CONDITIONAL PARETO OPTIMALITY OF STATIONARY EQUILIBRIUM IN A STOCHASTIC OVERLAPPING GENERATIONS MODEL*

by

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Abstract

In this paper, I describe and analyze a pure exchange overlapping generations model in which endowments and the dividends of an asset in fixed supply are exogenous and random; they follow a finite state Markov process. It is shown (i) that a stationary equilibrium exists and (ii) that no stationary allocation is Pareto superior to any stationary equilibrium allocation, where Pareto superiority is defined in terms of the conditional expected utilities agents maximize in the competitive setting. The main contribution of the paper is the optimality proof.

1 Introduction

In this paper, I describe and analyze a pure exchange overlapping generations model in which endowments and the dividends of an asset in fixed supply are exogeneous and random; they follow a finite state Markov process. It is shown (i) that a stationary equilibrium exists and (ii) that no stationary allocation is Pareto superior to any stationary equilibrium allocation, where Pareto superiority is defined in terms of the conditional expected utilities agents maximize in the competitive setting. (The implied optimality concept was labelled conditional Pareto optimality by Muench [2].) The main contribution of the paper is the optimality proof.

My model and analysis are closely related to Peled's [5]. His model is essentially a special case with zero dividends. Although I am able to appeal to an existence-of-equilibrium proof that he has presented, I cannot use his method of establishing optimality in the class of stationary allocations. According to his criterion, only a stochastic analogue of the golden rule allocation is optimal. In stochastic overlapping generations model with positive dividends, stationary equilibrium allocations are not golden rule allocations. Nevertheless, as I show, they are conditional Pareto optimal.

The rest of this paper proceeds as follows. I describe the physical environment in section 2. In section 3, I present the equilibrium conditions and, in section 4 I note that the existence of a stationary equilibrium follows from some of Peled's results. The optimality result is presented in section 5.

2 Physical Environment

The model is a discrete time, one-good, two-period overlapping generations model. The good is perishable and no storage technology is available. There is only one person in each generation. At each period $t \geq 1$, the aggregate endowment of the single good is $y_t + d_t$, where $y_t > 0$ is the endowment of the young person at period t and $d_t \geq 0$ is the dividends of the single asset in the model whose total supply is unity.

Preferences are identical for all individuals and are defined over lifetime consumption by $E_t\, U[c_1(t), c_2(t)]$, where $c_1(t)$ and $c_2(t)$ are the first and second period consumption by a person of generation t, respectively, and E_t is a mathematical expectation operator conditional on all available information at period t. The function $U[c_1, c_2]$ is continuously twice differentiable, strictly concave, and monotonically increasing. To ensure interior solutions to the individual utility maximization problem, we assume that the intertemporal marginal rate of substitution, $V[c_1, c_2] = U_1[c_1, c_2]/U_2[c_1, c_2]$, where $U_i[c_1, c_2]$ is the marginal utility of c_i, satisfies $V[c_1, c_2] \to \infty$ as $[c_1, c_2] \to [0, c]$, and $V[c_1, c_2] \to 0$ as $[c_1, c_2] \to [c, 0]$, for any $c > 0$.

We assume that endowments and dividends together are generated by the following finite ergodic Markov chain. Let S be a finite set of elements, $S = \{s_1, s_2, \ldots, s_N\}$, where all elements s_i are in $\mathbf{R}_+^2$, so that $s_i = \{\gamma_i, \delta_i\}$, $\gamma_i > 0$ and $\delta_i \geq 0$ for $i = 1, 2, \ldots, N$. At period t, $\{y_t, d_t\} \equiv s(t) \in S$ and $\text{Prob}\,\{s(t+1) = s_j \mid s(t) = s_i\} = \pi_{ij}$, where $\Sigma_j\, \pi_{ij} = 1$ for all i. The outcome s(t) is realized at period t, but prior to the appearance of the young generation in period t.

3 The Choice Problem and Equilibrium Condition

In this physical environment, there are two kinds of agents, one old and one young. The old prefers more consumption to less, derives no utility from holdings of the asset, and is assumed to own all of the asset at the first date. As a result, he (or she) will supply all asset holdings. The young at period t maximizes expected utility, $E_t\, U[c_1(t), c_2(t)]$, subject to

$$c_1(t) + p(t)\, a(t) \le y_t \tag{1}$$

$$c_2(t) \le \{p(t+1) + d_{t+1}\}\, a(t). \tag{2}$$

by choice of $(c_1(t), c_2(t), a(t))$, where $p(t)$ is the price of the asset at period t and $a(t)$ is the demand for the asset. As indicated by (2), the asset price $p(t)$ is an ex-dividends price at each date t. The young knows the current state of endowment, y_t, dividends, d_t, and the true transition probabilities at the time of his decision-making at period t.

The Kuhn-Tucker conditions for maximization for the choice problem of the young are (1) and (2) at equality and

$$- E_t\, U_1[c_1(t), c_2(t)]\, p(t) + E_t\, U_2[c_1(t), c_2(t)]\, \{p(t+1) + d_{t+1}\} \le 0, \tag{3}$$

which must hold equality if $a(t) > 0$.

In equilibrium, we require $a(t) = 1$ for all $t \ge 1$. Substituting this market clearing condition into the Kuhn-Tucker conditions for maximization gives the equilibrium conditions;

$$E_t\, U_1[c_1(t), c_2(t)]\, p(t) = E_t\, U_2[c_1(t), c_2(t)]\, \{p(t+1) + d_{t+1}\}, \tag{4}$$

$$c_1(t) + p(t) = y_t, \tag{5}$$

$$c_2(t) = p(t+1) + d_{t+1}. \tag{6}$$

4 The Existence of Stationary Equilibrium

We proceed by defining a stationary equilibrium and noting that one exists.

Definition 1. Let $p \equiv (p_1, p_2, \ldots, p_N) \in R_+^N$, $c_1 \equiv (c_{11}, c_{12}, \ldots, c_{1N}) \in R_+^N$ and $c_2 \equiv (c_{21}, c_{22}, \ldots, c_{2N}) \in R_+^N$. A **stationary equilibrium** is a vector (p, c_1, c_2) which for $i = 1, 2, \ldots, N$ satisfies

$$\Sigma_j\, \pi_{ij}\, U_1[c_{1i}, c_{2j}]\, p_i = \Sigma_j\, \pi_{ij}\, U_2[c_{1i}, c_{2j}]\, \{p_j + \delta_j\}, \tag{7}$$

$$c_{1i} + p_i = \gamma_i, \tag{8}$$

$$c_{2i} = p_i + \delta_i. \tag{9}$$

Note that p_i is the price of the asset at period t if $s(t) = s_i \equiv \{\gamma_i, \delta_i\}$, c_{1i} is the consumption of a person of generation t at period t if $s(t) = s_i$, and c_{2i} is the consumption of a person of generation t at period t+1 if $s(t+1) = s_i$.

By substituting (8) and (9) into (7), it follows that a stationary equilibrium price vector is a vector $p \in R_+^N$ that satisfies

$$\Sigma_j \pi_{ij} U_1[y_i - p_i, p_j + \delta_j]\, p_i - \Sigma_j \pi_{ij} U_2[y_i - p_i, p_j + \delta_j]\, \{p_j + \delta_j\} = 0,$$

for $i = 1, 2, \ldots, N$. (10)

In our particular set-up, there exists price vector.

Theorem 1. There exists $p \in R_{++}^N$ which solves (10).

Proof. Peled's proof in [3] (Appendix) applies, since our framework satisfies the assumptions of his proof. **Q.E.D.**

Peled's proof allows one to avoid the strong assumptions on preferences required if one attempts to prove Theorem 1 by applying Kakutani's fixed point theorem to the mapping from period t+1 price vectors to period t price vectors defined by (4)-(6). Without additional assumptions on preferences, that mapping is a nonconvex valued correspondence.

5 Conditional Pareto Optimality in the Class of Stationary Allocations

Consistent with our assumption that an agent at period t maximizes expected utility conditional on all information at period t, we use the following definition of Pareto superiority. [1]

Definition 2. The stationary allocation $\{c^*_{1i}, c^*_{2i}\}$ for $i = 1, 2, \ldots, N$ is **conditional Pareto superior** to the allocation $\{c_{1i}, c_{2i}\}$ for $i = 1, 2, \ldots, N$ if it satisfies the feasibility condition:

$$c^*_{1i} + c^*_{2i} \leq \gamma_i + \delta_i \qquad \text{for } i = 1, 2, \ldots, N, \tag{11}$$

and if

$$\Sigma_j \pi_{ij} U[c^*_{1i}, c^*_{2j}] \geq \Sigma_j \pi_{ij} U[c_{1i}, c_{2j}] \tag{12}$$

$$\Sigma_j \pi_{ij} U[\bar{c}_{1i}, c^*_{2j}] \geq \Sigma_j \pi_{ij} U[\bar{c}_{1i}, c_{2j}] \tag{13}$$

with strict inequality for some i in either (12) or (13), where $\bar{c}_{1i}$ is any given first period consumption of the members of generation 0.

Now, we prove that any stationary equilibrium is conditional Pareto optimal in this sense.

Theorem 2. There does not exist a stationary allocation which is conditional Pareto superior to the equilibrium.

Proof. Let the equilibrium allocation be given by $\{c_{1i}, c_{2i}\}$ for $i = 1, 2, \ldots, N$, and, by way of contradiction, let a proposed Pareto superior alternative allocation be $\{c^*_{1i}, c^*_{2i}\}$ for $i = 1, 2, \ldots, N$ which we call the "*" allocation from now on. Without loss of generality, we may assume that the "*" allocation satisfies feasibility, (11), with equality. Therefore, we can represent the difference between the equilibrium allocation and the "*" allocation by $(\Delta_1, \Delta_2, \ldots, \Delta_N)$ where $\Delta_i = c_{1i} - c^*_{1i} = c^*_{2i} - c_{2i}$. Also let $\Delta_{ij} = \Delta_j/\Delta_i$, $P_{ij} = (p_j+\delta_j)/p_i$, and let $e_{ij} = \Delta_{ij} - P_{ij}$. As a preliminary step, we establish the following properties of the e_{ij}.

Property 1: If $e_{is} > 0$ and $e_{sj} > 0$, then $e_{ij} > 0$.

Property 2: e_{is} and e_{si} both cannot be positive.

Since $\Delta_{ij} = (\Delta_{is})(\Delta_{sj})$, and since by hypothesis of Property 1, $\Delta_{is} > P_{is} > 0$ and $\Delta_{sj} > P_{sj} > 0$, we have: $\Delta_{ij} = \Delta_{is} \cdot \Delta_{sj} > P_{is} \cdot P_{sj}$. But $P_{is} \cdot P_{sj} = \{(p_s + \delta_s)/p_i\} \{(p_j + \delta_j)/p_s\} \geq \{(p_s + \delta_s)/p_i\} \{(p_j + \delta_j)/(p_s + \delta_s)\} = (p_j + \delta_j)/p_i = P_{ij}$. This proves Property 1. Now, since $e_{ii} = \Delta_{ii} - P_{ii} = 1 - (p_i - \delta_i)/p_i \leq 0$, this inequality contradicts $i = j$ which establishes Property 2.

Now, we can provide the rest of the argument. For the "*" allocation, there must exist i such that $\Delta_i > 0$. (Otherwise generation 0 is worse off, i.e. (13) is violated.) Then we have:

$$\Sigma_j \pi_{ij} U[c^*_{1i}, c^*_{2j}]$$

$$= \Sigma_j \pi_{ij} U[c_{1i} - \Delta_i, c_{2j} + \Delta_j]$$

$$= \Sigma_j \pi_{ij} U[c_{1i} - \Delta_i, c_{2j} + (\Delta_j/\Delta_i)\, \Delta_i]$$

$$\geq \Sigma_j \pi_{ij} U[c_{1i}, c_{2j}]$$

$$> \Sigma_j \pi_{ij} U[c_{1i} - \Delta_i, c_{2j} + \{(p_j+\delta_j)/p_i\}\, \Delta_i]. \quad (14)$$

Note that the first inequality follows from Pareto superiority, (12), while the second follows from revealed preference and strict concavity. In particular, the consumption bundle $[c_{1i} - \Delta_i, c_{2j}+\{(p_j+\delta_j)/p_i\} \Delta_i]$ for $j = 1, 2, \ldots, N$ was affordable given the equilibrium prices, but was not chosen. Instead, $\{c_{1i}, c_{2j}\}$ for $j = 1, 2, \ldots, N$ was chosen. It follows by strict concavity of the utility function that the latter is preferred to the former.

It follows from (14) that there exists a j such that $\Delta_j/\Delta_i > (p_j + \delta_j)/p_i$. This implies that Δ_j is positive, and that $e_{ij} > 0$ with $j \neq i$. (Obviously this inequality cannot hold for $j = i$.) Thus the existence of a state i with $\Delta_i > 0$ implies the existence of a state j, $j \neq i$ such that $e_{ij} > 0$ and $\Delta_j > 0$.

We can use this result to generate a sequence of states with positive Δ_j's and a corresponding sequence of positive e_{ij}'s. Because there are only N distinct states, there must be a state, say k, which is encountered at least twice in any subsequence of states of length N+1. The corresponding subsequence of e_{ij}'s of length N must contain a subsequence of the form, $\{e_{k\ell}, e_{\ell m}, e_{mn}, \ldots, e_{rs}, e_{sk}\}$. By construction, the second subscript of a term is the first subscript of the succeeding term and all terms are positive.

We now show using Properties 1 and 2 that this cannot be. Applying Property 1 to the first two terms of this sequence implies $e_{km} > 0$. Thus, the existence of that sequence implies the existence of a new sequence of the form $\{e_{km}, e_{mn}, \ldots, e_{rs}, e_{sk}\}$ where all terms are the same as those in the first one except that the first two terms have been replaced by $e_{km} > 0$. Repeating this argument no more than N-1 times, we end up with a sequence $\{e_{ks}, e_{sk}\}$ where $e_{ks} > 0$ and $e_{sk} > 0$. This contradicts Property 2. Q.E.D.

Note, finally, that since an economy with fixed stock of fiat money is the special case of our model with all dividends set at zero and since the optimality proof does not use the assumption that dividends are strictly positive, our result also applies to a monetary economy.[2]

References

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Footnotes

[1] See Muench [2], Peled [4], and Cass and Shell [1] for a discussion of this concept.

[2] Peled [4] has shown the conditional optimality of competitive equilibrium with a fixed stock of fiat money when stochastic endowments follow a Markov process.