IDENTIFICATION AND ESTIMATION OF A MODEL OF HYPERINFLATION WITH A CONTINUUM OF "SUNSPOT" EQUILIBRIA

by

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ABSTRACT

This paper constructs a model with two structural equations: the government budget constraint and a linear version of Cagan's portfolio balance equation. The model contains a continuum of equilibria with "sunspot equilibria." Closed forms for the solutions are found. Even though there is a continuum of equilibria, the model is overidentified econometrically, so that the model restricts time series data on price levels and currency stocks. We describe how the free parameters of the model can be estimated, including some parameters that serve to index particular members of the continuum of equilibria. The sunspot equilibria hold out some promise of explaining anomalies in the observed behavior of inflation and real balances during hyperinflations.

1. Introduction

In Sargent and Wallace [1973], we created a rational expectations model of the bivariate inflation-money creation process by solving the "inverse optimal predictor problem" for Cagan's [1956] adaptive expectations scheme. Our model was constructed by working backwards and finding a bivariate process for inflation and money creation that implied both that Cagan's portfolio balance equation held and that adaptive expectations were rational.[1/] That model is "successful" in several ways. First, the model explains the pattern of Granger causality in Cagan's data, in which inflation Granger causes money creation, but not *vice versa*. Second, a version of the model predicts the pattern of correlations across countries between Cagan's estimates of λ and α (see Sargent and Wallace [1973]). Third, the model predicts that the residuals in the regression equation fit by Cagan will be random walks, which explains the very high serial correlation that Cagan actually encountered (see Sargent [1977]). Fourth, the model predicts a pattern of results obtained by Rodney Jacobs [1975] when he reversed the direction of regression in Cagan's equation (see Sargent [1976]).

Against these "successes," the Sargent-Wallace model suffers some notable "failures." First, the model implies that both inflation and real balances will behave like processes with the highest autoregressive root being unity, so that while both may drift, they would not be expected to move systematically in one direction over time. However, for six out of the seven hyperinflations studied by Cagan, there is a noticeable tendency for real balances to fall and inflation to rise during the course of the hyperinflation. Figures 1a-1g graph the data. The apparent exception to this general tendency is Russia. The original Sargent-Wallace model does not explain this general pattern.

Second, there is a long-standing claim, which Cagan's estimates in his Figure 9 support, that the hyperinflations eventually proceeded at rates of inflation that exceeded the seignorage-maximizing stationary average inflation rates. (Again, Russia seems to be an exception to the general pattern in Cagan's table.) Our original model doesn't shed any light on this phenomenon.

Third, the model is weak in terms of economic motivation because it posits a stochastic process for money creation that was discovered on the statistical grounds that it solved the inverse optimal predictor problem. We argued informally that the extensive feedback from inflation to money creation embodied in this stochastic process might be interpreted as reflecting a feedback from prices through the government's budget constraint that would arise if the government were to attempt to finance a constant deficit through money creation. This heuristic argument is insufficient to permit interpreting the money creation process of the model in terms of economic parameters appearing in the government's budget constraint.

This paper describes a model that is designed to account for the two anomalous patterns in the data. The model explicitly incorporates a version of the government's budget constraint. The model consists of two linear expectational stochastic difference equations. The first equation is a linear version of Cagan's portfolio balance equation under rational expectations and summarizes behavior of the public. The second equation is the government's budget constraint with a particular and convenient parameterization of the part of the real government deficit that the government chooses to finance by creating base money. An equilibrium of the model is a solution of these two equations. The model possesses what Stanley Fischer [1983] has aptly dubbed a "slippery side of the Laffer curve." This feature of the model is the key in giving it the potential to account for the two aforementioned puzzles in the data. Equilibria of the model potentially possess two autoregressive roots, one larger than the other and each larger than unity, which correspond to the two stationary average gross inflation rates that would finance the average real government deficit. That there are two such stationary inflation rates (or tax rates on currency) that finance the deficit is a reflection of the Laffer curve. In time series generated by this model, the higher root will normally come to dominate as time passes. Along such paths, real balances gradually fall and inflation gradually rises on average. Furthermore, the inflation rate usually ends up exceeding the seignorage-maximizing rate of inflation.

In constructing and using our model, a number of technical issues had to be confronted. These issues are of interest in their own right, since they occur in a variety of other contexts. Among the issues encountered are the following ones: (a) uniqueness issues involving the connection between, on the one hand, the dimensionality of the equilibria that can be represented as depending on square summable linear combinations of current and lagged values of innovations to "fundamental" forcing variables; and, on the other hand, the dimensionality of additional solutions depending on "spurious indicators" (or sunspots); (b) the proper estimation of a model in which there are restrictions across the parameters of initial conditions, the exponentially growing coefficients on trend terms, and the remaining parameters in moving averages of white noises; (c) proper econometric identification and estimation of a model in which a "spurious indicator" may be impinging on the solution; (d) delineation of the econometric information and operating rules for the government that would be required in order to rule out or to correct nonoptimalities associated with "spurious indicator" equilibria.

The present paper confronts the preceding issues, and deduces the restrictions that our model places on time series data. A planned sequel to this paper will contain estimates of the model from time series drawn from various hyperinflationary episodes.

2. The Model

Our model is a stochastic version of the nonrandom one that we analyzed in our earlier paper [1981]. It is useful briefly to begin by reviewing that model, which consisted of the two equations

$$p(t) = \lambda p(t+1) + \gamma h(t) \tag{0a}$$

(0b) $\qquad h(t) = \frac{1}{1+n} h(t-1) + \xi p(t)$

where $0 < \lambda < 1$, $\gamma > 0$, and where $p(t)$ is the price level at t, $h(t)$ is per capita nominal balances at t, ξ is the constant real government deficit per capita that must be financed by printing new base money, and n is the rate of growth of the economy, assumed constant. It is assumed that $0 < \xi < \frac{\lambda}{\gamma}\left[\frac{1}{1+n} + \frac{1}{\lambda} - 2\sqrt{1/(1+n)\lambda}\right] \equiv \xi max$, where ξmax is the maximal constant deficit which it is feasible to finance through seignorage. Equation (0a) is a linear version of Cagan's portfolio balance equation, while (0b) is the government's budget constraint. The system is imagined to run over the period $t \geq 1, \ldots,$ and to be subject to a single initial condition for $h(0)$. There is no initial condition for the price level $p(1)$, it being the job of the model to determine a price path for $p(t)$, $t \geq 1$. Letting $\pi(t+1) = p(t+1)/p(t)$, Sargent and Wallace showed that (0a) and (0b) imply the difference equation

$$\pi(t+1) = \phi - (1/(1+n)\lambda)/\pi(t)$$

where $\qquad \phi \equiv (\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda)$.

The preceding equation is graphed in Figure 2, which is taken from Sargent and Wallace [1981]. Evidently, there are two stationary points π_1 and π_2 that satisfy $(1+n)^{-1} < \pi_1 < \pi_2 < \lambda^{-1}$. These stationary points correspond to two alternative stationary levels of the gross inflation rate $p(t+1)/p(t)$ that satisfy portfolio balance and that finance a constant per capita real deficit of ξ. Alternatively, direct calculations on (0) show that the solution is

$$\begin{bmatrix} p(t) \\ h(t) \end{bmatrix} = \begin{pmatrix} \dfrac{J_1}{1 - \lambda\pi_1} \\ J_1 \end{pmatrix} \pi_1^t + \begin{pmatrix} \dfrac{J_2}{1 - \lambda\pi_2} \\ J_2 \end{pmatrix} \pi_2^t$$

where J_1 and J_2 are chosen to satisfy the initial condition for $h(0)$. The absence of an initial condition for $p(1)$ means that there is a variety of choices of (J_1, J_2) pairs that work, each implying a different value for $p(1)$ but satisfying the initial condition for $h(0)$. Evidently, unless $J_2 = 0$ (which corresponds to a choice of $p(1)$ that starts the system at π_1 in Figure 2), the gross inflation rate converges to π_2 as $t \to \infty$. Figure 2 and the preceding equation both embody the "slippery side of the Laffer curve." There is a continuum of equilibria, indexed by either J_2 or the initial inflation rate $\pi(0)$ or the initial price level $p(1)$. All of the equilibria but one slide toward the higher of the two sustained gross inflation rates π_1 and π_2 that will finance the deficit. Along all of these "slippery" paths, the inflation rate increases over time, while real balances fall. Also, along these paths, the inflation rate eventually exceeds the revenue-maximizing rate. It is these features of these paths that seem potentially to give a version of our model the ability to match the anomalous facts cited at the beginning of this paper.

Our concern now is to study how such nonuniqueness surfaces in a stochastic version of the model. Our reasons for studying a stochastic version of the model are, first, that we want an econometrically implementable version of the model; and second, that presence of imperfect foresight by itself creates the possibility of additional equilibria that are random.

We now study the system

$$(1)(a) \qquad p(t) = \lambda E_t p(t+1) + \gamma h(t) + u(t)$$

$$(b) \qquad h(t) = \frac{1}{1+n} h(t-1) + \xi p(t) + \varepsilon(t),$$

where $0 < \lambda < 1$, $\gamma > 0$, $n > 0$, $\xi > 0$, and where $(u(t),\varepsilon(t))$ is a stochastic process with means of zero. We shall begin our analysis of the system at $t = 1$. As a normalization, we shall assume that $(u(s),\varepsilon(s)) = 0$ for $s < 0$, and shall take $h(0)$ as an initial condition. For $t > 1$, we assume that $(u(t),\varepsilon(t))$ has a moving average representation of the form

$$(2) \qquad \begin{pmatrix} u(t) \\ \varepsilon(t) \end{pmatrix} = \begin{bmatrix} a_{11}(L) & 0 & 0 \\ 0 & a_{22}(L) & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} ,$$

where $a_{11}(L)$ and $a_{22}(L)$ are each polynomials in the lag operator that are one-sided in nonnegative powers of L, and $w_1(t) = u(t) - E_{t-1}\, u(t)$, $w_2(t) = \varepsilon(t) - E_{t-1}\, \varepsilon(t)$. These last equalities imply that $w_1(t)$ is a white noise that is fundamental for $u(t)$ and that $w_2(t)$ is a white noise that is fundamental for $\varepsilon(t)$. In (2), $w_3(t)$ is a serially uncorrelated random process of mean zero that is uncorrelated at all leads and lags with $w_1(s)$ and $w_2(s)$. In particular, we assume that

$$(3) \qquad E\, w(t)w(t-s)^T = \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) & 0 \\ \sigma_{12}(t) & \sigma_{22}(t) & 0 \\ 0 & 0 & \sigma_{33}(t) \end{pmatrix} \text{ for } s = 0$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } s \neq 0,$$

where $w(t)^T = (w_1(t), w_2(t), w_3(t))^T$. For purposes of estimation, we would impose more structure on $Ew(t)w(t)^T$. In particular, we might eventually assume a version of

$$(4) \qquad Ew(t)w(t)^T = \mu^t V$$

where μ is a positive scalar, and V a positive semidefinite (3x3) matrix with a pattern of zeroes conforming with (3).

In (2), $w_3(t)$ plays the role of a "spurious" indicator that will be permitted to influence the solution for $(p(t), h(t))$, but which has no influence on the "fundamentals" $u(t)$ and $\varepsilon(t)$, the disturbances to portfolio balance and the nominal deficit, respectively.[2/] The agents in the model are supposed at time t to see $\{w_1(s), w_2(s), w_3(s), s=1, \ldots, t; p(1), h(0)\}$. Equivalently, the agents see $\{u(s), \varepsilon(s), w_3(s), s=1, \ldots, t; p(1), h(0)\}$. The econometrician will be imagined to possess a record of $\{p(t),\ h(t),\ t=1, \ldots, T\}$, from which he tries to estimate the structure.

The entire class of solutions of the pair of stochastic difference equations (1) can be represented in the form

$$(5) \qquad p(t) = d(L)w(t) + \pi_1^t F_1 + \pi_2^t F_2$$

$$h(t) = g(L)w(t) + \pi_1^t J_1 + \pi_2^t J_2, \quad t \geq 1$$

where $d(L)$ and $g(L)$ are each particular 1x3 polynomials in non-negative powers of L; F_1, F_2, J_1, and J_2 are particular random variables that represent the initial condition; and π_1 and π_2 are the same stationary inflation rates mentioned in the introduction. The J_j's are related to the initial condition for h(0) as follows:

(6) $$h(0) = J_1 + J_2.$$

The parameters F_1 and F_2 can be thought of as determining an initial price level $p(0) = F_1 + F_2$, which is regarded not as an initial condition given to the analyst, but as a random variable representing one dimension of the continuum of possible equilibrium stochastic processes.

To be a solution, $d(L)$ and $g(L)$ must satisfy a set of restrictions as must also F_1 and J_1, and F_2 and J_2. Given our assumption that at t, agents see $\{w_1(s),w_2(s),w_3(s),s=1,\ldots,t; p(1),h(0)\}$, Wiener-Kolmogorov prediction formulas for moving averages in terms of the $w(t)$'s hold for agents' expectations. Cross-equation restrictions in (5) can be derived by assuming (5) to be correct, then applying the Wiener-Kolmogorov prediction theory to (5), substituting the results into (1) and solving for the restrictions.[3/] After many computations, this process leads to the following concrete version of (5)

(7) $$\begin{bmatrix} p(t) \\ h(t) \end{bmatrix} = \frac{-\lambda^{-1}L}{(1-\pi_1 L)(1-\pi_2 L)} \begin{bmatrix} (1-\frac{1}{1+n}L)(a_{11}(L)-\lambda d_{01}L^{-1}), & -(1-\frac{1}{1+n}L)\lambda d_{02}L^{-1}+\gamma a_{22}(L), & -\lambda d_{03}L^{-1}(1-\frac{1}{1+n}L) \\ \xi(a_{11}(L)-\lambda d_{01}L^{-1}), & -\xi\lambda d_{02}L^{-1} + a_{22}(L)(1-\lambda L^{-1}), & -\xi\lambda d_{03}L^{-1} \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}$$

$$+ \pi_1^t \begin{pmatrix} \dfrac{\gamma J_1}{1-\lambda\pi_1} \\ J_1 \end{pmatrix} + \pi_2^t \begin{pmatrix} \dfrac{\gamma J_2}{1-\lambda\pi_2} \\ J_2 \end{pmatrix}.$$

In (7), $(d_{01},d_{02},d_{03}) = d_0$, the vector of coefficients on L^0 in $d(L)$. The parameters π_1 and π_2 are reciprocals of the zeroes of the characteristic polynomial of the deterministic part of system (1). That is, (π_1,π_2) satisfy $\left(1-(\frac{1}{\lambda}+\frac{1}{1+n}-\frac{\gamma\xi}{\lambda})L+\frac{1}{(1+n)\lambda}L^2\right) = (1-\pi_1L)(1-\pi_2L)$, and are given by

$$(8) \qquad \pi_1,\ \pi_2 = [(\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda) \pm \sqrt{(\lambda^{-1}+(1+n)^{-1}-\frac{\xi\gamma}{\lambda})^2-4/(1+n)\lambda}]/2.$$

This formula implies that when $0 < \xi < \xi_{max} \equiv \frac{\lambda}{\gamma}[\frac{1}{1+n}+\frac{1}{\lambda}-2\sqrt{\frac{1}{\lambda(1+n)}}]$, $(1+n)^{-1} < \pi_1 < \pi_2 < \lambda^{-1}$; that when $\xi = 0$, $\pi_1 = (1+n)^{-1}$, $\pi_2 = \lambda^{-1}$; and that when $\xi = \xi_{max}$, $\pi_1 = \pi_2 = \sqrt{(1+n)^{-1}\lambda^{-1}}$. Here ξ_{max} is the maximal value for the real deficit that is feasible.

Equation (7) represents the entire class of solutions of the stochastic difference equations (1) and (2), so far as their second moments are concerned. Equation (7) represents a solution of (1) and (2) for <u>any</u> values of the parameters d_{01}, d_{02}, d_{03}, J_1, and J_2. There is thus a multidimensional continuum of solutions to (1) and (3), a continuum that is conveniently indexed by the list of parameters $(d_{01},d_{02},d_{03},J_2)$. In indexing things in this way, we are regarding the setting of J_1 as reflecting a "choice of units:" note that variations in J_1 affect the entire time paths of p(t) and h(t) proportionately.

From the form of solution (7), it is evident that in general the (p(t),h(t)) process becomes of mean exponential order π_2. Components of mean exponential order π_2 are "activated" in two ways. First, if $J_2 \neq 0$, then the final term, which is a constant times π_2^t, occurs in the solution. Second, the presence of $(1-\pi_2 L)$ in the denominators of the polynomials operating on each component of $(w_1(t), w_2(t), w_3(t))$ implies that in general nonzero values of the random variables $w_1(t)$, $w_2(t)$, $w_3(t)$ will cause $(p(t), h(t))$ to become of mean exponential order π_2.

Notice that in general the spurious indicator $w_3(t)$ occurs in the solution for $(p(t), h(t))$ so long as $d_{03} \neq 0$, and that it gives rise to a component of mean exponential order π_2. The moving average in $w_3(t)$ appears in the solution despite the fact that it influences neither of the "fundamentals" $u(t)$ or $\varepsilon(t)$. In connection with this spurious indicator, there is a technical question about how we measure the extent of the multiplicity of solutions that is introduced by the freedom to make solutions depend on spurious indicators. As the reader can verify, moving averages in any number of "spurious" white noises $w_4(t)$, $w_5(t)$, ..., can be added to the solution (7), so long as the moving average polynomials on each such white noise are proportional to those on $w_3(t)$; e.g., the polynomials on $w_4(t)$ must be given by

$$\frac{-\lambda^{-1}L}{(1-\pi_1 L)(1-\pi_2 L)} \begin{bmatrix} -\lambda d_{04} L^{-1}(1-\frac{1}{1+n}L) \\ -\xi\lambda d_{04} L^{-1} \end{bmatrix}.$$

This means that given a particular $w_3(t)$ process, so far as sample paths are concerned, an indefinite number of additional solutions can be generated by adding moving averages in additional spurious indicators. However, each of these additional solutions will have population second moments that can be completely represented by equation (7) with $Ew_3(t)^2 = \sigma_{33}(t)$ chosen appropriately. This means that so far as concerns their second moments, all of these additional solutions are observationally equivalent with the solution that we have represented in (7). In this sense, the possibility of adding spurious indicators adds a one-dimensional multiplicity of solutions.[4/]

For simplicity, we now focus on the special case in which $a_{11}(L) = 1$, $a_{22}(L) = 1$, so that both $u(t)$ and $\varepsilon(t)$ are white noises. Now within the entire class of solutions given by (7), there is a singular solution which is of exponential order π_1. This particular solution emerges when $(d_{01}, d_{02}, d_{03}, J_2)$ are selected so as to deactivate the π_2 mode of this system. This is accomplished by setting $J_2 = 0$, $d_{03} = 0$, and by selecting d_{01} and d_{02} so that a multiplicative factor of $(1-\pi_2 L)$ appears in each of the numerator terms in the moving average polynomials operating on $w_1(t)$ and $w_2(t)$, respectively. The values of d_{01}, and d_{02} that accomplish this are given by

(9)
$$d_{01} = 1/\lambda\pi_2$$

$$d_{02} = \frac{-\gamma(1+n)}{\lambda(1-\pi_2(1+n))}$$

$$d_{03} = 0$$

$$J_2 = 0.$$

When $(d_{01}, d_{02}, d_{03}, J_2)$ are set at the values given by (9), the solution (7) simplifies to

$$(10)\qquad \begin{bmatrix} p(t) \\ h(t) \end{bmatrix} = \frac{\lambda^{-1}}{(1-\pi_1 L)} \begin{bmatrix} (1-\frac{1}{1+n}L)\frac{1}{\pi_2}, & -\gamma(1+n)/(1-\pi_2(1+n)) \\ \xi/\pi_2, & \lambda - (\xi\gamma(1+n))/(1-\pi_2(1+n)) \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{pmatrix} \frac{J_1\gamma}{1-\lambda\pi_1} \\ J_1 \end{pmatrix} \pi_1^t$$

In this solution, there appears no spurious indicator. Evidently, the solution is of mean exponential order π_1. Unless the parameters $(d_{01},d_{02},d_{03},J_2)$ satisfy (9), the solution is of mean exponential order π_2.

Some observations are in order about interpreting the multiplicity of solutions that we have described. One way to interpret this multiplicity of equilibria is as reflecting the incompleteness of the "feedback law" (1b) as a description of the evolution of $h(t)$. According to (1b), the monetary authority simply prints whatever new money is required to purchase $[\xi+\varepsilon(t)/p(t)]$ goods per capita at the ruling price level $p(t)$. This is an easy rule to implement, requiring little information about the economy, only the real amount of goods to be purchased and the price level. Furthermore, following this rule assures that the budget is financed.

An alternative way to have set up the model would be as follows. We could have imagined the monetary authority as having selected a contingency plan for the per capita money supply of the form

$$(11)\qquad h(t) = g(L)w(t) + J_1\pi_1^t + J_2\pi_2^t,$$

subject to the restrictions that portfolio balance prevails and that the government budget constraint be satisfied, i.e., that (1a) and (1b) hold. These restrict h(t) to obey a law given by the second row of (7), namely

$$(12)\qquad h(t) = \frac{\lambda^{-1}L}{(1-\pi_1 L)(1-\pi_2 L)}\left[\xi(1-\lambda d_{01}L^{-1})w_1(t)-(\xi\lambda d_{02}L^{-1}+(1-\lambda L^{-1}))w_2(t)-\xi\lambda d_{03}L^{-1}w_3(t)\right] + J_1\pi_1^t + J_2\pi_2^t.$$

Equation (12) represents the class of rules or strategies of class (11) among which it is feasible for the monetary authority to choose. The monetary authority's choice among such rules amounts to its choosing values of (d_{01}, d_{02}, d_{03}, and J_2). If the monetary authority has enough information, it is possible for it to select these parameters so that (9) is satisfied, thereby implementing the singular equilibrium described in (10). Notice that to formulate policy in this way, the authority has to know the parameters of the model, and to observe current and lagged $(w_1(t),w_2(t))$. To execute (1b), the authority needs much less information.

From this viewpoint, it seems that if a specification of a policy means specification of a strategy of form (11), then the equilibrium is unique. It also makes sense that uniqueness vanishes when policy is specified in the less restrictive way represented by (1b). In a way, this example illustrates a general point, namely the importance of the specification of strategy spaces in influencing matters of existence and uniqueness of equilibria.

However, the claim for uniqueness of equilibrium under a policy of the form (12) is somewhat artificial because it hinges critically on our having defined an equilibrium as a stochastic process $h(t)$, $p(t)$ that satisfies both (1a) and (1b). In particular, by requiring that (1b) be satisfied, we are insisting that the seignorage raised by money creation finance the real expenditure process $(\xi+\varepsilon(t)/p(t))$. This is a natural requirement to impose in defining equilibrium when the government's rule for creating base money is stated in the form (1b), but is one that seems artificial when the rule for creating base money is of the form of the contingency plan (12). When the rule for creating base money is of the form (12), a natural candidate for a definition of equilibrium is a stochastic process for $(p(t),h(t))$ that satisfies (1a) and (12), with the process for per capita real seignorage being given by $[h(t) - \frac{1}{1+n}h(t-1)]/p(t)$. Under this definition of equilibrium, for any choice of $h(t)$ process of the form (12), there is a single equilibrium that satisfies (1b). However, there is a continuum of additional equilibria in which the price level is eventually of mean exponential order λ^{-1}. For each such equilibrium, real balances tend to disappear, and (1b) is not satisfied. Such equilibria correspond to the speculative bubble equilibria familiar from overlapping generations models (for example, see Wallace [1980]), which occur when the government sets out an exogenous sequence of nominal balances, there existing many equilibria in which real balances tend to zero with the passage of time. In the present case, the government is once and for all setting forth a contingency plan of the form (12), which involves setting $h(t)$ as a function of exogenous variables, and refusing to feed back on $p(t)$.

From these considerations, we draw the conclusion that it is not easy to devise an operating rule for the government that eliminates the multiplicity of solutions of the form (7) without making the system vulnerable to multiple equilibria, elements of which ironically are eventually even more inflationary than those associated with (7).[5/]

3. Econometric Identification

Despite the existence of many equilibria, the view that the economy is operating along one of them turns out to restrict observations. More precisely, according to "order conditions," the model represented by (7) is overidentified even when the parameters $(d_{01}, d_{02}, d_{03}, J_2)$ that index the multiplicity of equilibria are among the free parameters to be estimated.[6/]

To discuss identification, we shall specialize (4) by setting $\mu = 1$ and assuming that

$$(13) \qquad Ew(t)w(t)^T = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv V.$$

Our having set $\sigma_{33} = 1$ is merely a normalization, since clearly only $d_{03}\sqrt{\sigma_{33}}$ is identified (see (7)). We shall consider the special case in which $a_{11}(L) = 1$, $a_{22}(L) = 1$. This is the case in which identification is least likely to occur.

When $a_{11}(L) = a_{22}(L) = 1$, representation (7) becomes

$$(\tilde{7})\qquad \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \frac{-\lambda^{-1}}{(1-\pi_1 L)(1-\pi_2 L)}$$

$$\begin{bmatrix} (1-\frac{1}{1+n}L)(L-\lambda d_{01}), & -(1-\frac{1}{1+n}L)\lambda d_{02} + \gamma L, & -\lambda d_{03}(1-\frac{1}{1+n}L) \\ \xi(L-\lambda d_{01}), & -\xi\lambda d_{02} + (L-\lambda), & -\xi\lambda d_{03} \end{bmatrix}$$

$$\begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} + \pi_1^t \begin{pmatrix} \frac{\gamma J_1}{1-\lambda\pi_1} \\ J_1 \end{pmatrix} + \pi_2^t \begin{pmatrix} \frac{\gamma J_2}{1-\lambda\pi_2} \\ J_2 \end{pmatrix} .$$

The form of $(\tilde{7})$ and the vector white noise characterization of $w(t)$ implies that $(1-\pi_1 L)(1-\pi_2 L)p(t)$ is a second-order moving average process, and that $(1-\pi_1 L)(1-\pi_2 L)h(t)$ is a first-order moving average process that is correlated with $(1-\pi_1 L)(1-\pi_2 L)p(t)$ lagged 0, -1, +1, and -2 times (here a lag of -2 refers to $(1-\pi_1 L)$ $(1-\pi_2 L)p(t+2)$). At all other leads and lags, $(1-\pi_1 L)(1-\pi_2 L)p(t)$ is uncorrelated with $(1-\pi_1 L)(1-\pi_2 L)h(t)$. These facts imply that a uniquely identified Wold representation for $(p(t),h(t))$ exists of the form

$$\begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \frac{1}{(1-\pi_1 L)(1-\pi_2 L)}$$

$$(14)\qquad \begin{bmatrix} c_{11}^0 + c_{11}^1 L + c_{11}^2 L^2, & c_{12}^0 + c_{12}^1 L + c_{12}^2 L^2 \\ c_{21}^1 L & , c_{22}^0 + c_{22}^1 L \end{bmatrix}$$

$$\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} + \begin{pmatrix} c_{31} \\ c_{32} \end{pmatrix} \pi_1^t + \begin{pmatrix} c_{41} \\ c_{42} \end{pmatrix} \pi_2^t$$

where $E\ a(t)a(t)^T = I$, and where $p(t) - E_{t-1}p(t)$ and $h(t) - E_{t-1}h(t)$ can each be expressed as linear combinations of the (2x1) vector $a(t) = (a_1(t), a_2(t))^T$. (Here $E_{t-1}(\cdot)$ denotes the linear least squares forecast of $(\cdot)$ conditional on values of $p(s)$, $h(s)$ for $s \leq t-1$.) The identifiable parameters of the model (i.e., a minimal set of parameters in terms of which the likelihood attains its maximal value) are the π_j's and the c_{ij}^k's in (14). There are 15 of these parameters.

Notice that the restriction $c_{21}^0 = 0$ is imposed in (14). This is normalization that selects from among all Wold representations the unique one for which $a_2(t) = h(t) - E_{t-1}h(t)$.

The theoretical model ($\tilde{7}$) is a collection of restrictions on (14). The nature of the restrictions can be explored by studying the spectral factorization identity that links the parameters of ($\tilde{7}$) and (14). The theoretical model has 12 deep free parameters, $(\lambda, \gamma, \xi, n, d_{01}, d_{02}, d_{03}, \sigma_{11}, \sigma_{12}, \sigma_{22}, J_1, J_2)$. These 12 parameters are linked to the 15 parameters of (14) by 15 restrictions which arise as follows. Begin by representing (7) as

$$(7') \qquad \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \frac{1}{(1-\pi_1 L)(1-\pi_2 L)} D(L)\ w(t) + \pi_1^t \begin{pmatrix} \frac{\gamma J_1}{1-\lambda\pi_1} \\ J_1 \end{pmatrix} + \pi_2^t \begin{pmatrix} \frac{\gamma J_2}{1-\lambda\pi_2} \\ J_2 \end{pmatrix}$$

where $D(L) = (1-\pi_1 L)(1-\pi_2)\begin{pmatrix} d(L) \\ g(L) \end{pmatrix}$.

Next, represent (14) as

$$(14') \qquad \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \frac{1}{(1-\pi_1 L)(1-\pi_2 L)} C(L)a(t) + \pi_1^t \begin{pmatrix} c_{31} \\ c_{32} \end{pmatrix} + \pi_2^t \begin{pmatrix} c_{41} \\ c_{42} \end{pmatrix}$$

where $C(L)$ is the (2x2) matrix of moving average polynomials on the right side of (14). The parameters of (7') and (14') are partially linked via the identity $D(L)w(t) = C(L)a(t)$. The implications of this identity are exhausted by the spectral factorization identity

$$(15) \qquad D(L)VD(L^{-1})^T = C(L)C(L^{-1})^T$$

where $\det C(z)$ has all of its zeroes on or outside the unit circle. Equation (15) supplies a total of nine equations restricting the elements of $D(L)$, V (corresponding to the nine identifiable parameters in $C(L)$).[7/] We obtain four more equations by equating coefficients on π_1^t and π_2^t in (7') and (14'), namely

$$(16) \qquad \begin{aligned} c_{31} &= \frac{\gamma J_1}{1-\lambda\pi_1} \qquad & c_{41} &= \frac{\gamma J_2}{1-\lambda\pi_2}. \\ c_{32} &= J_1 & c_{42} &= J_2 \end{aligned}$$

Finally, two more restrictions are implied by equation (8), which we repeat here for convenience

$$(8) \qquad \pi_1 = [(\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda) - \sqrt{(\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda)^2 - 4/(1+n)\lambda}\,]/2$$

$$\pi_2 = [(\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda) + \sqrt{(\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda)^2 - 4/(1+n)\lambda}\,]/2.$$

Equations (15), (16), and (8) supply a total of 15 independent restrictions on the 12 free parameters of the model $(\lambda, \gamma, \xi, n, d_{01}, d_{02}, d_{03}, \sigma_{11}, \sigma_{12}, \sigma_{22}, J_1, J_2) \equiv \theta$. According to these "order condition" considerations, the model is overidentified.

To this point, our discussion has assumed that $a_{11}(L) = a_{22}(L) = 1$. In general, richer specifications for $a_{11}(L)$ and $a_{22}(L)$ will lead to stronger overidentification. As (7) reveals, the higher is the order of $a_{11}(L)$ or $a_{22}(L)$, the more restrictions there are on the moving average representation for $(p(t),h(t))$.

The appendix describes identification in the singular case in which (9) holds so that the solution collapses to (10). We find that overidentification vanishes, but that identification still obtains. Thus, identification is more fragile than when the π_2 root is activated. This situation reflects the intuitive notion that when the π_2 root is activated, there occurs richer behavior of the time series, which contain more information about the parameters of the model.

4. Granger Causality

An advantage of the model described by Sargent and Wallace [1973] and Sargent [1977] is that it predicts that inflation Granger causes money creation, while money creation fails to Granger cause inflation. This predicted pattern of Granger causality has been found in data drawn from the hyperinflations (see Sargent and Wallace [1973]). It is useful to study what restrictions need to be imposed on the present model in order to produce such patterns of Granger causality.

First consider the special case of the model in which $a_{11}(L) = a_{22}(L) = 1$, $w_{3t} = 0$. Also assume that (9) holds, so that the π_2 mode of the system has been deactivated. In this special case, we saw that the solution for $p(t)$, $h(t)$ has the representation

$$(10)\qquad \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \left[\frac{\lambda^{-1}}{1-\pi_1 L} \begin{matrix} (1 - \frac{1}{1+n}L)\frac{1}{\pi_2}, -\gamma(1+n)/(1-\pi_2(1+n)) \\ \xi\pi_2,\ \lambda-(\xi\gamma(1+n))/(1-\pi_2(1+n)) \end{matrix} \right] \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{pmatrix} \gamma J_1/(1-\lambda\pi_1) \\ J_1 \end{pmatrix} \pi_1^t .$$

Note that in the representation for h(t), the polynomials in L on $w_1(t)$ and $w_2(t)$ are proportional to one another, while in the representation for p(t), the polynomials in $w_1(t)$ and $w_2(t)$ are not proportional. This implies that h(t) Granger causes p(t), while p(t) fails to Granger cause h(t).

This structure of Granger causality is a special feature of the singular case in which $(1-\pi_2 L)$ has been cancelled out. In the general case, p(t) and h(t) Granger cause each other, which can be proved by studying the structure of the solution (7). We now seek special cases in which p(t) Granger causes h(t), with no Granger causality extending from h(t) to p(t).

We begin with a case in which there is no spurious indicator impinging on the solution, so that in (7) $d_{03} = 0$. Suppose that $a_{11}(L)$ and $a_{22}(L)$ satisfy the restriction

$$a_{22}(L) = \frac{d_{02}}{\gamma d_{01}} (1 - \frac{1}{1+n}L)a_{11}(L).$$

The reader can verify that under the above restriction, the polynomials on $w_1(t)$ and $w_2(t)$ in d(L) are proportional to one another, while the polynomials in g(L) are not proportional to one another. Under this special condition, it follows that p(t) Granger causes h(t), but that h(t) fails to Granger cause p(t).

To motivate the next special case, notice that real balances demanded at a constant expected gross return on money of unity are given by $(1-\lambda)/\gamma$. Let us reparameterize the system by setting $\gamma = (1-\lambda)\theta$, where $\theta = (p/h)(1)$, the inverse of real balances at a gross inflation rate of unity. To achieve the following special case, we shall think of holding θ fixed as we vary λ, so that as $\lambda \to 1$, $\gamma \to 0$. In the limiting case with $\lambda = 1$, $\gamma = 0$, (7) implies

$$(1-\pi_1 L)(1-\pi_2 L)\begin{pmatrix} d(L) \\ g(L) \end{pmatrix} =$$

$$\begin{pmatrix} \dfrac{d_{01}-L}{1-L}, & \dfrac{d_{02}}{1-L}, & \dfrac{d_{03}}{1-L} \\ \dfrac{\xi(d_{01}-L)}{(1-L)(1-\frac{1}{1+n}L)}, & \dfrac{\xi d_{02}+(1-L)}{(1-L)(1-\frac{1}{1+n}L)}, & \dfrac{\xi d_{03}}{(1-L)(1-\frac{1}{1+n}L)} \end{pmatrix}$$

where we are using the fact that when $\gamma = 0$, $\pi_1 = (1+n)^{-1}$, $\pi_2 = 1$. In the special case that $w_{1t} = 0$, so that the portfolio balance schedule is exact, $d(L)$ is such that p_t is a martingale, so that $p(t)$ is not Granger-caused by $h(t)$. However, $p(t)$ Granger causes $h(t)$ so long as the spurious indicator is present. (The martingale characterization of $p(t)$ under these conditions can also be deduced directly from equation (1a).)

These examples constitute singular cases in which $h(t)$ fails to Granger cause $p(t)$. They indicate the presence of a range of examples close to these in which Granger causality extends from $h(t)$ to $p(t)$, but is difficult to detect in short samples. The model thus appears to be potentially capable of accommodating Granger causality patterns such as those detected in earlier work (Sargent and Wallace [1973]).

5. Estimation

We shall describe two estimators which are modifications of the time domain and frequency domain estimators described by Hansen and Sargent [1981]. The modifications involve properly taking account of the restrictions that exist across the initial conditions and the remaining parameters of the model.

Let us define $y(t)^T \equiv (p(t),h(t))^T$, and represent (7) as

$$y(t) = \frac{1}{(1-\pi_1 L)(1-\pi_2 L)} D(L)w(t) + H_1\pi_1^t + H_2\pi_2^t \tag{17}$$

where

$$D(L) \equiv (1-\pi_1 L)(1-\pi_2 L) \begin{pmatrix} d(L) \\ g(L) \end{pmatrix}$$

$$H_1 = \begin{pmatrix} \dfrac{\gamma J_1}{1-\lambda\pi_1} \\ J_1 \end{pmatrix} \qquad H_2 = \begin{pmatrix} \dfrac{\gamma J_2}{1-\lambda\pi_2} \\ J_2 \end{pmatrix}.$$

We assume that $Ew(t)w(t)^T = V$. It is to be understood that $D(L)$, π_1, π_2, H_1 and H_2 are all functions of the list of deep parameters of the model $\theta \equiv (\lambda,\gamma,\xi,n,d_{01},d_{02},d_{03},\sigma_{11},\sigma_{12},\sigma_{22},J_1,J_2)$.

The first step in constructing the time domain estimator is to replace $D(L)w(t)$ by its Wold representation $F(L)a(t)$ where

$$\underset{2\times 3}{D(L)}\ \underset{3\times 1}{w(t)} = \underset{2\times 2}{F(L)}\ \underset{2\times 1}{a(t)}$$

where

$$F(L) = I + F_1L + F_2L^2$$

$$\det F(z^0) = 0 \Rightarrow |z^0| > 1,$$

and

$$a(t) = y(t) - E(y(t)|y(t-1),\ldots,y(0)),$$

where $Ea(t)a(t)^T = \Omega$. To compute $F(L)$ we solve the spectral factorization equation

$$(18) \qquad D(L)VD(L^{-1})^T = F(L)\Omega F(L^{-1})^T$$

subject to the zeroes of det $F(z)$ not being inside the unit circle. Practically, given the parameters of $D(L)$ and V, this equation can be solved for F_1, F_2, Ω in a variety of ways, for example, by use of the Kalman filter.

The white noise vector $a(t)$ is fundamental for $y(t)$ and, therefore, is in the space spanned by current and lagged $w(t)$'s. It follows from $w(s) = 0$ for $s < 0$ that $a(s) = 0$ for $s < 0$. We can then represent (17) as

$$(19) \qquad y(t) = \frac{1}{(1-\pi_1L)(1-\pi_2L)} F(L)a(t) + H_1\pi_1^t + H_2\pi_2^t.$$

We propose to use this equation to calculate the $a(t)$ vector implied by a given set of parameter values θ. Suppose we have a sample on $y(t)$ for $t = 1, \ldots, T$. Writing (19) for $t = 0, 1$, and using $a(s) = 0$ for $s < 0$, we have

$$(20) \qquad y(0) = H_1 + H_2$$

$$y(1) = a(1) + H_1\pi_1 + H_2\pi_2$$

Multiplying (19) by $(1-\pi_1 L)(1-\pi_2 L)$ and solving for $a(t)$ gives

$$(21) \qquad a(t) = y(t) - (\pi_1+\pi_2)y(t-1) + \pi_1\pi_2 y(t-2)$$
$$- F_1 a(t-1) - F_2 a(t-2)$$

We can thus solve for $a(t)$ by beginning with (19), and then using recursions on (21):

$$\hat{a}(1) = y(1) - H_1\pi_1 - H_2\pi_2$$

$$\hat{a}(2) = y(2) - (\pi_1+\pi_2)y(1) + \pi_1\pi_2[H_1+H_2] - F_1\hat{a}(1)$$

$$(22) \qquad \hat{a}(3) = y(3) - (\pi_1+\pi_2)y(2) + \pi_1\pi_2 y(1) - F_1\hat{a}(2) - F_2\hat{a}(1),$$

$$\vdots$$

$$\hat{a}(t) = y(t) - (\pi_1+\pi_2)y(t-1) + \pi_1\pi_2 y(t-2) -$$
$$F_1\hat{a}(t-1) - F_2\hat{a}(t-2).$$

Equation (22) is to be understood as expressing estimated innovations as functions of the deep parameters θ that determine H_1, H_2, and $F(L)$.

In estimation problems without restrictions on the (2x1) vectors of initial conditions, H_1 and H_2, it is possible to choose them so that the first two sample disturbances $\hat{a}(1)$ and $\hat{a}(2)$ of (22) are set equal to zero. In practice, this is accomplished by first setting $\hat{a}(1)$ and $\hat{a}(2)$ to zero and ignoring the first two equations of (22). All of the free parameters of the model except the initial conditions H_1 and H_2 are then estimated, conditional on $\hat{a}(1) = \hat{a}(2) = 0$. As a final step, H_1 and H_2 are estimated by solving the first two equations of (22) for (H_1, H_2) with π_1 and π_2 taken at their estimated values. The restrictions across the rows of H_1 and H_2, and across H_1, H_2, and the remaining parameters, prevent this procedure from being applicable here. These restrictions, and the inapplicability of the standard procedure, reflect the information about the deep parameters carried by the initial conditions.[8/]

Assuming that the w(t) process is multivariate normal, the log likelihood function can be approximated by

$$L_T = -T \log 2\pi - T/2 \log \det \Omega - \frac{1}{2} \sum_{t=1}^{T} \hat{a}(t)^T \Omega^{-1} \hat{a}(t).$$

This is to be maximized with respect to the free parameters of θ subject to the $\hat{a}(t)$'s being given by (22) and F(L) and Ω satisfying (18).

Alternatively, approximate maximum likelihood estimates can be obtained by minimizing

$$\det T^{-1} \sum_{t=1}^{T} \hat{a}(t)\hat{a}(t)^T \tag{23}$$

with respect to the free parameter θ. The $\hat{a}(t)$'s are functions of the parameters θ via (22) and (18).

The frequency domain estimator has the virtue of avoiding the need to factor the spectral density matrix $D(L)VD(L^{-1})^T$ as in (18). To employ the frequency domain method, one uses (17) for $t = 0, 1$, to get

$$y(0) = H_1 + H_2$$

$$y(1) = D_0 w(1) + \pi_1 H_1 + \pi_2 H_2.$$

Solving for $D_0w(1)$ gives

$$(24) \qquad D_0w(1) = y(1) - \pi_1H_1 - \pi_2H_2$$

Then iterating on (17) gives

$$(25) \qquad D(L)w(2) = y(2) - (\pi_1+\pi_2)y(1) + \pi_1\pi_2[H_1+H_2]$$

$$D(L)w(t) = y(t) - (\pi_1+\pi_2)y(t-1) + \pi_1\pi_2y(t-2) \text{ for } t \geq 3.$$

For a given value of the parameter vector θ and the associated $D(L)w(t)$ series generated by solving (24)-(25), the periodogram is to be formed. Letting $z(t) \equiv D(L)w(t)$, we define

$$\tilde{z}(w_j) = \sum_{j=1}^{T} z(t)e^{-iw_jt}, \; w_j = \frac{2\pi j}{T}, \; j = 1, \ldots, T.$$

The periodogram is then defined as

$$(26) \qquad I(w_j) = T^{-1} z(w_j) \overline{z(w_j)}^T$$

where the bar denotes complex conjugation. It is important to note that the periodogram $I(w_j)$ of $D(L)w(t)$ is a function of the parameters of the model θ, and must be recalculated for each step in the nonlinear maximization of the criterion function (28) below.

Next, the theoretical spectral density matrix of $D(L)w(t)$ is given by

$$(27) \qquad s(z) = D(z)VD(z^{-1})^T$$

Approximate maximum likelihood estimates can be obtained by maximizing with respect to θ

$$(28) \qquad L = -\frac{1}{2}(2T) - \frac{1}{2}\sum_{j=1}^{T} \log \det S(e^{-iw_j}) - \frac{1}{2}\sum_{j=1}^{T} \mathrm{tr}[S(e^{-iw_j})^{-1} I(w_j)].$$

It is to be emphasized that both $S(e^{-iw_j})$ and $I(w_j)$ are functions of the parameter θ, and must be recomputed at each step of the maximization. The need to recompute $I(w_j)$ at each step of the optimization is a result of the fact that cross-equation restrictions exist on the parameters in H_1 and H_2, and that these restrictions contain information about θ. This distinguishes the current estimation problem from standard ones in which the absence of such cross-equation restrictions means that there are sufficient free parameters in H_1 and H_2 to make it appropriate to eliminate exponential terms before estimation, so that the counterpart of $I(w_j)$ need be computed only once.

6. Conclusions

This paper has described a model in which there is a four-dimensional continuum of equilibrium stochastic processes for the base money supply and price level. In many of these equilibria, there is a spurious indicator or "sunspot" variable that affects the base money supply and price level even though it fails to influence the fundamental forcing variables, namely, the random disturbances to portfolio balance and the government's budget constraint. Technically, the existence of equilibria depending on sunspots appears to be intricately tied to the existence of multiple equilibria depending on the fundamental driving processes.

Despite the existence of a four-dimensional continuum of equilibria, the model is econometrically overidentified. In other words, the view that the economy is operating along one member of the continuum of equilibria restricts the behavior of the time series for base money and the price level, and permits one to estimate the behavioral parameters that describe portfolio balance and the government budget constraint, and also the "nuisance parameters" that serve to select one from among the many equilibria. That the model is overidentified in this way gives empirical content to the hypothesis that hyperinflations have been characterized by a process of sliding down the slippery side of the Laffer curve. In a sequel to this paper, we intend to use the model econometrically to study this hypothesis.

Appendix on Identification

This appendix studies identification in the singular case in which (9) holds. In this case, the bivariate $(p(t),h(t))$ process evolves according to

$$(10)\qquad \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \frac{\lambda^{-1}}{1-\pi_1 L} \begin{bmatrix} (1 - \frac{1}{1+n} L)\frac{1}{\pi_2}, & -\gamma(1+n)/(1-\pi_2(1+n)) \\ \xi/\pi_2, & \lambda-(\xi\gamma(1+n))/(1-\pi_2(1+n)) \end{bmatrix} \begin{matrix} w_1(t) \\ w_2(t) \end{matrix}$$
$$+ \begin{pmatrix} \gamma J_1/(1-\lambda\pi_1) \\ J_1 \end{pmatrix} \pi_1^{\,t} .$$

We assume that the covariance matrix of $w(t)$ is given by

$$Ew(t)w(t)^T = V$$

where V is a positive definite matrix. Let G^{-1} be a lower triangular matrix that normalizes and diagonalizes V, i.e., $I = G^{-1}VG^{-1T}$. Define the transformed disturbance vector

$$\eta(t) = G^{-1}w(t).$$

Thus, $E\eta(t)\eta(t)^T = I$. Now using $w(t) = G\eta(t)$, we can express (10) as

$$(1-\pi_1 L)I \begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \lambda^{-1} \begin{bmatrix} (1 - \frac{1}{1+n}L)\frac{1}{\pi_2}, & \frac{-\gamma(1+n)}{1-\pi_2(1+n)} \\ \xi/\pi_2 \quad , & \lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)} \end{bmatrix}$$
$$\begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} + \begin{bmatrix} \gamma J_1/(1-\lambda\pi_1) \\ J_1 \end{bmatrix} \pi_1^{\,t}$$

or

$$\text{(A1)} \qquad (1-\pi_1 L)\begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \lambda^{-1}$$

$$\begin{bmatrix} (1 - \frac{1}{1+n}L)\frac{g_{11}}{\pi_2} - \frac{\gamma g_{21}(1+n)}{1-\pi_2(1+n)}, & \frac{-g_{22}\gamma(1+n)}{1-\pi_2(1+n)} \\ \xi g_{11}/\pi_2 + g_{21}(\lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)}), & g_{22}(\lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)}) \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}$$

$$+\begin{pmatrix} \gamma J_1/(1-\lambda\pi_1) \\ J_1 \end{pmatrix} \pi_1{}^t.$$

In this representation, the three parameters, g_{11}, g_{21}, g_{22} represent the covariance matrix of the original $w(t)$ process. The $\eta(t)$ process was constructed by orthonormalizing the $w(t)$ process with the matrix G. The matrix G thus summarizes all of the information in V.

Equation (A1) is a vector autoregressive moving average representation whose identifiable parameters can be displayed as follows. Represent (A1) as[9/]

$$(1-\pi_1 L)\begin{pmatrix} p(t) \\ h(t) \end{pmatrix} = \begin{pmatrix} c_{11}^0 + c_{12}^1 L, & c_{12} \\ c_{21} & , c_{22} \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} + \begin{pmatrix} c_{13} \\ c_{23} \end{pmatrix} \pi_1{}^t.$$

The identifiable parameters are π_1 and the c_{ij}^k's. From (A1) and the formula (8) for π_1, π_2, these parameters are linked to the deep parameters of the model by the following equations:

$$\text{(A2)} \qquad c_{11}^0 = \lambda^{-1}\left(\frac{g_{11}}{\pi_2} - \frac{\gamma g_{21}(1+n)}{1-\pi_2(1+n)}\right)$$

(A3) $$c_{11}^{1} = \frac{-\lambda^{-1}g_{11}}{(1+n)\pi_2}$$

(A4) $$c_{12} = \frac{-\lambda^{-1}g_{22}\gamma(1+n)}{1 - \pi_2(1+n)}$$

(A5) $$c_{21} = -\lambda^{-1}\left(\frac{\xi g_{11}}{\pi_2} + g_{21}\left(\lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)}\right)\right)$$

(A6) $$c_{22} = \lambda^{-1}g_{22}\left(\lambda - \frac{\xi\gamma(1+n)}{1-\pi_2(1+n)}\right)$$

(A7) $$\pi_1,\ \pi_2 = \{(\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda) \pm [(\lambda^{-1}+(1+n)^{-1}-\xi\gamma/\lambda)^2-4/(1+n)\lambda]^{1/2}\}/2$$

(A8) $$c_{13} = \gamma J_1/(1-\lambda\pi_1)$$

(A9) $$c_{23} = J_1.$$

The known variables (identifiable parameters) in the nine equations (A2)-(A9) are π_1, c_{11}^{0}, c_{11}^{1}, c_{12}, c_{21}, c_{22}, c_{13}, c_{23}. The unknown variables to be determined are λ, ξ, γ, n, π_2, g_{11}, g_{21}, g_{22}, J_1. We thus have nine equations to be solved for nine unknowns, which is promising from the viewpoint of identification.

To highlight the role of equations (A8) and (A9) in helping to achieve identification, we shall begin by ignoring them. This amounts to considering identification in the system from which the deterministic components $\pi_1{}^t\binom{J_1}{F_1}$ have been removed prior to estimation.[10] In this case, equations (A2)-(A7) form seven equations in the eight unknowns λ, ξ, γ, π_2, g_{11}, g_{21}, and g_{22} so that these parameters are in general underidentified. However, under the special assumption that $g_{21} = 0$, local identification obtains. The assumption that $g_{21} = 0$ is equivalent with the hypothesis that $w_1(t)$ and $w_2(t)$, the disturbances to portfolio balance and to the government budget, respectively, are orthogonal.

In the case that $g_{21} = 0$, identification can be thought to proceed as follows. Equations (A2) and (A3) imply that

$$(1+n) = -c_{11}^0/c_{11},$$

while equations (A5) and (A2) imply that

$$\xi = -c_{21}/c_{11}^0.$$

So n and ξ are identified.

After some algebra, (A6), (A4) and (A7) imply that

$$\text{(A10)} \qquad c_{22}/c_{12} = (1-\pi_1\lambda)/-\gamma$$

Given knowledge of n and ξ, equation (A10) together with (A7) for π_1, namely,

$$\pi_1 = \{(\lambda^{-1}+(1+n)^{-1}-\frac{\xi\gamma}{\lambda})-\sqrt{(\lambda^{-1}+(1+n)^{-1}-\frac{\xi\gamma}{\lambda})-4/(1+n)\lambda}\},$$

form two equations in γ and λ which possess a locally unique solution. Given π_1, λ, ξ, and n, π_2 can be obtained from (A7). Then g_{11} can be obtained from (A2), and g_{22} from (A4). This completes the discussion of identification in the special case in which $g_{21} = 0$ and $J_2 = 0$.

With $J_1 = 0$ and g_{21} an unknown to be identified, the parameters of the model become underidentified. In this case, we are one restriction short of having an identified system. When $J_1 \neq 0$, equations (A8) and (A9) add two equations but only one unknown to the system. This leaves us with a system of nine equations in the nine unknown parameters to be identified.

The preceding analysis shows that in the singular case in which the root π_2 has been eliminated from the system, identification is delicate. Identification hinges either on including the deterministic component (π_1^t) explicitly in the estimation process, or by a priori imposing orthogonality between $w_1(t)$ and $w_2(t)$.

Footnotes

1/This was thus an application of the same method that John Muth [1960] used to discover a univariate stochastic process for income that made Friedman's [1957] geometric lag formulation for forming permanent income consistent with rational expectations.

2/John Taylor [1977] has studied rational expectations models in which solutions exist that depend on a spurious indicator.

3/This is the solution strategy used by Whiteman [1983] and Saracoglu and Sargent [1978].

4/There are tight links between the dimensionality of solutions depending on the spurious indicators, the dimensionality of the multiplicity of solutions depending on the "fundamental" noises $w_1(t)$ and $w_2(t)$, and the fact that solution (7) is a second-order system in which two roots are being solved backwards and none forwards. It is no coincidence that the number of "extra" roots solved backwards (one) equals the dimensionality both of the multiplicity of solutions dependent on each of the fundamental noises (one), and the dimensionality of solutions depending on the spurious indicator. It is our conjecture that these links reflect the workings of an as yet unproved theorem that would extend Whiteman's theorem [1983, Chapter 5] to permit spurious indicators to impinge on the solution.

5/They are more inflationary because $\lambda^{-1} > \pi_2$.

6/The order conditions described in the text and appendix are suggestive, but are not sufficient for identification. In econometric practice, local identification can be checked numerically by inspecting the condition of the information matrix.

7/The (2x1) vector of innovations a(t) lies in the linear space spanned by current and past values of the (2x1) vector (p(t),h(t)). In general, even if $d_{03} = 0$, the (3x1) vector w(t) does not lie in the linear space spanned by current and past values of (p(t),h(t). The likelihood function is most readily viewed as a function of the a(t)'s (see Box and Jenkins [1970] or Hansen and Sargent [1981] for examples). The identity F(L)a(t) = D(L)w(t) and (15) are used to present the model in a form that exposes parameters that appear in the likelihood function. Note that a(t) may span a smaller space than w(t) for two reasons. A first occurs when $d_{03} \neq 0$. A second occurs when $d_{03} = 0$, and in which detD(z) has one or more zeroes inside the unit circle.

8/However, note in (22), that as t grows large, the dependence of $\hat{a}(t)$ on (H_1,H_2) becomes attenuated. The reason is that the zeroes of detF(z) in general lie outside the unit circle. This observation suggests that it will not be possible to estimate J_1 and J_2 consistently.

9/The off-diagonal parameters c_{12} and c_{21} are both uniquely identified because $c_{12}(L)$ as a polynomial has been restricted to be zero order. It is possible to obtain an alternative representation in which $c_{21}(L) = c_{21} = 0$. However, in such an alternative representation, it is necessary to make $c_{12}(L)$ a first-order polynomial $(c_{12}^0+c_{12}^1L)$ with $c_{12}^1 \neq 0$. (In the section on identification in the text, such a representation setting $c_{21}^0 = 0$ was used.)

[10] The structure of identification in this case is closely related to that which characterizes the model described by Sargent [1977].

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Figure 1a
Log of real balances and log p_t/p_{t-1} for the German hyperinflation

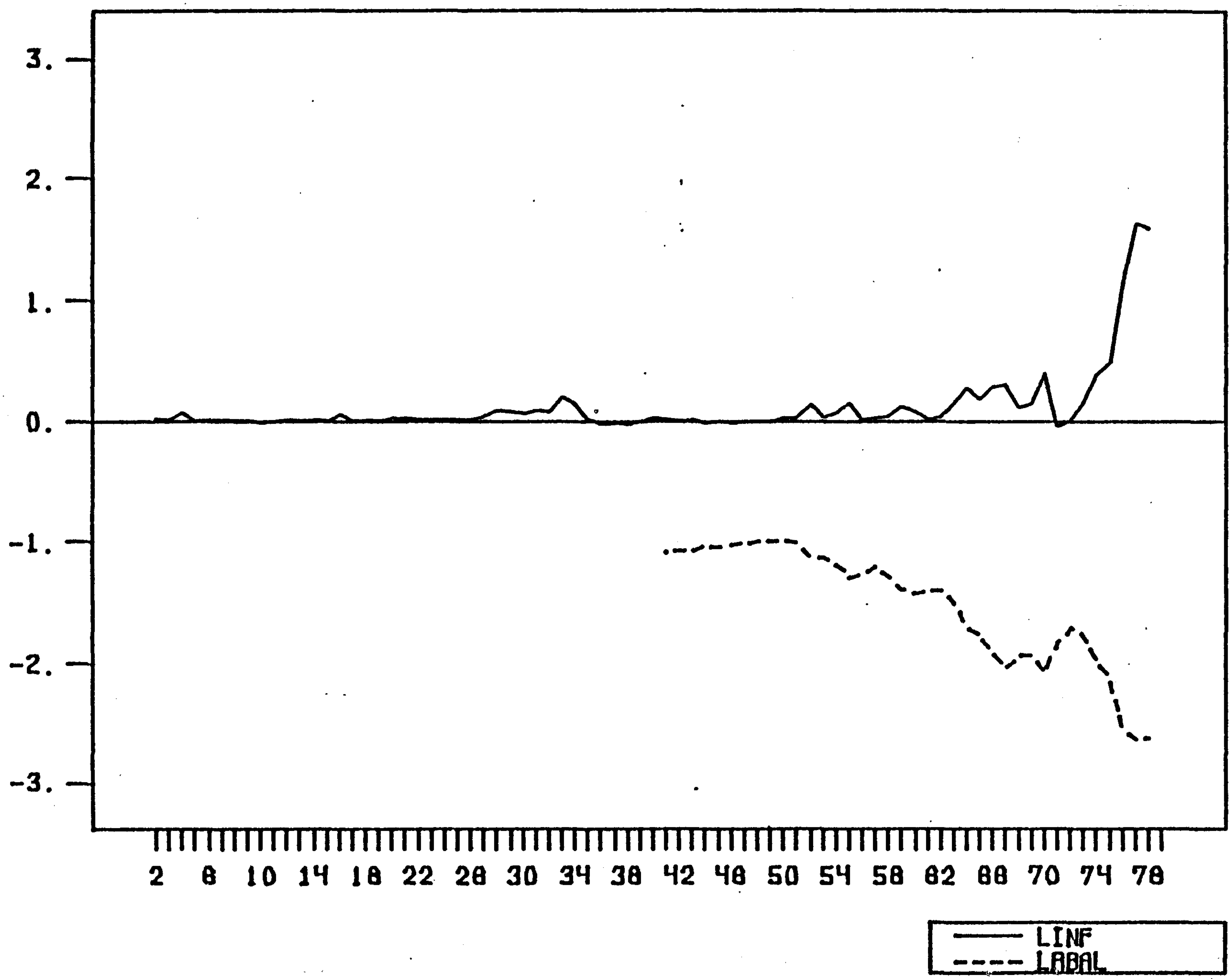


Figure 1b
Log of real balances and log p/p_{-1} for the Greek hyperinflation

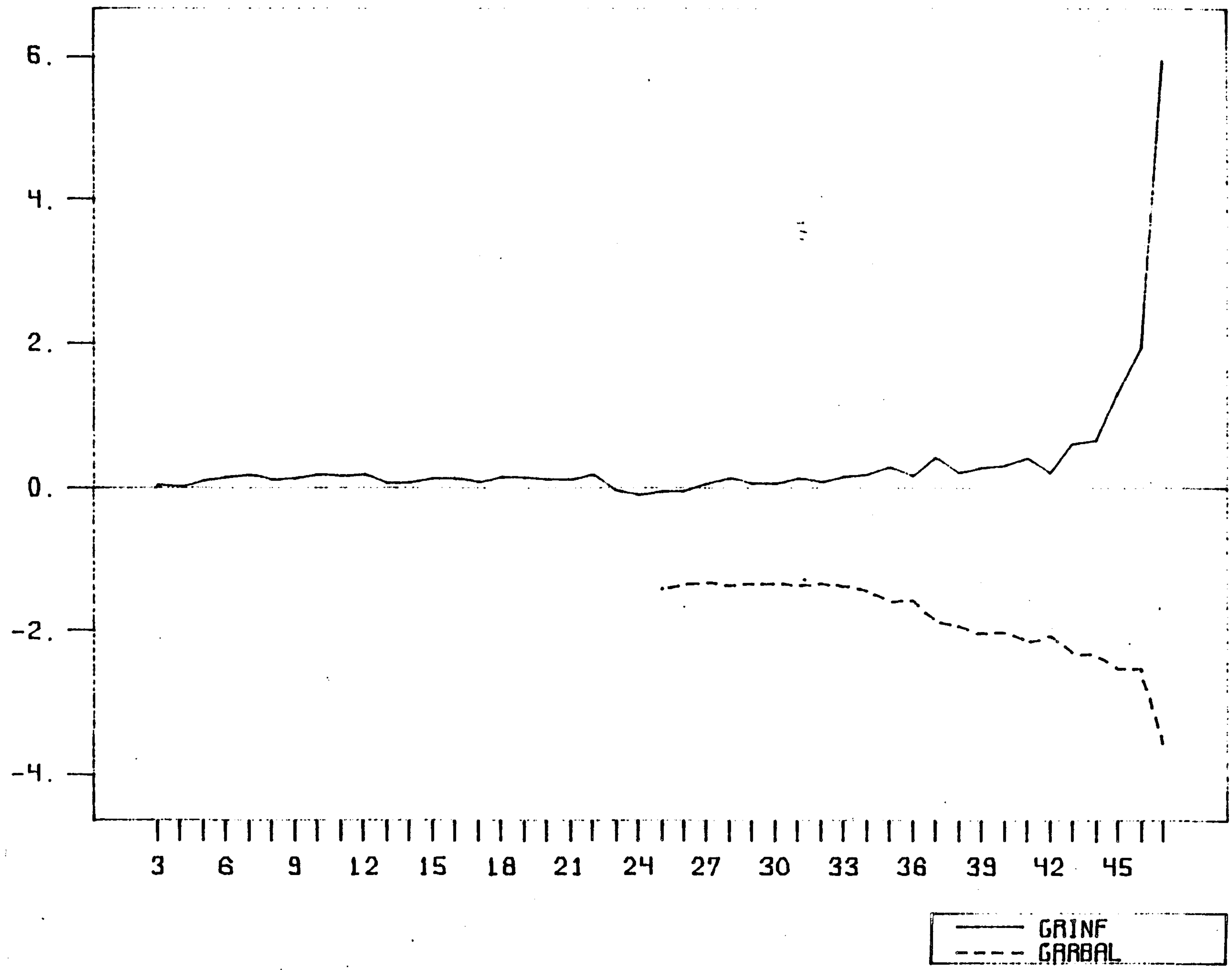


Figure 1c
Log of real balances and log p/p_{-1} for the first Hungarian hyperinflation

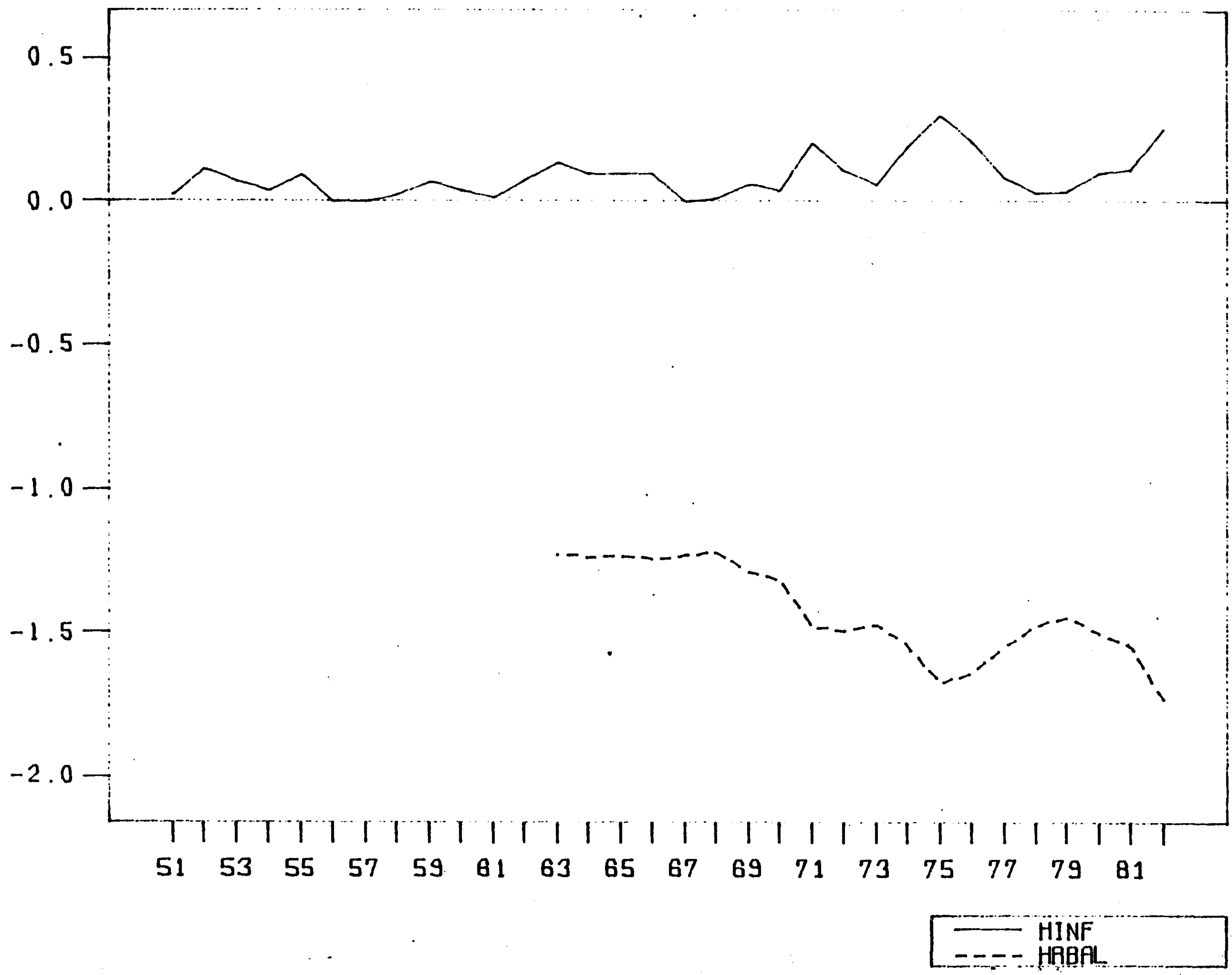


Figure 1d
Log of real balances and log p/p_{-1} for the second Hungarian hyperinflation

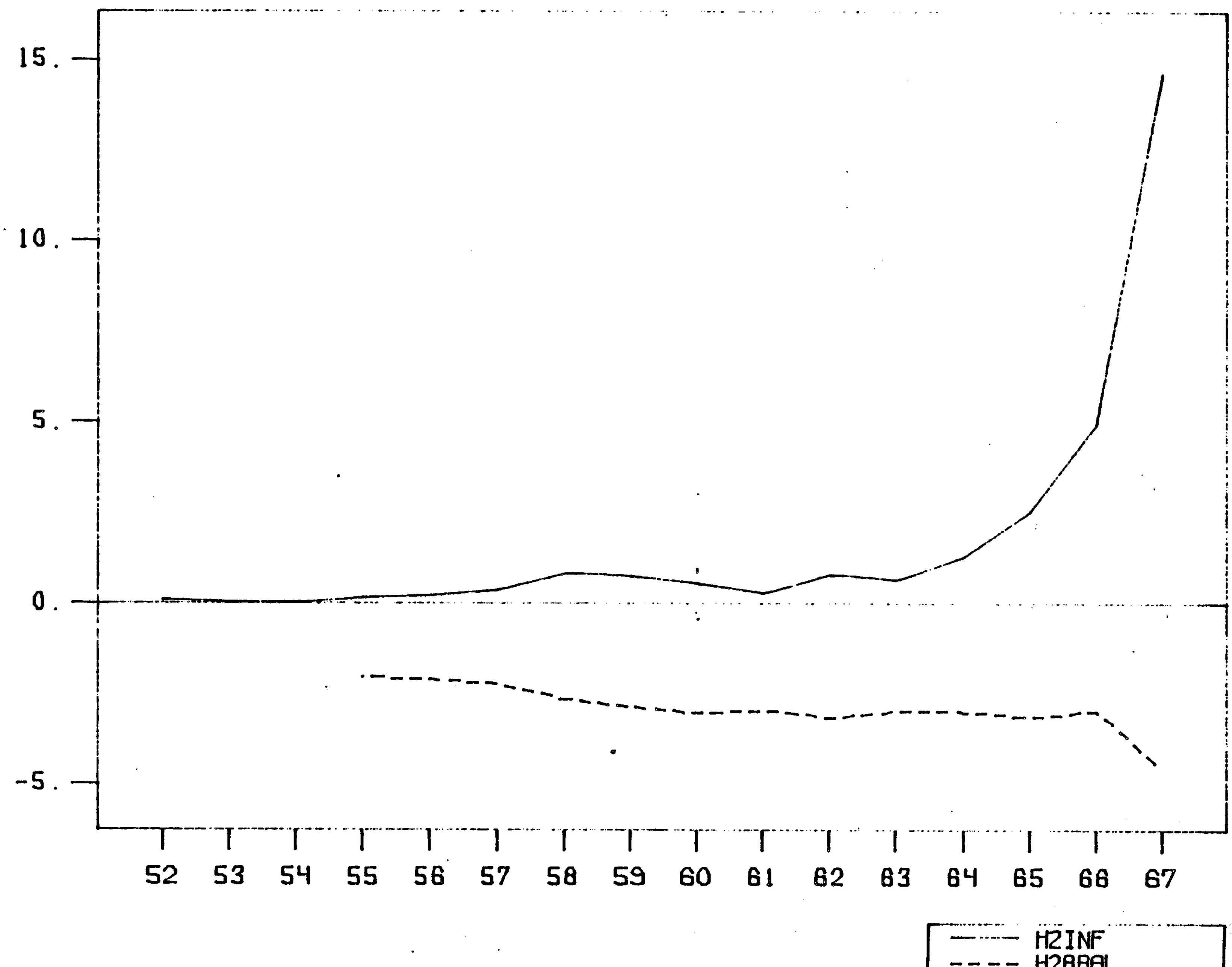


Figure 1e

Log of real balances and log p/p_{-1} for the Polish hyperinflation

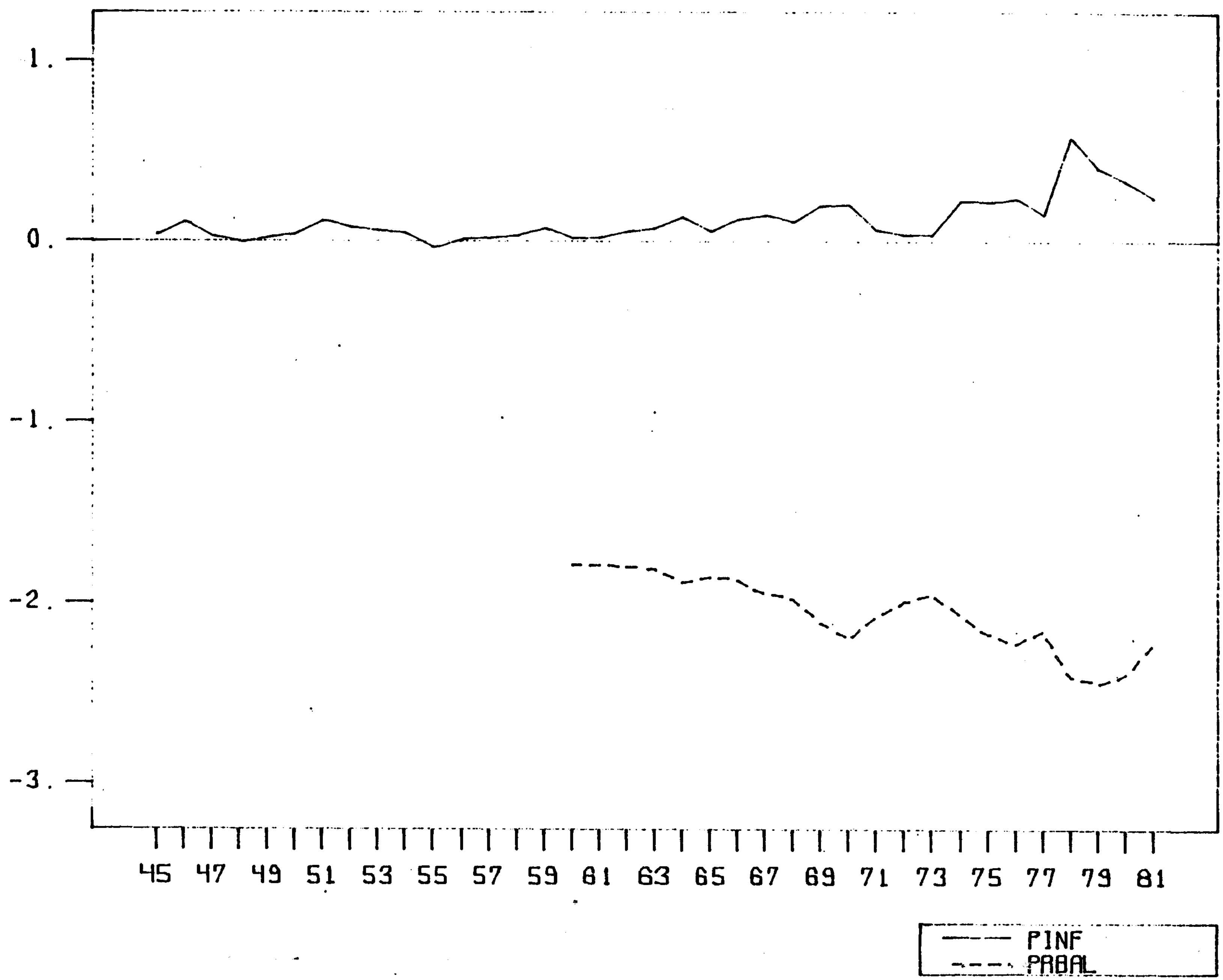


Figure 1f
Log of real balances and log p/p_{-1} for the Russian hyperinflation

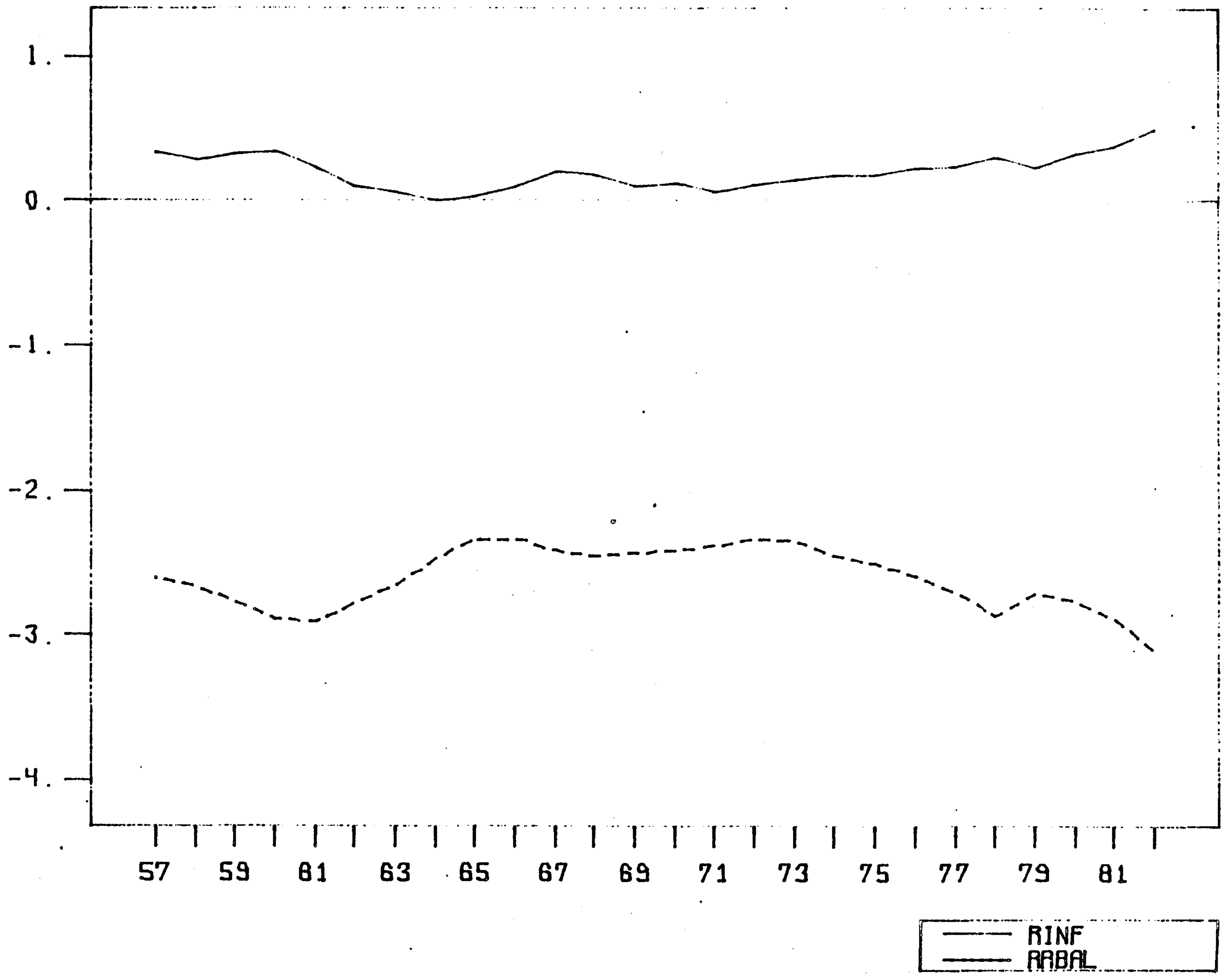
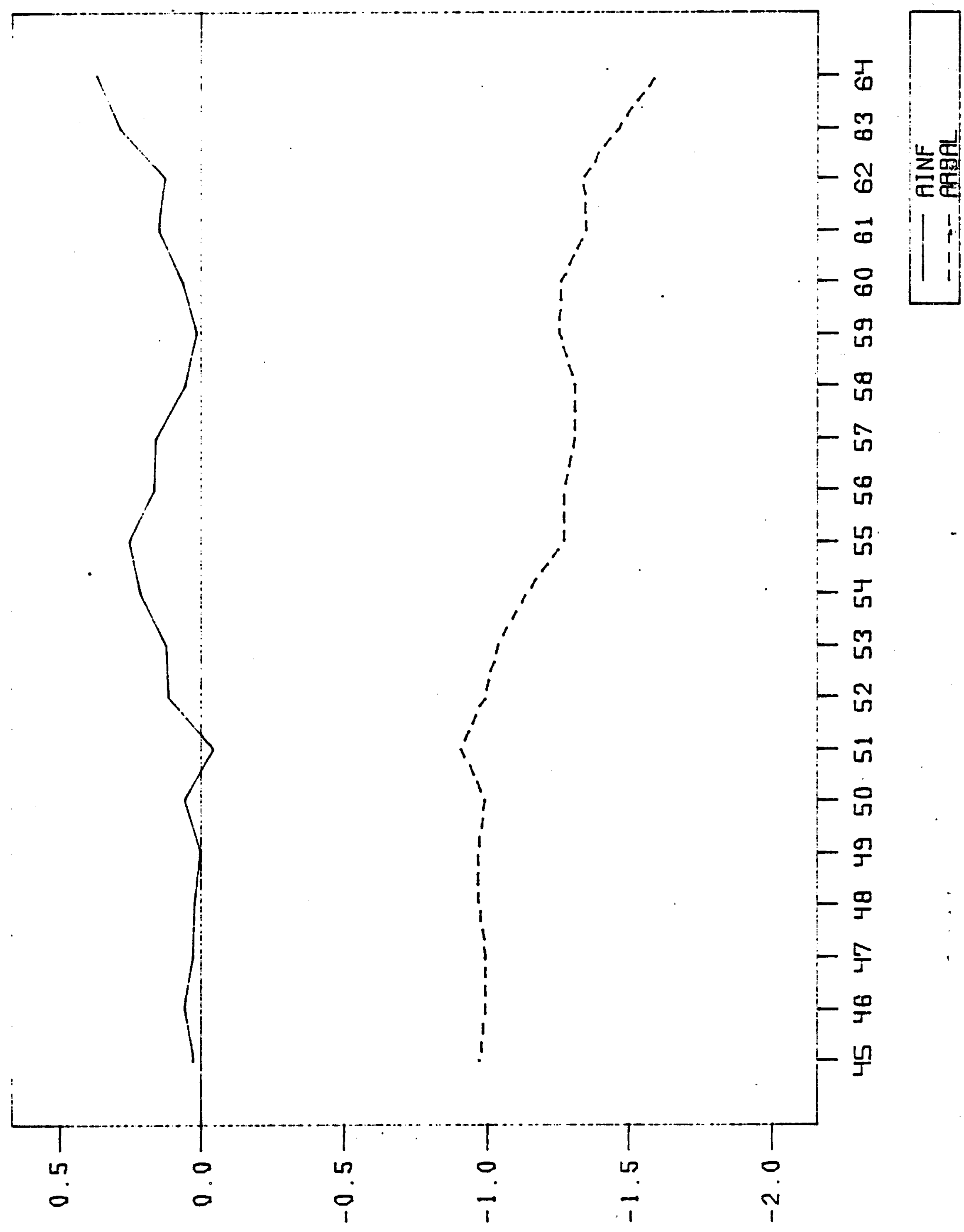


Figure 1g

Log of real balances and log p/p_{-1} for the Austrian hyperinflation



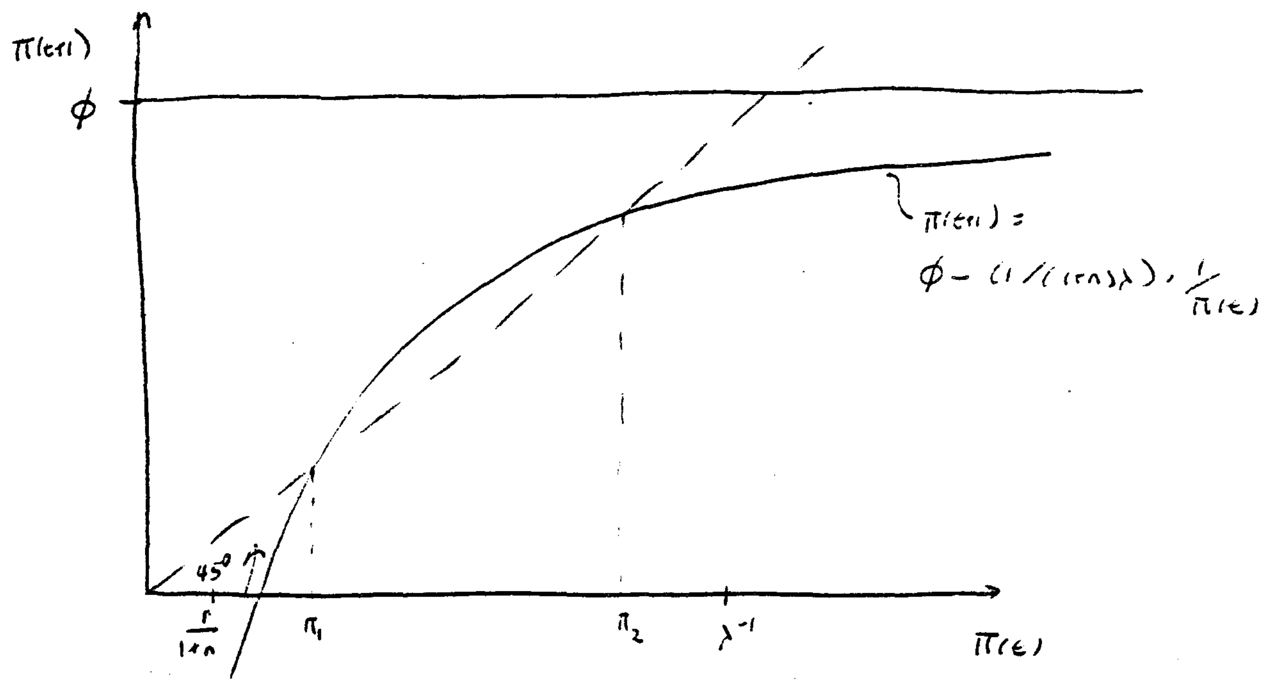


Figure 2