# EQUILIBRIA IN ABSTRACT ECONOMIES WITH A MEASURE SPACE OF AGENTS AND WITH AN INFINITE DIMENSIONAL STRATEGY SPACE

by

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ABSTRACT

The existence of an equilibrium for an abstract economy with a measure space of agents and with an infinite dimensional strategy space is proved. Agent's preferences need not be ordered, i.e., need not be transitive or complete, and therefore need not be representable by utility functions. The proof which follows closely the arguments in Yannelis-Prabhakar [26] is based on a Caratheodory-type selection theorem.

## 1. INTRODUCTION

The concept of Nash equilibrium, originated by Nash [21], has been extensively used in game theory and by extension in equilibrium analysis. In particular, Debreu [6] proved the existence of an equilibrium for an abstract economy (a generalized game) which contained the Nash equilibrium concept. The Debreu result was the main technical tool used to prove the existence of a competitive equilibrium for an economy (see Arrow-Debreu [1]). In the mid-seventies two major extensions of the Nash and Debreu equilibrium results were obtained. The first was Schmeidler's [23] generalization of Nash's theorem to games with a measure space of agents. The second was the Shafer-Sonnenschein [24] generalization of Debreu's [6] theorem to preference correspondences which need not be ordered, i.e., need not be transitive or complete, and therefore might not be representable by utility functions.[1]

Subsequent extensions of the work of Shafer-Sonnenschein [24] were made by Borglin-Keiding [4] and, of the work of Schmeidler [23], by Khan-Vohra [16].

A common characteristic of this later work was the finite dimensionality of the strategy space. However, recent contributions to equilibrium existence results have extended the work of Debreu [6], Shafer-Sonnenschein [24] and Borglin-Keiding [4] to infinite dimensional strategy spaces. In particular, the theorems of Debreu, Shafer-Sonnenschein, and Borglin-Keiding were generalized by Toussaint [25] and Yannelis-Prabhakar [26,27] to infinite dimensional strategy spaces and to any finite or infinite set of agents. However, in the works of Toussaint [25] and Yannelis-Prabhakar [26, 27] the set of agents does not have the structure of a measure space. Consequently, in this setting one cannot capture the meaning of either "negligible" agents or "average response strategy" as was done in Schmeidler [23] and subsequently in Khan [15], and Mas-Colell [20]. The purpose of the present paper is to simultaneously extend the results of Toussaint [25] and Yannelis-Prabhakar [26, 27] to a measure space of agents and the results of Khan-Vohra [16] and Schmeidler [23], to infinite dimensional strategy spaces. The main existence result in this paper was motivated by the recent attention given to economies with infinitely many commodities and a measure space of agents.

The equilibrium existence results for abstract economies discussed previously have been used to prove existence of competitive equilibria for either economies with finitely many agents and commodities (see for instance Arrow-Debreu [1] or Shafer-Sonnenschein [24]) or for economies with finitely many agents and infinitely many commodities (see for instance Toussaint [25]). We believe that our existence result, in addition to obtaining Nash equilibria in a very general setting, may serve as a useful mathematical tool for proving the existence of equilibrium in large square economies.

Finally we wish to comment on the technical contribution of the paper. Although the proof of our main result follows closely the arguments of Yannelis-Prabhakar [26, 27], the introduction of measurability assumptions forces us to overcome several nontrivial technical difficulties. In particular, we developed a new Caratheodory-type selection result which is given in Kim-Prikry-Yannelis [18].

The paper is organized as follows. Section 2 contains notation and definitions. Our main equilibrium existence theorem is stated in Section 3. Section 4 contains proofs of several Lemmata needed for the main existence result. The proof of the main existence theorem is given in Section 5. Finally, some concluding remarks are given in Section 6.

## 2. NOTATION AND DEFINITIONS

### 2.1 Notation

$2^A$ denotes the set of all subsets of the set A

$\mathbb{R}$ denotes the set of real numbers

conA denotes the convex hull of the set A

$c\ell A$ denotes the norm closure of the set A

$\setminus$ denotes the set theoretic subtraction

If $\phi : X \to 2^Y$ is a correspondence then $\phi|U : U \to 2^Y$ denotes the restriction of $\phi$ to U

### 2.2 Definitions

Let X, Y be two topological spaces. A correspondence $\phi : X \to 2^Y$ is said to be upper-semicontinuous (u.s.c.) if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y. The graph of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. The correspondence $\phi : X \to 2^Y$ is said to have a closed graph if the set $G_\phi$ is closed in $X \times Y$. A correspondence $\phi : X \to 2^Y$ is said to have open lower sections if for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X. Let $(T,\tau,\mu)$ be a complete finite measure space, i.e., $\mu$ is a real-valued, non-negative, countably additive measure defined in a complete $\sigma$-field $\tau$ of subsets of T such that $\mu(T) < \infty$. Let now X be a Banach space. $L_1(\mu,X)$ denotes the space of equivalence classes of X-valued Bochner integrable functions $f : T \to X$ normed by

$$\|f\| = \int_T \|f(t)\| d\mu(t).$$

A correspondence $\phi : T \to 2^X$ is said to be integrably bounded if there exists a map $g \in L_1(\mu)$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \phi(t)\} \leqslant g(t)$. The correspondence $\phi : T \to 2^X$ is said to have a measurable graph if $G_\phi \in \tau \otimes \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes Borel $\sigma$-algebra on $X$ and $\otimes$ denotes $\sigma$-product field. A Banach space $X$ has the Radon-Nikodym property with respect to $(T,\tau,\mu)$ if for each $\mu$-continuous vector measure $G : \tau \to X$ of bounded variation there exists $g \in L_1(\mu,X)$ such that $G(E) = \int_E g d\mu$ for all $E \in \tau$.

Let $X$ be a topological space and $Y$ be a linear topological space. Let $\phi : X \to 2^Y$ be a nonempty valued correspondence. A function $f : X \to Y$ is said to be a continuous selection from $\phi$ if $f(x) \in \phi(x)$ for all $x \in X$, and $f$ is continuous. Let $T$ be an arbitrary measure space. Let $\psi : T \to 2^Y$ be a nonempty valued correspondence. A function $f : T \to Y$ is said to be a measurable selection from $\psi$ if $f(t) \in \psi(t)$ for all $t \in T$, and $f$ is measurable. The above notions have been extensively used in the literature (see for instance Aumann [3] or Michael [21]). There is a growing literature on Caratheodory-type selections (see for instance Fryszkowski [11]). By this we mean the following. Let $Z$ be a topological space and $\phi : T \times Z \to 2^Y$ be a nonempty valued correspondence. A function $f : T \times Z \to Y$ is said to be a Caratheodory-type selection from $\phi$ if $f(t,z) \in \phi(t,z)$ for all $(t,z) \in T \times Z$ and $f(\cdot,z)$ is measurable for all $z \in Z$ and $f(t,\cdot)$ is continuous for all $t \in T$.

## 3. THE MAIN EQUILIBRIUM EXISTENCE THEOREM

### 3.1 Abstract Economies and Equilibrium

Let $(T,\tau,\mu)$ be a finite, positive, complete measure space. Let Y be a separable Banach space whose dual possesses the Radon-Nikodym property. For any correspondence $X : T \to 2^Y$, $L_1(\mu,X)$ will denote the subset of $L_1(\mu,Y)$ considering of those $x \in L_1(\mu,Y)$ which satisfy $x(t) \in X(t)$ for almost all t in T. Following the setting of Borglin-Keiding [4] and Yannelis-Prabhakar [26], we define an abstract economy in the usual way.

An abstract economy $\Gamma$ is a quadruple $[(T,\tau,\mu), X,P,A]$, where

(1) $(T,\tau,\mu)$ is a measure space of agents;

(2) $X : T \to 2^Y$ is a strategy correspondence;

(3) $P : T \times L_1(\mu,X) \to 2^Y$ is a preference correspondence such that $P(t,x) \subset X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$;

(4) $A : T \times L_1(\mu,X) \to 2^Y$ is a constraint correspondence such that $A(t,x) \subset X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$.

Notice that since P is a mapping from $T \times L_1(\mu,X)$ to $2^Y$, we have allowed for interdependent preferences. The interpretation of these preference correspondences is that $y \in P(t,x)$ means that agent t strictly prefers $y(t)$ to $x(t)$ if the given strategies of other agents are fixed. Notice that preferences need not be transitive or complete and therefore need not be representable by utility functions. However, it will be assumed that $x(t) \notin \mathrm{con}P(t,x)$ for all $x \in L_1(\mu,X)$ and for almost all t in T, which implies that $x(t) \notin P(t,x)$ for all $x \in L_1(\mu,X)$ and almost all t in T, i.e., $P(t,\cdot)$ is irreflexive for almost all t in T.

An <u>equilibrium</u>[2] for $\Gamma$ is an $x^* \in L_1(\mu,X)$ such that for almost all t in T the following conditions are satisfied:

(i) $x^*(t) \in c\ell A(t,x^*)$, and

(ii) $P(t,x^*) \cap c\ell A(t,x^*) = \phi$.

3.2 <u>The Main Existence Theorem</u>

We can now state the assumptions needed for the proof of the main theorem.

(A.1) $(T,\tau,\mu)$ is a finite, positive, complete, separable measure space.

(A.2) $X : T \to 2^Y$ is an integrably bounded correspondence with measurable graph such that for all $t \in T$, $X(t)$ is a nonempty, convex and weakly compact subset of Y, where Y is a separable Banach space whose dual possesses the Radon-Nikodym property.

(A.3) $A : T \times L_1(\mu,X) \to 2^Y$ is a correspondence such that:

(a) $\{(t,x,y) \in T \times L_1(\mu,X) \times Y : y \in A(t,x)\} \in \tau \otimes \mathcal{B}_w(L_1(\mu,X)) \otimes \mathcal{B}(Y)$ where $\mathcal{B}_w(L_1(\mu,X))$ is the Borel $\sigma$-algebra for the weak topology on $L_1(\mu,X)$ and $\mathcal{B}(Y)$ is the Borel $\sigma$-algebra for the norm topology on Y;

(b) it has weakly open lower sections, i.e., for each $t \in T$ and for each $y \in Y$, the set $A^{-1}(t,y) = \{x \in L_1(\mu,X) : y \in A(t,x)\}$ is weakly open in $L_1(\mu,X)$ ;

(c) for all $(t,x) \in T \times L_1(\mu,X)$, $A(t,x)$ is convex and has a nonempty norm interior in $X(t)$;

(d) for each $t \in T$, the correspondence $\bar{A}(t,\cdot) : L_1(\mu,X) \to 2^Y$, defined by $\bar{A}(t,x) = c\ell\, A(t,x)$ for all $(t,x) \in T \times L_1(\mu,X)$, is u.s.c. in the sense that the set $\{x \in L_1(\mu,X) : \bar{A}(t,x) \subset V\}$ is weakly open in $L_1(\mu,X)$ for every norm open subset V of Y.

(A.4) $P : T \times L_1(\mu,X) \to 2^Y$ is a correspondence such that:

(a) $\{(t,x,y) \in T \times L_1(\mu,X) \times Y : y \in \text{con}P(t,x)\} \in \tau \otimes \mathcal{B}_w(L_1(\mu,X)) \otimes \mathcal{B}(Y)$;

(b) it has weakly open lower sections, i.e., for each $t \in T$ and each $y \in Y$, $P^{-1}(t,y) = \{x \in L_1(\mu,X) : y \in P(t,x)\}$ is weakly open in $L_1(\mu,X)$;

(c) for all $(t,x) \in T \times L_1(\mu,X)$, $P(t,x)$ is norm open in $X(t)$;

(d) $x(t) \notin \text{con}P(t,x)$ for all $x \in L_1(\mu,X)$ and for almost all $t$ in $T$.

Below we state our main existence result.

Theorem 3.1: Let $\Gamma = [(T,\tau,\mu), X, P, A]$ be an abstract economy satisfying (A.1) - (A.4). Then $\Gamma$ has an equilibrium.

### 3.3 Discussion of the Assumptions

All the above assumptions are the standard ones needed for equilibrium in abstract economies (compare them with those in Borglin-Keiding [4], Toussaint [25], and Yannelis-Prabhakar [26]). However, as in any work with a measure space of agents (see for instance Aumann [2] or Khan-Vohra [16]) one must introduce several measurability assumptions. Of course all the measurability assumptions are quite natural and constitute no real economic restriction. Perhaps the assumptions which are worth discussing are (A.4)b. and c. In particular, having the weak topology on $L_1(\mu,X)$, which is the set of all joint strategies, signifies a natural form of myopic behavior on the part of the agents. Namely, an agent has to arrive at his decisions on the basis of knowledge of only finitely many (average) numerical characteristics of the joint strategies. However, there is no a priori upper bound on how many of these (average) numerical characteristics of the joint strategies an agent might seek in order to arrive at his decision. On the other hand, since each agents' strategy set is endowed with the norm topology this may be interpreted as signifying a very high degree of ability to discriminate between his own options. Of course, the agents' decisions depend on both of these observations, i.e., the ones of joint strategies made in the sense of the weak topology, as well as of his own options made with reference to the norm topology.

Although our choice of the weak topology on $L_1(\mu,X)$ was dictated by mathematical considerations (this is the only setting in which we are able to obtain a positive result), this setting seems to be more realistic than the one with a norm topology on $L_1(\mu,X)$. This latter setting would correspond to an extremely high degree of knowledge of the joint strategy on the part of each individual agent. However, in this latter setting by means of a counterexample we show that one cannot expect an equilibrium to exist.

### 3.4 <u>An Example of Non-Existence of Equilibrium</u>

As it was remarked in Section 3.3, having the weak topology on $L_1(\mu,X)$ was the only setting in which we were able to obtain a positive result. We now show that if we relax (A.4)b. to the assumption that P has norm open lower sections then our existence result fails.

<u>Example 3.1</u>: Consider an abstract economy with one agent. Let $Y = \ell_2$, where $\ell_2$ is the space of square summable real sequences. Denote by $\ell_2^+$ the positive cone of $\ell_2$. Let the strategy set X be equal to $\{z \in \ell_2^+ : \|z\| \leqslant 1\}$. Obviously X is convex and weakly compact.

Let $x = (x_0, x_1, x_2, \ldots) \in X$, and let $f : X \to X$ be a norm continuous mapping which does not have the fixed point property (for instance, let $f(x) = (1-\|x\|, x_0, x_1, x_2, \ldots)$, then $f : X \to X$ is a norm continuous function and it can be easily seen that $x \neq f(x)$). Denote by $B(f(x), \frac{\|x-f(x)\|}{2})$ an open ball in

$\ell_2$, centered at f(x) with radius $\frac{\|x-f(x)\|}{2}$. For each $x \in X$, let the preference correspondence be $P(x) = B(f(x), \frac{\|x-f(x)\|}{2}) \cap X$. Now, it can be easily checked that P has norm open lower and upper sections, is convex valued and irreflexive. Define the constraint correspondence $A : X \to 2^X$ by $A(x) = X$. Observe that for all $x \in X$, $f(x) \in P(x)$, i.e., P has no maximal element in X. Hence, there is no equilibrium in this one person abstract economy.

## 4. LEMMATA

We begin with a Caratheodory-type selection result whose proof can be found in Kim-Prikry-Yannelis [18].

Selection Theorem: Let $(T,\tau,\mu)$ be a complete measure space and Y be a separable Banach space and Z be a complete, separable metric space. Let $X : T \to 2^Y$ be a correspondence with measurable graph and $\phi : T \times Z \to 2^Y$ be a convex valued correspondence (possibly empty) with measurable graph, such that:

(i) for each $t \in T$, $\phi(t,x) \subset X(t)$ for all $x \in Z$,

(ii) for each t, $\phi(t,\cdot)$ has open lower sections in Z, i.e., for each $t \in T$, and each $y \in Y$, $\phi^{-1}(t,y) = \{x \in Z : y \in \phi(t,x)\}$ is open in Z.

(iii) for each $(t,x) \in T \times Z$, $\phi(t,x)$ has a nonempty interior in X(t).

Let $U = \{(t,x) \in T \times Z : \phi(t,x) \neq \phi\}$ and for each $x \in Z$, let $U_x = \{t \in T : (t,x) \in U\}$ and for each $t \in T$, let $U^t = \{x \in Z : (t,x) \in U\}$. Then for each $x \in Z$, $U_x$ is a measurable set in T and there exists a Caratheodory-type selection from $\phi|U$, i.e., there exists a function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$ and for each $x \in Z$, $f(\cdot,x)$ is measurable on $U_x$ and for each $t \in T$, $f(t,\cdot)$ is continuous on $U^t$. Moreover, $f(\cdot,\cdot)$ is jointly measurable.

Lemma 4.1: Let $(T,\tau)$ be a measurable space, Z be an arbitrary topological space and $W_n$, n=1,2,... be correspondences from T into Z with measurable graphs. Then $\bigcap_{n=1}^{\infty} W_n(\cdot)$ has measurable graph.

Proof: See Castaing-Valadier [5, Theorem III.40].

<u>Lemma 4.2</u>: Let $(T,\tau,\mu)$ be a complete finite measure space, $X$ be a topological space and $Y$ be a complete separable metric space. Let $\phi : T \times X \to 2^Y$ be a correspondence such that for each $x \in X$, $\phi(\cdot,x)$ has a measurable graph in $T \times Y$. Define the correspondence $\psi : T \times X \to 2^Y$ by $\psi(t,x) = c\ell\phi(t,x)$ for all $(t,x) \in T \times X$. Then for each $x \in X$, $\psi(\cdot,x)$ has a measurable graph.

<u>Proof</u>: See Castaing-Valadier [5, Theorem III.40].

<u>Lemma 4.3</u>: Let $(T,\tau,\mu)$ be a complete finite measure space and $X$ and $Y$ be topological spaces. Let $U$ be a subset of $T \times X$ and for each $x \in X$, $U_x = \{t \in T : (t,x) \in U\}$ be a measurable set in $T$. Let $f : U \to Y$ be a function such that for each $x \in X$, $f(\cdot,x)$ is measurable on $U_x$. Let $\phi : T \times X \to 2^Y$ be a correspondence such that for each $x \in X$, $\phi(\cdot,x)$ has a measurable graph in $T \times Y$. Define the correspondence $\psi : T \times X \to 2^Y$ by

$$\psi(t,x) = \begin{cases} \{f(t,x)\} & \text{if } (t,x) \in U \\ \phi(t,x) & \text{if } (t,x) \notin U. \end{cases}$$

Then for each $x \in X$, $\psi(\cdot,x) : T \to 2^Y$ has a measurable graph in $T \times Y$.

<u>Proof</u>: For all $x \in X$, $G_{\psi(\cdot,x)} = \{(t,y) \in T \times Y : y \in \psi(t,x)\} = A \cup B$ where $A = \{(t,y) \in T \times Y : y \in f(t,x) \text{ and } t \in U_x\}$ and $B = \{(t,y) \in T \times Y : y \in \phi(t,x) \text{ and } t \notin U_x\}$. Since $A,B$ are in $\tau \otimes \mathcal{B}(Y)$, we conclude that $G_{\psi(\cdot,x)} = A \cup B \in \tau \otimes \mathcal{B}(Y)$, i.e., $\psi(\cdot,x)$ has a measurable graph in $T \times Y$. This completes the proof of the Lemma.

Lemma 4.4: Let X, Y be linear topological spaces and $\phi : X \to 2^Y$ be a correspondence with open lower sections. Define the correspondence $\psi : X \to 2^Y$ by $\psi(x) = \mathrm{con}\phi(x)$ for all $x \in X$. Then $\psi$ has open lower sections.

Proof: See Yannelis-Prabhakar [26, Lemma 5.1].

Lemma 4.5: Let X, Y be topological spaces and $U \subset X$ be open in X. Let $\phi : X \to 2^Y$ be an u.s.c. correspondence and $f : U \to Y$ be a continuous selection from $\phi|U$. Then the correspondence $\psi : X \to 2^Y$ defined by

$$\psi(x) = \begin{cases} \{f(x)\} & \text{if } x \in U \\ \phi(x) & \text{if } x \notin U \end{cases}$$

is u.s.c.

Proof: See Yannelis-Prabhakar [26, Lemma 6.1].

Lemma 4.6: (Aumann) Let $(T, \tau, \mu)$ be a complete measure space, Y be a complete, separable metric space, and $F : T \to 2^Y$ be a correspondence with measurable graph. Then there is a measurable function $f : T \to Y$ such that $f(t) \in F(t)$ for all $t \in T$.

Proof: See Aumann [3], or Hildenbrand [13].

Lemma 4.7: The weak topology of a weakly compact subset of a separable Banach space is a metric topology.

Proof: See Dunford-Schwartz [9, Theorem 3, p. 434].

Lemma 4.8: Let $(T,\tau,\mu)$ be a finite positive complete separable measure space, and $Y$ be a reflexive separable Banach space. Let $X : T \to 2^Y$ be an integrably bounded correspondence such that for all $t \in T$, $X(t)$ is a nonempty, convex and norm closed subset of $Y$.[3] Then $L_1(\mu,X)$ is weakly compact and metrizable.

Proof: From Diestel-Uhl [7, p. 101] we know that for a bounded subset $K$ of $L_1(\mu,Y)$ to be relatively weakly compact, it is sufficient that the set $K$ be uniformly integrable. Since $X(\cdot)$ is integrably bounded, $L_1(\mu,X)$ is bounded and uniformly integrable and therefore $L_1(\mu,X)$ is a relatively weakly compact subset of $L_1(\mu,Y)$. Since $L_1(\mu,X)$ is convex and norm closed, as a consequence of the separation theorem it is weakly closed. Therefore, $L_1(\mu,X)$ is a weakly compact subset of $L_1(\mu,Y)$. Since $L_1(\mu,X)$ is a weakly compact subset of $L_1(\mu,Y)$ which is a separable Bannach space, by Lemma 4.7, $L_1(\mu,X)$ is metrizable. This completes the proof of the Lemma.

Remark 4.1: The reflexivity assumption in Lemma 4.8 can be relaxed to the assumption that $Y$ is a separable Banach space whose dual has the Radon-Nikodym property (see Khan [15, Theorem 7.12]), provided that for all $t \in T$, $X(t)$ is weakly compact.

Lemma 4.9: Let $(T,\tau,\mu)$ be a complete finite measure space, $Y$ be a Banach space and $X : T \to 2^Y$ be an integrably bounded, nonempty valued correspondence. Let $\theta : T \times L_1(\mu,X) \to 2^Y$ be a correspondence such that $\theta(t,x) \subset X(t)$ for all $(t,x)$ and for each $x \in L_1(\mu,X)$, $\theta(\cdot,x)$ has a measurable graph, and for each $t \in T$, $\theta(t,\cdot) : L_1(\mu,X) \to 2^Y$ is u.s.c. in the sense that the set $\{x \in L_1(\mu,X) : \theta(t,x) \subset V\}$ is weakly open in $L_1(\mu,X)$ for every norm open subset $V$ of $Y$. Then the correspondence $F : L_1(\mu,X) \to 2^{L_1(\mu,X)}$ defined by

$$F(x) = \{y \in L_1(\mu,X) : \text{for almost all } t \in T,\ y(t) \in \theta(t,x)\}$$

is u.s.c. in the sense that the set $\{x \in L_1(\mu,X) : F(x) \subset V\}$ is weakly open in $L_1(\mu,X)$ for every norm open subset $V$ of $L_1(\mu,X)$.

Proof: Let $B$ and $\tilde{B}$ be the closed unit balls in $Y$ and $L_1(\mu,Y)$ respectively. Then clearly $L_1(\mu,B) \subset \mu(T) \cdot \tilde{B}$ where $L_1(\mu,B)$ denotes a subset of $L_1(\mu,Y)$ such that $f \in L_1(\mu,B)$ implies that $f(t) \in B$ for almost all $t$ in $T$. Let $\{x_n\}$ be a sequence converging weakly to $x$ and $\varepsilon > 0$ be given. Since for each $t \in T$, $\theta(t,\cdot)$ is u.s.c., we can find minimal $N_t$ such that

$$(4.1) \quad \theta(t,x_n) \subset \theta(t,x) + \frac{\varepsilon}{3\mu(T)} B \text{ for all } n \geqslant N_t$$

Let $\frac{\varepsilon}{3\mu(T)} = \delta_1$. For fixed $x$ and $n$, the correspondences $\theta(\cdot,x) : T \to 2^Y$ and $\theta(\cdot,x_n) : T \to 2^Y$ have measurable graphs. Indeed

$$\{t \in T : \theta(t,x_n) \subset \theta(t,x) + \delta_1 B\} = \{t \in T : \theta(t,x_n) \setminus \{\theta(t,x) + \delta_1 B\} \neq \phi\} =$$

$$\mathrm{proj}_T\{(t,y) \in T \times Y : (t,y) \in G_{\theta(\cdot,x_n)} \cap (G_{\theta(\cdot,x)} + \delta_1 B)^c\} \in \tau,$$

(where c denotes complement).

Since $\{t \in T : N_t = m\} = \bigcap_{n \geqslant m} \{t \in T : \theta(t,x_n) \subseteq \theta(t,x) + \delta_1 B\} \cap \{t \in T : \theta(t,x_{m-1})$

$\not\subseteq \theta(t,x) + \delta_1 B\} \in \tau$, $N_t$ is a measurable function of $t$.

Since $X(\cdot)$ is integrably bounded, there exists $g \in L_1(\mu,\mathbb{R})$ such that $\sup\{\|x\| : x \in X(t)\} \leqslant g(t)$. Pick $\delta_2$ such that if $\mu(A) < \delta_2$ $(A \subset T)$, then $\int_A g(t)dt < \frac{\varepsilon}{3}$. Choose $n_0$ such that $\mu(\{t \in T : N_t \geqslant n_0\}) < \delta_2$. Let $T_0 = \{t : N_t \geqslant n_0\}$. To complete the proof we must show $F(x_n) \subset F(x) + \varepsilon\tilde{B}$ for all $n \geqslant n_0$. Let, $n \geqslant n_0$. Choose $y \in F(x_n)$. We will show that $y \in F(x) + \varepsilon\tilde{B}$. Since $y \in F(x_n)$, $y(t) \in \theta(t,x_n)$ for almost all $t$ in $T$. For $t \in T_0$, let $z_1(\cdot)$ be a measurable selection from $\theta(\cdot,x)$. For $t \notin T_0$, $n > N_t$ since $n_0 > N_t$. Therefore, $y(t) \in \theta(t,x) + \delta_1 B$ by (4.1). Since the correspondences $\{y(\cdot)\} + \delta_1 B$ and $\theta(\cdot,x)$ have a measurable graph in $T \times Y$, we can select a measurable function $z_2 : T \to Y$ such that

$z_2(t) \in (\{y(t)\} + \delta_1 B) \cap \theta(t,x)$ for all $t \in T \setminus T_0$ Define $z : T \to Y$ by

$$z(t) = \begin{cases} z_1(t) & \text{if } t \in T_0 \\ z_2(t) & \text{if } t \in T \setminus T_0 \end{cases}$$

Then $z \in F(x)$ and

$$\begin{aligned} \|z-y\| &= \int_{T_0} \|z(t) - y(t)\| d\mu(t) + \int_{T \setminus T_0} \|z(t)-y(t)\| d\mu(t) \\ &< 2 \int_{T_0} g(t) d\mu(t) + \int_T \delta_1 d\mu(t) \\ &< \frac{2}{3} \varepsilon + \delta_1 \mu(T) \\ &= \frac{2}{3} \varepsilon + \frac{\varepsilon}{3\mu(T)} \cdot \mu(T) = \varepsilon. \end{aligned}$$

Since $z \in F(x)$ and $\|z-y\| < \varepsilon$, $y \in F(x) + \varepsilon\tilde{B}$ as it was to be shown. This completes the proof of the Lemma.

<u>Remark 4.2</u>: Notice that it follows directly from Lemma 4.9 that $F : L_1(\mu,X) \to 2^{L_1(\mu,X)}$ is weakly u.s.c., i.e., the set $\{x \in L_1(\mu,X) : F(x) \subset V\}$ is weakly open in $L_1(\mu,X)$ for every weakly open subset $V$ of $L_1(\mu,X)$.

## 5. PROOF OF MAIN EXISTENCE THEOREM

Define $\psi : T \times L_1(\mu,X) \to 2^Y$ by $\psi(t,x) = \mathrm{con}P(t,x)$ for all $(t,x) \in T \times L_1(\mu,X)$. Then by Lemma 4.4 for each $t \in T$, $\psi(t,\cdot)$ has weakly open lower sections, and it is norm open valued in $X(t)$. Define $\phi : T \times L_1(\mu,X) \to 2^Y$ by $\phi(t,x) = A(t,x) \cap \psi(t,x)$ for all $(t,x) \in T \times L_1(\mu,X)$. Then it can be easily checked that $\phi$ is convex valued, has a nonempty norm interior in $X(t)$ and for each $t \in T$, $\phi(t,\cdot)$ has weakly open lower sections. Moreover, by Lemma 4.1, $\phi$ has a measurable graph. Let $U = \{(t,x) \in T \times L_1(\mu,X) : \phi(t,x) \neq \emptyset\}$. For each $x \in L_1(\mu,X)$, let $U_x = \{t \in T : \phi(t,x) \neq \emptyset\}$ and for each $t \in T$, let $U^t = \{x \in L_1(\mu,X) : \phi(t,x) \neq \emptyset\}$. By the Caratheodory-type Selection Theorem, there exists a function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$, and for each $x \in L_1(\mu,X)$, $f(\cdot,x)$ is measurable on $U_x$ and for each $t \in T$, $f(t,\cdot)$ is continuous on $U^t$ where $L_1(\mu,X)$ is endowed with the weak topology and $Y$ with the norm topology. Moreover, for each $x \in L_1(\mu,X)$, $U_x$ is a measurable set. Define $\theta : T \times L_1(\mu,X) \to 2^Y$ by

$$\theta(t,x) = \begin{cases} \{f(t,x)\} & \text{if } (t,x) \in U \\ \\ c\ell A(t,x) & \text{if } (t,x) \notin U. \end{cases}$$

By Lemma 4.2, for each $x \in L_1(\mu,X)$, the correspondence $c\ell A(\cdot,x) : T \to 2^Y$ has a measurable graph. Therefore, by Lemma 4.3, for each $x \in L_1(\mu,X)$, $\theta(\cdot,x) : T \to 2^Y$ has a measurable graph. Notice that since for each $t \in T$, $\phi(t,\cdot)$ has weakly open lower sections, for each $t \in T$ the set $U^t$ is weakly open in $L_1(\mu,X)$. Consequently, by Lemma 4.5, $\theta(t,\cdot) : L_1(\mu,X) \to 2^Y$ is u.s.c. in the sense that the set $\{x \in L_1(\mu,X) : \theta(t,x) \subset V\}$ is weakly open in

$L_1(\mu,X)$ for every norm open subset V of Y. Moreover, $\theta$ is convex and nonempty valued. Define $F : L_1(\mu,X) \to 2^{L_1(\mu,X)}$ by $F(x) = \{y \in L_1(\mu,X)$ : for almost all t in T, $y(t) \in \theta(t,x)\}$. Since for each $x \in L_1(\mu,X)$, $\theta(\cdot,x)$ has a measurable graph, F is nonempty valued by Lemma 4.6.

Since $\theta$ is convex valued, so is F. By Lemma 4.9 and Remark 4.2, F is weakly u.s.c. Furthermore, since $X(\cdot)$ is integrably bounded and has a measurable graph, $L_1(\mu,X)$ is nonempty as a consequence of Aumann's measurable selection theorem; also since $X(\cdot)$ is convex valued, $L_1(\mu,X)$ is convex. By Lemma 4.8 and Remark 4.1, $L_1(\mu,X)$ is weakly compact. Therefore, by the Fan fixed point theorem (Fan [10, Theorem 1]), there exists $x^* \in L_1(\mu,X)$ such that $x^* \in F(x^*)$, i.e., $x^*(t) \in \theta(t,x^*)$ for almost all t in T. We now show that the fixed point is by construction an equilibrium for $\Gamma$. Suppose that for a non-null set of agents S, $(t,x^*) \in U$ for all $t \in S$. Then by the definition of $\theta$ $x^*(t) = f(t,x^*) \in \phi(t,x^*) \subset \text{con}P(t,x^*)$ for all $t \in S$, a contradiction to assumption (A.2)d. Therefore, $(t,x^*) \notin U$ for almost all t in T and so for almost all $t \in T$, $x^*(t) \in c\ell A(t,x^*)$ and $\phi(t,x^*) = A(t,x^*) \cap \text{con}P(t,x^*) = \phi$. But, $A(t,x^*) \cap \text{con}P(t,x^*) = \phi$ implies that $A(t,x) \cap P(t,x^*) = \phi$. Since by assumption (A.3)c, $P(t,x)$ is norm open in $X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$, the fact that $A(t,x^*) \cap P(t,x^*) = \phi$, implies that $c\ell A(t,x^*) \cap P(t,x^*) = \phi$, i.e., $x^*$ is an equilibrium for $\Gamma$. This completes the proof of the main existence theorem.

## 6. REMARKS

Remark 6.1: If the strategy space $Y = \mathbb{R}^{\ell}$, Theorem 3.1 can be used in a straightforward way to prove the existence of a competitive equilibrium for an economy with a measure space of agents, allowing for preferences which need not be ordered and may be interdependent (see Khan-Vohra [16, Theorem 3, p. 137] for a detailed argument). It should be noted that our continuity assumptions on preference correspondences are weaker than those in Khan-Vohra. Consequently, when Theorem 3.1 is applied to exchange economies with a finite dimensional commodity space, it provides a slight generalization of Theorem 3 in Khan-Vohra [16, p. 137]. Furthermore, it should be remarked that such a result provides an extension of Aumann's [2] existence result to economies with externalities in consumption.

Remark 6.2: Notice that the separability of the strategy space Y is crucial to the arguments for two reasons. First, it is used to obtain the Caratheodory-Type Selection theorem (see [17]) and second, is used in the proof of the equilibrium result to show that $L_1(\mu,X)$ is metrizable. We do not know whether or not the separability of Y can be relaxed either from the selection theorem or the main existence result.

Remark 6.3: In Theorem 3.1 it was assumed that Y*, i.e., the dual of Y possesses the Radon-Nikodym Property. This was needed to show that $L_1(\mu,X)$ is weakly compact (Lemma 4.8 and Remark 4.1). However, this assumption can be relaxed if one adds the following assumption in (A.2); for all $t \in T$, $X(t) \subset K$, where K is a relatively weakly compact subset of Y. Then using Diestel's Theorem (see Diestel [7, Theorem 2 and Remark]) one can conclude that $L_1(\mu,X)$ is weakly compact.

FOOTNOTES

1. The result of Shafer-Sonnenschein [24] was motivated by a competitive equilibrium existence result of Mas-Colell [19] for economies where agents' preferences were allowed to be non-ordered.
2. This notion of equilibrium is the same with that of Shafer-Sonnenschein [24]. Also it was used in Toussaint [25] and Yannelis-Prabhakar [26, 27].
3. Recall that it is a consequence of the separation theorem that the weak and norm topologies coincide on closed convex sets. Consequently X(t) is a weakly closed subset of Y as well.

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