ON CORES OF WEAKLY BALANCED GAMES

WITHOUT ORDERED PREFERENCES

by

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ABSTRACT

A new concept of balancedness for games in normal form is introduced, called weak balanacedness. It is shown that the α-core of a weakly balanced game with an infinite dimensional strategy space and without ordered preferences is nonempty. Using this result we prove core existence theorems for economies (either exchange economies of coalitional production economies) with infinitely many commodities and without ordered preferences, by converting the economy to a game and showing that the derived game is weakly balanced. Surprisingly, no convexity assumption on preferences is needed to demonstrate that the game derived from the economy is weakly balanced.

1. INTRODUCTION

Two game theoretic solution concepts that have been used extensively in general equilibrium analysis are the Nash equilibrium (a noncooperative notion) and the core (a cooperative notion). These concepts can be traced back to Cournot and Edgeworth, respectively. Both concepts are very useful in their own right (i.e. in games), but have also proved useful in obtaining results for economies. Although the core is the primary focus of this paper, we will also discuss the development of the Nash equilibrium concept. Many of the factors that have motivated extensions of Nash equilibrium existence results also provide motivation for our core existence results. In addition, our core results have implications for the Nash equilibrium.

The existence of a Nash equilibrium for a game was first proved by Nash [19] and then by Debreu [10]. The latter result has played an important role in proving the existence of a competitive equilibrium for an economy. In particular, Arrow-Debreu [2] used the Debreu game theoretic result to prove the existence of a competitive equilibrium for an economy in the following way.[1] First, they demonstrated that an economy can be converted to a game, and then used the Debreu result to establish that the existence of a Nash equilibrium for a game implies the existence of a competitive equilibrium for an economy. This provided an important link between game theoretic and general equilibrium models of economic behavior.

Recently, the Nash equilibrium existence result has been generalized in two important ways. The first generalization has been to prove the existence of a Nash equilibrium for a game without ordered preferences (i.e. players' preferences need not be transitive or complete, hence need not be representable by utility functions). This game theoretic research was motivated by a result on the existence of competitive equilibria in economies without ordered preferences by Mas-Colell [17], and is important because many instances of behavior which appear to be nontransitive have been documented by experimental studies.[2] Shafer-Sonnenschein [22] and Borglin-Keiding [8], among others, have obtained Nash equilibrium existence results for games with preferences that need not be ordered. These results have led to new competitive equilibrium existence theorems in economies where agents' preferences need not be ordered (see for instance Kim-Richter [19] and their references).

The second generalization of the Nash result has been to prove the existence of a Nash equilibrium in games with an infinite dimensional strategy space. This research has been motivated by work on economies with infinite dimensional commodity spaces by Bewley [4], and is important because such commodity spaces arise very naturally in many settings (e.g. problems involving infinite time horizons, uncertainty about the possibly infinite number of states of the world, or commodities with an infinite variety of characteristics). Toussaint [24] and Yannelis-Prabhakar [25] have obtained Nash equilibrium existence results for games with infinite dimensional strategy spaces. These results have led to new competitive equilibrium existence theorems for economies where agents' preferences need not be ordered and with infinitely many commodities (see for instance Toussaint [24] or Yannelis-Zame [26]).

The core is an alternative game theoretic solution concept which is widely used in economics. The classical definition of the core (i.e., the "selfish core") is the set of all feasible outcomes that cannot be improved upon by any coalition. This notion does not allow for interdependent preferences as the preference of each agent is defined only on the agent's own strategy set. Another notion of the core, called the α-core, was introduced by Aumann [3] and allows for interdependent preferences. The α-core of a game is the set of all feasible outcomes such that no coalition of agents can change its strategy to make each player in the coalition better off, independently of the choices of the complementary coalition. Under this definition, each agent's preferences are defined over the product of all individual strategy sets. Two other notions of the core, the β-core and the strong equilibrium, are closely related to the α-core. However, these concepts are not of primary interest in this paper and consequently will be discussed in Section 8.

The first core existence result for a game, where players preferences were representable by utility functions and the strategy space was finite dimensional, was proved by Scarf [20]. Scarf used this pioneering game theoretic result to obtain core existence results for economies by following an approach similar to Arrow-Debreu [2] (i.e. by converting the economy to a game and then showing that the nonemptiness of the core of the game implied the nonemptiness of the core of the economy). This existence result was obtained for the "selfish core," and was extended to the α-core in Scarf [21]. However, the latter result also required agents' preferences to be representable by utility functions and the strategy space to be finite dimensional.

Recently, Border [6] extended Scarf's [20] core existence result to a game where agents' preferences need not be ordered. Border obtained his result for the "selfish core," but his approach does not cover the α-core or extend to infinite dimensional strategy spaces (see Section 3.6 for a discussion of the latter point). These two extensions are of interest for the following reasons. The α-core provides information about settings where agents' preferences may be interdependent, and many economic problems can be characterized by such preferences (e.g. externalities in consumption). Infinite dimensional spaces also arise very naturally in many economic settings (see the previous discussion), and recently several nonexistence results for the core in economies with infinite dimensional commodity spaces have been obtained. In particular, Araujo [1], Jones [15] and Mas-Colell [18] have shown that the core may be empty in very well behaved economies with an infinite dimensional commodity space. Consequently, it is important to know under what conditions core existence results can be obtained in such settings.

The primary purpose of this paper is to provide a set of assumptions which guarantee the nonemptiness of the α-core for a game where players' preferences need not be ordered and the strategy space is infinite dimensional. Nonemptiness of the "selfish core" follows directly from this result, as do α-core existence results for either exchange economies or coalitional production economies with infinitely many commodities. These α-core existence results also allow us to prove the existence of a Nash equilibrium for a game where agents' preferences need not be ordered and the strategy space is infinite dimensional. Hence generalizations of the Debreu [10] and Shafer-Sonnenschein [22] results, similar with those in Toussaint [24] and Yannelis-Prabhakar [25], follow from our α-core existence result. This provides an interesting link between existence results for these two important, but different (i.e. cooperative vs. noncooperative) game theoretic solution concepts.

Finally, we wish to comment briefly on our methodology. To allow for preferences that need not be ordered and an infinite dimensional strategy space, we must follow arguments that are necessarily different from those adopted by Scarf [20, 21] and Border [6]. In particular, we introduce a new concept of balancedness, called weak balancedness, which is of fundamental importance for our results. This notion of balancedness is weaker than the one used by Boehm [5] and Border [6], and is different than the one used by Scarf [20] and subsequently by Shapley [23] and Ichiishi [13, 14]. An important aspect of our approach is that the game derived from an exchange economy satisfies the weak balancedness condition directly; thus, no convexity assumption on preferences is needed. In contrast, Scarf [20] required agents' utility functions to be quasi-concave (i.e. preferences to be convex) to show that the game derived from an exchange economy was balanced.

The paper is organized in the following way. Section 2 contains some notation, definitions and a statement of the main technical tools that are needed for the proof of our main theorem which is stated in Section 3. The proof of this theorem is given in Section 4. Sections 5 and 6 contain core existence results for exchange economies and coalitional production economies respectively. Section 7 shows how one can deduce a generalized version of the Debreu [10] and Shafer-Sonnenschein [22] result from our main theorem. Finally, Section 8 contains strong equilibria and β-core existence results.

2. PRELIMINARIES

2.1 Notation

2^A denotes the set of all subsets of the set A.

conA denotes the convex hull of the set A.

$\mathbb{R}^n$ denotes the n-fold product of the set of real numbers $\mathbb{R}$.

$|S|$ denotes the number of elements in the set S.

If $\phi : X \to 2^Y$ is a correspondence, $\phi|_A$ denotes the restriction of ϕ to A, i.e., $\phi|_A : A \to 2^Y$.

$\backslash$ denotes the set theoretic subtraction.

intA denotes the interior of A.

2.2 Definitions

Let X, Y be two topological spaces. Let $\phi : X \to 2^Y$ be a correspondence. The correspondence $\phi^{-1} : Y \to 2^X$ defined by $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is called the lower section of ϕ. We say that $\phi : X \to 2^Y$ has open lower sections if for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X. A binary relation $\wp$ on X is a subset of $X \times X$. We read $x \wp y$ as 'x is strictly preferred to y'. Define the correspondence $P : X \to 2^X$ by $P(x) = \{y \in X : y \wp x\}$ and the correspondence $P^{-1} : X \to 2^X$ by $P^{-1}(y) = \{x \in X : y \in P(x)\}$. We call P a preference correspondence, and P(x) denotes its upper section and $P^{-1}(y)$ its lower section. $P : X \to 2^X$ has an open graph if the set $\{(x, y) \in X \times X : y \in P(x)\}$ is open in $X \times X$. If there exists $x^* \in X$ such that $P(x^*) = \emptyset$ we say that x^* is a maximal element in X.

2.3 Existence of Maximal Elements

We now state the Knaster-Kuratowski-Mazurkiewicz Lemma as extended by Fan [12]. We call this the K-K-M-F Lemma. It has been shown in [27] that the K-K-M-F Lemma is equivalent to Browder's fixed point theorem.

Lemma (K-K-M-F): Let X be an arbitrary set in a Hausdorff linear topological space Y. For each $x \in X$, let $F(x)$ be a closed set in Y such that the following two conditions are satisfied:

(i) the convex hull of any finite subset $\{x_1, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$, and

(ii) $F(x)$ is compact for at least one $x \in X$.

Then $\bigcap_{x \in X} F(x) \neq \phi$.

The following consequence of the K-K-M-F Lemma or the Browder fixed point theorem is by now well known (see for instance [7, p.33] or [27]).

Corollary 2.1: Let X be a nonempty, compact, convex subset of a Hausdorff linear topological space and $P : X \to 2^X$ be a preference correspondence satisfying:

(i) $x \notin \text{con}P(x)$ for all $x \in X$, and

(ii) if $x \in P^{-1}(y)$ there exists y' (possibly $y' = y$) such that $x \in \text{int}P^{-1}(y')$.

Then there exists $x^* \in X$ such that $P(x^*) = \phi$.

The above Corollary will be the major mathematical tool used to prove our Main Theorem.

We conclude this section by stating the following simple fact (see for instance [9, p. 24]).

Fact 2.1: Let X be a topological space and A, B be subsets of X. Then, $\text{int}(A \cap B) = \text{int}A \cap \text{int}B$.

3. THE MAIN THEOREM

3.1 Games in Normal Form

Following Border's [6] setting we may define a game as follows:

A game G (in normal form) is a quintuple (I, X_i, P_i, F, F^S), where

(1) $I = \{1,2,\ldots,N\}$ is the set of agents;

(2) X_i is the strategy set of agent i;

(3) $P_i : X \to 2^X$, (where $X = \prod_{i\in I} X_i$) is the preference correspondence of agent i;

(4) $F \subset X$ denotes the set of jointly feasible strategies; and

(5) $F^S : X \to 2^{\prod_{i\in S} X_i}$ denotes the feasible strategy correspondence of coalition S.

Notice that since the preference correspondence of each agent i, i.e., P_i is a mapping from $X = \prod_{i\in I} X_i$ to 2^X, we have allowed for interdependent preferences. The interpretation of these preference correspondences is as follows:

We read $y \in P_i(x)$ as "y is strictly preferred to x", i.e., one may define $P_i : X \to 2^X$ by $P_i(x) = \{y \in X : y \mathcal{P}_i x\}$.

3.2 The α-Core

If $S \subset I$ then $(y^S, z^{I\setminus S})$ denotes the vector w in $\prod_{i\in I} X_i = X$ such that:

$$w_i = \begin{cases} y_i & \text{if } i \in S \\ z_i & \text{if } i \notin S. \end{cases}$$

The α-core of the game $G = (I, X_i, P_i, F, F^S)$ is the set of all $x \in F$ satisfying:

(i) It is not true that there exist $B \subset I$ and $y^B \in F^B(x)$

such that $(y^B, z^{I \setminus B}) \in P_i(x)$ for all $i \in B$ and for any $z^{I \setminus B} \in \prod_{i \notin B} X_i$.

The above notion of α-core was introduced by Aumann [3]. The non-emptiness of the α-core for a game where agents' preferences are representable by utility functions and the strategy space is finite dimensional was first proved by Scarf [21].

3.4 The Selfish Core

Let $G = (I, X_i, \bar{P}_i, F, F^S)$ be a game where $\bar{P}_i : X_i \to 2^{X_i}$ is defined by $\bar{P}_i(x_i) = \{y_i \in X_i : y_i \mathcal{P}_i x_i\}$, i.e., preference correspondences are now selfish.

The selfish core of the game $G = (I, X_i, \bar{P}_i, F, F^B)$ is the set of all $x \in F$ satisfying: [3]

(i) It is not true that there exist $S \subset I$ and $y^S \in F^S(x)$ such that $y_i^S \in \bar{P}_i(x_i)$ for all $i \in S$.

The above notion of core is the one used in Scarf [20] and Border [6].

3.4 Weakly Balanced Games

Let β be a family of subsets of I. For each $i \in I$, let $\beta(i) = \{B \in \beta : i \in B\}$. The family β is said to be balanced if there are non-negative numbers (weights) $\{\lambda_B : B \in \beta\}$ such that for each $i \in I$, $\sum_{B \in \beta(i)} \lambda_B = 1$. The game $G = (I, X_i, P_i, F, F^S)$ is said to be balanced if whenever β is a balanced family with weights $\{\lambda_B : B \in \beta\}$, and $x^B \in F^B(z)$ for each $z \in X$ and for each $B \in \beta$, then $x \in F$, where $x_i = \sum_{B \in \beta(i)} \lambda_B x_i^B$. The above notion of balancedness is the one introduced by Boehm [5]. The same concept was also used by Border [6]. We now introduce a different notion of balancedness.

The game $G = (I, X_i, P_i, F, F^B)$ is said to be <u>weakly balanced</u> if for each $S \subset I$ and each $z \in X$, $y^S \in F^S(z)$ and $w_i \in F^{\{i\}}(z)$ for $i \notin S$, then $x \in F$, where

$$x_i = \begin{cases} y_i^S & \text{if} \quad i \in S \\ w_i & \text{if} \quad i \notin S. \end{cases}$$

Notice that the balancedness condition is a stronger requirement than weak balancedness. Indeed, more formally we can prove the following simple proposition.

<u>Proposition 3.1</u>: If a game $G = (I, X_i, P_i, F, F^S)$ is balanced then it is weakly balanced.

<u>Proof</u>: Suppose that G is a balanced game. We will show that G is weakly balanced. For any $S \subset I$, let $\beta_1 = \{\{i\} : i \notin S\}$, $\beta_2 = \{S\}$. Set $\beta = \beta_1 \cup \beta_2$. Then β is a balanced family with weights $\lambda_B = 1$ for all $B \in \beta$.

To see this simply observe that:

$$\beta(i) = \begin{cases} \{\{i\}\} & \text{if} \quad i \notin S \\ \{S\} & \text{if} \quad i \in S, \end{cases}$$

and $\sum_{B \in \beta(i)} \lambda_B = 1$.

For each $B \in \beta$ define

$$x^B = \begin{cases} y^B & \text{if} \quad B = S \\ w^B & \text{if} \quad B = \{i\} \text{ for } i \notin S. \end{cases}$$

Since G is balanced, if $x^B \in F^B(z)$ for all $B \in \beta$ and all $z \in X$ then $x \in F$, where $x_i = \sum_{B \in \beta(i)} \lambda_B x_i^B$. Now it can be easily checked that:

$$x_i = \sum_{B \in \beta(i)} \lambda_B x_i^B = \begin{cases} y_i^S & \text{if} \quad i \in S \\ w_i & \text{if} \quad i \notin S, \end{cases}$$

$y^S \in F^S(z)$, $w_i \in F^{\{i\}}(z)$ for all $i \notin S$ and clearly $x \in F$, i.e., G is weakly balanced. This completes the proof of the Proposition.

3.5 The Main Result

We can now state the main result of the paper.

Main Theorem: Let $G = (I, X_i, P_i, F, F^S)$ be a game satisfying the following assumptions:

(a.1) For each $i \in I$, X_i is a nonempty, convex subset of a Hausdorff linear topological space,

(a.2) for each $i \in I$, $F^{\{i\}}$ is nonempty valued,

(a.3) F is a compact, convex, nonempty subset of X;

(a.4) for each $i \in I$, P_i has open lower sections,

(a.5) for all $i \in I$, $x \notin \text{con}P_i(x)$ for all $x \in X$,

(a.6) G is weakly balanced.

Then G has a nonempty α-core.

An immediate consequence of Theorem 3.1 is the following generalization of the main result in Border [6].

Corollary 3.1: Let $G = (I, X_i, \bar{P}_i, F, F^S)$ be a game where preference correspondences are selfish, i.e., $\bar{P}_i : X_i \to 2^{X_i}$. Suppose that G satisfies (a.1) - (a.6). Then G has a nonempty selfish core.

Proof: For each $i \in I$, define $P_i : X \to 2^X$ by $P_i(x) = \bar{P}_i(x_i) \times \prod_{j \neq i} X_j$. The result now follows from the Main Theorem.

Corollary 3.2: Let $G = (I, X_i, P_i, F, F^S)$ be a game satisfying (a.1) - (a.5). Moreover, assume that G is balanced. Then G has a nonempty α-core.

Proof: Since G is balanced, by Proposition 3.1 it is weakly balanced. The result now follows from the Main Theorem.

3.6 Remarks

Remark 3.1: Notice that the continuity assumption in Corollary 3.1, i.e., $\bar{P}_i$ has open lower sections, is weaker than that in Border [6 , p. 1538] who requires $\bar{P}_i$ to have an open graph. Also, we have completely dropped the first part of Border's assumption 2 , i.e., for each nonempty $B \subset I$ the correspondence $F^B : X \to 2^{\prod_{i \in B} X_i}$ is continuous with compact values. Moreover, the strategy set X_i of each agent i, is a subset of any Hausdorff linear topological space whereas Border [6] assumes that for each $i \in I$, X_i is a subset of $\mathbb{R}^n$. Also, as was pointed out in Yannelis [27], Border's proof cannot be extended to infinite dimensional spaces. It fails due to the fact that the convex hull of an upper semicontinuous (u.s.c.) correspondence need not be u.s.c. if the dimensionality of the space is infinite. Finally, our assumption (a.6), i.e., the weak balancedness of the game G, is a weaker requirement than the balancedness assumption made by Border [6]. Therefore, it is clear that Corollaries 3.1 and 3.2 are generalizations of Border's [6] results.

Remark 3.2: Our Main Theorem can also be seen as an extension (though not generalization) of Scarf's [21] α-core existence result. He assumes that preferences are representable by continuous quasi-concave utility functions and that the strategy set of each agent is a compact, convex, nonempty subset of $\mathbb{R}^n$. In particular, as we will see in Section 5 when our Main Theorem is applied to exchange economies, it provides a significant generalization of the α-core existence result for exchange economies than one can obtain using Scarf's [21] game theoretic results.

Remark 3.3: Note that no compactness assumption on the strategy sets X_i is made in the Main Theorem, contrary to Scarf's [21] α-core existence result. This is quite important because the Main Theorem can be applied to prove α-core existence results for exchange economies without assuming that the consumption set of each agent is compact. In particular, in an exchange economy framework one only needs to ensure that the set of all feasible allocations is compact; this is free, for instance if for each i, $X_i = \mathbb{R}^n_+$. To the contrary, even in $\mathbb{R}^n$ if one wants to prove an α-core existence theorem for an exchange economy using Scarf's [21] result without assuming that the consumption set of each agent is compact, then a truncation of the original economy together with a limiting argument seems to be needed.

4. PROOF OF THE MAIN THEOREM

Suppose otherwise, i.e., the α-core of G is empty, then for all $x \in F$ there exist $S_x \subset I$ and $y^{S_x} \in F^{S_x}(x)$, such that:

(4.1) $(y^{S_x}, z^{I \setminus S_x}) \in P_i(x)$ for all $i \in S_x$ and for any $z^{I \setminus S_x} \in \prod_{i \notin S_x} X_i$.

For each $i \in I$ define $\psi_i : X \to 2^X$ by $\psi_i(x) = \text{con}P_i(x)$. Since by assumption (a.4), P_i has open lower sections, it follows from Lemma 5.1 in [25, p.239] that ψ_i has open lower sections. Consider the restriction of ψ_i to F, i.e., $\psi_{i|F}$. From (4.1) it follows that:

(4.2) for all $x \in F$, $(y^{S_x}, z^{I \setminus S_x}) \in P_i(x) \subset \text{con}P_i(x) = \psi_{i|F}(x)$ for all $i \in S_x$ and for any $z^{I \setminus S_x} \in \prod_{i \notin S_x} X_i$.

For each $i \in I$ define $\Phi_i : F \to 2^F$ by $\Phi_i(x) = \psi_{i|F}(x) \cap F$. It can be easily seen that Φ_i has open lower sections. Furthermore, from assumption (a.5) we have that $x \notin \text{con}\Phi_i(x) = \Phi_i(x)$ for all $x \in F$. We can now show that:

(4.3) For all $x \in F$ there exists $v \in X$ such that $v \in \Phi_i(x)$ for all $i \in S_x$.

First notice that by virtue of assumption (a.2) for each $x \in F$, we may choose $w_i \in F^{\{i\}}(x)$ for all $i \notin S_x$.

Define $v \in X$ as follows:

$$(4.4) \quad v_i = \begin{cases} y_i^{S_x} & \text{if} \quad i \in S_x \\ w_i & \text{if} \quad i \notin S_x \end{cases}$$

Since $y^{S_x} \in F^{S_x}(x)$ and $w_i \in F^{\{i\}}(x)$ for all $i \notin S_x$, it follows from the weak balancedness assumption (a.6) that $v \in F$. Taking into account (4.2) we have that for all $x \in F$ and all $i \in S_x$, $v = (y^{S_x}, w^{I \setminus S_x}) \in \psi_{i|F}(x)$. Therefore, for all $x \in F$ and all $i \in S_x$, $v \in \Phi_i(x)$, and this proves (4.3).

Define the correspondence $\theta : F \to 2^F$ by $\theta(x) = \bigcap_{i \in S_x} \Phi_i(x)$. It follows from (4.3) that:

(4.5) For all $x \in F$, $\theta(x) \neq \phi$.

Notice that $x \notin \mathrm{con}\theta(x) = \theta(x)$ for all $x \in F$. Moreover θ satisfies condition (ii) of Corollary 2.1. In fact,

$$
\begin{aligned}
x \in \theta^{-1}(z) &\Leftrightarrow x \in \bigcap_{i \in S_x} \Phi_i^{-1}(z) \\
&\Leftrightarrow x \in \Phi_i^{-1}(z) \quad \text{for all } i \in S_x \\
&\Leftrightarrow x \in \mathrm{int}\Phi_i^{-1}(z) \quad \text{for all } i \in S_x, \text{ (since } \Phi_i \text{ has open lower sections)} \\
&\Leftrightarrow x \in \bigcap_{i \in S_x} \mathrm{int}\, \Phi_i^{-1}(z) \\
&\Leftrightarrow x \in \mathrm{int} \bigcap_{i \in S_x} \Phi_i^{-1}(z), \text{(by Fact 2.1)} \\
&\Leftrightarrow x \in \mathrm{int}\theta^{-1}(z).
\end{aligned}
$$

Therefore, θ satisfies condition (ii) of Corollary 2.1. Furthermore, by assumption (a.3) F is a compact, convex, nonempty subset of X. Hence, by Corollary 2.1 there exists $x^* \in F$ such that $\theta(x^*) = \phi$, a contradiction to (4.5). Since we have obtained a contradiction to our original supposition that the core of G is empty, the proof of the Main Theorem is now complete.

5. THE CORE OF AN EXCHANGE ECONOMY

5.1 Exchange Economies

Let $I = \{1, 2, \ldots, N\}$ be a finite set of agents. For each $i \in I$, let X_i be a nonempty subset of an ordered Hausdorff linear topological space E. An exchange economy in E is a set $\mathcal{E} = \{(X_i, P_i, e_i) : i = 1, 2, \ldots, N\}$ of triples where,

(1) X_i is the consumption set of agent i;

(2) $P_i : X \to 2^X$ (where $X = \prod_{i \in I} X_i$) is the preference correspondence of agent i;

(3) e_i is the initial endowment of agent i, where $e_i \in X_i$ for all $i \in I$.

An allocation is a vector $x = (x_1, x_2, \ldots, x_N) \in \prod_{i \in I} X_i = X$. An allocation x is said to be feasible if $\sum_{i \in I} x_i = \sum_{i \in I} e_i$.

5.2 The α-Core of an Exchange Economy

An α-core allocation of $\mathcal{E}$ is a vector $x \in X$ such that:

(i) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$, and

(ii) it is not true that there exist $S \subset I$ and $y^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i^S = \sum_{i \in S} e_i$ and $(y^S, z^{I \setminus S}) \in P_i(x)$ for all $i \in S$ and for any $z^{I \setminus S} \in \prod_{i \notin S} X_i$.

Denote by $\mathcal{C}(\mathcal{E})$ the set of all α-core allocations of $\mathcal{E}$.

5.3 The Selfish Core of an Exchange Economy

Let $\mathcal{E} = \{(X_i, \bar{P}_i, e_i) : i = 1, 2, \ldots, N\}$ be an exchange economy, where $\bar{P}_i : X_i \to 2^{X_i}$ is defined by $\bar{P}_i(x_i) = \{y_i \in X_i : y_i \mathcal{P}_i x_i\}$, i.e., preference correspondences are selfish. We may define the notion of selfish core as follows:

A selfish core allocation of $\mathcal{E}$ is a vector $x \in X$ such that:

(i) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$, and

(ii) it is not true that there exist $S \subset I$ and $y^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i^S = \sum_{i \in S} e_i$ and $y_i^S \in \bar{P}_i(x_i)$ for all $i \in S$.

Denote by $\mathcal{C}_s(\mathcal{E})$ the set of all selfish core allocations of $\mathcal{E}$.

5.4 The Game Derived from an Exchange Economy is Weakly Balanced

Theorem 5.1: The game derived from an exchange economy $\mathcal{E} = \{(X_i, P_i, e_i) : i = 1, 2, \ldots, N\}$ is weakly balanced.

Proof: We convert the exchange economy $\mathcal{E}$ to a game as follows: For each $i \in I$, $I = \{1, 2, \ldots, N\}$ and each $z \in X$, let

$$F^{\{i\}}(z) = \{x_i \in X_i : x_i = e_i\}.$$

Let

$$F = \{\mathbf{x} \in X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}.$$

For any $S \subset I$ and $z \in X$ let,

$$F^S(z) = \{x^S \in \prod_{i \in S} X_i : \sum_{i \in S} x_i^S = \sum_{i \in S} e_i\}.$$

Thus, we have converted $\mathcal{E}$ to a game $G = (I, X_i, P_i, F, F^S)$. We can now show that the game G is weakly balanced. To this end, let $y^S \in F^S(z)$, i.e.,

$$(5.1) \quad \sum_{i \in S} y_i^S = \sum_{i \in S} e_i .$$

Let $w_i \in F^{\{i\}}(z)$ for all $i \notin S$, i.e.,

(5.2) $w_i = e_i$ for all $i \notin S$.

Define $x \in X$ as follows:

$$x_i = \begin{cases} y_i^S & \text{if } i \in S \\ w_i & \text{if } i \notin S. \end{cases}$$

We must show that $x \in F$. Indeed,

$$\sum_{i \in I} x_i = \sum_{i \in S} y_i^S + \sum_{i \notin S} w_i$$

$$= \sum_{i \in S} e_i + \sum_{i \in I \setminus S} e_i \qquad \text{(by (5.1) and (5.2))}$$

$$= \sum_{i \in S} e_i + \sum_{i \in I} e_i - \sum_{i \in S} e_i$$

$$= \sum_{i \in I} e_i .$$

Therefore, $x \in F$ and consequently G is weakly balanced. This completes the proof of the Theorem.

Remark 5.1: It is rather surprising that no convexity assumption on the preference correspondences is needed in Theorem 5.1. To the contrary, in Scarf's [20,21] framework, in order to show that the game derived from

from an exchange economy is balanced one has to assume that the utility function of each agent is quasi concave, i.e., preferences are convex, (see [20, p. 55]).

We can now apply our Main Theorem to exchange economies and obtain generalizations of Scarf's [20,21] core existence results for exchange economies.

5.5 An α-Core Existence Theorem for Exchange Economies

We can now obtain as corollaries of our Main Theorem, α-core existence results for exchange economies.

Theorem 5.2: Let $\mathcal{E} = \{(X_i, P_i, e_i) : i = 1, 2, \ldots, N\}$ be an exchange economy in E. Let τ be a Hausdorff vector space topology on E which is weaker than the Hausdorff topology on E and has the property that all order intervals $[0, y] = \{z \in E : 0 \leqslant z \leqslant y\}$ in E are τ-compact.[4] Assume that for each $i \in I$, $I = \{1, 2, \ldots, N\}$

(c.1) $X_i = E^+$, where E^+ denotes the positive cone of E,

(c.2) $e_i \in X_i$,

(c.3) P_i has τ-open lower sections,

(c.4) $x \notin \text{con}P_i(x)$ for all $x \in X$.

Then there exists an α-core allocation of $\mathcal{E}$, i.e., $\mathcal{C}(\mathcal{E}) \neq \phi$.

Proof: We convert the exchange economy $\mathcal{E}$ to a game as follows: For each $i \in I$ and each $z \in X$ let

$$F^{\{i\}}(z) = \{x_i \in X_i : x_i = e_i\}.$$

For any $S \subset I$ and $z \in X$ let

$$F^S(z) = \{x^S \in \prod_{i \in S} X_i : \sum_{i \in S} x_i^S = \sum_{i \in S} e_i\}.$$

Let

$$F = \{x \in X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}.$$

We then have a game $G = (I, X_i, P_i, F, F^S)$. It can be easily seen that G satisfies all the assumptions of the Main Theorem. Indeed, from (c.2) it follows that for each $i \in I$ and each $z \in X, F^{\{i\}}(z) \neq \phi$, i.e., $F^{\{i\}}$ is nonempty valued. Denote by e the aggregate initial endowment, i.e., $e = \sum_{i \in I} e_i$. Notice that F is τ-closed and lies in the order interval $[0, e]^N = \{w \in X : 0 \leqslant w_i \leqslant e, \text{ for all } i \in I\}$, which is τ-compact. Thus F is τ-compact, and is clearly convex and nonempty. Moreover, by virtue of Theorem 5.1 we have that G is weakly balanced. Therefore, G has a nonempty α-core and this implies that $\mathcal{C}(\mathcal{E}) \neq \phi$. The proof of the Theorem is now complete.

Remark 5.1: A direct proof of Theorem 5.2, i.e., without showing that the game derived from an exchange economy is weakly balanced and then appeal to the Main Theorem, is given in Yannelis [27].

The following result follows immediately from Theorem 5.2.

Corollary 5.1: Let $\mathcal{E} = \{(X_i, \bar{P}_i, e_i) : i = 1, 2, \ldots, N\}$ be an exchange economy in E. Let τ be a Hausdorff vector space topology on E which is weaker than the Hausdorff topology on E, and has the property that all order intervals $[0, y] = \{z \in E : 0 \leqslant z \leqslant y\}$ in E are τ-compact. Suppose that for each $i \in I$, $I = \{1, 2, \ldots, N\}$

(c.5) $X_i = E^+$,

(c.6) $e_i \in X_i$,

(c.7) $\bar{P}_i$ has τ-open lower sections,

(c.8) $x_i \notin \mathrm{con}\bar{P}_i(x_i)$ for all $x_i \in X_i$.

Then there exists a selfish core allocation of $\mathcal{E}$, i.e., $\mathcal{C}_s(\mathcal{E}) \neq \phi$.

Remark 5.2: One can easily construct examples of economies which satisfy all the assumptions of Corollary 5.1, but no quasi equilibria exists (see, for instance the examples given in [18], [26] and [27]). Therefore, Corollary 5.1 cannot be obtained by means of competitive equilibrium existence theorems.

6. THE CORE OF A COALITIONAL PRODUCTION ECONOMY

6.1 Coalitional Production Economies

The model described in this section is that of Boehm [5]. The commodity space is any ordered Hausdorff linear topological space E. A coalitional production economy $\mathcal{E}$ is a sextuple $(I, X_i, P_i, e_i, Y, Y^S)$ where

(1) $I = \{1, 2, \ldots, N\}$ is the set of agents;

(2) $X_i \subset E$ is the consumption set of agent i;

(3) $P_i: X \to 2^X$ (where $X = \prod_{i \in I} X_i$), is the preference correspondence of agent i;

(4) e_i is the initial endowment of agent i, where $e_i \in X_i$ for all $i \in I$;

(5) $Y^S \subset E$ is the production set of coalition S;

(6) $Y \subset E$ is the aggregate production set.

6.2 The α-Core of a Coalitional Production Economy

An allocation is a vector $x = (x_1, x_2, \ldots, x_N)$ in $\prod_{i \in I} X_i = X$. An allocation $x \in X$ is said to be feasible if

$$\sum_{i \in I} x_i - \sum_{i \in I} e_i \in Y.$$

An allocation $x \in X$ is said to be an α-core allocation of $\mathcal{E}$ if:

(i) $\sum_{i \in I} x_i - \sum_{i \in I} e_i \in Y$, and

(ii) it is not true that there exist S and $z^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} z_i^S - \sum_{i \in S} e_i \in Y^S$ and $(z^S, w^{I \setminus S}) \in P_i(x)$ for all $i \in S$ and for any $w^{I \setminus S} \in \prod_{i \notin S} X_i$.

6.3 The Selfish Core of a Coalitional Production Economy

Let now $\mathcal{E} = (I, X_i, \bar{P}_i, e_i, Y, Y^S)$ be a coalitional production economy where $\bar{P}_i : X_i \to 2^{X_i}$ is defined by $\bar{P}_i(x) = \{y_i \in X_i : y_i \mathcal{P}_i x_i\}$, i.e., preference correspondences are selfish. An allocation $x \in X$ is said to be a selfish core allocation of $\mathcal{E}$ if:

(i) $\sum_{i \in I} x_i - \sum_{i \in I} e_i \in Y$, and

(ii) it is not true that there exist $S \subset I$ and $z^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} z_i^S - \sum_{i \in S} e_i \in Y^S$ and $z_i^S \in \bar{P}_i(x_i)$ for all $i \in S$.

6.4 An α-Core Existence Theorem for Coalitional Production Economies

In this section we will assume that the set of all feasible allocations for $\mathcal{E}$, i.e., $F = \{x \in X : \sum_{i \in I} x_i - \sum_{i \in I} e_i \in Y\}$ is τ-compact, where τ is a Hausdorff vector space topology on the commodity space E, which is weaker than the Hausdorff topology on E and has the property that τ-bounded subsets of E are relatively τ-compact.

It is important to note that if $E = \mathbb{R}^\ell$, the assumptions which guarantee that F is compact are by now standard (see for instance Border [6, 7] or Boehm [5]). However, a similar set of assumptions can guarantee that if E is endowed with the topology τ, F is τ-compact (see for instance Duffie [11] or Zame [28]). Therefore, the assumption that F is τ-compact should not be considered strong; it is just a consequence of standard assumptions on production sets, and it is made for simplicity only. No generality is lost.

Theorem 6.1: Let $\mathcal{E} = (I, X_i, P_i, e_i, Y, Y^S)$ be a coalition production economy, satisfying the following assumptions.

(A.1) $X_i = E^+$ for all $i \in I$, $I = \{1, 2, \ldots, N\}$, (A.2) $x \notin \mathrm{con}P_i(x)$ for all $x \in X = \prod_{i \in I} X_i$ and for all $i \in I$,

(A.3) P_i has τ-open lower sections for all $i \in I$,

(A.4) $0 \in Y^{\{i\}}$ for all $i \in I$,

(A.5) F is τ-compact and convex,

(A.6) for each balanced family β of coalitions with weights $\{\lambda_B : B \in \beta\}$, $\sum_{B \in \beta} \lambda_B y^B \in Y$.

Then the α-core of $\mathcal{E}$ is nonempty.

Proof: We convert the economy $\mathcal{E}$ to a game G and show that the α-core of the game G is nonempty. We then show that the nonemptiness of the α-core of G implies the nonemptiness of the α-core of $\mathcal{E}$. For each $i \in I$ and each $z \in X$ define

$$F^{\{i\}}(z) = \{x_i \in X_i : x_i - e_i \in Y^{\{i\}}\}.$$

For any $S \subset I$ and $z \in X$ let

$$F^S(z) = \{x^S \in \prod_{i \in S} X_i : \sum_{i \in S} x_i^S - \sum_{i \in S} e_i \in Y^S\}.$$

Let

$$F = \{x \in X : \sum_{i \in I} x_i - \sum_{i \in I} e_i \in Y\}.$$

Thus we have converted the economy $\mathcal{E}$ to a game $G = (I, X_i, P_i, F, F^S)$. It is easy to check that all the assumptions of Corollary 3.2 of the Main Theorem are satisfied. In fact assumptions (a.1), (a.3), (a.4) and (a.5) are implied by (A.1), (A.2), (A.3) and (A.5). Assumption (A.4) implies that for each $i \in I$, $F^{\{i\}}(z) \neq \phi$ and consequently assumption (a.2) of Corollary 3.2 is satisfied. We now follow a standard argument to show that the balancedness assumption of Corollary 3.2 is satisfied as well. To this end let β be a balanced family of coalitions with weights $\{\lambda_S : S \in \beta\}$ and for each $z \in X$, let $x^S \in F^S(z)$ for each $S \in \beta$. Then there is a $y^S \in Y^S$ such that $\sum_{i \in S} x_i^S = \sum_{i \in S} e_i + y^S$. For each $i \in I$ define $x_i = \sum_{S \in \beta(i)} \lambda_S x_i^S$. We will show that $x = (x_1, \ldots, x_N)$ is in F. In fact,

$$\sum_{i \in I} x_i = \sum_{i \in I} \sum_{S \in \beta(i)} \lambda_S x_i^S = \sum_{S \in \beta} \lambda_S \sum_{i \in S} x_i^S$$

$$= \sum_{S \in \beta} \lambda_S \left(\sum_{i \in S} e_i + y^S \right)$$

$$= \sum_{i \in I} e_i \left(\sum_{S \in \beta(i)} \lambda_S \right) + \sum_{S \in \beta} \lambda_S y^S$$

$$= \sum_{S \in \beta} \lambda_S y^S + \sum_{i \in I} e_i .$$

It follows from assumption (A.6) that $\sum_{i \in I} x_i - \sum_{i \in I} e_i \in Y$, i.e., $x \in F$. Therefore, by Corollary 3.2, G has a nonempty α-core. It is now easily seen that the nonemptiness of an α-core for G implies that $\mathcal{E}$ has a nonempty α-core as well. The proof of Theorem 6.1 is now complete.

Corollary 6.1: Let $\mathcal{E} = (I, X_i, \bar{P}_i, e_i, Y, Y^S)$ be a coalitional production economy where $\bar{P}_i$ is a selfish preference correspondence. Suppose that $\mathcal{E}$ satisfies (A.1)-(A.6) of Theorem 6.1. Then $\mathcal{E}$ has a nonempty selfish core.

Proof: For each $i \in I$ define $P_i : X \to 2^X$ by $P_i(x) = \bar{P}_i(x_i) \times \prod_{j \neq i} X_j$. The result now follows from Theorem 6.1.

Notice that Corollary 6.1 generalizes Border's [6] Proposition. In fact, one doesn't need to assume a stronger continuity assumption than (A.3), i.e., the graph of $\bar{P}_i$ is open in $X_i \times X_i$ and assumption (4) in Border's Proposition [6 , p. 1540] i.e., for each $B \subset I$, Y^B is closed, can be dropped. Moreover, the dimensionality of the commodity space need not be finite.

6.5 Additive Production Sets

If for every $S \subset I$, $Y^S = \sum_{i \in S} Y_i$ we say that production sets are additive.

It can be easily shown that if production sets are additive, then the game derived from a coalitional production economy is weakly balanced. More formally we have the following result.

Theorem 6.2: Let $\mathcal{E} = (I, X_i, P_i, e_i, Y, Y^S)$ be a coalitional production economy with additive production sets. Then the game derived from the coalitional production economy $\mathcal{E}$, is weakly balanced.

Proof: Convert the economy $\mathcal{E}$ to a game G exactly as was done in the proof of Theorem 6.1. Then adopting the same argument used in the proof of Theorem 5.1 the reader can easily verify that the game G derived from $\mathcal{E}$ is weakly balanced.

We can now obtain the following α-core existence result for a coalitional production economy, without using the balancedness assumption (A.6) of Theorem 6.1.

Theorem 6.3: Let $\mathcal{E} = (I, X_i, P_i, e_i, Y, Y^S)$ be a coalitional production economy with additive production sets. Suppose that $\mathcal{E}$ satisfies assumption (A.1)-(A.5) of Theorem 6.1. Then the α-core of $\mathcal{E}$ is nonempty.

Proof: By virtue of Theorem 6.2 the game derived from the economy $\mathcal{E}$ is weakly balanced. Hence, the result follows from the Main Theorem.

The following result is a straightforward consequence of Theorem 6.2.

Corollary 6.2: Let $\mathcal{E} = (I, X_i, \bar{P}_i, e_i, Y, Y^S)$ be a coalitional production economy with additive production sets where $\bar{P}_i$ is a selfish preference correspondence. Suppose that $\mathcal{E}$ satisfies the assumptions (A.1)-(A.5) of Theorem 6.1. Then $\mathcal{E}$ has a nonempty selfish core.

Remark 6.1: One can easily construct examples of economies which satisfy all the assumptions of Corollaries 6.1 and 6.2 but no quasi-equilibrium exists (see for instance Zame [28]).

7. NASH EQUILIBRIUM

We will now indicate how a generalization of the results in [10], [19] and [22] on the existence of Nash equilibrium can be obtained from our Main Theorem.

A society (or abstract game) $\Gamma = \{(X_i, Q_i) : i = 1, 2, \ldots, N\}$ is a family or ordered pairs (X_i, Q_i) where:

(1) X_i is the strategy set of agent i; and

(2) $Q_i : \prod_{j=1}^{N} X_j \to 2^{X_i}$ is the preference correspondence of agent i.

The interpretation of the preference correspondence Q_i is as follows: We read $y_i \in Q_i(x)$ as "agent i strictly prefers y_i to x_i if the given strategies of other agents are fixed." More simply one may define $Q_i : \prod_{j=1}^{N} X_j \to 2^{X_i}$ by $Q_i(x_1, \ldots, x_N) = \{y_i \in X_i : (x_1, \ldots, y_i, \ldots, x_N) \mathcal{P}_i (x_1, \ldots, x_N)\}$.

A Nash equilibrium for Γ is an $x^* \in X = \prod_{i=1}^{N} X_i$ such that for each i, $Q_i(x^*) = \phi$.

Theorem 7.1: Let $\Gamma = \{(X_i, Q_i) : i=1, \ldots, N\}$ be a society satisfying for each i the following assumptions:

(G.1) X_i is a compact, convex, nonempty subset of a Hausdorff linear topological space,

(G.2) Q_i has open lower sections,

(G.3) $x_i \notin \mathrm{con}Q_i(x)$ for all $x \in X$.

Then Γ has a Nash equilibrium.

Proof: We convert the society Γ to a game G as it was defined in Section 3 as follows: For each $i \in I$, $I = \{1, 2, \ldots, N\}$ define $F^{\{i\}} : X \to 2^{X_i}$ by

$F^{\{i\}}(x) = X_i$. Set $F = F^N(x) = \prod_{i\in I} X_i$. For each $i \in I$, define $P_i : X \to 2^X$ by $P_i(x) = Q_i(x) \times \prod_{j\neq i} X_j$. Set each coalition $S = \{i\}$. We then have a game $G = (I, X_i, P_i, F, F^{\{i\}})$. It can be easily checked that G satisfies all the assumptions of the Main Theorem, and consequently G has a nonempty α-core, i.e., there exists $x^* \in F = \prod_{i\in I} X_i$, satisfying:

(7.1) It is not true that there exist $S = \{i\}$ and $y^{\{i\}} \in F^{\{i\}}(x^*)$ such that $(y^{\{i\}}, z^{I\setminus\{i\}}) \in P_i(x^*)$ for any $z^{I\setminus\{i\}} \in \prod_{j\neq i} X_j$.

But (7.1) implies that $Q_i(x^*) = \phi$ for all $i \in I$, i.e., x^* is a Nash equilibrium for Γ. This completes the proof of the Theorem.

Theorem 7.1 can be now used to prove directly (see for instance Toussaint [24] for a detailed argument) the existence of an equilibrium in an abstract economy _a la_ Debreu [10] and Shafer-Sonnenschein [22], with an infinite dimensional strategy space.

8. STRONG EQUILIBRIA AND THE β-CORE

8.1 Strong Equilibria

The following concept called strong equilibrium was introduced in Aumann [3].

The strong equilibrium of $G = (I, X_i, P_i, F, F^S)$ is the set of all $x \in F$ satisfying:

(i) It is not true that there exist $B \subset I$ and $y^B \in F^B(x)$ such that $(y^B, x^{I \setminus B}) \in P_i(x)$ for all $i \in B$.

It is easy to see that the notion of strong equilibrium is sharper than that of α-core.

8.2 β-Core

Another closely related idea to the α-core is the notion of β-core introduced by Aumann [3], which is defined as follows:

The β-core of the game $G = (I, X_i, P_i, F, F^S)$ is the set of all $x \in F$ satisfying:

(i) It is not true that there exists $B \subset I$ such that for every $z^{I \setminus B} \in \prod_{i \notin B} X_i$ there exists $y^B \in F^B(x)$ such that $(y^B, z^{I \setminus B}) \in P_i(x)$ for all $i \in B$.

Notice that the notion of strong equilibrium is contained in the β-core which in turn is contained in the α-core.

8.3 An Existence Theorem

We can now prove a strong equilibrium existence result.

Theorem 8.1: Let $G = (I, X_i, P_i, F, F^S)$ be a game satisfying the assumptions (a.1), (a.3)-(a.6) of the Main Theorem. Moreover, assume that for each $i \in I$, $x_i \in F^{\{i\}}(x)$ for all $x \in X$. Then G has a strong equilibrium.

Proof: The proof is identical with that of the Main Theorem with only one exception. One must define (4.4) in Section 4 as follows:

$$v_i = \begin{cases} y_i^{S_x} & \text{if} \quad i \in S_x \\ x_i & \text{if} \quad i \notin S_x. \end{cases}$$

Since $y^{S_x} \in F^{S_x}(x)$ and by assumption $x_i \in F^{\{i\}}(x)$ for all $x \in X$ and all $i \notin S_x$, it follows from the weak balancedness assumption that $v \in F$. The proof now can be completed following the same argument with that in Section 4.

Remark 8.1: Since the concept of strong equilibria is contained in the β-core, the assumptions of Theorem 7.1 ensure the existence of a β-core.

Remark 8.2: The assumption, for all $i \in I$, $x_i \in F^{\{i\}}(x)$ for all $x \in X$, used in Theorem 8.1 is much stronger than (a.2) of the Main Theorem. In fact it is too strong to be justified in the economic context of Section 5. In that sense Theorem 8.1 may be considered as a weak existence result.

Remark 8.3: The existence of a strong equilibrium has been demonstrated by Ichiishi [14] as well. His result is applicable to labor managed economies. However, his assumptions appear to be too strong to be justified in the economic context of Section 5. Our Theorem 8.1 does not imply his and vice versa.

FOOTNOTES

1. The equilibrium notion that Debreu [10] uses is actually sharper than the Nash equilibrium introduced in [19]. Moreover, the game form introduced by Debreu [10] was called an abstract economy in Arrow-Debreu [2]. In fact, this is the terminology that we will adopt in Section 7.

2. In "Economic Theory of Choice and the Preference Reversal Phenomenon," American Economic Review, 69:4, Grether and Plott document systematic preference reversals by financially motivated subjects. These results are consistent with findings from human experiments reported by psychologists and findings from animal experiments reported by economists.

3. The notion of selfish core is known in the literature as the core. We chose to call the core a selfish core, in order to stress the fact that preferences are selfish.

4. In concrete spaces the topology τ will vary according to the underlying ordered Hausdorff linear topological space E. For instance if the commodity space is the Lebesgue space L_p, $1 \leqslant p < \infty$ the compatible topology will be the weak topology. In fact, order intervals in L_p, $1 \leqslant p < \infty$ are weakly compact. If the commodity space is L_∞, the compatible topology will be the weak* topology. If the commodity space is the space of real sequences ℓ_p, $1 \leqslant p < \infty$, the norm topology itself is compatible. For more details as well as for references about the choice of the topology in concrete spaces, see [26] or [27].

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