STRATEGY SPACE REDUCTION IN THE MASKIN - WILLIAMS THEOREM: SUFFICIENT CONDITIONS FOR NASH IMPLEMENTATION*

by

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ABSTRACT

Any social choice correspondence satisfying monotonicity and no veto power with at least three participants is Nash implementable. This theorem by Maskin, of which an extended version was proved by Williams, requires a rather large strategy space. Each participant announces every participant's preferences and an alternative. This paper presents a significantly smaller strategy space when the number of participants is large. Each participant announces his own preferences, his neighbor's preferences, an alternative, and an integer between zero and the number of participants less one. With this specification of the strategy spaces, the Maskin-Williams Theorem remains valid without imposing any restrictions on the size of the alternative set or the environment set. That is, a complete proof for the original Maskin theorem is provided.

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1. Introduction

After a society reaches an agreement on the set of socially optimal alternatives, is it possible for the planner to design a rule or a mechanism to achieve one of the optimal alternatives without destroying the participants' incentives? This is a problem regarding the implementation of a given social choice correspondence. The Gibbard-Satterthwaite theorem, a classical result of this line of research, gives a negative answer. Suppose that there are at least three alternatives and the social choice correspondence is single-valued. If it is possible to construct a mechanism in which the true preference announcement is a dominant strategy and the announcement achieves the optimal alternative, then the social choice function is dictatorial.

In 1977, Maskin [3] found such a mechanism with a formulation which differs from the Gibbard-Satterthwaite approach in three respects. Maskin employed the Nash equilibrium concept, used an implementation condition different from the truthful implementation which appears in the work of Gibbard and Satterthwaite, and admitted multi-valued social rules, which we shall call social choice correspondences. Under these changes, Maskin showed that any social choice correspondence satisfying the monotonicity and no veto power conditions with at least three participants is Nash implementable. This sufficiency theorem for Nash implementability, of which an extended version was proved by Steven Williams [11,12], uses a rather large strategy space. Roughly speaking, in Maskin's version each participant announces his own preferences, every other participant's preferences, and a socially optimal alternative. Williams found a counterexample to Maskin's construction and proposed a revised construction by using a slightly larger strategy space. Saijo [5] found a strategy space smaller than Williams' which also yields Nash implementability.

Recently, due to an unpublished communication whose authorship is at present unknown to the author of this paper, another construction by using a slightly different strategy space is proposed. Each participant announces his preferences, every other participant's preferences, any alternative, and an integer between zero and the number of participants less one. With this specification of the strategy space, Maskin's theorem is true.(1) Inspired by the anonymous author's strategy space, we shall present a significantly small strategy space when the number of participants is large. Each participant announces his own preferences, his next neighbor's preferences, an alternative, and an integer between zero and the number of participants less one. Under this specification of the strategy space, Maskin's theorem, which we shall call the Maskin-Williams theorem, is true. Our strategy space specification will shed new light on the interpretation of the Nash equilibrium concept in Nash implementation. It is widely believed that the Nash equilibrium requires each agent to know the preferences of all other agents (see Sonnenschein [p.16, 9] and Maskin [Section 3 Revelation Principle, 4]).(2) In the Maskin-Williams framework, our strategy space specification is the first attempt that does not require each participant to announce every other participant's preferences.

In the next section, we shall introduce notation and definitions. In section 3, assuming that there are at least three participants, we shall construct a game form which will yield Nash implementability under the monotonicity and no veto power conditions. Detailed comments on the assumptions which we will use will be also found in this section in comparison with previous results. In fact, our Theorem 1 provides a first complete proof on Maskin's theorem in his original form without imposing any restriction on the size of the alternative set and the environment set. Several remarks will be given in the final section.

2. Notation and Definitions

Suppose that there are n participants, and let I be the set of indices of participants, $I = \{1,2,\cdots,n\}$. Let $F: E \twoheadrightarrow A$ be a _social choice correspondence_ (or a _target correspondence_, or a _goal correspondence_) such that $F(e)$ is nonempty for each $e \in E$, where E is the _set of environments_ and A is the _set of social alternatives_. We shall assume that the cardinal number of A is strictly greater than one and that E is nonempty.

We shall regard E as a subset of $\underline{E}^n$, where $\underline{E}^n = X_{i\in I}\, \underline{E}$ and $\underline{E}$ is the class of all complete preference orderings on A. We say that the pair (E,A) has the _coordinate property_ if E is the Cartesian product $X_{i\in I}\, E_i$ of E_i with $i \in I$, where _participant i's characteristic set_ E_i, a class of complete preference orderings on A for participant i, is a subset of $\underline{E}$. If the domain E of a social choice correspondence F has the coordinate property, then we say that F has the coordinate property. More fully, we shall denote the social choice correspondence (F;E,A).

Let $g: S \to A$ be a _game form_ (or an _outcome function_, or a _mechanism_) with $S = X_{i\in I}\, S_i$, where S_i is the ith _participant's strategy space_ and S is called the _strategy space_. We shall denote it (g;S,A). Let $L(a,e_j)$ be participant j's weak lower contour set with preference e_j at $a \in A$:

$$L(a,e_j) = \{ b \in A : a \text{ is preferred to } b \text{ or } a \text{ is indifferent to } b \text{ by } e_j \}.$$

Definition 1: A strategy $\bar{s} \in S$ is a Nash equilibrium of the game form $(g;S,A)$ for the environment $e = (e_i) = (e_1,\cdots,e_n)$ if and only if: for each j

$$\{a \in A : a = g(s_j,\bar{s}_{-j}) \text{ for some } s_j \in S_j\} \subset L(g(\bar{s}),e_j),$$

where $(s_j,\bar{s}_{-j})$ is the list $(\bar{s}_1,\cdots,\bar{s}_{j-1},s_j,\bar{s}_{j+1},\cdots,\bar{s}_n)$ obtained by replacing the jth element of $(\bar{s}_1,\bar{s}_2,\cdots,\bar{s}_j,\cdots,\bar{s}_n)$ by s_j, and $\subset$ denotes nonstrict inclusion.

Let N_g: $E \twoheadrightarrow S$ be the correspondence whose value at $e \in E$ is the set of all Nash equilibria for e when $(g;S,A)$ is the game form used.

Definition 2: The game form $(g;S,A)$ implements the social choice correspondence $(F;E,A)$ in Nash equilibria if and only if: for any $e \in E$,

$$g\cdot N_g(e) = F(e),$$

where $g\cdot N_g(e) = \{a \in A : a = g(s) \text{ for some } s \in N_g(e)\}$.

The implementation requires that every socially optimal alternative $a \in F(e)$ for the given environment e should be attainable by the game form. That is, a social choice correspondence is Nash implementable if it is possible to decompose the Nash equilibrium correspondence and the game form. We might think that if strategy space S is large enough, any social choice correspondence is Nash implementable. Unfortunately, this is not true. There exists a rich class of social choice correspondences which are not Nash implementable. For example, Sen [8] and Saijo [6] show that any non-constant single-valued social choice correspondence on unrestricted domain is not Nash implementable.

Definition 3: The social choice correspondence $(F;E,A)$ satisfies the <u>monotonicity</u> (M) condition if and only if: for any $e, e' \in E$ and any $a \in F(e)$,

$$L(a,e_i) \subset L(a,e_i') \text{ for all } i \text{ implies } a \in F(e').$$

Monotonicity of social choice correspondence can be explained as follows. Suppose that an alternative a is socially optimal with a preference profile $e = (e_1,e_2,\cdots,e_n)$, i.e., $a \in F(e)$. Now keeping the ranking of all alternatives by preference e_j, and putting up the alternative a in a higher position in the ranking than before (or keeping the same position), we can make a new ranking e_j' for all j. Since initially, a is socially optimal by e and under a new preference profile $e' = (e_1',e_2',\cdots,e_n')$, every participant states the relative position of a among the alternatives is improved (or the same as before), the alternative a should be a socially optimal alternative under the new profile e', i.e., $a \in F(e')$.

Definition 4: The social choice correspondence $(F;E,A)$ satisfies the <u>no veto power</u> (NVP) condition if and only if: for any $a \in A$ and any $e \in E$,

$$\#\{i \in I : L(a,e_i) = A\} \geq n - 1 \text{ implies } a \in F(e).$$

If an alternative a is regarded as the top ranked alternative for at least $n - 1$ participants under a preference profile e, a should be socially optimal, i.e., $a \in F(e)$. That is, if there exists a participant who does not claim a as his best alternative, then he should not have veto power. A detailed discussion of monotonicity and no veto power can be found in Maskin [3,4].

3. Strategy Space Reduction

Information is decentralized. It is assumed that each participant's true characteristic is known himself only. On the other hand, the social choice correspondence $(F;E,A)$, the game form $(g;S,A)$ and the strategies proposed are public information. Since throughout this paper, we shall assume that E has the coordinate property, each participant i can observe the size and the shape of the other participant's characteristic set E_j, but he cannot observe which element is participant j's true characteristic unless $\#E_j = 1$.

The designer's task is to construct a game form, under an appropriate equilibrium concept, which will implement the social choice correspondence. The designer does not get a specific social choice correspondence, but he is told to deal with the class of all social choice correspondences satisfying monotonicity and no veto power. Our choice of the equilibrium concept is the Nash equilibrium. We should notice that the Nash equilibrium requires information about the strategies taken by others, but not the preference profile.

Game form or mechanism designers work on developing superior game forms or mechanisms. The game forms proposed so far require that every participant should announce at least a preference profile. Every participant in Maskin [3] and Williams [12] should announce a preference profile and a socially optimal alternative evaluated by the announced preference profile, i.e., the ith participant's strategy space is

(M-W) $$S_i = \{(e_1,e_2,\cdots,e_n;a) \in E_1 \times E_2 \times \cdots \times E_n \times A : a \in F(e)\}.$$

Maskin's new version [4] of the ith participant's strategy space is

(M) $$S_i = E_1 \times E_2 \times \cdots \times E_n \times A.$$

That is, every participant should announce a preference profile and an alternative, where the alternative announcement is not necessarily socially optimal. Without any restriction on the size of E_i, we can still find a counterexample to Maskin's construction even with this new version of the strategy space.[8] Although the unidentified author did not notice this flaw, he extended participant i's strategy space to deal with an arbitrary size A:

$$(U) \qquad S_i = E_1 \times E_2 \times \cdots \times E_n \times A \times \{0,1,\cdots,n-1\}.$$

Under this specification of the strategy space, the Maskin-Williams theorem is true without adding any restriction on A and E. All strategy spaces above have a common feature. Every participant should announce at least a preference profile. This paper provides a game form which does not require each participant to announce every other participant's preference. We shall now specify our strategy space of the i-th participant as follows:

$$(1) \quad S_i = E_i \times E_{i+1} \times A \times \{0,1,\cdots,n-1\} \text{ for } i = 1,\cdots,n-1; \text{ and}$$
$$S_n = E_n \times E_1 \times A \times \{0,1,\cdots,n-1\}.$$

By comparison, it is clear that (U) is larger than (M), (M) is larger than (M-W), and (U) is larger than (1). The comparison among (M), (M-W) and (1) is not clear in the sense of the set inclusion since (1) has $\{0,1,\cdots,n-1\}$ parts. We believe that the game form proposed here is "superior" to other game forms because of its preference announcement part. This point shall be discussed again in the final section.

We shall denote an element of S_i by $s_i = (e_i^i, e_i^{i+1}, a_i, m_i)$, where $i + 1 = 1$ if $i = n$. e_i^j is the announcement by participant i about participant j's preferences. Let us pay attention to the characteristic announcement part of each participant's strategy. Put all participants on a circle, facing toward its center. Then (1) says that every participant is required to announce his own characteristic and his left side participant's characteristic if the numbering of the participants is clockwise. A similar idea on the "cyclic" announcement of strategies can be found in Hurwicz [1] and Walker [10]. For notational simplicity, we shall denote

(α-1) $\quad a_i = a^*$ for all i;

(α-2) $\quad a_i = a^*$ for all $i \neq j$ and $a_j \neq a^*$;

(β-1) $\quad$ (i) $\quad e_i^i = e_{i-1}^i$ for all i or

(ii) $\quad e_i^i = e_{i-1}^i$ for all i except j or

(iii) $e_i^i = e_{i-1}^i$ for all i except j and $j+1$;

(β-2) $\quad e_i^i = e_{i-1}^i$ for all i except $j+1$; and

(γ) $\quad a^* \in F(e_1^1,\cdots,e_{j-1}^{j-1},e_{j-1}^j,e_{j+1}^{j+1},\cdots,e_n^n)$,

where $i - 1 = n$ if $i = 1$. From (α-1) to (γ), script "j" is related to each other. For example, consider (α-2), (β-2) and (γ). Participant j who announces a different alternative a_j in (α-2) is exactly the same participant j in (β-2) and (γ). Construct a game form as follows:

<u>Rule I</u>: If [(α-1) and (β-1) and (γ)] is true, then $g(s) = a^*$.

<u>Rule II</u>: If there exists a participant, say j, such that(4)

(α-2) and [(β-1) or (β-2)] and (γ),

then

$$g(s_j, s_{-j}) = \begin{cases} a_j & \text{if } a_j \in L(a^*, e_{j-1}^j) \\ a^* & \text{otherwise.} \end{cases}$$

<u>Rule III</u>: If neither Rule I nor Rule II is applicable, then

$$g(s_1, s_2, \cdots, s_n) = a_t,$$

where $t = (\Sigma_{k=1}^{n} m_k)(\text{mod } n) + 1.$[5]

Summarizing the game form constructed by Rules I, II and III, we have the following table.

	(β-1) & (γ)	(β-2) & (γ)	neither
(α-1)	Rule I	Rule III	Rule III
(α-2)	Rule II	Rule II	Rule III
neither	Rule III	Rule III	Rule III

Table 1: Game form constructed by Rules I, II & III.

The following three properties summarize some special features of the game form.

Property 1: Regardless of the preference and integer announcement, if every participant announces a^* as an alternative, the outcome by the game form is always a^*.

If every participant announces the same alternative, we should apply Rule I or III. In either case, the same alternative will be the outcome. Hence in Table 1, the row element labelled under (α-1) always gives a^* regardless (β) or (γ). We shall refer to this situation as unanimity.[6]

In Rule II, participant j who announces an alternative different from the others' is referred to as the deviator.

Property 2: If participant j is the deviator, his preference announcement $(e_j^j,\ e_j^{j+1})$ does <u>not</u> enter the evaluation of the preference announcement profile

$$(e_1^1, \cdots, e_{j-1}^{j-1}, e_{j-1}^{j}, e_{j+1}^{j+1}, \cdots, e_n^n).$$

Therefore $(e_j^j,\ e_j^{j+1})$ does not affect the evaluation of the social choice correspondence (F;E,A) when Rule II is applied.

This feature is fundamental to our game form construction. Even though the deviator changes his strategy, he cannot let Rule III apply. As Rule II indicates, the outcome will be trapped in the weak lower contour set $L(a^*, e_{j-1}^{j})$. Notice also that the evaluation of the weak lower contour set is not done by his own characteristic announcement e_j^j, but by his next neighbor's announcement e_{j-1}^{j}. Hence $L(a^*, e_{j-1}^{j})$ will not change its shape even though participant j changes his strategies.

Property 3: In Rule III, participant i, say, can designate himself or some other participant as a "dictator" to choose any alternative when other participants' integer announcements are given, i.e., when $\Sigma_{k \neq i}^{n}\ m_k$ is given.

In Rule III, given all other participants' strategies, participant i can designate not only his own alternative, but also some other participant's alternative announcement.

It can now be verified that this game form implements (F;E,A) in Nash equilibria:

Theorem 1: Suppose that the number of participants is at least three; i.e., $n \geq 3$. If the social choice correspondence (F;E,A) has the coordinate property (i.e., $E = X_{i \in I} E_i$) and if (F;E,A) satisfies the monotonicity (M) and no veto power (NVP) conditions, then game form (g;S,A) defined by Rules I, II, and III with the strategy space (1) implements (F;E,A) in Nash equilibria.

Remark 1: Several comments are in order on the assumptions of Theorem 1.

1. Both Maskin [3,4] and Williams [11,12] assumed implicitly that (F;E,A) has the coordinate property. Saijo [Theorem 7, 5] reported that if we use, roughly speaking, the strategy space that Williams used, we do not need to assume the coordinate property. Since we use two participants' characteristic sets as a part of a participant's strategy space, we have to change the characteristic set in a participant's strategy space to some meaningful set if we abandon the coordinate property. It might be possible to avoid the coordinate property in Theorem 1, but we shall not pursue this generalization.

2. Williams pointed out that Maskin implicitly used the following assumption: every alternative is socially attainable. That is, $\{a \in A : a \in F(e)$ for some $e \in E\} = A$, which is called the "no taboo" condition in [5].(7) In the construction of game forms, Maskin [3] and Williams [11,12] required that each participant be able to announce any alternative as a socially optimal alternative. At present, there are at least two ways to avoid the "no taboo" condition. Saijo [Theorem 7, 5] found that in Williams' construction, the "extended" social choice correspondence satisfies the "no taboo" condition, so that the original social choice correspondence need not satisfy it. Another method is due to Maskin [4]. He proposed that each participant can announce any alternative. Since the choice of alternative in (1) is not restricted, our specification of the strategy space employs Maskin's method. Because each participant can announce any alternative in our specification, we can avoid the "no taboo" condition.

3. Maskin [3,4] implicitly assumed that the number of alternatives is finite. Williams [11,12] succeeded in extending A to an arbitrary size by adding one condition [see (6) in 12]. Just after Williams' breakthrough, the unidentified author proposed a different way of handling A of arbitrary size. His/her way is to extend the strategy space. The announcement of an integer between zero and the number of participants less one takes care of the size problem on A. Since our specification of the strategy space employs the announcement of an integer, the size problem on A is also avoided.

4. The importance of the cardinal number of each participant's characteristic set in <u>Maskin's</u> <u>construction</u> was first pointed out by Williams [11,12] (see also footnote 10). His counterexample to <u>Maskin's</u> <u>construction</u> of the game form assumed the fact that one participant's characteristic set is a singleton. In contrast to the Maskin-Williams type constructions of the game form, we do not impose any size restrictions on a participant's characteristic set. The Maskin theorem (Theorem 5 in [3]) does not have any restrictions on the size of characteristic sets. The source of the difference again stems from the integer announcements in our strategy specification.

Since the equality of the implementability condition $g \cdot N_g(e) = F(e)$ of the social choice correspondences is considered as the set inclusions for both directions, we shall divide the proof of Theorem 1 into two parts. Lemma 1 proves $F(e) \subset g \cdot N_g(e)$ and Lemmata 2 and 3 prove $g \cdot N_g(e) \subset F(e)$.

Lemma 1: Suppose that the number of participants is at least three; i.e., $n \geq 3$. If the social choice correspondence $(F;E,A)$ has the coordinate property (i.e., $E = X_{i \in I} E_i$), then game form $(g;S,A)$ defined by Rules I, II, and III with the strategy space (1) gives $F(e) \subset g \cdot N_g(e)$ for all $e \in E$.

I.e.,

(2) $F(e) \subset \{a \in A : a = g(s) \text{ for some } s \in N_g(e)\}$ for all $e \in E$.

Remark 2: In Lemma 1, we do not require any specific property such as monotonicity or no veto power of the social choice correspondence. That is, the inclusion $F(e) \subset g \cdot N_g(e)$ can be obtained just by formulating the game form appropriately.

Proof of Lemma 1. For any $e = (e_i) \in E$ and any $a \in F(e)$, let $s_i = (e_i^i, e_i^{i+1}, a, m_i)$ with $e_i^i = e_i$ and $e_i^{i+1} = e_{i+1}$ for all i, and, for each i, let m_i be any integer between 0 and n - 1. Then clearly we have $a = a_i$ for all i, $e_i^i = e_{i-1}^i$ for all i and $a \in F(e_1^1, \cdots, e_{j-1}^{j-1}, e_{j-1}^j, e_{j+1}^{j+1}, \cdots, e_n^n)$. I.e., ($\alpha$-1), ($\beta$-1)-(i) and ($\gamma$) are satisfied. Hence, by Rule I, we have $g(s) = a$. Suppose that participant i deviates from his strategy s_i to $s_i' = ((e_i^i)', (e_i^{i+1})', a_i', m_i')$ while other participants' strategies remain unchanged. To show that s is a Nash equilibrium for e, we must show $g(s_i', s_{-i}) \in L(a, e_i)$ for all i. If $a_i' = a$, we have the unanimity case (see Property 1). Hence $g(s_i', s_{-i}) = a$. Since, by definition, the weak lower contour set at a for any preference always includes a itself and $e_{i-1}^i = e_i$, we have $a \in L(a, e_{i-1}^i) = L(a, e_i)$. Therefore, assume $a_i' \neq a$. Participant i's preference announcement deviation $((e_i^i)', (e_i^{i+1})')$ does not affect the evaluation of the preference profile since $e_i^i = e_{i-1}^i$ for all i. That is,

$$
\begin{aligned}
(e_1^1, \cdots, e_{j-1}^{j-1}, e_{j-1}^j, e_{j+1}^{j+1}, \cdots, e_n^n) &= (e_1^1, \cdots, e_{j-1}^{j-1}, e_j^j, e_{j+1}^{j+1}, \cdots, e_n^n) \\
&= (e_1^1, \cdots, e_{i-1}^{i-1}, e_{i-1}^i, e_{i+1}^{i+1}, \cdots, e_n^n).
\end{aligned}
$$

Hence (γ) is satisfied. It is clear that (β-1) or (β-2) is satisfied.[8] That is, participant i is the "dictator." Therefore, by applying Rule II, we have

$g(s_i', s_{-i}) \in L(a, e_{i-1}^{\,i}) = L(a, e_i)$ when $a_i' \neq a$. ■

Lemma 2: Suppose that the number of participants is at least three; i.e., $n \geq 3$. Suppose that

$$\text{not } [n = 3 \text{ and } \#A = 2].$$

If the social choice correspondence $(F;E,A)$ has the coordinate property (i.e., $E = X_{i \in I} E_i$) and if $(F;E,A)$ satisfies the monotonicity (M) and no veto power (NVP) conditions, then game form $(g;S,A)$ defined by Rules I, II, and III with the strategy space (1) gives

$$g \cdot N_g(e) \subset F(e) \text{ for all } e \in E. \tag{3}$$

Remark 3: Notice that both monotonicity and no veto power are used to show (3), but not (2). This inclusion is important since it says that any Nash equilibrium outcome is socially optimal. This condition and the nonemptiness of the set of Nash equilibria together are sometimes called Nash implementability. Some authors refere to our definition 2 as "full Nash implementability" (see Maskin [3,4]). The case $n = 3$ and $\#A = 2$ cannot be treated the same way as in Case 2 in the following proof of Lemma 2. Lemma 3 will take care of this case.

Proof of Lemma 2. Take any e and any $\bar{a} \in g \cdot N_g(e)$ so that there is a Nash equilibrium $\bar{s}$ for e with $g(\bar{s}) = \bar{a}$. To show that $\bar{a} \in F(e)$, we shall consider two cases. In Case 1, every participant announces the same $\bar{a}$ (the unanimity case). In Case 2, there are at least two participants who announce different alternatives.

Case 1: Suppose that $\bar{s} = (\bar{s}_i) = ((e_i^i, e_i^{i+1}, \bar{a}, m_i))$ is a Nash equilibrium for e in which every participant announces the same $\bar{a}$.

Suppose participant i deviates from $\bar{s}_i$ to $s_i' = ((e_i^i)', (e_i^{i+1})', a_i', m_i')$. There are two subcases, depending on whether (γ) is satisfied or not. Let

$$J = \{k \in I : \bar{a} \in F(e_1^1, \cdots, e_{k-1}^{k-1}, e_{k-1}^{k}, e_{k+1}^{k+1}, \cdots, e_n^n)\}.$$

Subcase 1.1: (γ) is satisfied. That is, $J \neq \phi$.

There are three subsubcases, depending on the number of "breaks" in the preference announcement "chain."

Subsubcase 1.1.1: No "break," i.e., $e_i^i = e_{i-1}^i$ for all i.

Applying exactly the same argument as in Lemma 1, we have $g(s_i', \bar{s}_{-i}) \in L(\bar{a}, e_{i-1}^i) = L(\bar{a}, e_i^i)$. Since the choice of a_i' is arbitrary when we apply Rule II, we have

$$L(\bar{a}, e_i^i) = g(S_i, \bar{s}_{-i}). \tag{4}$$

where $g(S_i, \bar{s}_{-i}) = \{b \in A: b = g(s_i, \bar{s}_{-i})$ for some $s_i \in S_i\}$. On the other hand, since by hypothesis $\bar{s}$ is a Nash equilibrium for e,

$$g(S_i, \bar{s}_i) \subset L(\bar{a}, e_i) \text{ for all } i. \tag{5}$$

Hence (4) and (5) together show $L(\bar{a}, e_i^i) \subset L(\bar{a}, e_i)$ for all i. Since $\bar{a} \in F(e_1^1, e_2^2, \cdots, e_n^n)$, we have $\bar{a} \in F(e)$ by the monotonicity condition of (F;E,A).

Subsubcase 1.1.2: One "break," i.e., $e_j^j \neq e_{j-1}^j$ and $e_i^i = e_{i-1}^i$ for all $i \neq j$.

There are two cases depending on whether or not the "break" matches with (γ).

1.1.2-(i): $e_j^j \neq e_{j-1}^j$ and $e_i^i = e_{i-1}^i$ for all $i \neq j$, and $j \in J$.

First, we shall prove that $g(S_i,\bar{s}_{-i}) = A$ for all $i \notin \{j, j-1\}$. Any participant i other than j and j-1 can change his strategy from $\bar{s}_i$ to $s_i' = (e_i^i,e_i^{i+1},a_i',m_i')$, where $a_i' \neq \bar{a}$ and m_i' is chosen so that $i = (\Sigma_{r\neq i}^{n} m_r + m_i')(\mathrm{mod}\ n) + 1$. Then clearly ($\alpha$-1) is violated. Furthermore, (α-2) and [(β-1) or (β-2)] are violated since participant i is not the deviator. That is, the deviation of the alternative announcement and the "break" of preference announcement do not much. So, Rule III applies. Therefore, we have $g(s_i',\bar{s}_{-i}) = a_i'$. Since participant i other than j and j-1 can choose any a_i' in $A\backslash\{\bar{a}\}$ and $g(\bar{s}) = \bar{a}$, we have $g(S_i,\bar{s}_{-i}) = A$.

Consider participant j. If $a_j' = \bar{a}$, $(s_j',\bar{s}_{-j})$ satisfies (α-1), (β-1) and (γ). Therefore applying Rule I, we have $g(s_j',\bar{s}_{-j}) = \bar{a}$. If $a_j' \neq \bar{a}$, $(s_j',\bar{s}_{-j})$ satisfies (α-2), (β-1) and (γ). Hence, by Rule II, $g(s_j',\bar{s}_{-j}) \in L(\bar{a},e_{j-1}^{j})$. Since the choice of a_j' is arbitrary, we have $g(S_j,\bar{s}_{-j}) = L(\bar{a},e_{j-1}^{j})$.

Consider participant j-1. There are two cases, depending on whether $\bar{a} \in F(e_1^1,\cdots,e_{j-2}^{j-2},e_{j-2}^{j-1},e_j^j,\cdots,e_n^n)$ or not.

Suppose $\bar{a} \in F(e_1^1,\cdots,e_{j-2}^{j-2},e_{j-2}^{j-1},e_j^j,\cdots,e_n^n)$. Then $(s_{j-1}',\bar{s}_{-(j-1)})$ satisfies (γ) at participant j-1. If $a_{j-1}' = \bar{a}$, by Rule I, $g(s_{j-1}',\bar{s}_{-(j-1)}) = \bar{a}$. Suppose $a_{j-1}' \neq \bar{a}$. Since $e_i^i = e_{i-1}^i$ for all $i \neq j$ and the possible breaks are $(e_{j-1}^{j-1})' \neq e_{j-2}^{j-1}$ and/or $e_j^j \neq (e_{j-1}^j)'$, either (β-1) or (β-2) is satisfied at participant j-1. Therefore, Rule II is applicable at participant j-1, and hence $g(s_{j-1}',\bar{s}_{-(j-1)}) \in L(\bar{a},e_{j-2}^{j-1})$. Since $e_{j-2}^{j-1} = e_{j-1}^{j-1}$ and the choice of a_{j-1}' is arbitrary, we have $g(S_{j-1},\bar{s}_{-(j-1)}) = L(\bar{a},e_{j-1}^{j-1})$.

Suppose $\bar{a} \notin F(e_1^1,\cdots,e_{j-2}^{j-2},e_{j-2}^{j-1},e_j^j,\cdots,e_n^n)$. Then ($\gamma$) is violated. Choose m_{j-1}' so that $j-1 = (\Sigma_{r\neq j-1}^{n} m_r + m_{j-1}')(\mathrm{mod}\ n) + 1$. Then by Rule III, we have $g(s_{j-1}',\bar{s}_{-(j-1)}) = a_{j-1}'$. The choice of a_{j-1}' is arbitrary, $g(S_j,\bar{s}_{-(j-1)}) = A$.

Summarizing the above results, we have

$$g(S_i, \bar{s}_{-i}) = A \quad \text{for all } i \neq j,\ j-1;$$

$$g(S_j, \bar{s}_{-j}) = L(\bar{a}, e_{j-1}^{j}); \text{ and}$$

(6)
$$g(S_{j-1}, \bar{s}_{-(j-1)}) = \begin{cases} L(\bar{a}, e_{j-1}^{j-1}) & \text{if } \bar{a} \in F(e_1^1, \cdots, e_{j-2}^{j-2}, e_{j-2}^{j-1}, e_j^j, \cdots, e_n^n) \\ A & \text{if } \bar{a} \notin F(e_1^1, \cdots, e_{j-2}^{j-2}, e_{j-2}^{j-1}, e_j^j, \cdots, e_n^n). \end{cases}$$

We are now ready to show $\bar{a}$ is in $F(e)$. There are two cases, depending on whether $\bar{a} \in F(e_1^1, \cdots, e_{j-2}^{j-2}, e_{j-2}^{j-1}, e_j^j, \cdots, e_n^n)$ or not.

Consider the case with $\bar{a} \in F(e_1^1, \cdots, e_{j-2}^{j-2}, e_{j-2}^{j-1}, e_j^j, \cdots, e_n^n)$. Since $\bar{s}$ is a Nash equilibrium for e, by (6) we have,

$$L(\bar{a}, e_i) = A \text{ for all } i \text{ other than } j,\ j-1;$$

$$L(\bar{a}, e_{j-1}^{j}) \subset L(\bar{a}, e_j); \text{ and}$$

$$L(\bar{a}, e_{j-1}^{j-1}) \subset L(\bar{a}, e_{j-1}).$$

Since $L(\bar{a}, e_i^i) \subset L(\bar{a}, e_i)$ for all participant i other than j and j-1, $L(\bar{a}, e_{j-1}^{j}) \subset L(\bar{a}, e_j)$ for participant j, $L(\bar{a}, e_{j-1}^{j-1}) \subset L(\bar{a}, e_{j-1})$ for participant j-1, and $\bar{a} \in F(e_1^1, \cdots, e_{j-1}^{j-1}, e_{j-1}^{j}, e_{j+1}^{j+1}, \cdots, e_n^n)$, by using the monotonicity condition, we have $\bar{a} \in F(e)$.

Consider the case with $\bar{a} \notin F(e_1^1, \cdots, e_{j-2}^{j-2}, e_{j-2}^{j-1}, e_j^j, \cdots, e_n^n)$. Then for all $i \neq j$,

(7)
$$g(S_i, \bar{s}_i) = A.$$

Then, since $\bar{s}$ is a Nash equilibrium for e, we have

(8)
$$A \subset L(g(\bar{s}), e_i) \text{ for all } i \text{ other than } j.$$

Hence, we have

$$(9) \qquad \#(r \in I : L(g(\bar{s}), e_r) = A) \geq n - 1,$$

so that the hypothesis of the no veto power condition is satisfied. Therefore, we have $\bar{a} = g(\bar{s}) \in F(e)$.

<u>1.1.2-(ii)</u>: $e^j_j \neq e_{j-1}^j$ and $e^i_i = e_{i-1}^i$ for all $i \neq j$, and $j \notin J$.

We shall claim that $g(S_i, \bar{s}_{-i}) = A$ for all $i \neq j-1$. Consider participant j. Since $j \notin J$, (γ) is not satisfied at participant j. Hence setting m'_j such that $j = (\Sigma_{r \neq j}^{n} m_r + m'_j)(\text{mod } n) + 1$, participant j can get any alternative a'_j since by Rule III, $g(s'_j, \bar{s}_{-j}) = a'_j$. Therefore, we have $g(S_j, \bar{s}_{-j}) = A$. For participant i with $i \in I \backslash (j, j-1)$, the "break" does not match with Rule I or II, choosing m'_i appropriately, participant i can designate himself as the "dictator" so that we have $g(S_i, \bar{s}_{-i}) = A$. Since $\bar{s}$ is a Nash equilibrium for e, we have (8). That is, we get (9), and by no veto power, $\bar{a} \in F(e)$, which proves that $\bar{a} \in F(e)$ in Subsubcase 1.1.2.

<u>Subsubcase 1.1.3</u>: At least two "breaks."

There are two cases depending on whether there are exactly two "contiguous breaks" or not. In order to get (9), whether J is empty or not does not matter.

<u>1.1.3-(i)</u>: There are exactly two "contiguous breaks," i.e., $e^j_j \neq e_{j-1}^j$, $e^{j+1}_{j+1} \neq e^{j+1}_j$ and $e^i_i = e_{i-1}^i$ for all $i \neq j, j+1$.

From the argument of Subsubcase 1.1.2, it is now clear that we have $g(S_i, \bar{s}_{-i}) = A$ for all $i \neq j$.

<u>1.1.3-(ii)</u>: Not 1.1.3-(i), i.e., there are non-contiguous "breaks" or at least three "breaks."

Choose any participant i. Suppose that participant i deviates from $\bar{s}_i$ to $(e_i^i, e_i^{i+1}, a_i', m_i')$, where he keeps the preference announcement the same as before. Choose any $a_i' \in A$, and set m_i' such that $i = (\Sigma_{r \neq i}^{n} m_r + m_i')(\text{mod } n) + 1$. Then, $g(s_i', \bar{s}_{-i}) = a_i'$. Hence we have $g(S_i, \bar{s}_{-i}) = A$ for all i.

In both 1.1.3-(i) and (ii), since $\bar{s}$ is a Nash equilibrium for e, we have (8). That is, we get (9), and by no veto power, $\bar{a} \in F(e)$. This takes care of the case with $J \neq \phi$.

Subcase 1.2: (γ) is not satisfied. That is, $J = \phi$.

Applying exactly the same argument as 1.1.3-(ii), we have $g(S_i, \bar{s}_{-i}) = A$ for all i. Then through the same argument as (7), (8), and (9), we have $\bar{a} \in F(e)$. This takes care of Case 1.

In the following, we shall show that, in Case 2, (9) is in fact satisfied.

Case 2: Suppose $\bar{s} = (\bar{s}_i) = ((e_i^i, e_i^{i+1}, a_i, m_i))$ is a Nash equilibrium for e such that there are two participants j and k who announce different alternatives, i.e., $a_j \neq a_k$.

We shall claim that $g(S_i, \bar{s}_{-i}) = A$ for all $i \in I \backslash \{j,k\}$. There are two cases depending on the number of alternatives. Subcase 2.1 is for $\#A \geq 3$ and Subcase 2.2 is for $\#A = 2$. Participant i other than j or k can change his strategy from $\bar{s}_i$ to $s_i' = ((e_i^i)', (e_i^{i+1})', a_i', m_i')$.

Subcase 2.1: $\#A \geq 3$.

Since $\#A \geq 3$, we can choose any $a_i' \in A \backslash \{a_j, a_k\}$. Since there are three distinct alternatives a_i', a_j and a_k, Rule III should be applied. Choose m_i' such that $i = (\Sigma_{r \neq i}^{n} m_r + m_i')(\text{mod } n) + 1$. Then $g(s_i', \bar{s}_{-i}) = a_i'$. Therefore, any point in $A \backslash \{a_j, a_k\}$ is covered.

Now we must "cover" a_j and a_k. Choose any $a_i' \in A\setminus\{a_j,a_k\}$ and set m_i' such that $j = (\Sigma_{r\neq i}^{n} m_r + m_i')(\text{mod } n) + 1$. Then by Rule III, we have $g(s_i',\bar{s}_{-i}) = a_j$. Similarly, set m_i' such that $k = (\Sigma_{r\neq i}^{n} m_r + m_i')(\text{mod } n) + 1$. Then again by Rule III, $g(s_i',\bar{s}_{-i}) = a_k$. Hence a_k and a_j are "covered."

Since the above argument assumes that the number of alternatives is at least three, we must consider the case with $A = \{a_j,a_k\}$.

<u>Subcase 2.2</u>: $\#A = 2$.

If $n \geq 4$, choosing a_i' appropriately, participant i can make $\#\{v : a_v = a_j\} \geq 2$ and $\#\{v : a_v \neq a_j\} \geq 2$. Choosing m_i' appropriately, participant i can designate participant j's alternative a_j. Then by Rule III, $g(s_i',\bar{s}_{-i}) = a_j$. Apply the exactly same argument to a_k.

We have now $g(S_i,\bar{s}) = A$ for all $i \in I\setminus\{j,k\}$. Since $a_j \neq a_k$, we have $a_i \neq a_j$ or $a_i \neq a_k$ for any $j \in I\setminus\{j,k\}$. Without loss of generality, assume $a_i \neq a_j$. Then by using the same argument as above, we have $g(S_k,\bar{s}_{-k}) = A$.

Since we have $g(S_i,\bar{s}_{-i}) = A$ for all $i \neq j$ in Case 2 and $\bar{s}$ is a Nash equilibrium for e, we get (8), and hence we have (9), so that the hypothesis of the no veto power condition is satisfied. Hence we have $\bar{a} = g(\bar{s}) \in F(e)$. This takes care of Case 2, and hence this completes the proof. ■

We shall now consider the case with $n = 3$, and $\#A = 2$. Let $A = \{a,b\}$. Then there are three possible preferences on A:

(i) a is strictly preferred to b;

(ii) a is indifferent to b; and

(iii) b is strictly preferred to a.

In the proof of the following lemma, we shall abbreviate the preferences of participants as follows: "$>$" represents (i), "$=$" represents (ii), and "$<$" represents (iii).

Lemma 3: Suppose that $n = 3$, and $\#A = 2$, and let $(F;E,A)$ be a social choice correspondence satisfying the coordinate property (i.e., $E = X_{i \in I} E_i$) and if $(F;E,A)$ satisfies the monotonicity (M) and no veto power (NVP) conditions, then game form $(g;S,A)$ defined by Rules I, II and III with the strategy space (1) gives

$$(3) \qquad g \cdot N_g(e) \subset F(e) \text{ for all } e \in E.$$

<u>Proof</u>. Let $A = \{a,b\}$. Since the "unanimity" case has been proven in Lemma 2, we shall consider the case in which there are two participants who announce different alternatives.

Let $\bar{s}$ be a Nash equilibrium for e. We must show $g(\bar{s}) \in F(e)$. Suppose toward a contradiction that $g(\bar{s}) \notin F(e)$. Without loss of generality, suppose that $g(\bar{s}) = b$. Since $g(\bar{s}) \notin F(e)$, $F(e) = \{a\}$. Since $(F;E,A)$ satisfies no veto power, $F(e) = \{a\}$ if and only if

$$(10) \qquad \{j \in I : L(a,e_j) = A\} \geq 2 \text{ and } \{j \in I : L(b,e_j) = A\} \leq 1.$$

By taking into account all of the possible cases, we can conclude that the social choice correspondence $(F;E,A)$ satisfies (10) if and only if the preference profile $e = (e_1,e_2,e_3)$ is either

$$(>,>,>), \; (>,>,=) \text{ or } (>,>,<),$$

where the order of the preferences in the preference profiles is arbitrary. For each preference profile, we shall show that every strategy in which two participants announce different alternatives is not a Nash equilibrium for e.

Clearly, this is a contradiction since we assume that $\bar{s}$ is a Nash equilibrium for e. To claim that every strategy in which two participants announce different alternatives is not a Nash equilibrium for e, we shall assume that E is a singleton, i.e., E = {e}. Notice that the set of Nash equilibria will shrink if each participant's strategy space gets large. If the size of E, more precisely, the size of E_i and/or E_{i+1}, becomes large, then the size of participant i's strategy space also becomes large. Therefore, if we cannot find any Nash equilibrium for e assuming E = {e}, a fortiori, the set of Nash equilibria for e with #E ≥ 2 should be empty.

The following table shows all eighteen possible cases in which two participants announce different alternatives. The left column shows all six possible preference announcement profiles. The upper row shows all three possible true preference profiles. The entry of the table expresses the reason why this combination does not produce a Nash equilibrium. For example, suppose that the true preference profile is (>,>,=) and the alternative announcement profile is (a,b,a). Then the entry is R-1. Below the table, reason 1 shows why the strategy cannot be a Nash equilibrium for e = (>,>,=).

			(>,>,>)	(>,>,=)	(>,>,<)
a	a	b	R-1	R-3	R-3
a	b	a	R-1	R-1	R-1
a	b	b	R-2	R-2	R-2
b	a	a	R-1	R-1	R-1
b	a	b	R-2	R-2	R-2
b	b	a	R-2	R-2	R-2

Table 2: Reasons why non-unanimous and non-implementable Nash equilibria do not exist.

Reason 1 (R-1): A participant who announces b can be better off just by announcing a, so that the unanimous case will occur.

Reason 2 (R-2): Since this is not the unanimity case, and since $b \notin F(e)$, Rule II cannot be applied, but Rule III should be applied. Hence, participant 2 can change the outcome just by changing his integer announcement so that participant 2 designates the participant who announces alternative a.

Reason 3 (R-3): Since $a \in F(e)$, Rule II should be applied. Then $g(\bar{s}) = b$, which we assumed. Now participant 2 can change his alternative announcement to b. Then go to Reason 2. ∎

Remark 4: We should notice that Lemma 3's proof does not claim the non-existence of the non-unanimous Nash equilibria, but it claims that there does not exist any non-unanimous Nash equilibrium such that the game form does not implement the social choice function when $n = 3$ and $\#A = 2$. For example, let the true preference profile be $(\succ,\succ,\succ)$ (Figure 3 illustrates this situation). Suppose that $e_1^2 \neq e_2^2$, that is, there is a "break" in the preference announcements between participants 1 and 2, and the alternative announcement profile is (b,b,a). Furthermore, suppose $(m_1 + m_2 + m_3)(\text{mod } 3) + 1 = 3$. Let this strategy profile be $\bar{s}$. Then by Rule III, $g(\bar{s}) = a$. Clearly, by no veto power, $F(\succ,\succ,\succ) = \{a\}$, or $F(\succ,\succ,\succ) = \{a,b\}$. Since every participant strictly prefers a to b, $\bar{s}$ is a Nash equilibrium and the alternative announcement is non-unanimous. Since $g(\bar{s}) \in F(\succ,\succ,\succ)$, condition (3) is satisfied.

Remark 5: The case with $n = 3$ and $\#A = 2$ cannot be handled with the same way as Case 2 in Lemma 2 which claims $g(S_i,\bar{s}_{-i}) = A$ for all $i \neq j$. For example, let

$A = \{a,b\}$ and $E = \{e\} = \{(=,>,<)\}$. Assume $F(e) = \{a,b\}$. This social choice correspondence clearly satisfies monotonicity and no veto power. Choose any alternative from $g(N_g(e))$, say a. Let

$$\bar{s} = ((=,>,a,m_1),(>,<,a,m_2),(<,=,b,m_3)),$$

where m_i $(i = 1,2,3)$ is an arbitrary number between 0 and 2. The outcome for this strategy $\bar{s}$ is a by applying Rule II since $L(a,e_3) = \{a\}$ and $b \notin \{a\}$. Then $\bar{s}$ is a Nash equilibrium for e, which will be confirmed later. Notice that since e is fixed, only effective way to change the outcome is to change the alternative announcement and/or the integer announcement. However, the integer announcement does not change the outcome through the game form since any combination of two alternatives by three participants with $F(E) = A$ falls into Rule I or II. For example, suppose that participant 1 changes his alternative announcement from a to b. Then the alternative announcement profile is (b,a,b). Since both a and b are in $F(e)$, regardless of the integer announcement, this falls into Rule II. Since $L(b,e_2) = \{b\}$ and $a \notin \{b\}$, the outcome by applying Rule II is b. Since participant 1's choice in $\{a,b\}$ is $\{a,b\}$, he will not be "better off" by changing his strategy. If participant 2 changes his alternative announcement from a to b, since $L(b,e_1) = \{a,b\}$ and $a \in \{a,b\}$, the outcome will be a. Hence participant 2 cannot gain by his strategy change. Clearly the change from b to a by participant 2 is not beneficial since the outcome remains the same. Hence $\bar{s}$ is a Nash equilibrium for e. Notice that $g(S_2,\bar{s}_{-2}) = \{a\}$, and hence it does not cover $\{a,b\}$. That is, this case cannot be treated the same as Case 2 in Lemma 2. However, notice that this case does not cause any trouble since $F(e) = A$, (3) is always true.

<u>Remark 6</u>: It seems that the non-unanimous case in the proof of Lemma 3 does not

use the monotonicity condition. However, there is a natural way to define a social choice correspondence just by using no veto power, and the social choice correspondence satisfies monotonicity. Define

$$F(e_1,e_2,e_3) = \{c \in A : \#\{j \in I : L(c,e_j) = \{a,b\}\} \geq 2\}.$$

It is easy to confirm that this social choice correspondence satisfies both monotonicity and no veto power. Of course, since we used monotonicity in the unanimous case in Lemma 2, the co-existence of a Nash implementable and non-monotonic social choice correspondence is not claimed. In fact, by Maskin's theorem [3], monotonicity is a necessary condition for Nash implementability. Hence there does not exist any Nash implementable and non-monotonic social choice correspondence. We shall here give a simplest example such that a non-monotonic social choice correspondence which is not Nash implementable. Let $E_1 = \{>\}$, $E_2 = \{>\}$ and $E_3 = \{>,=\}$, and let $F(>,>,>) = \{a,b\}$ and $F(>,>,=) = \{a\}$. Then $(F;E,A)$ does not satisfy monotonicity, but it does satisfy no veto power. Suppose that the true preference profile is $(>,>,=)$ and consider a strategy with $\bar{s} = ((>,>,b,m_1),(>,>,b,m_2),(>,>,b,m_3))$, where m_i is any integer number between zero and two. Then $\bar{s}$ is a Nash equilibrium for $(>,>,=)$ and $g(\bar{s}) = b \notin F(>,>,=)$, and hence it is not Nash implementable.[9]

<u>Remark 7</u>: The integer announcements in our construction are important. If we do not admit the integer announcements, it is easy to construct a counterexample to <u>our construction</u> of the game form. Let $n = 3$ and $A = \{a,b\}$. Suppose that $E = \{e\} = \{(>,>,<)\}$ and $F(>,>,<) = \{a\}$. Since E is a singleton, the preference announcements do not matter. Since integer announcements are not allowed, the only possible way is just to announce an alternative. Let the alternative announcement be (a,a,b). Since $a \in F(>,>,<)$ and $b \notin L(a,<)$,

the outcome is b. Even when either participant 1 or 2 changes the alternative announcement from a to b, the outcome is still b. Participant 3 has no incentive to change his strategy since he prefers b to a. Hence this strategy is a Nash equilibrium for $(>,>,<)$, but the social choice correspondence is not Nash implementable by this game form.

Proof of Theorem 1: It is now trivial from Lemmata 1, 2 and 3. ■

Remark 8: The skeleton of the proof is based on Maskin's original proof [3]. The "covering" argument was first found by Williams [11,12].[10]

The size of each participant's strategy space is not independent of the number of participants in the society. However, only the announcement of an integer between zero and n - 1 depends on the number of participants. On the other hand, the unidentified author's specification of a participant's strategy space depends on the number of participants in two respects. One is the announcement of an integer, and the other is the announcement of the whole preference profile. That is,

$$S_i = E_1 \times E_2 \times \cdots E_n \times A \times \{0,1,\cdots,n-1\} \tag{U}$$

for each i. The dependence of our strategy spaces on the number of participants is significantly weaker than the unidentified author's strategy spaces when the number of participants is larger.

The game form constructed by Rule I, Rule II and Rule III is not the only game form which uses two participant characteristic sets, the set of alternatives, and the integer between zero and n - 1 as its strategy space. In the same spirit, we can construct some other game forms which use almost the same strategy space as (1). For example, every participant can announce his left side neighbor's preferences, right side neighbor's preferences, an alternative and an integer. More precisely, consider the following game form.

(1′) $S_1 = E_n \times E_2 \times A \times \{0,1,\cdots,n-1\}$;

$S_i = E_{i-1} \times E_{i+1} \times A \times \{0,1,\cdots,n-1\}$ for $i = 2,3,\cdots,n-1$; and

$S_n = E_{n-1} \times E_1 \times A \times \{0,1,\cdots,n-1\}$.

We shall denote an element of S_i by $s_i = (e_i^{i-1}, e_i^{i+1}, a_i, m_i)$, where $i - 1 = n$ if $i = 1$ and $i + 1 = 1$ if $i = n$. e_i^j is the announcement by participant i about participant j's preferences. The conditions on the game form will be changed so we shall introduce slightly different notation:

(α-1) $a_i = a^*$ for all i;

(α-2) $a_i = a^*$ for all $i \neq j$ and $a_j \neq a^*$;

(β-1′) $e_{i-1}^{i} = e_{i+1}^{i}$ for all i or

$e_{i-1}^{i} = e_{i+1}^{i}$ for all i except j-1 or

$e_{i-1}^{i} = e_{i+1}^{i}$ for all i except j-1 and j+1;

(β-2′) $e_{i-1}^{i} = e_{i+1}^{i}$ for all i except j+1; and

(γ′) $a^* \in F(e_1^n, \cdots, e_{j-2}^{j-1}, e_{j-1}^{j}, e_{j+2}^{j+1}, \cdots, e_{n-1}^{n})$,

where again $i - 1 = n$ if $i = 1$ and $i + 1 = 1$ if $i = n$. Construct a game form as follows:

<u>Rule I′</u>: If [(α-1) and (β′) and (γ′)] is true, then $g(s) = a^*$.

<u>Rule II′</u>: If there exists a participant, say j, such that

(α-2) and [(β-1′) or (β-2′)] and (γ′),

then

$$g(s_j, s_{-j}) = \begin{cases} a_j & \text{if } a_j \in L(a^{*}, e_j^{j-1}) \\ a^{*} & \text{otherwise.} \end{cases}$$

Rule III′: If neither Rule I′ nor Rule II′ is applicable, then

$$g(s_1, s_2, \cdots, s_n) = a_t,$$

where $t = (\Sigma_{k=1}^{n} m_k)(\text{mod } n) + 1$.

Under this game form constructed by (1′), Rule I′, Rule II′ and Rule III′, we still have Theorem 1. The proof requires almost no modification. In the same spirit, we can construct some other game forms in which the strategy space contains two participant characteristic sets and Theorem 1 is still true.

From the point of view of the size of the strategy space, the game form constructed by Rules I, II and III (game form 1) and the game form constructed by Rules I′, II′ and III′ (game form 2) are identical. Let us now add one more criterion to the comparison of game forms. In game form 1, any participant must announce only one preference other than his own. On the other hand, in game form 2, he must announce two. That is, if the designer wants to construct a game form which "minimizes" the number of announcements of preferences of other participants, he prefers game form 1 to game form 2.

4. Concluding Remarks

1. We have established that our strategy space defined by (1) is significantly smaller when the number of participants is large. However, our strategy space still depends on the number of agents. Each participant is required to announce an integer between 0 and n-1. One of the open questions

is whether it is possible to provide a strategy space which is independent of the number of participants to get Nash implementability. As we noticed in remark 3 just after the statement of Theorem 1, the integer announcement is related to the size of the alternative set A. An immediate possible solution is to employ Williams' method, i.e., condition (6) in [12]. However, this also depends on the number of participants in a certain way. His condition (6) is,

(11) There exists a mapping P_A from A^n to A such that for any $(\bar{a}_i)$ in A^n and for each j, the mapping $P_A(\cdot, \bar{a}_{-j}): A \to A$ is surjective.

Williams employed (11) as one of the assumptions. Notice that (11) does not say anything about the strategy space. Therefore, one should ask whether it is possible to get Nash implementability with the participant strategy space

$$S_i = E_i \times E_{i+1} \times A$$

assuming (11). The author's conjecture is that it is not, but it has not been confirmed.

2. Is our strategy space (1) "minimum"? In order for this question to be meaningful, we have to establish some measurement of the size of the strategy space. The theory of mechanism design has two fundamental branches. One is implementation theory and the other is realization theory. Realization theory has developed some measures for the comparison of "message" spaces (see the references in [11]). First, we have to ask ourselves whether or not we can apply the same method to compare sets. If not, we have to develop a measurement theory in implementation.

3. In order to get Nash implementability in our framework, it is believed

that it is necessary for each participant's strategy space to include the set of environments E (see Sonnenschein [p.16, 9] and Maskin [section 3, 4]). As Theorem 1 shows, this is not true. We have not confirmed that employing E itself as each participant's strategy space is sufficient for Nash implementability. We add something more to E to get Nash implementation (see (U)). Our strategy space specification in (1) uses some part of E with an additional set $A \times \{0,1,\cdots,n-1\}$, but it does not use all of E. Hence we need new interpretations of the Revelation Principle by using (1).

4. Recently, a new generalized approach to Nash implementation theory has been proposed by Hurwicz [2]. He uses the set of <u>choice function</u> profiles instead of the set of preference profiles. Possibly the simplest definition of the choice function is:

Definition 5: Let $\underline{A} = \{B \subset A: 1 \leq \#B \leq 2\}$. A function $c_i: \underline{A} \to \underline{A}$ is called a <u>choice function of participant i</u> if and only if: for all $\{a,b\} \in \underline{A}$,

(i) $c_i(\{a,b\}) \neq \emptyset$; and

(ii) $c_i(\{a,b\}) \subset \{a,b\}$.

Although the concept of choice function does not presuppose any complete ordering, if there is a binary relation, it should satisfy both totality and reflexivity, but not transitivity. Hurwicz shows that most of the results in Nash implementation theory can be obtained by using the choice function approach. That is, in Nash implementation, we do not need the transitivity of each participant's preferences. Therefore, without a surprise, our Theorem 1 is also true without each participant's preferences. Even though we change the definition of participant j's weak lower contour set with e_j at a from $L(a,e_j)$ to

$$L(a,c_j) = \{d \in A : d \in c_i((a,b)) \text{ for some } (a,b) \in \underline{A}\},$$

Definitions 1 through 4 are still meaningful. That is, to define important concepts in Nash implementation theory we do not need transitivity of each participant's preferences.

FOOTNOTES

1. The unidentified author just specified the game form. A detailed proof of the unidentified author's claim can be found in Saijo [7].

2. The author wishes to thank Steven Williams who pointed out some of the impacts of our strategy space specification on the interpretation of the Nash equilibrium concept and these references.

3. A counterexample to the Maskin-Williams type construction will be given in Remark 7 later.

4. Note that there cannot be more than one such j. Hence, Rule II is well defined.

5. Note that in $t = (\Sigma_{k=1}^{n} m_k)(\mathrm{mod}\ n) + 1$ and $0 \leq m_k \leq n - 1$, we have the following property:

(i) $1 \leq t \leq n$;

(ii) For given t with $1 \leq t \leq n$ and for given $\Sigma_{k\neq j}^{n} m_k$, there exists m_j' with $0 \leq m_j' \leq n - 1$ such that

$$t = (\Sigma_{k\neq j}^{n} m_k + m_j')(\mathrm{mod}\ n) + 1; \text{ and}$$

(iii) t is unique in (ii).

6. To understand the game form and the unanimity case well, consider the following "boundary" case between Rule I and Rule III.

(α-1) $a_i = a^x$ for all i;

(β) $e_i^i = e_{i-1}^i$ for all i except j;

(γ) $a^x \in F(e_1^1, \cdots, e_{j-1}^{j-1}, e_{j-1}^{j}, e_{j+1}^{j+1}, \cdots, e_n^n)$; and

(γ)' $a^x \in F(e_1^1, \cdots, e_{j-2}^{j-2}, e_{j-2}^{j-1}, e_j^j, \cdots, e_n^n)$.

Even though [(α-1) & (β) & (γ)'], that is, [(α-1) & (β-2) & (γ)] is satisfied, we should not apply Rule III. This case does satisfy [(α-1) & (β) & (γ)]; that is, [(α-1) & (β-1)-(i) & (γ)]. Hence we should apply Rule I, and we will have a^x as an outcome. However, since (α-1) is satisfied for both cases, this is a case of unanimity. Therefore, both Rules I and III produce a^x as an outcome. A possible trouble is the case that starting from the above strategy, a participant deviates from his strategy to another strategy while other participants' strategies remain unchanged. This is not a problem either. In this situation, a possible trouble-maker is participant j-1 (see Figure 1: rectangles show the preference announcements and the right hand characters of the rectangles represent the alternative announcements.). By changing participant (j-1)'s strategy with $a'_{j-1} \neq a^{*}$, Rule II is applicable, and the outcome is determined uniquely. See also (6) in Subsubcase 1.1.2 in the proof of Lemma 2.

7. In the social choice literature, the "no taboo" condition is known as the "citizen sovereignty" condition.

8. Figure 2 shows the situation in which (β-2) condition is actually used. Since $a'_i \neq a$, (α-2) is satisfied. Since $e_{i+1}^{i+1} \neq (e_i^{i+1})'$, ($\beta$-2) is satisfied for participant i. (γ) is also satisfied since participant i's announcement of preferences does not affect the preference profile evaluation. Hence this is the case with (α-2), (β-2) and (γ).

9. $\bar{s}$ is in fact a Nash equilibrium for $(>,>,=)$. Since $\#E_1 = \#E_2 = 1$, participant 1 can only change the alternative and/or the integer announcements. Suppose that participant 1 changes the alternative announcement from b to a. Since $b \in F(e_3^1, e_2^2, e_3^3) = F(>,>,>) = \{a,b\}$ and $a \notin L(b, e_3^1)$, by applying Rule II, we have $g(s'_1, \bar{s}_{-1}) = b$. Notice also that the integer announcement by participant 1 does not affect the outcome. Participant 2 has slightly more freedom than participant 1. Since $E_3 = \{>,=\}$, participant 2 can change preference announcement e_2^3 from ">" to "=." However, this change will not affect the outcome. If participant 2 announces b, then this is the unanimity case and hence $g(s'_2, \bar{s}_2) = b$. Therefore, suppose $a'_2 = a$. But since $b \in F(e_1^1, e_1^2, e_3^3) = F(>,>,>) = \{a,b\}$ and $a \notin L(b, e_1^2)$, by Rule II, we have $g(s'_2, \bar{s}_{-2}) = b$. If participant 2 does not change his preference announcement, the situation is the same as participant 1. Participant 3 has no incentive to deviate from his strategy since he is indifferent between a and b.

10. Since Williams did not use the integer announcement as a part of participant's strategies, he introduced an additional assumption on the environment set:

If $n \geq 4$ and $a \in A$, then $F^{-1}(a)$ contains at least two distinct environments; if $n = 3$ and $a \in A$, then $F^{-1}(a)$ contains at least three distinct environments.

Here, we do not need to impose any restriction on E since the above mentioned trouble by Williams can be avoided by designating some other participant's alternative using the integer announcement. In an earlier version of this paper, there was a restriction which says $\#E_i \geq 2$ for all i. I am very grateful to Prof. Hurwicz who suggested that any restriction on E is unnecessary.

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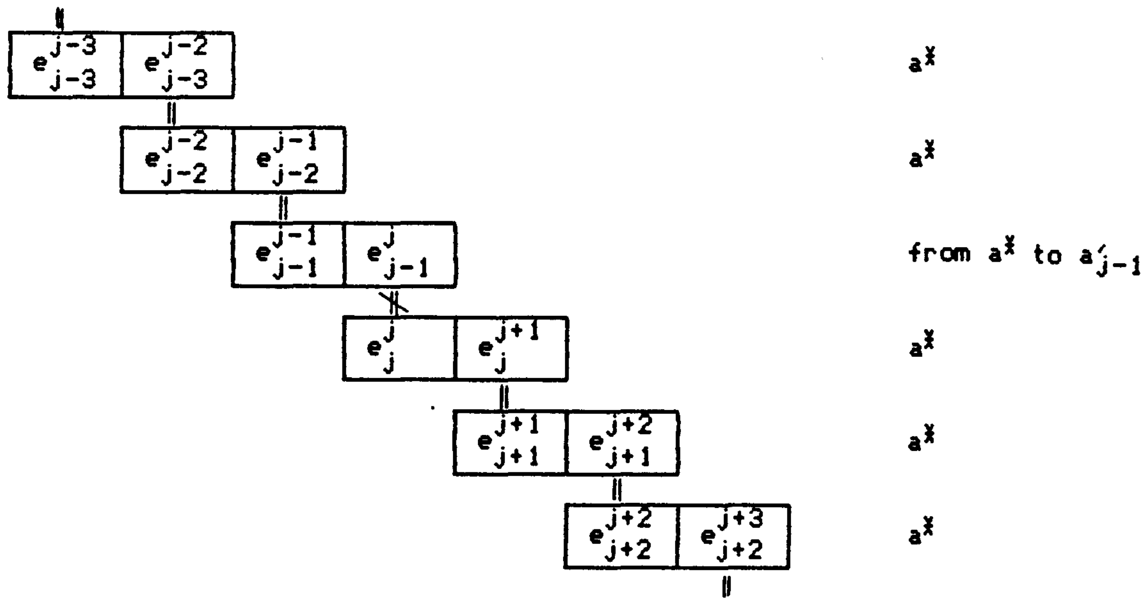
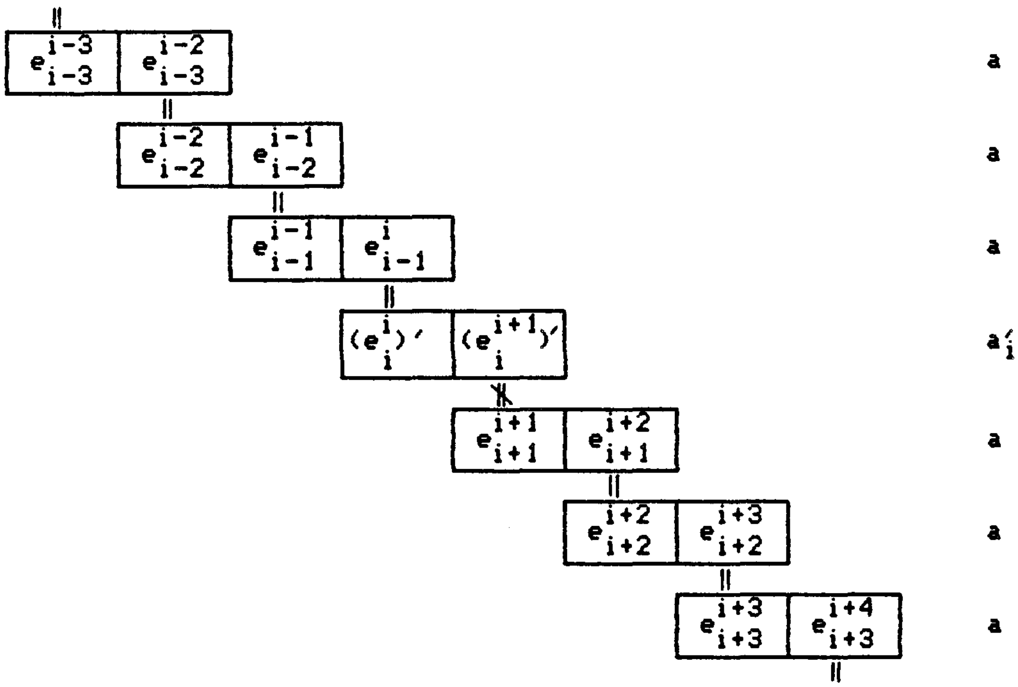


$a^* \in F(e_1^1, \cdots, e_{j-1}^{j-1}, e_{j-1}^{j}, e_{j+1}^{j+1}, \cdots, e_n^n)$ and

$a^* \in F(e_1^1, \cdots, e_{j-2}^{j-2}, e_{j-2}^{j-1}, e_j^j, \cdots, e_n^n)$.

Figure 1: A "boundary" case between Rule I and Rule II.



<u>Figure 2</u>: Participant i is the deviator with (α-2), (β-2) and (γ).

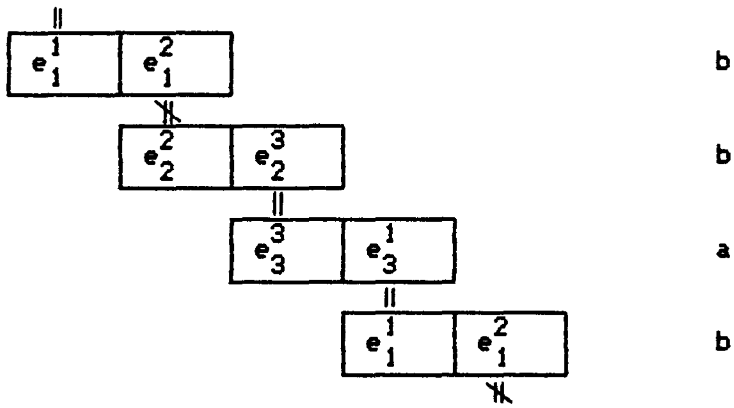


Figure 3: A non-unanimous Nash equilibrium.