ON A CARATHEODORY-TYPE SELECTION THEOREM

by

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ABSTRACT

We offer a new Caratheodory-type selection theorems. This result arose naturally from the authors' [9] study of equilibria in abstract economies (generalized games) with a measure space of agents.

1. INTRODUCTION

In a seminal paper, Nash [12] proved the existence of an equilibrium (or Nash-equilibrium), for an n-person noncooperative game. The concept of Nash equilibrium has been perhaps one of the most useful ideas in economic analysis. Specifically, Debreu [4] building on Nash's ideas, proved the existence of an equilibrium for an n-person noncooperative game (or abstract economy), which contained the Nash equilibrium concept. The result of Debreu was the main technical tool used to prove the existence of a competitive equilibrium for an economy. Several extensions of the Nash-Debreu existence results were made by Borglin-Keiding [2], Khan [7], Khan-Papageorgiou [8], Mas-Colell [10], Schmeidler [13], Shafer-Sonnenschein [14], Toussaint [15], and Yannelis-Prabhakar [16], [17].

Recently, the authors [9], motivated by the above work, attempted to extend the previous results to abstract economies with a measure space of agents and with an infinite dimensional strategy space. In the process of extending an argument of Yannelis-Prabhakar [16], which is based on a continuous selection result of a Michael-type, we encountered the following problem.

Let T be a measure space, Z be a complete separable metric space and Y be a separable Banach space. Let $\phi : T \times Z \to 2^Y$ be a convex valued (possibly empty) correspondence. Let $U = \{(t,x) \in T \times Z : \phi(t,x) \neq \emptyset\}$. Under appropriate assumptions, does there exist a <u>Caratheodory-type Selection</u> from the correspondence $\phi : U \to 2^Y$, i.e., a function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all (t,x) and $f(\cdot,x)$ is measurable for each x and $f(t,\cdot)$ is continuous for each t.

Although Fryszkowski [5] has proved a Caratheodory-type Selection theorem, neither his result, nor his arguments can be used to answer the above question, for two main reasons. First, our correspondence ϕ is not assumed to be closed valued, an assumption which is crucial to his proofs, and second, the domain of our selection is restricted to U which is not a product space. Therefore, to provide an answer to the above question, a different approach seems to be needed. We have applied the Caratheodory-type Selection theorem in [9] to resolve the problem of existence of equilibrium in an abstract economy with a measure space of agents and with an infinite dimensional strategy space.

The paper is organized in the following way. Section 2 contains definitions and notation. The main result is stated in Section 3. Section 4 contains several Lemmata needed for the proof of the main result. Finally, the proof of the main theorem is given in Section 5.

2. NOTATION AND DEFINITIONS

2.1 Notation

2^A denotes the set of all subsets of the set A

$c\ell A$ denotes the closure of the set A

$\setminus$ denotes the set theoretic subtraction

If $\phi : X \to 2^Y$ is a correspondence then $\phi|U : U \to 2^Y$ denotes the restriction of ϕ to U

diam denotes diameter

dist denotes distance

proj denotes projection.

2.2 Definitions

Let X, Y be two topological spaces. The graph of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. The correspondence $\phi : X \to 2^Y$ is said to have a closed graph if the set G_ϕ is closed in $X \times Y$. A correspondence $\phi : X \to 2^Y$ is said to have open lower sections if for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X. An open cover of a topological space X is a collection $U = \{u_a : a \in A\}$ of open subsets of X whose union is X, i.e., $\bigcup_{a \in A} u_a = X$. An open cover $U = \{u_a : a \in A\}$ is locally finite if every $x \in X$ has a neighborhood intersecting only finitely many $u \in U$. Let (T,τ,μ) be a complete finite measure space, i.e., μ is a real-valued, non-negative, countably additive measure defined in a complete σ-field τ of subsets of T such that $\mu(T) < \infty$. The correspondence $\phi : T \to 2^X$ is said to have a measurable graph if $G_\phi \in \tau \otimes \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes Borel σ-algebra on X and $\otimes$ denotes σ-product field.

3. MAIN RESULT

Let X be a topological space and Y be a linear topological space. Let $\phi : X \to 2^Y$ be a nonempty valued correspondence. A function $f : X \to Y$ is said to be a continuous selection from ϕ if $f(x) \in \phi(x)$ for all $x \in X$, and f is continuous. Let T be an arbitrary measure space. Let $\psi : T \to 2^Y$ be a nonempty valued correspondence. A function $f : T \to Y$ is said to be a measurable selection from ψ if $f(t) \in \psi(t)$ for all $t \in T$, and f is measurable. The above notions have been extensively used in the literature, (see for instance Aumann [1] or Michael [11]). There is a growing literature on Caratheodory-type selections (see for instance Fryszkowski [5]). By this we mean the following. Let Z be a topological space and $\phi : T \times Z \to 2^Y$ be a nonempty valued correspondence. A function $f : T \times Z \to Y$ is said to be a Caratheodory-type selection from ϕ if $f(t,z) \in \phi(t,z)$ for all $(t,z) \in T \times Z$ and $f(\cdot,z)$ is measurable for all $z \in Z$ and $f(t,\cdot)$ is continuous for all $t \in T$.

Below we state our Caratheodory-type selection theorem.

Selection Theorem: Let (T,τ,μ) be a complete measure space, Y be a separable Banach space and Z be a complete separable metric space. Let $X : T \to 2^Y$ be a correspondence with measurable graph, and $\phi : T \times Z \to 2^Y$ be a convex valued correspondence (possibly empty) with measurable graph, such that:

(i) for each $t \in T$, $\phi(t,x) \subset X(t)$ for all $x \in Z$.

(ii) for each t, $\phi(t,\cdot)$ has open lower sections in Z, i.e., for each $t \in T$, and each $y \in Y$, $\phi^{-1}(t,y) = \{x \in Z : y \in \phi(t,x)\}$ is open in Z.

(iii) for each $(t,x) \in T \times Z$, $\phi(t,x)$ has a nonempty interior in $X(t)$.

Let $U = \{(t,x) \in T \times Z : \phi(t,x) \neq \emptyset\}$ and for each $x \in Z$, $U_x = \{t \in T : (t,x) \in U\}$ and for each $t \in T$, $U^t = \{x \in Z : (t,x) \in U\}$. Then for each $x \in Z$ U_x is a measurable set in T and there exists a Caratheodory-type selection from $\phi|_U$, i.e., there exists a function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$ and for each $x \in Z$, $f(\cdot,x)$ is measurable on U_x and for each $t \in T$, $f(t,\cdot)$ is continuous on U^t. Moreover, $f(\cdot,\cdot)$ is jointly measurable.

Before we prove our Selection Theorem we will need some Lemmata.

4. LEMMATA

Lemma 4.1: (Aumann). If (T,τ,μ) is a complete finite measure space, Y is a complete, separable metric space, and $F : T \to 2^Y$ is a correspondence with measurable graph, then there is a measurable function $f : T \to Y$ such that $f(t) \in F(t)$ for all $t \in T$.

Proof: See Aumann [1].

Lemma 4.2: (Projection Theorem). Let (T,τ,μ) be a complete measure space and Y be a complete, separable metric space. If G belongs to $\tau \otimes \mathcal{B}(Y)$, its projection $\mathrm{proj}_T(G)$ belongs to τ.

Proof: See Theorem III.23 in Castaing-Valadier [3].

Lemma 4.3: Let (T,τ,μ) be a complete measure space, and Y be a complete separable metric space. Let $X : T \to 2^Y$ be a correspondence with measurable graph. Then there exist $\{f_k : k = 1,2,...\}$ such that:

(i) for all k, f_k is a measurable function from T into Y,

(ii) for each $t \in T$, $\{f_k(t) : k = 1,2,...\}$ is a dense subset of $X(t)$.

Proof: For each $n = 1, 2,...,$ let $\{E_\ell^n : \ell = 1,2,...\}$ be an open cover of Y such that $\mathrm{diam}(E_\ell^n) < \frac{1}{2^n}$. For each $n, \ell = 1,2,...,$ define $T_\ell^n = \{t \in T : X(t) \cap c\ell E_\ell^n \neq \phi\}$. Since $T_\ell^n = \mathrm{proj}_T \{(t,y) \in T \times Y : y \in X(t) \cap c\ell E_\ell^n\}$ and $X(\cdot) \cap c\ell E_\ell^n$ has a measurable graph in $T \times Y$, $T_\ell^n \in \tau$ by Lemma 4.2. It can be easily checked that $\bigcup_{\ell=1}^{\infty} T_\ell^n = T$.

For each $n, \ell = 1,2,...,$ define the correspondence $X_\ell^n : T \to 2^Y$ by

$$X_\ell^n(t) = \begin{cases} X(t) \cap c\ell E_\ell^n & \text{if } t \in T_\ell^n \\ X(t) & \text{if } t \notin T_\ell^n . \end{cases}$$

Since graph of $X_\ell^n = \{(t,y) \in T_\ell^n \times Y : y \in X(t) \cap c\ell E_\ell^n\} \cup \{(t,y) \in T \setminus T_\ell^n \times Y : y \in X(t)\}$, the correspondence X_ℓ^n has a measurable graph. By Lemma 4.1, for each n, ℓ = 1, 2,..., there exists a measurable function $f_\ell^n : T \to Y$ such that $f_\ell^n(t) \in X_\ell^n(t)$ for all $t \in T$. Fix t in T. Let $y \in X(t)$. Since for each n, $\{E_\ell^n : \ell = 1,2,\ldots\}$ is an open cover of Y, for each n, there is some ℓ such that $y \in X(t) \cap c\ell E_\ell^n$. Therefore $\{f_\ell^n(t) : n, \ell = 1,2,\ldots\}$ is dense in X(t). Changing index, $\{f_k : k = 1,2,\ldots\}$ are measurable functions from T into Y and $\{f_k(t) : k = 1,2,\ldots\}$ is dense in X(t) for each $t \in T$. This completes the proof of the lemma.

Lemma 4.4: Let $(S_i, \mathcal{Q}_i)$ for i = 1,2 be measurable spaces, $h: S_1 \to S_2$ be a measurable function and $A \in \mathcal{Q}_1 \otimes \mathcal{Q}_2$. Then

$$\mathrm{Proj}_{S_1}(G_h \cap A) \in \mathcal{Q}_1.$$

Proof: (a) If $A = A_1 \times A_2$ where $A_i \in \mathcal{Q}_i$, i = 1, 2, then $\mathrm{Proj}_{S_1}(G_h \cap A) = A_1 \cap h^{-1}(A_2) \in \mathcal{Q}_1$.

(b) If $\mathrm{Proj}_{S_1}(G_h \cap A) \in \mathcal{Q}_1$, then $\mathrm{Proj}_{S_1}(G_h \cap A^c) \in \mathcal{Q}_1$, where $A^c = S_1 \times S_2 \setminus A$. For, $\mathrm{Proj}_{S_1}(G_h \cap A^c) = S_1 \setminus \mathrm{Proj}_{S_1}(G_h \cap A)$.

(c) If $\mathrm{Proj}_{S_1}(G_h \cap A_n) \in \mathcal{Q}_1$ for all n = 1, 2,..., then $\mathrm{Proj}_{S_1}(G_h \cap (\bigcup_{n=1}^{\infty} A_n)) \in \mathcal{Q}_1$. For, $\mathrm{Proj}_{S_1}(G_h \cap (\bigcup_{n=1}^{\infty} A_n)) = \bigcup_{n=1}^{\infty} \mathrm{Proj}_{S_1}(G_h \cap A_n)$.

Therefore, $\mathrm{Proj}_{S_1}(G_h \cap A) \in \mathcal{Q}_1$ for all $A \in \mathcal{Q}_1 \otimes \mathcal{Q}_2$.

Lemma 4.5: Let (T_i, τ_i) for i = 1,2,3 be measurable spaces, $y : T_1 \to T_3$ be a measurable function and $\phi : T_1 \times T_2 \to 2^{T_3}$ be a correspondence with measurable graph, i.e., $G_\phi \in \tau_1 \otimes \tau_2 \otimes \tau_3$. Let $W : T_1 \to 2^{T_2}$ be defined by

$$W(t) = \{x \in T_2 : y(t) \in \phi(t,x)\}.$$

Then W has a measurable graph, i.e., $G_W \in \tau_1 \otimes \tau_2$.

Proof: Define $h : T_1 \times T_2 \to T_3$ by

$h(t,x) = y(t)$ for all $t \in T_1$ and $x \in T_2$. Let $S_1 = T_1 \times T_2$, $\mathcal{Q}_1 = \tau_1 \otimes \tau_2$, $S_2 = T_3$, $\mathcal{Q}_2 = \tau_3$ and $A = G_\phi$. Then $h : S_1 \to S_2$ is a measurable function and $A \in \mathcal{Q}_1 \otimes \mathcal{Q}_2$. So, by Lemma 4.4,

$$G_W = \{(t,x) : (t,x,h(t,x)) \in A\} \in \mathcal{Q}_1 = \tau_1 \otimes \tau_2.$$

Lemma 4.6: Let (T,τ) be a complete measure space, Z be an arbitrary topological space and W_n, $n = 1,2,\ldots$ be correspondences from T into Z with measurable graphs. Then the correspondences $\bigcup_n W_n(\cdot)$, $\bigcap_n W_n(\cdot)$ and $Z \setminus W_n(\cdot)$ have measurable graphs.

Proof: See Theorem III.40 in Castaing-Valadier [3].

Lemma 4.7: Let (T,τ) be a measurable space, Z be a separable metric space and $W : T \to 2^Z$ be a correspondence with measurable graph. Then for every $x \in Z$, $\mathrm{dist}(x,W(\cdot))$ is a measurable function.

Proof: See Theorem III.9 in Castaing-Valadier [3].

Lemma 4.8: Let (T,τ) be a complete measure space, Z be a complete, separable metric space, and $W : T \to 2^Z$ be a correspondence with measurable graph. Then the correspondence $V : T \to 2^Z$ defined by

$$V(t) = \{x \in Z : \mathrm{dist}(x,W(t)) \geqslant \lambda\}, \text{ (where } \lambda \text{ is any real number)}$$

has a measurable graph.

<u>Proof</u>: See Lemma 4.7 above and Theorem 6.4 in Himmelberg [6].

<u>Lemma 4.9</u>: Let (T,τ) be a measurable space, Z be a separable metrizable space, Y be a metrizable space and $f : T \times Z \to Y$ be a function which is measurable in $t \in T$ and continuous in $x \in Z$. Then f is jointly measurable.

<u>Proof</u>: See Lemma III.14 in Castaing-Valadier [3].

5. PROOF OF MAIN THEOREM

Let $\phi_x(t) \equiv \phi(t,x)$ for all $x \in Z$. Notice that for each $x \in Z$, $\phi_x(\cdot)$ has a measurable graph in $T \times Y$. Observe that

$$U_x = \{t \in T : \phi_x(t) \neq \emptyset\}$$

$$= \mathrm{proj}_T \{(t,y) \in T \times Y : (t,y) \in G_{\phi_x}\}.$$

By Lemma 4.2, $U_x \in \tau$. By Lemma 4.3 there exist measurable functions $\{y_n(\cdot) : n= 1,2,\ldots\}$ such that for each t, $\{y_n(t)\}$ is a countable dense subset of $X(t)$. For each $t \in T$, let $W_n(t) = \{x \in Z : y_n(t) \in \phi(t,x)\}$. By assumption (ii) $W_n(t)$ is open in Z. Since for each $(t,x) \in T \times Z$, $\phi(t,x)$ has a nonempty interior in $X(t)$ and $\{y_n(t) : n = 1,2,\ldots\}$ is dense in $X(t)$, $\{W_n(t) : n = 1,2,\ldots\}$ is a cover of the set $\{x \in Z : (t,x) \in U\}$. By Lemma 4.5, $W_n(\cdot)$ has a measurable graph. For each $m = 1,2,\ldots$ define the operator $(\ )_m$ by

$$(W)_m = \{w \in W : \mathrm{dist}(w, Z \setminus W) \geqslant \frac{1}{2^m}\}.$$

For $n=1,2,\ldots$ and t in T let $V_n(t) = W_n(t) \setminus \bigcup_{k=1}^{n-1} (W_k(t))_n$. Obviously, $V_n(t)$ is open in Z. It can be easily checked that $\{V_n(t) : n = 1,2,\ldots\}$ is a locally finite open cover of the set $\{x \in Z : (t,x) \in U\}$. Since $W_n(\cdot)$ has a measurable graph, $V_n(\cdot)$ has a measurable graph by Lemmas 4.6 and 4.8. Let $\{g_n(t,x) : n = 1,2,\ldots\}$ be the partition of unity subordinated to the open cover $\{V_n(t) : n = 1,2,\ldots\}$, for instance, for each $n = 1,2,\ldots$, let

$$g_n(t,x) = \frac{\mathrm{dist}(x, Z \setminus V_n(t))}{\sum_{k=1}^{\infty} \mathrm{dist}(x, Z \setminus V_k(t))}.$$

Then $\{g_n(t,\cdot) : n = 1,2,\ldots\}$ is a family of continuous functions $g_n :$ $U \to [0,1]$ such that $g_n(t,x) = 0$ for $x \notin V_n(t)$ and $\sum_{n=1}^{\infty} g_n(t,x) = 1$ for all $(t,x) \in U$. Define $f : U \to Y$ by $f(t,x) = \sum_{n=1}^{\infty} g_n(t,x)y_n(t)$. Since $\{V_n(t) : n=1,2,\ldots\}$ is locally finite, each x has a neighborhood N_x which intersects only finitely many $V_n(t)$. Hence, $f(t,\cdot)$ is a finite sum of continuous functions on N_x and it is therefore continuous on N_x. Consequently, $f(t,\cdot)$ is continuous. Furthermore, for any n such that $g_n(t,x) > 0, x \in V_n(t) \subset W_n(t) = \{z \in Z : y_n(t) \in \phi(t,z)\}$, i.e., $y_n(t) \in \phi(t,x)$. So $f(t,x)$ is a convex combination of elements $y_n(t)$ from the convex set $\phi(t,x)$. Consequently, $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$. Since $V_n(\cdot)$ has a measurable graph, $\mathrm{dist}(x, Z \setminus V_n(\cdot))$ is a measurable function by Lemmas 4.6 and 4.7. Therefore for each n and x, $g_n(\cdot,x)$ is a measurable function. Since for each $n, y_n(\cdot)$ is a measurable function, it follows that $f(\cdot,x)$ is measurable for each x, i.e., $f(t,x)$ is a Caratheodory-type selection from $\phi|_U$. To complete the proof we must show that $f(\cdot,\cdot)$ is jointly measurable. But this is a consequence of Lemma 4.9. The proof of the Selection Theorem is now complete.

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