixed point theorems.

All the above results are deterministic, i.e., players cannot accommodate any kind of uncertainty or randomness in their responses to potential changes in their primitive environment. In reality however, there are many factors which go beyond the control of players and cannot be influenced by their actions. In that sense, it seems natural to assume that player's payoff functions depend not only on the strategies, but on the states of nature of the world as well. In other words payoff functions can be random. This is the type of the game we will consider in this paper. Of course with random payoff functions the equilibrium strategy vector will be random as well, and therefore the equilibrium will change from one state of the environment to another.

It is the purpose of this paper to prove two random equilibrium existence results for a quite general form of random games. Our results provide as corollaries random versions of the theorems of Nash, Fan and Browder. It should be noted that our proofs do not subassume any of the ordinary equilibrium existence results of Nash or Fan or Browder. Our arguments start from a rudimentary level and provide a different way to prove the deterministic results of the above authors. As the deterministic results of Nash, Fan and Browder are based on deterministic fixed point theorems, the proofs of our random equilibrium existence theorems are based on random fixed points. In particular, we make use of a random version of Fan's [4, Theorem 6, p. 238] coincidence theorem. In addition we employ Aumann-type measurable selection theorems and some recent Caratheodory-type selections results proved in Kim-Prikry and the author [8, 9]. Finally we would like to note that recently Nowak [11, 12, 13] employing measure theoretic arguments has obtained random versions of the minimax and inequality results of Fan. Although, his techniques are different from ours we have obtained some results in a rather similar fashion.

The paper is organized as follows: Section 2 contains several preliminary results of measure theoretic character. Moreover, a random version of Fan's coincidence theorem is established. The main results of the paper are stated in Section 3. Finally, Section 4 contains the proofs of the main results.

# 2. PRELIMINARIES

## 2.1 Notation

$2^A$ denotes the set of all subsets of the set A

conA denotes the convex hull of the set A

$\setminus$ denotes the set theoretic subtraction

$\mathbb{R}^\ell$ denotes the $\ell$-fold Cartesian product of the set of real numbers $\mathbb{R}$

proj denotes projection

If $\phi : X \to 2^Y$ is a correspondence, then $\phi|_U : U \to 2^Y$ denotes the restriction of $\phi$ to U

$\emptyset$ denotes the empty set.

## 2.2 Upper and Lower Semicontinuous Correspondences

Let X and Y be sets. The graph $G_\phi$ of a correspondence $\phi : X \to 2^Y$ is the set $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. If X and Y are topological spaces, $\phi : X \to 2^Y$ is said to have an open graph if the set $G_\phi$ is open in $X \times Y$; $\phi : X \to 2^Y$ is said to be lower semicontinuous (l.s.c.) if the set $\{x \in X : \phi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y and; $\phi : X \to 2^Y$ is said to be upper-semicontinuous (u.s.c.) if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y. It can be easily checked that if a correspondence has an open graph, then it is l.s.c., but the reverse is not true.

We will need the following facts.

(F.2.1) Let X be a topological space and Y be a linear topological space. If the correspondence $\phi : X \to 2^Y$ is l.s.c. then the correspondence $\psi : X \to 2^Y$ defined by $\psi(x) = \text{con}\, \phi(x)$ is also l.s.c. [9, Proposition 3.6, p. 366].

(F.2.2) Let X be a topological space and $\{Y_i : i \in I\}$, (I can be any finite or infinite set) be a family of compact spaces. Let $Y = \prod_{i \in I} Y_i$. If for each $i \in I$, the correspondence $F_i : X \to 2^{Y_i}$ is u.s.c. and closed valued then the correspondence $F : X \to 2^Y$ defined by $F(x) = \prod_{i \in I} F_i(x)$ is also u.s.c. [3, Lemma 3, p. 124].

## 2.3 Auxilliary Measure Theoretic Facts

Let X, Y be topological spaces and $\phi : X \to 2^Y$ be a nonempty valued correspondence. A continuous selection for $\phi$ is a continuous function $f : X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in X$.

Let $(\Omega, \mathcal{Q})$ be a measurable space, Y be a topological space and $\phi : \Omega \to 2^Y$ be a nonempty-valued correspondence. A measurable selection for $\phi$ is a measurable function $f : \Omega \to Y$ such that $f(\omega) \in \phi(\omega)$ for all $\omega \in \Omega$.

We now define the concept of a Caratheodory selection which combines the notion of continuous selection and measurable selection.

Let $(X, \mathcal{Q})$ be a measurable space and Y and Z be topological spaces. Let $\phi : X \times Z \to 2^Y$ be a (possibly empty-valued) correspondence. Let $U = \{(x,z) \in X \times Z : \phi(x,z) \neq \emptyset\}$. A Caratheodory selection for $\phi$ is a function $f : U \to Y$ such that $f(x,z) \in \phi(x,z)$ for all $(x,z) \in U$ and; for each $x \in X$, $f(x,\cdot)$ is continuous on $U^x = \{z \in Z : (x,z) \in U\}$ and for each $z \in Z$, $f(\cdot,z)$ is measurable on $U^z = \{x \in X : (x,z) \in U\}$.

If $(X, \mathcal{Q})$ and $(Y, \mathcal{B})$ are measurable spaces and $\phi : X \to 2^Y$ is a correspondence, $\phi$ is said to have a measurable graph if $G_\phi$ belongs to the product $\sigma$-algebra $\mathcal{Q} \otimes \mathcal{B}$. We are usually interested in the situation where $(X, \mathcal{Q})$ is a measurable space, Y is a topological space and $\mathcal{B} = \mathcal{B}(Y)$ is the Borel $\sigma$-algebra of Y. For a correspondnece $\phi$ from a measurable space into a topological space, if we say that $\phi$ has a measurable graph, it is understood that the topological space is endowed with its Borel $\sigma$-algebra (unless specified otherwise). In the same setting as above i.e., $(X,\mathcal{A})$ a measurable space and Y a topological space, $\phi$ is said to be <u>lower measurable</u> if $\{x : \phi(x) \cap V \neq \phi\} \in \mathcal{A}$ for every V open in Y.

The following facts will be useful in the sequel.

(F.2.3) Let $(\Omega,\mathcal{A},\mu)$ be a complete finite measure space X be a separable metric space and $\phi : \Omega \to 2^X$ be a nonempty valued correspondence having a measurable graph, i.e., $G_\phi \in \mathcal{A} \otimes \mathcal{B}(X)$. Then there exists a measurable selection for $\phi$ [2, Theorem III.22, p. 22, or 6, Theorem 5.2, p. 60].

(F.2.4) Let $(\Omega,\mathcal{A},\mu)$ be a complete finite measure space, X be a complete separable metric space and $\phi : \Omega \times X \to 2^{\mathbb{R}^\ell}$ be a convex (possibly empty-) valued correspondence such that

(i) $\phi(\cdot,\cdot)$ is lower measurable with respect to the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}(X)$, and

(ii) for each $\omega \in \Omega$, $\phi(\omega,\cdot)$ is l.s.c.

Then there exists a Caratheodory selection for $\phi$ [7, Theorem 3.2].

(F.2.5) The previous fact remains true if $\phi$ is a correspondence from $\Omega \times X$ into $2^Y$, where Y is a separable Banach space and (i) and (ii) are replaced by

(i') $G_\phi \in \mathcal{A} \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$ and

(ii') for each $\omega \in \Omega$, $\phi(\omega,\cdot)$ has an open graph, i.e., for each $\omega \in \Omega$ the set $G_{\phi(\omega,\cdot)} = \{(x,y) \in X \times Y : y \in \phi(\omega,x)\}$ is open in $X \times Y$ [8, Main Theorem].

(F.2.6) Let $\Omega$ be a measurable space $\{Y_i : i \in I\}$, (where I is a countable set) be a family of second countable topological spaces. Let $Y = \prod_{i \in I} Y_i$. If for each $i \in I$, $F_i : \Omega \to 2^{Y_i}$ is lower measurable then the correspondence $F : \Omega \to 2^Y$ defined by $F(\omega) = \prod_{i \in I} F_i(\omega)$ is also lower measurable [6, Proposition 2.3, p. 55].

(F.2.7) Let $\Omega$ be a measurable space, X be a separable metric space and for each $i \in I$, (where I is a countable set) $F_i : \Omega \to 2^X$ is a lower measurable and closed valued correspondence. Suppose that for each $\omega \in \Omega$, $F_i(\omega)$ is compact for at least one $i \in I$. Then the correspondence $F : \Omega \to 2^X$ defined by $F(\omega) = \bigcap_{i \in I} F_i(\omega)$ is lower measurable [6, Theorem 4.1, p. 58].

If $(X,\mathcal{A})$, $(Y,\mathcal{B})$ and $(Z,\mathcal{F})$ are measurable spaces, $U \subseteq X \times Z$ and $f : U \to Y$, we call f <u>jointly measurable</u> if for every $B \in \mathcal{B}$, $f^{-1}(B) = U \cap A$ for some $A \in \mathcal{A} \otimes \mathcal{F}$. It is a standard result that if Z is a separable metric space, Y is metric and $f : X \times Z \to Y$ is such that for each fixed $x \in X$, $f(x,\cdot)$ is continuous and for each fixed $z \in Z$, $f(\cdot,z)$ is measurable, then f is jointly measurable (where $B = \mathcal{B}(Y)$, $\mathcal{F} = \mathcal{B}(Z)$). It turns out, that in several instances U is a proper subset of $X \times Z$, and this situation is more delicate. However, in this more delicate situation it can be shown that f is still jointly measurable. In particular, we have the following fact.

(F.2.8) Let $(\Omega,\mathcal{A})$ be a measurable space, X be a separable metric space, Y be a metric space and $U \subseteq \Omega \times X$ be such that

(i) for each $\omega \in \Omega$ the set $U^{\omega} = \{x \in X : (\omega,x) \in U\}$ is open in X, and

(ii) for each $x \in X$ the set $U^x = \{\omega \in \Omega : (\omega,x) \in U\}$ belongs to $\mathcal{A}$.

Let $f : U \to Y$ be a function such that for each $\omega \in \Omega$, $f(\omega,\cdot)$ is continuous on $U^{\omega}$ and for each $x \in X$, $f(\cdot,x)$ is measurable on $U^{x}$. Then $f$ is jointly relatively measurable with respect to the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}(X)$, i.e. for every open subset $V$ of $Y$, $\{(\omega,x) \in U : f(\omega,x) \in V\} = U \cap A$ for some $A \in \mathcal{A} \otimes \mathcal{B}(X)$ [7, Lemma 4.12].

(F.2.9) Let $(\Omega,\mathcal{A})$ be a measurable space and $X$ be a complete separable metric space. If $0$ belongs to $\mathcal{A} \otimes \mathcal{B}(X)$ its projection $\mathrm{proj}_{\Omega}(0)$ belongs to $\mathcal{A}$, [2, Theorem III.23, p. 75].

## 2.4 The Random Coincidence Theorem

The result below is a random version of Fan's Coincidence Theorem, [4, Theorem 6, p. 238].

Theorem 2.4.1: Let $X$ be a compact convex nonempty subset of a locally convex separable and metrizable linear topological space $Y$ and let $(\Omega,\mathcal{F},v)$ be a complete finite measure space. Let $\gamma : \Omega \times X \to 2^{Y}$ and $\mu : \Omega \times X \to 2^{Y}$ be two nonempty, convex, closed and at least one of them is compact valued correspondences such that:

(i) $\mu(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are lower measurable,

(ii) for each fixed $\omega \in \Omega$, the correspondences $\mu(\omega,\cdot) : X \to 2^{Y}$ and $\gamma(\omega,\cdot) : X \to 2^{Y}$ are u.s.c.

(iii) for every $\omega \in \Omega$ and every $x \in X$, there exist three points $y \in X$, $u \in \gamma(\omega,x)$, $v \in \mu(\omega,x)$ and a real number $\lambda > 0$ such that $y - x = \lambda(u-v)$.

Then there exists a measurable function $x^{*} : \Omega \to X$ such that $\gamma(\omega,x^{*}(\omega)) \cap \mu(\omega,x^{*}(\omega)) \neq \phi$ for almost all $\omega \in \Omega$.

<u>Proof</u>: Define the correspondence $W : \Omega \times X \to 2^Y$ by $W(\omega,x) = \gamma(\omega,x) \cap \mu(\omega,x)$. Since $\gamma(\cdot,\cdot)$ and $\mu(\cdot,\cdot)$ are closed valued and lower measurable and at least one of them is compact valued, it follows from (F.2.7) that $W(\cdot,\cdot)$ is lower measurable. Define the correspondence $\phi : \Omega \to 2^X$ by

$$\phi(\omega) = \{x \in X : W(\omega,x) \neq \emptyset\}.$$

Observe that

$$\begin{aligned} G_\phi = \{(\omega,x) \in \Omega \times X : x \in \phi(\omega)\} &= \{(\omega,x) \in \Omega \times X : W(\omega,x) \neq \emptyset\} \\ &= \{(\omega,x) \in \Omega \times X : W(\omega,x) \cap Y \neq \emptyset\}, \end{aligned}$$

and the latter set belongs to $\mathfrak{F} \otimes \mathcal{B}(X)$ since $W(\cdot,\cdot)$ is lower measurable. Therefore, $G_\phi \in \mathfrak{F} \otimes \mathcal{B}(X)$. It follows from Fan's Coincidence Theorem, (Fan [4, Theorem 6, p. 238]) that for each $\omega \in \Omega$, $\phi(\omega) \neq \emptyset$. Therefore, the correspondence $\phi : \Omega \to 2^X$ satisfies all the conditions of (F.2.3), (the Aumann Measurable Selection Theorem) and consequently, there exists a measurable function $x^* : \Omega \to X$ such that $x^*(\omega) \in \phi(\omega)$ for almost all $\omega$ in $\Omega$, i.e., $\gamma(\omega,x^*(\omega)) \cap \mu(\omega,x^*(\omega)) \neq \emptyset$ for almost all $\omega$ in $\Omega$. This completes the proof of the Theorem.

An immediate corollary of the above theorem is a random version of the Kakutani-Fan fixed point theorem (see [3, Theorem 1, p. 122]).

<u>Corollary 2.4.1</u>: Let $X$ be a compact, convex, non-empty subset of a locally convex separable and metrizable linear topological space $Y$ and let $(\Omega,\mathfrak{F},v)$ be a complete finite measure space. Let $\gamma : \Omega \times X \to 2^X$ be a nonempty, convex, closed valued correspondence such that for each fixed $\omega \in \Omega$, $\gamma(\omega,\cdot)$ is u.s.c. Then $\gamma(\cdot,\cdot)$ has a <u>random fixed point</u>, i.e., there exists a measurable function $x^* : \Omega \to X$ such that $x^*(\omega) \in \gamma(\omega,x^*(\omega))$ for almost all $\omega$ in $\Omega$.

Proof: Let $\mu(\omega,x) = \{x\}$. Clearly for each fixed $\omega \in \Omega$, $\mu(\omega,\cdot)$ u.s.c. and $\mu(\cdot,\cdot)$ is convex, nonempty, compact valued. Let $x \in X$ and $\omega \in \Omega$. By choosing $u \in \gamma(\omega,x)$, $v = x \in \mu(\omega,x)$ and $\lambda \in (0,1)$ assumption (iii) of Theorem 2.4.1 is satisfied (simply notice that since X is convex $y = x + \lambda(u-v) = \lambda u + (1-\lambda)x \in X$). Hence, by the previous theorem there exists a measurable function $x^* : \Omega \to X$ such that $\gamma(\omega,x^*(\omega)) \cap \mu(\omega,x^*(\omega)) \neq \phi$ for almost all $\omega \in \Omega$, i.e., $x^*(\omega) \in \gamma(\omega,x^*(\omega))$ for almost all $\omega \in \Omega$.

## 3. THE MAIN RESULTS

### 3.1 Random Games and Random Equilibria

Let $(\Omega, \mathfrak{F}, \mu)$ be a complete finite measure space. We interpret $\Omega$ as the states of nature of the world and assume that $\Omega$ is large enough to include all the events that we consider to be interesting. $\mathfrak{F}$ will denote the $\sigma$-field of events. Denote by I the set of players. I can be any finite or countably infinite set. A random game $E = \{(X_i, P_i) : i \in I\}$ is a set of ordered pairs $(X_i, P_i)$ where

(1) $X_i$ is the strategy set of player i, and

(2) $P_i : \Omega \times X \to 2^{X_i}$ (where $X = \prod_{i \in I} X_i$) is the preference (or choice) correspondence of player i.

We read $y_i \in P_i(\omega, x)$ as player i strictly prefers $y_i$ to $x_i$ at the state of nature $\omega$, if the (given) components of other players are fixed.

A random equilibrium for E is a measurable function $x^* : \Omega \to X$ such that for all $i \in I$, $P_i(\omega, x^*(\omega)) = \phi$ for almost all $\omega \in \Omega$.

Notice that each player in the game described above is characterized by a strategy set and a preference correspondence. We now follow the original formulation by Nash [10] (and his generalizations by Fan [5] and Browder [1] among others) where preference correspondences are replaced by payoff functions, i.e., real valued functions defined on $\Omega \times X$.

Let $\Gamma = \{(X_i, u_i) : i \in I\}$ be a Nash-type random game, i.e.,

(i) $X_i$ is the strategy set of player i, and

(ii) $u_i : \Omega \times X \to \mathbb{R}$, (where $X = \prod_{i \in I} X_i$) is the payoff function of player i.

Let $\tilde{X}_i = \prod_{j \neq i} X_j$ and denote the points of $\tilde{X}_i$ by $\tilde{x}_i$.

A random Nash equilibrium for $\Gamma$ is a measurable function $x^* : \Omega \to X$ such that for all i,

$$u_i(\omega, x^*(\omega)) = \max_{y_i \in X_i} u_i(\omega, y_i \tilde{x}_i^*(\omega)) \text{ for almost all } \omega \in \Omega.$$

3.2 Main Theorems

Theorem 3.1: Let $E = \{(X_i, P_i) : i \in I\}$ be a random game satisfying for each i the following assumptions:

(a.1.1) $X_i$ is a compact, convex, nonempty subset of $\mathbb{R}^\ell$,

(a.1.2) $\text{con}P_i(\cdot,\cdot)$ is lower measurable, i.e., for every open subset V of $X_i$, the set $\{(\omega,x) \in \Omega \times X : \text{con}P_i(\omega,x) \cap V \neq \phi\}$ belongs to $\mathfrak{F} \otimes \mathfrak{B}(X)$,

(a.1.3) for every measurable function $x : \Omega \to X$, $x_i(\omega) \notin \text{con}P_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$,

(a.1.4) for each fixed $\omega \in \Omega$, $P_i(\omega,\cdot)$ is l.s.c.

Then there exists a random equilibrium for E.

As a Corollary of Theorem 3.1 we obtain a generalized random version of Nash's [10, Theorem 1, p. 288] equilibrium existence result.

Corollary 3.1: Let $\Gamma = \{(X_i, u_i) : i \in I\}$ be a Nash-type random game satisfying for each i the following assumptions.

(c.1.1) $X_i$ is a compact, convex, nonempty subset of $\mathbb{R}^\ell$, (c.1.2) for each fixed $\omega \in \Omega$, $u_i(\omega, \cdot)$ is continuous,

(c.1.3) for each fixed $x \in X$, $u_i(\cdot, x)$ is measurable,

(c.1.4) for each $\omega \in \Omega$ and each $\tilde{x}_i \in \tilde{X}_i = \prod_{j \neq i} X_j$, $u_i(\omega, x_i, \tilde{x}_i)$ is a quasi-concave function of $x_i$ on $X_i$.

Then there exists a random Nash equilibrium for $\Gamma$.

Theorem 3.2: Let $E = \{(X_i, P_i) : i \in I\}$ be a random game satisfying for each i the following assumptions.

(a.2.1) $X_i$ is a compact, convex, nonempty subset of a separable Banach space,

(a.2.2) $\text{con}P_i(\cdot, \cdot)$ has a measurable graph, i.e., the set $\{(\omega, x, y_i) \in \Omega \times X \times X_i : y_i \in \text{con}P_i(\omega, x)\} \in \mathcal{F} \otimes \mathcal{B}(X) \otimes \mathcal{B}(X_i)$,

(a.2.3) for every measurable function $x : \Omega \to X$, $x_i(\omega) \notin \text{con}P_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$,

(a.2.4) for each $\omega \in \Omega$, $P_i(\omega, \cdot)$ has an open graph in $X \times X_i$.

Then E has a random equilibrium.

The following Corollary of Theorem 3.2 extends Corollary 3.1 to strategy sets which can be subsets of arbitrary separable Banach spaces. We thus have a random version of Nash's result [10, Theorem 1, p. 288] in separable Banach spaces. It should be noted that Corollary 3.2 may be seen as a random generalization of the deterministic equilibrium existence results of Fan [5, Theorem 4, p. 192] and Browder [1, Theorem 14, p. 277], but only if the underlying strategy space is separable. It is worth noting that Fan and Browder allow only for a finite number of players whereas in our setting the set of players can be any finite or countably infinite set.

Corollary 3.2: Replace assumption (c.1.1) in Corollary 3.1 by

(c.1.1') $X_i$ is a nonempty, compact, convex subset of a separable Banach space.

Then the conclusion of Corollary 3.1 remains true.

A couple of comments are in order. Notice that the continuity assumption in Theorem 3.1, i.e., (a.1.4) is weaker than the continuity assumption (a.2.4) of Theorem 3.2. The reason we need a weaker continuity assumption is that the proof of Theorem 3.1 makes use of (F.2.4) which is a Caratheodory selection result for a correspondence which is lower measurable in one variable and l.s.c. in the other. However, in the proof of Theorem 3.2 a different Caratheodory selection result is used, i.e., (F.2.5), which requires a stronger continuity assumption. Moreover, observe that Corollary 3.1 follows directly from Corollary 3.2. Nevertheless, we choose to state Corollary 3.1 since its proof by means of Theorem 3.1 is slightly different than the proof of Corollary 3.2 which follows from Theorem 3.2. Finally it is important to note that the proofs of Theorems 3.1 and 3.2 do not subassume any deterministic equilibrium existence results. To the contrary, our arguments "start from scratch" and provide alternative ways to prove the equilibrium results of Nash, Fan and Browder.

# 4. PROOF OF THE MAIN THEOREMS

## 4.1 Lemmata

We begin by proving two Lemmata which are going to be needed in the sequel.

Lemma 4.1: Let $(S,\mathcal{A})$ be a measurable space and X, Y be separable metric spaces. Let $\phi : S \times X \to 2^Y$ be a lower measurable (possibly empty-valued) correspondence. Suppose that for each fixed $s \in S$, $\phi(s,\cdot)$ is l.s.c. Let $O = \{(s,x) \in S \times X : \phi(s,x) \neq \emptyset\}$, and let $f : O \to Y$ be a Caratheodory selection for $\phi$. Then the correspondence $\theta : S \times X \to 2^Y$ defined by

$$\theta(s,x) = \begin{cases} \{f(s,x)\} & \text{if } (s,x) \in O \\ \\ Y & \text{if } (s,x) \notin O \end{cases}$$

is lower measurable.

Proof: We begin by making a couple of observations. First notice that, since $\phi(\cdot,\cdot)$ is lower measurable the set $O = \{(s,x) \in S \times X : \phi(s,x) \neq \emptyset\} = \{(s,x) \in S \times X : \phi(s,x) \cap Y \neq \emptyset\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(X)$. By (F.2.9) for each $x \in X$ the set

$$\begin{aligned} O^x = \{s \in S : (s,x) \in O\} &= \mathrm{proj}_S(\{(s,x) \in S \times X : \phi(s,x) \neq \emptyset\} \cap (S \times \{x\})) \\ &= \mathrm{proj}_S(O \cap (S \times \{x\})), \end{aligned}$$

belongs to $\mathcal{A}$. Moreover, notice that since for each fixed $s \in S$, $\phi(s,\cdot)$ is l.s.c. it follows that for each $s \in S$ the set $O^s = \{x \in X : (s,x) \in O\}$ is open in X. Since for each fixed $s \in S$, $f(s,\cdot)$ is continuous on $O^s$ and for each fixed $x \in S$, $f(\cdot,x)$ is measurable on $O^x$, by (F.2.8) $f(\cdot,\cdot)$ is jointly measurable. It can be easily now seen that for every open subset V of Y the set $A = \{(s,x) \in S \times X : \theta(s,x) \cap V \neq \emptyset\} = B \cup C$

where $B = \{(s,x) \in O : f(s,x) \in V\}$ and $C = \{(s,x) \in S \times X \setminus O : Y \cap V \neq \phi\}$. Clearly, $B \in \mathcal{A} \otimes \mathcal{B}(X)$ and $C \in \mathcal{A} \otimes \mathcal{B}(X)$ and therefore $A = B \cup C$ belongs to $\mathcal{A} \otimes \mathcal{B}(X)$. Consequently, $\theta(\cdot,\cdot)$ is lower measurable as was to be shown.

Lemma 4.2: Let $(S,\mathcal{A})$ be a measurable space, Z be a separable metric space and Y be a metric space. Let $g : S \times Z \to Y$ be a function such that for each fixed $s \in S$, $g(s,\cdot)$ is continuous and for each fixed $x \in Z$, $g(\cdot,z)$ is measurable. Define the correspondence $K : S \to 2^Z$ by

$$K(s) = \{z \in Z : g(s,z) > 0\}.$$

Then, (a) $G_K \in \mathcal{A} \otimes \mathcal{B}(Z)$, i.e., $K(\cdot)$ has a measurable graph, and (b) $K(\cdot)$ is lower measurable.

Proof: (a) Since for each fixed $s \in S$, $g(s,\cdot)$ is continuous and for each fixed $z \in Z$, $g(\cdot,z)$ is jointly measurable, it follows from a standard result that $g(\cdot,\cdot)$ is jointly measurable. Observe that,

$$\begin{aligned} g^{-1}((0,+\infty)) &= \{(s,z) \in S \times Z : g(s,z) > 0\} \\ &= \{(s,z) \in S \times Z : z \in K(s)\} \\ &= G_K, \end{aligned}$$

and the latter set belongs to $\mathcal{A} \otimes \mathcal{B}(Z)$ since $g(\cdot,\cdot)$ is jointly measurable.

(b) We must show that the set $\{s \in S : K(s) \cap V \neq \phi\}$ belongs to $\mathcal{A}$ for every open subset V of Z. As it was remarked above, $g(\cdot,\cdot)$ is jointly measurable, i.e., g is measurable with respect to the product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}(Z)$. Let D be a countable dense subset of Z, and let $U = (0, +\infty)$. Observe that,

$$\{s : K(s) \cap V \neq \phi\} = \{s : g(s,z) \in U \text{ for some } z \in V\}$$
$$= \{s : g(s,d) \in U \text{ for some } d \in D\}$$
$$= \bigcup_{d \in D} \{s : g(s,d) \in U\},$$

and the latter set belongs to $\mathcal{Q}$ since for each fixed $z \in Z$, $g(\cdot,z)$ is measurable. This completes the proof of the Lemma.

4.2 Proof of Theorem 3.1

For each $i \in I$ define the correspondence $\phi_i : \Omega \times X \to 2^{X_i}$ by $\phi_i(\omega,x) = \text{con}P_i(\omega,x)$. Since by assumption (a.1.4) for each fixed $\omega \in \Omega$, $P_i(\omega,\cdot)$ is l.s.c. it follows from (F.2.1) that for each fixed $\omega \in \Omega$, $\phi_i(\omega,\cdot)$ is l.s.c. Furthermore, by assumption (a.1.2), $\phi_i(\cdot,\cdot)$ is lower measurable and clearly convex valued. For $i \in I$ let $O_i = \{(\omega,x) \in \Omega \times X : \phi_i(\omega,x) \neq \phi\}$. For each $\omega \in \Omega$ let $O_i^{\omega} = \{x \in X : (\omega,x) \in O_i\}$ and for each $x \in X$ let $O_i^{x} = \{\omega \in \Omega : (\omega,x) \in O_i\}$. It follows from (F. 2.4) that there exists a Caratheodory selection for $\phi_i$, i.e., there exists a function $f_i : O_i \to X_i$ such that $f_i(\omega,x) \in \phi_i(\omega,x)$ for all $(\omega,x) \in O_i$ and for each $x \in X$, $f_i(\cdot,x)$ is measurable on $O_i^x$ and for each $\omega \in \Omega$, $f_i(\omega,\cdot)$ is continuous on $O_i^{\omega}$. For each $i \in I$ define the correspondence $F_i : \Omega \times X \to 2^{X_i}$ by

$$F_i(\omega,x) = \begin{cases} \{f_i(\omega,x)\} & \text{if } (\omega,x) \in O_i \\ X_i & \text{if } (\omega,x) \notin O_i. \end{cases}$$

By Lemma 4.1, $F_i(\cdot,\cdot)$ is lower measurable, and it is obviously closed, convex, nonempty valued. Since for each fixed $\omega \in \Omega$, $\phi_i(\omega,\cdot)$ is l.s.c. the set $O_i^{\omega} = \{x \in X : (\omega,x) \in O_i\} = \{x \in X : \phi_i(\omega,x) \neq \phi\} = \{x \in X : \phi_i(\omega,x) \cap X_i \neq \phi\}$

is open in the relative topology of X, and consequently for each fixed $\omega \in \Omega$, $F_i(\omega,\cdot)$ is u.s.c. Define the correspondence $F : \Omega \times X \to 2^X$ by $F(\omega,x) = \prod_{i \in I} F_i(\omega,x)$. Since $F_i(\cdot,\cdot)$ is lower measurable it follows from (F.2.6) that $F(\cdot,\cdot)$ is lower measurable as well. Obviously $F(\cdot,\cdot)$ is closed, convex and nonempty valued. By (F.2.2) for each fixed $\omega \in \Omega$, $F(\omega,\cdot) : X \to 2^X$ is u.s.c. Therefore, $F(\cdot,\cdot)$ satisfies all the conditions of Corollary 2.4.1 and consequently there exists a random fixed point, i.e., there exists a measurable function $x^* : \Omega \to X$ such that $x^*(\omega) \in F(\omega,x^*(\omega))$ for almost all $\omega \in \Omega$. We now show that the random fixed point is by construction a random equilibrium for the game E. Notice that for each $i \in I$, if $(\omega,x^*(\omega)) \in O_i$, then by the definition of $F_i$, $x_i^*(\omega) = f_i(\omega,x^*(\omega)) \in \mathrm{con}P_i(\omega,x^*(\omega))$, a contradiction to assumption (a.1.3). Thus, for all $i \in I$, $(\omega,x^*(\omega)) \notin O_i$ for almost all $\omega \in \Omega$, i.e., for all $i \in I$, $\mathrm{con}P_i(\omega,x^*(\omega)) = \phi$ for almost all $\omega \in \Omega$ which in turn implies that for all $i \in I$, $P_i(\omega,x^*(\omega)) = \phi$ for almost all $\omega \in \Omega$, i.e., $x^* : \Omega \to X$ is a random equilibrium for E. This completes the proof of the Theorem.

### 4.3 Proof of Corollary 3.1

For each $i \in I$, define the correspondence $Q_i : \Omega \times X \to 2^{X_i}$ by $Q_i(\omega,x) = \{y_i \in X_i : h_i(\omega,x,y_i) > 0\}$, where $h_i(\omega,x,y_i) = u_i(\omega,y_i,\tilde{x}_i) - u_i(\omega,x)$. Setting $S = \Omega \times X$, $Z = X_i$, $\mathcal{A} = \mathcal{F} \otimes \mathcal{B}(X)$, $g(s,z) = h_i(\omega,x,y_i)$, $K(s) = Q_i(\omega,x)$ for $s = (\omega,x)$ in Lemma 4.2(b) we can conclude that $Q_i(\cdot,\cdot)$ is lower measurable. It follows from assumption (c.1.4) that, $Q_i(\cdot,\cdot)$ is convex valued, and clearly for any measurable function $x : \Omega \to X$, $x_i(\omega) \notin \mathrm{con}Q_i(\omega,x(\omega)) = Q_i(\omega,x(\omega))$ for almost all $\omega \in \Omega$. Hence, the random game $E = \{(X_i,Q_i) : i \in I\}$ satisfies all the assumptions of Theorem 3.1 and therefore E has a random equilibrium, i.e., there exists a measurable function $x^* : \Omega \to X$ such that for all i, $Q_i(\omega,x^*(\omega)) = \phi$ for almost all $\omega \in \Omega$. But this implies that for all i, $u_i(\omega,x^*(\omega)) = \max_{y_i \in X_i} u_i(\omega,y_i,\tilde{x}_i^*(\omega))$, for almost all $\omega \in \Omega$, i.e., $x^* : \Omega \to X$ is a random Nash equilibrium for the game $\Gamma = \{(X_i,u_i) : i \in I\}$. This completes the proof of the Corollary.

## 4.4 Proof of Theorem 3.2

For each $i \in I$ define the correspondence $\phi_i : \Omega \times X \to 2^{X_i}$ by $\phi_i(\omega,x) = \mathrm{con}P_i(\omega,x)$. Since by assumption (a.2.4) for each $\omega \in \Omega, P_i(\omega,\cdot)$ has an open graph in $X \times X_i$ it can be easily checked that so does $\phi_i(\omega,\cdot)$ for each $\omega \in \Omega$.

Let $O_i = \{(\omega,x) \in \Omega \times X : \phi_i(\omega,x) \neq \phi\}$. Since $\phi_i(\cdot,\cdot)$ has a measurable graph (recall assumption (a.2.2)) and it is convex valued appealing to (F.2.5) we can ensure the existence of a Caratheodory selection for $\phi_i$. One can now proceed as in the proof of Theorem 3.1 to complete the proof.

## 4.5 Proof of Corollary 3.2

The proof is identical with that of Corollary 3.1 except with the fact that one now has to use Lemma 4.2(b) to show that $Q_i(\cdot,\cdot)$ has a measurable graph.

## FOOTNOTES

1. Notice that this notion of equilibrium is non-cooperative. No communication between players is allowed.

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