# A NOTE ON THE EXISTENCE OF AN OPTIMAL SOLUTION FOR CONCAVE INFINITE HORIZON ECONOMIC MODELS

by

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ABSTRACT

The purpose of this paper is to establish conditions for the existence of an optimal solution in concave maximization problems in infinite horizon economic models.

# A Note On:

# The Existence of an Optimal Solution for Concave Infinite Horizon Economic Models

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This problem was suggested to me by Prof. J. Jordan.
While this paper was in progress it was brought to the author's notice by Prof. R. Marimon that an optimization result such as the above exists in a book by I. Ekeland and T. Turnbull: 'Infinite Dimensional Optimization and Convexity'. Our set up differs from theirs in two respects. There the optimization problem is assumed to be valid either 1) in a finite horizon economy or 2) in an infinite horizon economy with utility being discounted at a positive rate. Further, utilities are assumed bounded in the $L_\infty$ norm along any feasible path as also the time derivative of the path itself. Our optimization problem makes no such assumptions. However, our optimization problem is a concave optimization problem. For helpful suggestions I am grateful to Professors L. Hurwicz, M.K. Richter, D. Kahn and P. Rejto.

## Introduction

The purpose of this paper is to establish conditions for the existence of an optimal solution in concave maximization problems in infinite horizon economic models. Such models have been used extensively in the study of optimal economic growth and planning.

The general type of problem we treat is:

$$\text{(S)} \qquad \begin{aligned} &\operatorname{Sup} \int_0^\infty l(x, \overset{\circ}{x}, t)dt \\ &\text{s.t.} (x, \overset{\circ}{x}) \in A \subseteq R^{2m} \\ &x(o) = x_o \end{aligned}$$

where A is convex with int $(A) \neq \phi$ and l a concave upper semi-continuous function which is bounded above by an integrable function (see Section 1 for full details). We note that there is a connection between this type of optimization problem and general equilibrium theory with an infinite number of commodities as treated for instance in Bewley [1972]. We could consider (S) as an equilibrium problem of one consumer where his choice variable is the control path and given the initial endowment he chooses the optimal control path. For a specific version of our problem, Peleg and Ryder (1972) get existence without a restrictive boundedness assumption on A, which appears in our analysis.

## Section 1:

Let $A \subseteq R^{2m}$ be a convex set with Int. $(A) \neq \phi$. Suppose $l: A \times R_+ \to R$ satisfies:

### Assumption (*):

(a) l is upper semi-continuous (hereafter referred to as u.s.c.) and for each $t \in R_+$, $l(\cdot,t)$ is concave. $l: A \times R_+ \to R$ is upper-semicontinuous if and only if $\limsup_{(t',y') \to (t,y)} l(t',y') \leq l(t,y)$

(b) There exists an integrable function $\beta: R_+ \to R$ such that $l(y,t) \leq \beta(t)$ for all $y \in A$.

(c) If $y \in \partial A$ (the topological boundary of A) and $y \notin A$ and if $\lim y_n = y$ with $y_n \in A$ then $\limsup_{n \to \infty} l(y_n,t) = -\infty$ for every $t \in R_+$.

Remark 1: If $l(y,t) = e^{-\delta t} m(y)$ and m is a bounded continuous function on a compact set A then Assumption (*) is satisfied. Since A is not necessarily closed Assumption (*) is also satisfied if $m: (0,1] \times R \to R$ is given by $m(y^1,y^2) = \log y^1$.

Some notation will help us state and study problem (S). Let $L_\infty$ denote the space of (equivalence classes of) measurable functions $w: R_+ \to R^m$ such that $\text{ess}_t \sup |w(t)| < +\infty$, endowed with the norm $|w| = \text{ess}_t \sup |w(t)|$. Let $E = \{z \in L_\infty /$ there exists $M>0$ such that $\sup_t |\int_0^t x(s)ds| < M\}$. E is a linear subspace of $L_\infty$ and may thus be endowed with the sup norm.

Let us consider the problem:

$$\text{(P)} \qquad \text{Sup} \int^\infty l(x_o + \int_0^t z(s)ds,\ z(t),\ t)dt$$
$$\text{subject to } (x_o + \int_0^t z(s)ds,\ z(t)) \in A$$

where the sup is taken among all measurable functions $z: R_+ \to R^m$, such that $\text{ess}_t \sup |z(t)| < +\infty$ and $\text{ess}_t \sup |\int_0^t z(t)dt| < +\infty$.

Let $v: R^{2m} \times R_+ \to [-\infty, +\infty)$ be the extended real function given by:

$$v(y,t) = l(y,t) \text{ if } y \in A$$

$$v(y,t) = -\infty \text{ if } y \notin A$$

Assumptions (*)(a) and (c) imply that $v(\cdot,t)$ is a concave upper-semicontinuous function, i.e. $\lim_{n\to\infty} y_n = y$, implies $\lim \sup_{n\to\infty} v(y_n,t) \leq v(y,t)$

For every $w \in L_\infty$, $z \in E$, let $J(w,z) = \int_0^\infty v(w(t),z(t),t)dt$.

Since v is u.s.c. $\{(y',t') \in A \times R_+ / v(t',y') \leq a\}$ is closed $\forall a \in R$ and hence v is a measurable function.

Assumption (*) implies that $J: L_\infty \times E \to R$ is a well defined concave functional. In fact since $v(y,t)$ is u.s.c. in y and Int. $(A) \neq \phi$, $-v$ is a normal convex integrand in the language of Rockafellar (1968). Furthermore if we define for each $p \in R^{2m}$, $v^*(p,t) = \operatorname{Inf}_y \{-\langle p,y\rangle - v(y,t)\}$ (the concave conjugate of v) then $-\beta(t) \leq v^*(o,t) \leq -v(x_o + \int_0^t u(s)ds, u(t), t)$.

It follows that J is a well defined concave functional (Rockafellar (1968) Theorem 1 page 532). Thus we may rewrite (P) as

$$\sup_{(w,z) \in L_\infty \times E} J(w,z)$$

$$\text{s.t. } w(t) = x_o + \int_0^t z(s)ds$$

We start by showing that J is an u.s.c. functional. Here all limits are taken with respect to the strong topology.

Lemma 1: If $\lim_{n\to\infty} (w_n,z_n) = (w,z) \in L_\infty \times E$ then
$$\lim \sup_{n\to\infty} J(w_n,z_n) \leq J(w,z)$$

Proof: The proof will parallel a proof of an analogous theorem in Araujo and Scheinkman [1980]. We may without loss of generality assume that any subsequence $n_k$ such that $\lim_{k\to\infty} J(w_{n_k}, z_{n_k}) = \limsup_{n\to\infty} J(w_n, z_n)$ is such that $(x_o + (z_{n_k}(s)ds,\ z_{n_k}(t))$ $\varepsilon A$ a.e., and $w_{n_k}(t) = x_o + \int_0^t z_{n_k}(s)ds$, for otherwise the inequality is trivial. Since $v(y,t) \leq \beta(t)$ it follows from Fatou's lemma that

$$\limsup_{k\to\infty} J(w_{n_k}, z_{n_k})$$

$$\leq \int_0^\infty \limsup_{k\to\infty} v(x_o + \int_0^t z_{n_k}(s)ds,\ z_{n_k}(t),t)\ dt$$

$$\leq \int_0^\infty v(x_o + \int_0^t z(s)ds, z(t),t)dt$$

Since v is u.s.c. and for each $t<\infty$, $\lim_{k\to\infty} z_{n_k}(t) = z(t)$ and $\lim_{k\to\infty} x_o + \int_0^t z_{n_k}(s)ds = x_o + \int_0^t z(s)ds.$

Q.E.D.

Now we proceed to our main theorem.

Theorem 1: If A is a bounded subset of $R^{2m}$, and there exists an integrable function $\beta: R_+ \to R$ such that $l(y,t) \leq \beta(t)\ \forall y \varepsilon A$, then S admits an optimum solution in $L_\infty$ X E.

Proof: If A is a bounded subset of $R^{2m}$, then $\|(w_n, z_n\| \to \infty$ implies $\max\ \{\|w_n\|,\ \|z_n\|\} \to +\infty$

$$\max\ \{\text{ess}_t \sup \|w_n(t)\|,\ \text{ess}_t \sup_t \|z_n(t)\|\} \to +\infty$$

there exists $N \varepsilon \mathbb{N}$, such that for all $n \geq N$

$$\text{meas}\ \{t \varepsilon R_+ \mid (w_n(t), z_n(t))\ \varepsilon R^{2m}\ A\} > 0$$

$$\therefore \int_0^{+\infty} v(w_n(t), z_n(t),t)dt$$

$$= \int_{\{t:R_+/(w_n(t),z_n(t))\varepsilon A\}} v(w_n(t), z_n(t),t)dt$$

$$+ \int_{\{t:R_+/(w_n(t),z_n(t))\ \varepsilon R^{2m}\setminus A\}} v(w_n(t), z_n(t),t)dt$$

$$= -\infty \quad \forall n \geq N$$

$$\therefore \int_0^{+\infty} v(w_n(t), z_n(t), t)\,dt \to -\infty$$

Further, $(x, \overset{\circ}{x}) \in L_\infty \times E$

$(\underset{\sim}{x}, \overset{\circ}{\underset{\sim}{x}}) \in L_\infty \times E$

implies that

$$(x, \overset{\circ}{x}) + (1-\lambda)(\underset{\sim}{x}, \overset{\circ}{\underset{\sim}{x}}) \in L_\infty \times E$$

$$\text{and } \int_0^\infty [\lambda v(x, \overset{\circ}{x}, t) + (1-\lambda) v(\underset{\sim}{x}, \overset{\circ}{\underset{\sim}{x}}, t)]dt$$

$$\leq \int_0^\infty v(\lambda x + (1-\lambda)\underset{\sim}{x},\ \lambda \overset{\circ}{x} + (1-\lambda)\overset{\circ}{\underset{\sim}{x}}, t)dt$$

(by concavity of v)

$\therefore J(x, \overset{\circ}{x}) = \int_0^\infty v(x(t), \overset{\circ}{x}(t), t)dt$ is a concave function defined on $L_\infty \times E$.

Let $(x_n, \overset{\circ}{x}_n)$ be a maximizing sequence of J; that is a sequence of elements of $L_\infty \times E$ such that

$$J(x_n, \overset{\circ}{x}_n) \to \sup_{(x, \overset{\circ}{x}) \in L_\infty \times E} J(x, \overset{\circ}{x}) = \alpha$$

Note that $\alpha$ belongs a priori to $(-\infty, +\infty]$; we will see from what follows that $\alpha \neq +\infty$. The sequence $(x_n, \overset{\circ}{x}_n)$ is bounded in $\{(x, \overset{\circ}{x}) / J(x, \overset{\circ}{x}) \geq \alpha\}$. This is because the sequence $J(x_n, \overset{\circ}{x}_n)$ is bounded below. Thus we can extract from $(x_n, \overset{\circ}{x}_n)$ a subsequence $(x_{n_i}, \overset{\circ}{x}_{n_i})$, which converges weakly in $L_\infty \times E$ to an element $(x, \overset{\circ}{x})$ belonging to $L_\infty \times E$. (See Appendix)

By Corollary 1.2.2. of Ekeland and Temam [1976] or Corollary 38.12 of Berberian [1974] and the fact $\{(\underline{x}, \overset{\circ}{\underline{x}}) \in L_\infty \times E / J(\underline{x}, \overset{\circ}{\underline{x}}) \geq a\}$ is convex $\forall a \in R$, J is u.s.c. on $L_\infty \times E$ for the weak topology of $L_\infty \times E$ and hence

$$J(x, \overset{\circ}{x}) \geq \limsup_{n_i \to \infty} J(x_{n_i}, \overset{\circ}{x}_{n_i}) = \alpha,$$

$\therefore (x,\overset{o}{x})$ is a solution of the optional control problem and $\alpha \neq +\infty$.

Note:

(1) The above lemma and theorem are easily seen to be valid if instead of Assumption (*) a, we had the following assumption (*)a':

Assumption (*) a': $l(\cdot,t):A\to R$ is u.s.c. for all $t\epsilon R_+$ and $l:A \times R_+ \to R$ is a measurable function

(2) The space E may be identified with the space of bounded, finitely-additive set functions on $R^m$ which are absolutely continuous with respect to the lebesgue measure. This is the dual of $L_\infty (R_+,R^m)$.

Appendix:

Proposition 1: Let K be a bounded set in E.

$$\text{If } \lim_{\text{meas}(D)\to 0} \int_D f(t)dt = 0$$

$\forall$ f in K, then K is weakly sequentially compact.

Proof: Suppose K is bounded and that the integrals $\int_D f(t)dt$ satisfy the above condition $\forall$ f in K.

Let $f_n \epsilon K$ and suppose that $\|f_n\|_E \leq C$ for $n=1,2,\ldots$ where $\|f_n\|_E = |\int_0^\infty f_n(t)dt|$

By the Cantor diagonal process, choose a subsequence $\{g_n\}$ of $\{f_n\}$ such that the limit

$$\lambda(D) = \lim_{n\to\infty} \int_D g_n(t)dt$$

exists for every D in $B([0,\infty])$, (i.e. the Borel $\sigma$-algebra on $[0,\infty]$). This limit exists since $B([0,\infty])$ is countably generated, and so the Cantor diagonal process is operative.

$\therefore$ $g_n$ is weakly convergent in E.

Q.E.D.

(The proof of this theorem closely parallels the proof of Theorem 9 in Dunford and Schwarz [1957], v. 1, page 292).

<u>Proposition 2</u>: $(\dot{x}_j)_{j\epsilon N}$ is weakly convergent and A bounded implies $(x_j,\dot{x}_j')_{j\epsilon N}$ is weakly convergent.

<u>Proof</u>: $x_j(t) = x_o + \int_0^t x_j(s)ds \;\forall\; t\geq 0$

and the fact that $\dot{x}_j(\cdot) \to f(\cdot)$ (weakly), where $f\epsilon E$, implies

$$x_j(t) = x_o + \int_0^t x_j(s)ds \to x_o + \int_0^t f(s)ds \qquad \text{a.e.}$$

Define $\dot{x}(t) = f(t)$.

Then $x(t) = x_o + \int_0^t x(s)ds$

$$\therefore\; x_j(\cdot) \to x(\cdot) \qquad \text{a.e.}$$

Since A is a bounded subset of $R^{2m}$, $\text{proj}_1 A = \{x\epsilon R^m/\text{there exists } \dot{x}\epsilon R^m \text{ with } (x,\dot{x})\epsilon A\}$ is bounded.

$\therefore$ sup $\{\|x\| \,/\, x \in \mathrm{proj}_1 A\} < +\infty$, where this norm is the Euclidean norm.

$\therefore \{\|x_j\|_{L^\infty}\}_{j \in N}$ is bounded above.

$\therefore$ $x_j(\cdot) \to x(\cdot)$ a.e. implies that

$x_j(\cdot) \to x(\cdot)$ weakly.

$\therefore$ $(x_j(\cdot), \dot{x}_j(\cdot)) \to (x(\cdot), \dot{x}(\cdot))$ weakly.

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