EQUILIBRIA IN NONCOOPERATIVE MODELS OF COMPETITION

by

Nicholas C. Yannelis

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

ABSTRACT

An equilibrium in a game theoretic setting a la Debreu [8] and Shafer-Sonnenschein [33] with broader structure is proved. In particular, our framework is general enough to encompass both the Aumann [2,3] economy of perfect competition and the nonordered preferences setting of Mas-Colell [25]. Moreover, since the dimensionality of the strategy space may be infinite it contains Bewley-type [5] results and may be useful in obtaining existence results for economies with a measure space of agents and infinitely many commodities.

1. INTRODUCTION

The classical model of exchange under perfect competition is the Arrow-Debreu-McKenzie model. The existence of an equilibrium for this model was proved in Arrow-Debreu [1] and McKenzie [23]. The heart of the proof of the Arrow-Debreu equilibrium result is an equilibrium theorem for an abstract economy given in Debreu [8], which in turn is a generalization of the Nash [28] noncooperative equilibrium result. The prominent features of the classical model are: First, its finiteness, i.e., both the set of agents and the number of commodities are finite. Secondly, agents behave in a transitive and complete fashion, i.e., agents' preferences are assumed to be transitive and complete and consequently are representable by utility functions.

Three major extensions of the Arrow-Debreu-McKenzie model have been made. The first is a generalization of the set of agents to a measure space of agents by Aumann [2, 3]. Aumann argued that the Arrow-Debreu-McKenzie model is clearly at odds with itself as the finitude of agents means that each agent is able to exercise some influence. Aumann resolves this problem by assuming that the set of agents is an atomless measure space, and consequently the influence of each agent is "negligible." In this sense the Aumann model, captures precisely the meaning of perfect competition. Bewley [5] provides the second major extension of the Arrow-Debreu-McKenzie model. Bewley amends the classical model to permit the dimensionality of the commodity space to be infinite. This extension is of great importance since infinite dimensional commodity spaces arise very naturally in general equilibrium analysis. In particular, an infinite dimensional commodity space may be desirable in problems involving infinite time horizons, uncertainty about an infinite number of states of the world, or infinite varieties of commodity characteristics. The third important contribution is a substantial improvement of the Arrow-Debreu-McKenzie model made by Mas-Colell [25]. In particular, Mas-Colell [25] builds on an idea of Sonnenschein [34] and shows that even if preferences are not transitive or complete (i.e., preferences need not be ordered), still an equilibrium exists. This result of Mas-Colell has been further improved by Shafer-Sonnenschein [33] and subsequently by Borglin-Keiding [6], Gale-Mas-Colell [13], Kim-Richter [20], McKenzie [24] and Shafer [32] among others.

The purpose of this paper is to prove the existence of an equilibrium in a game theoretic setting (abstract economy), a la Debreu [8] and Shafer-Sonnenschein [33] with a broader structure. In fact, our setting is general enough to include the three major extensions of the classical model mentioned above. It encompasses both the Aumann [2, 3] economy of perfect competition and the nonordered preferences setting of Mas-Colell [25]. Moreover, since the dimensionality of the strategy space may be infinite it contains Bewley-type [5] results and may be useful in obtaining existence results for economies with a measure space of agents and infinitely many commodities.

Our generalization of the Debreu-Shafer-Sonnenschein existence of an equilibrium result for an abstract game or economy with a measure space of agents has several implications. First it extends the Aumann [3] and Schmeidler [31] results, to allow agents' preferences to be both nonordered and interdependent (i.e., it allows for externalities in consumption).

Secondly, it may be seen as a first step in providing a synthesis of the Aumann [3] model of perfect competition with the Bewley [5] model of an infinite dimensional commodity space. Finally, our result extends Schmeidler's [30] theorem on the existence of Nash equilibrium with a continuum of players to a more general class of games where agents' preferences need not be ordered, and therefore need not be representable by utility functions; it also extends the Khan-Vohra [18] equilibrium in abstract economies result to infinite dimensional strategy spaces.

The paper is organized in the following way. Section 2 contains some notation and definitions. The main existence theorem of the paper as well as its relationship with the literature is given in Section 3. Several technical Lemmata and Facts needed for the proof of the main existence theorem are concentrated in Section 4. The proof of the main result is given in Section 5. Finally, some concluding remarks are given in Section 6.

2. NOTATION AND DEFINITIONS

2.1 Notation

2^A denotes the set of all subsets of the set A

$\mathbb{R}$ denotes the set of real numbers

$\mathbb{R}^\ell$ denotes the ℓ-fold product of $\mathbb{R}$

conA denotes the convex hull of the set A

$c\ell A$ denotes the closure of the set A

$\backslash$ denotes the set theoretic subtraction

If $\phi : X \to 2^Y$ is a correspondence then $\phi|U : U \to 2^Y$ denotes the restriction of ϕ to U

proj denotes projection

2.2 Definitions

Let X, Y be two topological spaces. A correspondence $\phi : X \to 2^Y$ is said to be upper-semicontinuous (u.s.c.) if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y. The graph of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. The correspondence $\phi : X \to 2^Y$ is said to have a closed graph if the set G_ϕ is closed in $X \times Y$. A correspondence $\phi : X \to 2^Y$ is said to be lower-semicontinous (l.s.c.) if the set $\{x \in X : \phi(x) \cap V \neq \phi\}$ is open in X for every open subset V of Y. A correspondence $\phi : X \to 2^Y$ is said to have open lower sections if for each $y \in Y$, the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X. If for each $x \in X$, $\phi(x)$ is open in Y, ϕ is said to have open upper sections. Let (T,τ,μ) be a complete finite measure space, i.e., μ is a real-valued, non-negative, countably additive measure defined in a complete σ-field τ of subsets of T such that $\mu(T) < \infty$. $L_1(\mu,\mathbb{R}^\ell)$ denotes the space of equivalence classes of $\mathbb{R}^\ell$-valued Bochner integrable functions $f : T \to \mathbb{R}^\ell$ normed by

$$\|f\| = \int_T \|f(t)\| d\mu(t).$$

A correspondence $\phi : T \to 2^{\mathbb{R}^\ell}$ is said to be <u>integrably bounded</u> if there exists a map $g \in L_1(\mu)$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \phi(t)\} \leqslant g(t)$. The correspondence $\phi : T \to 2^{\mathbb{R}^\ell}$ is said to have a <u>measurable graph</u> if $G_\phi \in \tau \otimes \mathcal{B}(\mathbb{R}^\ell)$, where $\mathcal{B}(\mathbb{R}^\ell)$ denotes Borel σ-algebra and $\otimes$ denotes σ-product field. A correspondence $\phi : T \to 2^X$ is said to be <u>lower measurable</u> if the set $\{t \in T : \phi(t) \cap V \neq \emptyset\} \in \tau$ for every open subset V of X. Notice that, if T is a complete measure space, X is a complete separable metric space and if the correspondence $\phi : T \to 2^X$ has a measurable graph, then ϕ is lower measurable. Moreover, if ϕ is closed valued and lower measurable then ϕ has a measurable graph, (see [7, Theorem III.30, p. 80] or [15, Proposition 4, p. 61]).

Let now X be a topological space and Y be a linear topological space. Let $\phi : X \to 2^Y$ be a nonempty valued correspondence A function $f : X \to Y$ is said to be a <u>continuous selection</u> from ϕ if $f(x) \in \phi(x)$ for all $x \in X$, and f is continuous. Let T be an arbitrary measure space. Let $\psi : T \to 2^Y$ be a nonempty valued correspondence. A function $f : T \to Y$ is said to be a <u>measurable selection</u> from ψ if $f(t) \in \psi(t)$ for all $t \in T$, and f is measurable.

We now define the concept of a Caratheodory-type Selection which roughly speaking combines the notions of continuous selection and measurable selection. Let Z be a topological space and $\phi : T \times Z \to 2^Y$ be a nonempty valued correspondence. A function $f : T \times Z \to Y$ is said to be a <u>Caratheodory-type selection</u> from ϕ if $f(t,z) \in \phi(t,z)$ for all $(t,z) \in T \times Z$ and $f(\cdot,z)$ is measurable for all $z \in Z$ and $f(t,\cdot)$ is continuous for all $t \in T$.

3. THE MAIN THEOREM

3.1 Abstract Economies and Equilibrium

Let (T,τ,μ) be a finite, positive, complete measure space. For any correspondence $X : T \to 2^{\mathbb{R}^\ell}$, $L_1(\mu,X)$ will denote the subset of $L_1(\mu,\mathbb{R}^\ell)$ consisting of those $x \in L_1(\mu,\mathbb{R}^\ell)$ which satisfy $x(t) \in X(t)$ for almost all t in T. Following the Debreu [8], Arrow-Debreu [1] and Shafer-Sonnenschein [33] setting, we define an abstract economy as follows:

An abstract economy Γ is a quadruple $[(T,\tau,\mu), X,P,A]$, where

(1) (T,τ,μ) is a measure space of agents;

(2) $X : T \to 2^{\mathbb{R}^\ell}$ is a strategy correspondence;

(3) $P : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ is a preference correspondence such that $P(t,x) \subset X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$;

(4) $A : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ is a constraint correspondence such that $A(t,x) \subset X(t)$ for all $(t,x) \subset T \times L_1(\mu,X)$.

Observe that since P is a mapping from $T \times L_1(\mu,X)$ to $2^{\mathbb{R}^\ell}$, we have allowed for interdependent preferences. The interpretation of these preference correspondences is that $y \in P(t,x)$ means that agent t strictly prefers $y(t)$ to $x(t)$ if the given strategies of other agents are fixed. Notice that $L_1(\mu,X)$ is the set of all joint strategies. As in [30] and [18] we endow $L_1(\mu,X)$ throughout the paper with the weak topology. This signifies a natural form of myopic behavior on the part of the agents. In particular, an agent has to arrive at his decisions on the basis of knowledge of only finitely many (average) numerical characteristics of the joint strategies.

An equilibrium for Γ is an $x^* \in L_1(\mu,X)$ such that for almost all t in T the following conditions are satisfied:

(i) $x^*(t) \in A(t,x^*)$, and

(ii) $P(t,x^*) \cap A(t,x^*) = \phi$.

3.2 The Main Theorem

We can now state the assumptions needed for the proof of the main theorem.

(A.1) (T,τ,μ) is a finite, positive, complete, separable measure space.[1]

(A.2) $X : T \to 2^{\mathbb{R}^\ell}$ is a correspondence such that:

(a) it is integrably bounded and for all $t \in T$, $X(t)$ is a nonempty, convex, closed subset of $\mathbb{R}^\ell$;

(b) for every open subset V of $\mathbb{R}^\ell$, $\{t \in T : X(t) \cap V \neq \phi\} \in \tau$.

(A.3) $A : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ is a correspondence such that:

(a) for each $t \in T$, $A(t,\cdot) : L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ is continuous;

(b) for all $(t,x) \in T \times L_1(\mu,X)$, $A(t,x)$ is convex and nonempty;

(c) for every open subset V of $\mathbb{R}^\ell$, $\{(t,x) \in T \times L_1(\mu,X) : A(t,x) \cap V \neq \phi\} \in \tau \otimes \mathcal{B}_w(L_1(\mu,X))$, where $\mathcal{B}_w(L_1(\mu,X))$ is the Borel σ-algebra for the weak topology on $L_1(\mu,X)$.

(A.4) $P : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ is a correspondence such that:

(a) for each $t \in T$, $P(t,\cdot) : L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ has an open graph in $L_1(\mu,X) \times \mathbb{R}^\ell$;

(b) $x(t) \notin \mathrm{con}P(t,x)$ for all $x \in L_1(\mu,X)$ for almost all t in T;

(c) for every open subset V of $\mathbb{R}^\ell$, $\{(t,x) \in T \times L_1(\mu,X) : \mathrm{con}P(t,x) \cap V \neq \phi\} \in \tau \otimes \mathcal{B}_w(L_1(\mu,X))$.

We can now state our main theorem.

Theorem 3.1: Let $\Gamma = [(T,\tau,\mu), X,P,A]$ be an abstract economy satisfying (A.1) - (A.4). Then Γ has an equilibrium.

3.3 Comparisons with Related Results

It may be instructive to compare our assumptions with those of Shafer-Sonnenschein [33]. First notice that (A.2)a implies that $X(t)$ is a compact subset of $\mathbb{R}^{\ell}$ for almost all t in T. Assumptions (A.3) a, b and (A.4) a, b are the same as those of Shafer-Sonnenschein [33] and consequently, are the corresponding Shafer-Sonnenschein [33] assumptions in a measure theoretic framework. Assumptions (A.2)b, (A.3) c and (A.4) c are the measurability conditions and are natural in models with a measure space of agents; they constitute no real economic restriction.

Let us now compare our assumptions with those of Khan-Vohra [18]. Apart from the measurability assumptions, all other conditions are identical. In particular, Khan-Vohra [18] assume that the correspondences X, A, P have measurable graphs rather than assuming lower measurability. Therefore, our main existence theorem is closely related to theirs but the methods of proofs are different. Specifically, the Khan-Vohra [18] approach follows the Shafer-Sonnenschein [33] construction of a utility indicator. Our proof is based on selection-type arguments given in Yannelis-Prabhakar [37]. The approach adopted by Khan-Vohra [18] does not extend to infinite dimensional strategy spaces. It fails due to the fact that the convex hull of an u.s.c. correspondence in an infinite dimensional strategy space need not be u.s.c. (see [29, exercise 27, p. 72]). In contrast our selection-type arguments can be directly extended to separable Banach strategy spaces (see Remark 6.4 in Section 6).

We now compare our assumptions with those of Khan-Papageorgiou [19] and Kim-Prikry-Yannelis [21]. Our continuity assumption (A.4) a, on the preference correspondence P, are stronger than those in [19], [21] which require that P have open upper and lower sections. In particular, it is known that if a preference correspondence satisfies (A.4) b and it has open upper and lower sections, it may not have an open graph. However, our assumptions (A.3) a, b on the constraint correspondence A are weaker than those in [19] and [21]. In particular, in [19], [21] it is assumed that:[2]

(i) for all $t \in T$, $A(t,\cdot)$, is u.s.c.;

(ii) there exists a correspondence $B : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ such that:

   (a) $c\ell B(t,x) = A(t,x)$ for all $(t,x) \in T \times L_1(\mu,X)$;

   (b) B has open lower sections; and

   (c) B is convex, nonempty valued.

We now show that (i) and (ii) a, b, c are stronger than (A.3) a, b. More formally we can prove the following proposition.

Proposition 3.1: Assumptions (i) - (ii) a, b, c imply (A.3) a, b, but the reverse need not be true.

Proof: We first show that (i) - (ii) a, b, c $\Rightarrow$ (A.3) a, b. Since B has open lower sections, i.e., for each $(t, y) \in T \times \mathbb{R}^\ell$, $B^{-1}(t,y) = \{x \in L_1(\mu,X) : y \in B(t,x)\}$ is weakly open in $L_1(\mu,X)$, it follows from Proposition 4.1 in [36, p. 237] that for each $t \in T$, $B(t,\cdot)$ is l.s.c. By Fact 4.3 (see next Section) for each $t \in T$, $c\ell B(t,\cdot)$ is l.s.c. Since $c\ell B = A$ and for each $t \in T$, $A(t,\cdot)$ is u.s.c. it follows that for each $t \in T$, $A(t,\cdot)$ is continuous. Since B is convex nonempty valued so is A.

To show that the reverse need not be true we construct the following simple counterexample. Suppose that there is one agent. Let $X \subset \mathbb{R}^1$ and for each $x \in X$, let $A(x) = \{x\}$. Notice that A satisfies (A.3) a, b. However there does not exist mapping $B : X \to 2^X$ satisfying (ii) a, b, c. Indeed, the only mapping B which is convex, nonempty valued, $c\ell B = A$ and $c\ell B$ is u.s.c., is A itself. However, $A^{-1}(y) = \{x : y \in A(x)\} = \{y\}$ is not open for every $y \in X$. The proof of Proposition 4.1 is now complete.

Apart from the above differences we may also note that in [19] it was assumed that the measure space is a locally compact subset of a metric space with a countably generated σ-field. The latter assumption is stronger than (A.1). Moreover, the measurability assumptions in [19] and [21] were made on the graphs of the correspondences X, P, A. Furthermore, notice that our main existence result extends the equilibrium theorems for abstract economies in Toussaint [35] and Yannelis-Prabhakar [36, 37] to a measure space of agents. Also, it generalizes Schmeidler's [30] result to nonordered preferences. Finally, it should be noted that a different approach to equilibrium in abstract games with a continuum of agents has been followed by Green [14] and Mas-Colell [26].

We can now turn to some technical Lemmata needed for the proof of our main result.

4. LEMMATA AND FACTS

Fact 4.1: Let X be a linear topological space.

(i) If $A \subset X$ is open in X and $a \neq o$ is a real number then aA is open in X.

(ii) If $A \subset X$ is open in X and B is any set in X then A + B is open in X.

Proof: Trivial.

Lemma 4.1: Let X, Y be any two linear topological spaces and $\phi : X \to 2^Y$ be a correspondence such that $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$ is open in $X \times Y$. Define $\psi : X \to 2^Y$ by $\psi(x) = \text{con } \phi(x)$ for all $x \in X$. Then $G_\psi = \{(x,y) \in X \times Y : y \in \psi(x)\}$ is open in $X \times Y$.

Proof: Let $(x_o, y_o) \in G_\psi$; we must show that there exist A_o open in X and B_o open in Y such that $(x_o, y_o) \in A_o \times B_o \subset G_\psi$. Since $(x_o, y_o) \in G_\psi$, there exist $y_1, \ldots, y_n$ in $\phi(x_o)$ and reals $a_1, \ldots, a_n$ such that $a_i > 0$, $\sum_{i=1}^n a_i = 1$ and $y_o = \sum_{i=1}^n a_i y_i$. Thus, $(x_o, y_i) \in G_\phi$ and since G_ϕ is open in $X \times Y$ there exist A_i open in X and B_i open in Y such that $(x_o, y_i) \in A_i \times B_i \subset G_\phi$. Define $A_o = \bigcap_{i=1}^n A_i$ and $B_o = \sum_{i=1}^n a_i B_i$. Then A_o is open in X and by Fact 4.1, B_o is open in Y. Note that $(x_o, y_o) \in A_o \times B_o$. To complete the proof we must show that $A_o \times B_o \subset G_\psi$. Let $(x, y) \in A_o \times B_o$, then $y = \sum_{i=1}^n a_i z_i$ where $z_i \in B_i$ for all $i = 1, \ldots, n$. Since $x \in A_o$, $x \in A_i$ and so $(x, z_i) \in A_i \times B_i$. Since $A_i \times B_i \subset G_\phi$, $z_i \in \phi(x)$ for all $i = 1, \ldots, n$ and so $y \in \psi(x)$, i.e., $(x, y) \in G_\psi$. Hence $(x_o, y_o) \in A_o \times B_o \subset G_\psi$. This completes the proof of the Lemma.

<u>Lemma 4.2</u>: Let X, Y be any topological spaces, and $\phi : X \to 2^Y$, $\psi : X \to 2^Y$ be correspondences such that

(i) $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$ is open in $X \times Y$

(ii) ψ is l.s.c.

Then the correspondence $\theta : X \to 2^Y$ defined by $\theta(x) = \phi(x) \cap \psi(x)$ is l.s.c.[3]

<u>Proof</u>: Let V be an open subset of Y and K be the set $\{x \in X : \theta(x) \cap V \neq \phi\}$. Let $x_o \in K$, we must find an open set U in X such that $x_o \in U \subset K$. Since $\theta(x_o) \cap V \neq \phi$ we can choose $y_o \in \theta(x_o) \cap V$. Thus, $(x_o, y_o) \in G_\phi$ and since G_ϕ is open in $X \times Y$ there exist A open in X and B open in Y such that $(x_o, y_o) \in A \times B \subset G_\phi$. Since ψ is l.s.c. the set $E = \{x \in X : \psi(x) \cap B \cap V \neq \phi\}$ is open in X and $x_o \in E$ since $y_o \in \psi(x_o) \cap B \cap V$. Let $U = A \cap E$. Then U is open in X and $x_o \in U$. To complete the proof we must show that $U \subset K$. Let $z \in U$, then $z \in E$ and $z \in A$. Since $z \in E$, $\psi(z) \cap B \cap V \neq \phi$. Choose $w \in \psi(z) \cap B \cap V$. Then $(z,w) \in A \times B \subset G_\phi$ and so $w \in \phi(z)$. Hence, $w \in \phi(z) \cap \psi(z) \cap V$, i.e., $z \in K$. Consequently, $x_o \in U \subset K$, and this completes the proof of the Lemma.

<u>Remark 4.1</u>: Michael [27, Proposition 2.5, p. 366] has proved the following related result:

Let X, Y be two topological spaces and $\phi : X \to 2^Y$, $\psi : X \to 2^Y$ be correspondences such that:

(i) ϕ is l.s.c. and for all $x \in X$, $\phi(x)$ is open in Y,

(ii) ψ is l.s.c.,

(iii) for all $x \in X$, $\phi(x) \cap \psi(x) \neq \phi$.

Then the correspondence $\theta : X \to 2^Y$ defined by $\theta(x) = \phi(x) \cap \psi(x)$ is l.s.c.

However, we will show in Section 6 (Remark 6.3) by means of a counter-example that in Lemma 4.2 assumption (i) cannot be replaced by the assumption that ϕ is l.s.c. and open valued.

Fact 3.2: Let A, B be nonempty subsets of a topological space X. Suppose that A is open in X. Then $A \cap B \neq \phi$ if an only if $A \cap c\ell\, B \neq \phi$.

Proof: Trivial.

Fact 3.3: Let X, Y be two topological spaces and $\phi : X \to 2^Y$ be a l.s.c. correspondence. Then $c\ell\phi : X \to 2^Y$ is l.s.c.

Proof: We must show that the set $A = \{x \in X : c\ell\, \phi(x) \cap V \neq \phi\}$ is open in X for every open subset V of Y. By assumption the set $E = \{x \in X : \phi(x) \cap V \neq \phi\}$ is open in X for every open subset V of Y. Let $x_o \in A$, i.e., $c\ell\phi(x_o) \cap V \neq \phi$. By Fact 3.2 $c\ell\phi(x_o) \cap V \neq \phi$ if and only if $\phi(x_o) \cap V \neq \phi$. Hence, $x_o \in A \Leftrightarrow x_o \in E$, i.e., $A = E$. Consequently A is open in X for every open subset V of Y and this completes the proof of Fact 3.3.

Fact 3.4: Let (T,τ,μ) be a measure space, and X be any topological space. The correspondence $\phi : T \to 2^X$ is lower measurable if and only if $c\ell\phi : T \to 2^X$ is lower measurable.

Proof: The proof is trivial. Simply notice that by virtue of Fact 3.2, for every **open** subset V of X, $\{t \in T : \phi(t) \cap V \neq \phi\} = \{t \in T : c\ell\phi(t) \cap V \neq \phi\}$.

We now state a Caratheodory-type selection result whose proof can be found in Kim-Prikry-Yannelis [22, Theorem 3.2].

<u>Caratheodory-Type Selection Theorem</u>: Let (T,τ,μ) be a complete measure space, Z be a complete, separable metric space. Let $\phi : T \times Z \to 2^{\mathbb{R}^\ell}$ be a convex (possibly empty) valued correspondence such that:

(i) $\phi(\cdot,\cdot)$ is lower measurable,

(ii) for each $t \in T$, $\phi(t,\cdot)$ is l.s.c.

Let $U = \{(t,x) \in T \times Z : \phi(t,x) \neq \emptyset\}$ and for each $t \in T$, let $U^t = \{x \in Z : (t,x) \in U\}$ and for each $x \in Z$, let $U_x = \{t \in T : (t,x) \in U\}$.

Then there exists a Caratheodory-type selection from $\phi_{|U}$, i.e., there exists a function $f : U \to \mathbb{R}^\ell$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$ and for each $x \in Z$, $f(\cdot,x)$ is measurable on U_x and for each $t \in T$, $f(t,\cdot)$ is continuous on U^t. Furthermore, $f(\cdot,\cdot)$ is jointly measurable.

<u>Lemma 4.3</u>: Let (T,τ,μ) be a finite positive complete separable measure space, and $X : T \to 2^{\mathbb{R}^\ell}$ be an integrably bounded correspondence with measurable graph, such that for all $t \in T$, $X(t)$ is a nonempty, convex, closed subset of $\mathbb{R}^\ell$. Then $L_1(\mu,X)$ is nonempty, convex, weakly compact and metrizable.

<u>Proof</u>: Since the correspondence $X : T \to 2^{\mathbb{R}^\ell}$ has a measurable graph, Aumann's measurable selection theorem [4] assures that $L_1(\mu,X)$ is nonempty. Since $X(\cdot)$ is convex valued, $L_1(\mu,X)$ is convex. Notice that since $X(\cdot)$ is integrably bounded $L_1(\mu,X)$ is bounded and uniformly integrable. Hence, from Dunford's Theorem [10, p. 76 and p. 101] it follows that $L_1(\mu,X)$ is a relatively weakly compact subset of $L_1(\mu,\mathbb{R}^\ell)$. Since $L_1(\mu,X)$ is convex and norm closed by Theorem 17.1 in [16, p. 154], it is weakly closed. Therefore, $L_1(\mu,X)$ is a weakly compact subset of $L_1(\mu,\mathbb{R}^\ell)$. It follows from Theorem 3 in Dunford-Schwartz [11, p. 434] that $L_1(\mu,X)$ is metrizable. This completes the proof of the Lemma.

<u>Remark 4.2</u>: Lemma 4.3 remains true if the correspondence X maps points from T into Y, where Y is a separable Banach space, provided that X is convex, nonempty, weakly closed valued with measurable graph and for all $t \in T$, $X(t) \subset K$, where K is a weakly compact subset of Y.

Lemma 4.4: Let (T,τ,μ) be a complete finite measure space, and Y be a separable Banach space. Let $X : T \to 2^Y$ be an integrably bounded, nonempty, convex, weakly closed valued correspondence with measurable graph such that for all $t \in T$, $X(t) \subset K$ where K is a weakly compact subset of Y. Let $\phi : T \times L_1(\mu,X) \to 2^Y$ be a correspondence[4] such that $\phi(t,x) \subset X(t)$ for all $(t,x) \in T \times L_1(\mu,X)$ and for each $x \in L_1(\mu,X)$, $\phi(\cdot,x)$ has a measurable graph, and for each $t \in T$, $\phi(t,\cdot)$ is u.s.c. in the sense that the set $\{x \in L_1(\mu,X) : \phi(t,x) \subset V\}$ is weakly open in $L_1(\mu,X)$ for every norm open subset V of Y. Then the correspondence $\Phi : L_1(\mu,X) \to 2^{L_1(\mu,X)}$ defined by

$$\Phi(x) = \{y \in L_1(\mu,X) : \text{for almost all } t \in T,\ y(t) \in \phi(t,x)\}$$

is nonempty valued and weakly u.s.c.

Proof: Several proofs of this Lemma have been given in [17], [18], [19], [21], [30]. The proof given below is based on an argument given in [19] and seems to be the simplest. Notice first that nonempty valueness of Φ is a direct consequence of the Aumann measurable selection theorem [4], (simply observe that for each $x \in L_1(\mu,X)$, $\phi(\cdot,x)$ has a measurable graph). We now show that Φ is u.s.c. Denote by B the closed unit ball in Y. Since by Lemma 4.3 and Remark 4.2, $L_1(\mu,X)$ with the weak topology is compact and metrizable, it suffices to show that the graph of Φ, i.e., G_Φ is closed. To this end let (x_n, y_n) be a sequence converging weakly to (x,y) where $(x_n,y_n) \in G_\Phi$, i.e., $y_n \in \Phi(x_n)$. We must show that $y \in \Phi(x)$. Since $y_n \in \Phi(x_n)$, we have that $y_n(t) \in \phi(t, x_n)$ for almost all $t \in T$. By Corollary 17.2 in [16, p.154], there exists $z_n(\cdot) \in \text{con} \bigcup_{n_o \geq n} y_{n_o}(\cdot)$ such that $z_n(\cdot)$ converges in norm to $y(\cdot)$. Without loss of generality we may assume that $z_n(t)$ converges in norm to $y(t)$ (otherwise pass to a subsequence) for all $t \in T \setminus S$ where S is a negligible set of agents. Fix an agent t in $T \setminus S$. Since $\phi(t,\cdot)$ is u.s.c. for every small positive number ε there exists n such that for all $n_o \geq n$ we have that

$\phi(t,x_{n_o}) \subseteq \phi(t,x) + \varepsilon B$. But then $\mathrm{con} \bigcup_{n_o \geq n} \phi(t,x_{n_o}) \subseteq \phi(t,x) + \varepsilon B$ which implies that $z_n(t) \in \phi(t,x) + \varepsilon B$ and so $y(t) \in \phi(t,x) + \varepsilon B$. Therefore, $y(t) \in \phi(t,x)$ by letting ε converge to zero. Since t is any arbitrary agent in $T \setminus S$, $y(t) \in \phi(t,x)$ for almost all t in T, i.e., $y \in \Phi(x)$. Hence, G_Φ is closed, as was to be shown. This completes the proof of the Lemma.

5. PROOF OF MAIN THEOREM

Define $\psi : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ by $\psi(t,x) = \text{con}P(t,x)$ for all $(t,x) \in T \times L_1(\mu,X)$. By Lemma 4.1 for each $t \in T$, $\psi(t,\cdot)$ has an open graph in $L_1(\mu,X) \times \mathbb{R}^\ell$ where $L_1(\mu,X)$ is endowed with the weak topology. Moreover, it follows from assumption (A.4)c that ψ is lower measurable. Define $\theta : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ by $\theta(t,x) = A(t,x) \cap \psi(t,x)$ for all $(t,x) \in T \times L_1(\mu,X)$. Then θ is convex valued and it follows from Lemma 4.2 that for each $t \in T$, $\theta(t,\cdot)$ is l.s.c. in the sense that the set $\{x \in L_1(\mu,X) : \theta(t,x) \cap V \neq \phi\}$ is weakly open in $L_1(\mu,X)$ for every open subset V of $\mathbb{R}^\ell$. Since $c\ell\theta(t,x) = c\ell(A(t,x) \cap \psi(t,x)) = A(t,x) \cap c\ell\psi(t,x)$, it follows from Proposition III.4 in [7] that $c\ell\theta(\cdot,\cdot)$ is lower measurable. By Fact 3.4, $\theta : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ is lower measurable as well. Let $U = \{(t,x) \in T \times L_1(\mu,X) : \phi(t,x) \neq \phi\}$. For each $x \in L_1(\mu,X)$, let $U_x = \{t \in T : \theta(t,x) \neq \phi\}$ and for each $t \in T$, let $U^t = \{x \in L_1(\mu,X) : \theta(t,x) \neq \phi\}$. It follows from the Caratheodory-type selection theorem that there exists a function $f : U \to \mathbb{R}^\ell$ such that $f(t,x) \in \theta(t,x)$ for all $(t,x) \in U$ and for each $x \in L_1(\mu,X)$, $f(\cdot,x)$ is measurable on U_x and for each $t \in T$, $f(t,\cdot)$ is continuous on U^t. Furthermore, $f(\cdot,\cdot)$ is jointly measurable. Notice that for each $x \in L_1(\mu,X)$, $U_x = \{t \in T : \theta(t,x) \neq \phi\} = \{t \in T : \theta(t,x) \cap \mathbb{R}^\ell \neq \phi\} = \text{proj}_T \{(t,x) \in T \times L_1(\mu,X) : \theta(t,x) \cap \mathbb{R}^\ell \neq \phi\}$. Since $\theta(\cdot,\cdot)$ is lower measurable, it follows from Theorem 11 in [15, p. 44] that for each $x \in L_1(\mu,X)$, U_x is a measurable set. Define the correspondence $\phi : T \times L_1(\mu,X) \to 2^{\mathbb{R}^\ell}$ by

$$\phi(t,x) = \begin{cases} \{f(t,x)\} & \text{if } (t,x) \in U \\ A(t,x) & \text{if } (t,x) \notin U. \end{cases}$$

Since for each $t \in T$, $\theta(t,\cdot)$ is l.s.c., for each $t \in T$, the set $U^t = \{x \in L_1(\mu,X) : \theta(t,x) \neq \phi\} = \{x \in L_1(\mu,X) : \theta(t,x) \cap \mathbb{R}^\ell \neq \phi\}$ is weakly open in $L_1(\mu,X)$. Hence, by Lemma 6.1 in [36, p. 241] $\phi(t,\cdot)$ is u.s.c. in the sense that the set $\{x \in L_1(\mu,X) : \phi(t,x) \subset V\}$ is weakly open in $L_1(\mu,X)$ for every open subset V of $\mathbb{R}^\ell$. Since A is closed valued it follows from (A.3)c that for each $x \in L_1(\mu,X)$, $A(\cdot,x)$ has a measurable graph. It can be easily seen that for each $x \in L_1(\mu,X)$, $\phi(\cdot,x)$ has a measurable graph. In fact, for all $x \in L_1(\mu,X)$, $G_{\phi(\cdot,x)} = \{(t,y) \in T \times \mathbb{R}^\ell : y \in \phi(t,x)\} = C \cup D$ where $C = \{(t,y) \in T \times \mathbb{R}^\ell : y \in f(t,x) \text{ and } t \in U_x\}$ and $D = \{(t,x) \in T \times \mathbb{R}^\ell : y \in A(t,x) \text{ and } t \notin U_x\}$. Since C, D are in $\tau \otimes \mathcal{B}(\mathbb{R}^\ell)$, we have that $C \cup D = G_{\phi(\cdot,x)}$ is in $\tau \otimes \mathcal{B}(\mathbb{R}^\ell)$. Obviously ϕ is convex and nonempty valued. Define $\Phi : L_1(\mu,X) \to 2^{L_1(\mu,X)}$ by $\Phi(x) = \{y \in L_1(\mu,X) : \text{for almost all } t \text{ in } T \; y(t) \in \phi(t,x)\}$. By Lemma 4.4, Φ is nonempty valued and weakly u.s.c. Since ϕ is convex valued so is Φ. Moreover by Lemma 4.3, $L_1(\mu,X)$ is nonempty, convex and weakly compact. Hence, by Fan's fixed point theorem [12, Theorem 1, p.122]) there exists $x^* \in L_1(\mu,X)$ such that $x^* \in \Phi(x^*)$, i.e., $x^*(t) \in \phi(t,x^*)$ for almost all t in T. Suppose that for a non-negligible set of agents S, $(t, x^*) \in U$ for all $t \in S$. Then by the definition of ϕ, $x^*(t) = f(t,x^*) \in \theta(t,x^*) \subset \text{conP}(t,x^*)$ for all $t \in S$, a contradiction to (A.4)b. Therefore, $(t,x^*) \notin U$ for almost all t in T and consequently for almost all $t \in T$, $x^*(t) \in A(t,x^*)$ and $\theta(t,x^*) = \text{conP}(t,x^*) \cap A(t,x^*) = \phi$ which implies that $P(t,x^*) \cap A(t,x^*) = \phi$, i.e., x^* is an equilibrium for Γ. This completes the proof of the main theorem.

6. CONCLUDING REMARKS

Remark 6.1: Our main existence theorem can be used to prove directly the existence of a competitive equilibrium for an economy with a continuum of agents whose preferences may be interdependent and need not be transitive or complete (see for instance Khan-Vohra [18, Theorem 3, p. 137]). Therefore, an extension of the Aumann [3] and Schmeidler [31] results to economies with non-ordered and interdependent preferences can be obtained.

Remark 6.2: Notice that in the Aumann [3] model the convexity assumption on preferences is not required, since the Lyapunov theorem convexifies the aggregate demand set. However, without transitivity and completeness the convexity assumption on preferences cannot be relaxed (see Mas-Colell [25, p.243]). Moreover, even if preferences are transitive, complete, and interdependent, the convexity assumption still cannot be relaxed. In fact, as Khan-Vohra [18] pointed out, with externalities in consumption there is no convexifying effect on aggregation. Therefore, it appears that the convexity assumption (A.4)b cannot be relaxed in models with a continuum of agents and interdependent preferences.

Remark 6.3: Assumption (A.4)a, i.e., for each $t \in T$, $P(t,\cdot)$ has an open graph in $L_1(\mu,X) \times \mathbb{R}^{\ell}$ cannot be relaxed to open upper and lower sections in our framework. In particular if (A.4)a is weakened to open lower and upper sections, the correspondence $\theta : T \times L_1(\mu,X) \to 2^{\mathbb{R}^{\ell}}$ defined in Section 5 by $\theta(t,x) = A(t,x) \cap \text{con}P(t,x)$, need not be l.s.c. in x. Hence, Lemma 4.2 fails, and the proof of the main existence theorem does not go through. The following simple example illustrates this.

Example: Consider the following mappings:

$$P(x) = \begin{cases} \mathbb{R} & \text{if } x \leqslant 0 \\ \mathbb{R} \setminus \{x\} & \text{if } x > 0 \end{cases}$$

and $A(x) = \{x\}$. Note that for any $x \in \mathbb{R}$, $P(x)$ is always open in $\mathbb{R}$ and for any $y \in \mathbb{R}$, $P^{-1}(y) = \{x : y \in P(x)\}$ is open in $\mathbb{R}$. Further, P is l.s.c. since the set $\{x : P(x) \cap V \neq \phi\} = \mathbb{R}$ is open in $\mathbb{R}$ for every V open subset of $\mathbb{R}$. Also, A is continuous, i.e., u.s.c. and l.s.c. However, the correspondence $\theta(x) = P(x) \cap A(x)$ is not l.s.c. Indeed, note that for $V = \mathbb{R}$ the set $\{x : \theta(x) \cap V \neq \phi\} = \{x : -\infty < x \leqslant 0\} = (-\infty, 0]$ is not open in $\mathbb{R}$.

Remark 6.4: We now indicate how our main existence theorem can be extended to separable Banach strategy spaces. One must modify assumptions (A.2) and (A.3)b as follows:

(A.2)' The correspondence $X : T \to 2^Y$, (where Y is a separable Banach space) is integrably bounded, nonempty convex, weakly closed valued, lower measurable and for all $t \in T$, $X(t) \subset K$ where K is a relatively weakly compact subset of Y.

(A.3)b' For each $(t,x) \in T \times L_1(\mu,X)$, $A(t,x)$ is convex and has a nonempty interior in $X(t)$.

Note from (A.2)' it follows that $L_1(\mu,X)$ is weakly compact (Diestel [9, Theorem 2 and Remark, p. 89]). Hence the argument used to prove weak compactness of $L_1(\mu,X)$ in Lemma 4.3, becomes redundant. Notice that all other Lemmata and facts in Section 4 are true for a separable Banach space Y. Moreover, the Caratheodory-type selection result is true for any separable Banach space provided that the correspondence $\phi : T \times X \to 2^Y$ has a nonempty interior for all $(t,x) \in U$ (see [22, Theorem 3.2]). The proof of the main existence result remains the same.

One only needs to check that from assumption (A.3)b' it follows that the correspondence $\theta : T \times L_1(\mu, X) \to 2^Y$ (defined in Section 5) has a nonempty interior in X(t) and consequently a trivial modification of our Caratheodory-type selection theorem assures that there exists a Caratheodory selection $f : U \to Y$ from $\theta|_U$. The rest of the proof remains unchanged.

Remark 6.5: In a subsequent paper we hope to show how the main existence result of this paper can be used to obtain a generalization of Bewley's [5] result to economies with a measure space of agents. The fact that the abstract economy approach can be used to prove Bewley's existence result (recall that the set of agents in the Bewley model is finite) has been already demonstrated in Toussaint [35].

FOOTNOTES

1. The reason we assume that (T,τ,μ) is a separable measure space is that we want $L_1(\mu)$ to be separable.

2. The work in [19] and [21] follows closely the Borglin-Keiding [6] abstract economy setting rather than Shafer-Sonnenschein [33]. It is exactly for this reason that the results in [19] and [21] do not generalize Shafer-Sonnenschein [33]. In contrast, Khan-Vohra [18] and the present paper, constitute direct generalization of the Shafer-Sonnenschein [33] result.

3. Green [14] has proved a related Lemma [14, Lemma 3, p. 984]. His result is implied by ours.

4. $L_1(\mu,X)$ will not denote the subset of $L_1(\mu,Y)$ consisting of those $x \in L_1(\mu,Y)$ which satisfy $x(t) \in X(t)$ for almost all $t \in T$. Notice that following the previous notation $L_1(\mu,Y)$ denotes the space of equivalence classes of Y-valued Bochner integrable functions.

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