INTRANSITIVE INDIFFERENCE AND REVEALED PREFERENCE

by

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ABSTRACT

In recent years, preferences without transitive indifference have been much discussed in consumer theory. Surprisingly, we can show that their singleton valued choice functions turn out to be indistinguishable from those of classical transitive preferences.

Let $\succsim$ be a preference relation on any set X, with choice function h. We prove that $\succsim$ is reflexive, total, and semitransitive (or pseudotransitive) if and only if h satisfies the Strong Axiom of Revealed Preference. It follows that choice behavior generated by classical transitive preferences is indistinguishable from that generated by the more general preferences discussed in [2], [3], [4], and [10].

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1. Introduction

In recent years, there have appeared several important papers discussing numerical representations of preference, without the assumption of transitive indifference (e.g., Bridges [2], Chipman [3], Fishburn [4], Scott [10]). The following weak forms of transitivity have been introduced:

$\succsim$ is <u>semitransitive</u>[2] if and only if $x \succsim y \succ z \succ w$ or $x \succ y \succ z \succsim w$ implies $x \succ w$ for all $x,y,z,w \in X$.

$\succsim$ is <u>pseudotransitive</u>[3] if and only if $x \succ y \succsim z \succ w$ implies $x \succ w$ for all $x,y,z,w \in X$

where $\succ$ is the asymmetric part of $\succsim$.

The work of Chipman [3] and Fishburn [4] is based on semitransitivity, and that of Bridges [2] and Fishburn [5] on pseudotransitivity. On the other hand, Scott [10] showed that if $\succsim$ is semitransitive and pseudotransitive on a finite set, then it is representable by u in the sense that $x \succ y$ if and only if $u(x) \geq u(y) + 1$. He also showed the converse.

However, it is not known how much intransitive indifference affects the

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1. I am grateful to Professors M.K. Richter and N.C. Yannelis for their helpful comments.

2. Chipman [3] and Fishburn [4] call this semitransitivity of $\succ$. However, in order to emphasize that this condition is stronger than transitivity of $\succ$ and weaker than transitivity of $\succsim$, we call this semitransitivity of $\succsim$.

3. See Bridges [2].

individual choice behavior. Along this line, the following problems are examined in this paper.

(1) Given a choice function, can we tell whether it could be generated by a semitransitive preference?

(2) Given a choice function, can we tell whether it could be generated by a pseudotransitive preference?

(3) Can we distinguish between choice functions generated by classical transitive preferences and those generated by the more general preferences discussed in [2], [3], [4], or [10]?

The answers to the first two problems will turn out to be positive and that of the third, negative. To answer these questions, we give revealed preference axioms characterizing semitransitive and pseudotransitive preferences. As a result, we can see that semitransitive, pseudotransitive and transitive preferences all lead to identical choice behavior.

2. Definitions

In the following, X will be any set (finite or infinite) and $\succsim$ will be a reflexive binary relation on X. The relations $\succ$ and $\sim$ are defined as usual:

$x \succ y$ if and only if $x \succsim y$ and not $(y \succsim x)$
(so $\succ$ is the asymmetric part of $\succsim$)

$x \sim y$ if and only if $x \succsim y$ and $y \succsim x$ (so $\sim$ is the symmetric part of $\succsim$)

("$x \succsim y$" is interpreted as "x is as good as y", and "$x \succ y$" is interpreted as "x is strictly preferred to y").

It is easily seen that $\succ$ is irreflexive.

We say that $\succsim$ is

<u>total</u> if $x \neq y \Rightarrow [x \succsim y$ or $y \succsim x]$

<u>transitive</u> if $[x \succsim y$ and $y \succsim z] \Rightarrow x \succsim z$

<u>semitransitive</u> if $[x \succsim y \succ z \succ w$ or $x \succ y \succ z \succsim w] \Rightarrow x \succ w$

<u>pseudotransitive</u> if $[x \succ y \succsim z \succ w] \Rightarrow x \succ w$

for all $x,y,z,w \in X$.

Note that $\succsim$ is semitransitive if and only if the following condition holds:

$$[x \succ y \text{ and } y \succ z] \Rightarrow [x \succ w \text{ or } w \succ z] \text{ for all } x,y,z,w \in X.$$

Note that $\succsim$ is pseudotransitive if and only if the following condition holds:

$$[x \succ y \text{ and } z \succ w] \Rightarrow [x \succ w \text{ or } z \succ y] \text{ for all } x,y,z,w \in X.$$

We assume there is a distinguished family $\mathcal{B}$ of subsets B of X. By analogy with classical consumer theory, we call each element B of $\mathcal{B}$ a <u>budget</u>. We call $(X,\mathcal{B})$ a <u>choice space</u>. A <u>choice</u> on $(X,\mathcal{B})$ is a function defined on $\mathcal{B}$ that assigns to each $B \in \mathcal{B}$ a subset $h(B)$ of B. The intuition, of course, is that $h(B)$ is the set of alternatives chosen by a decision-maker from the alternatives in the "budget" B.

Next we define two notions of revelation. If, for some budget $B \in \mathcal{B}$ and some $x,y \in X$, we have $x \in h(B)$, $y \in B$ and $x \neq y$, then we say that x is <u>directly revealed preferred to</u> y and we write xSy. If xSy or, if for some finite sequence $u_1,\ldots,u_m$ of elements of X, we have $xSu_1S,\ldots,Su_mSy$, then we say that x is <u>revealed preferred to</u> y and we write xHy.

Corresponding to these two notions of revelation, we have two axioms.

The <u>Weak Axiom</u> (WA): S is asymmetric.

The <u>Strong Axiom</u> (SA): H is asymmetric.

We will use the notion of rationality made precise in [9] as follows. A choice h is called <u>rational</u> if there exists a preference $\succsim$ on X such that for every $B \in \mathcal{B}$,

$$h(B) = \{x \in B: \forall y_{y \in B}\ x \succsim y\}.$$

Then $\succsim$ is said to <u>rationalize</u> h. If there exists such a preference $\succsim$ that is reflexive, total and transitive (resp. semitransitive, pseudotransitive), then h is said to be <u>transitive</u> (resp. semitransitive, pseudotransitive)-<u>rational</u>.

We call a choice h <u>strongly-rational</u> if there exists a preference on X such that, for every $B \in \mathcal{B}$,

$$h(B) = \{x \in B: \forall y_{y \in B}\ x \succsim y \text{ and } \forall y_{y \in B, x \neq y} \text{ not } (y \succsim x)\}.$$

Then $\succsim$ is said to <u>strongly rationalize</u> h. If there exists such a preference $\succsim$ that is reflexive, total and transitive (resp. semitransitive, pseudotransitive), then h is said to be <u>transitive</u> (resp. semitransitive, pseudotransitive) -<u>strongly-rational</u>.

3. Results

When is a choice h semitransitive-rational? When is a choice h pseudotransitive-rational? The answers are:

<u>Theorem 1</u>: Let a choice h be singleton-valued. Then h is semitransitive-rational if and only if it satisfies the SA.

<u>Theorem 2</u>: Let a choice h be singleton-valued. Then h is pseudotransitive-rational if and only if it satisfies the SA.

To prove the above theorems, we use several lemmas.

<u>Lemma 1</u>: If $\succsim$ is semitransitive, then $\succ$ is transitive. Moreover if $\succsim$

is total, then $x \succ y \succsim z$ implies $x \succsim z$ for all $x,y,z \in X$.

<u>Proof</u>: If $x \succ y$ and $y \succ z$, then $x \succ y \succ z \succsim z$. Since $\succsim$ is semitransitive, $x \succ z$. Now, suppose $x \succ y \succsim z$ and $z \succ x$. Then $z \succ x \succ y \succsim z$. By semitransitivity of $\succsim$, $z \succ z$ which contradicts to irreflexivity of $\succ$. Since $\succsim$ is total, $x \succsim z$. Q.E.D.

<u>Remark 1</u>: There are reflexive, total binary relations $\succsim$ which are not semitransitive but whose asymmetric parts $\succ$ are transitive. For example, let $X = \{m\colon m$ is a positive integer$\}$ and define $\succsim$ by:

$$m \succsim n \text{ if } \begin{array}{ll} m + 1 \geq n, & m \neq 2 \text{ or} \\ m + 3 \geq n, & m = 2. \end{array}$$

Then asymmetric part $\succ$ is:

$$m \succ n \text{ if } \begin{array}{ll} m > n + 1, & n \neq 2 \text{ or} \\ m > n + 3, & n = 2. \end{array}$$

It is easy to show that $\succ$ is transitive. On the other hand, $5 \succ 3 \succ 1 \succsim 2$ but not $(5 \succ 2)$. So $\succsim$ is not semitransitive.

Hence from Lemma 1 and Remark 1, semitransitivity of $\succsim$ is stronger than transitivity of the asymmetric part $\succ$ of $\succsim$.

<u>Lemma 2</u>: If $\succsim$ is pseudotransitive, then $\succ$ is transitive. Moreover, if $\succsim$ is total, then $x \succ y \succsim z$ implies $x \succsim z$ for all x,y,z in X.

<u>Proof</u>: This can be proved similarly to Lemma 1. Q.E.D.

<u>Remark 2</u>: There are reflexive, total binary relations $\succsim$ which are not pseudotransitive but whose asymmetric parts are transitive. For example, let

$X = \{m: m$ is a positive integer$\}$ and define $\succsim$ by:

$$m \succsim n \text{ if } m + 1 \geq n \text{ or } m = 1, \; n = 4.$$

Then asymmetric part $\succ$ is:

$$m \succ n \text{ if } m > n + 1, \; n \neq 1 \text{ or } m > n + 1, \; m \neq 4.$$

It is easy to show that $\succ$ is transitive. On the other hand, $4 \succ 2 \succsim 3 \succ 1$ but not $(4 \succ 1)$. So $\succsim$ is not pseudotransitive.

Hence from Lemma 2 and Remark 2, pseudotransitivity of $\succsim$ is stronger than transitivity of the asymmetric part $\succ$ of $\succsim$.

Lemma 3: Let h be a singleton-valued choice. If $\succsim$ is reflexive, total and semitransitive, then "$\succsim$ rationalizes h" implies "$\succsim$ strongly rationalizes h".

Proof: Suppose, by way of contradiction, that for some $B \in \mathcal{B}$, there exists $x_1 \in B$ such that $x_1 \succsim y$, $x_1 \neq y$ and $\{y\} = h(B)$. Then there exists $x_2 \in B$ such that $x_2 \succ x_1$. Note that $x_2 \neq x_1$ since $\succ$ is irreflexive. Also $x_2 \neq y$ since otherwise $y \succ x_1$, which contradicts $x_1 \succsim y$. Since $x_2 \succ x_1 \succsim y$, $x_2 \succsim y$ by Lemma 1.

As $x_2 \neq y$ and $x_2 \succsim y = h(B)$, there exists $x_3 \in B$ such that $x_3 \succ x_2$. Note that $x_3 \neq x_2$ since $\succ$ is irreflexive. Also $x_3 \neq x_1$ since otherwise $x_1 \succ x_2$ which contradicts to $x_2 \succ x_1$. Moreover $x_3 \neq y$, for otherwise $y \succ x_2 \succ x_1$ hence $y \succ x_1$, contradicting $x_1 \succsim y$. Therefore $x_3 \succ x_2 \succ x_1 \succsim y$ and $x_1, x_2, x_3, y \in B$. By semitransitivity of $\succsim$, $x_3 \succ y$ which contradicts the fact that $\succeq$ rationalizes h. Q.E.D.

Lemma 4: If h is semitransitive-strongly-rational, then h satisfies the SA.

Proof: Suppose, by way of contradiction, that $x_1Sx_2S,\ldots,Sx_nSx_1$. Strong-rationality then requires $x_1 \succ x_2 \succ \ldots \succ x_1$, hence $x_1 \succ x_1$ be Lemma 1. But this contradicts the irreflexivity of $\succ$. Q.E.D.

Lemma 5: Let h be a singleton-valued choice. If h is pseudotransitive-rational, then h satisfies the SA.

Proof: Suppose, by way of contradiction, that $x_1Sx_2S,\ldots,Sx_nSx_1$. Since x_iSx_{i+1} and h is singleton-valued, there exists y_i such that $x_i \succsim y_i \succ x_{i+1}$ (choose $y_i = x_i$ if $x_i \succ x_{i+1}$). Then $x_1 \succsim y_1 \succ x_2 \succsim y_2 \succ x_3 \succsim \cdots \succsim y_n \succ x_1$. Repeated appeals to the pseudotransitivity of $\succsim$ yields $x_1 \succsim y_1 \succ x_1$, which is a contradiction. Q.E.D.

Lemma 6: If h satisfies the SA, then h is transitive-strongly-rational.

Proof: Let P be the binary relation on X defined by: $xPy \Leftrightarrow xHy$. Then P is transitive by definition of H, and irreflexive by the SA. Then there exists a total, irreflexive, transitive extension $\bar{P}$ of P on X, by the axiom of choice. Let $x \succsim y$ if $x\bar{P}y$ or $x = y$. Then $\succsim$ is reflexive, total and transitive. And the asymmetric part $\succ$ of $\succsim$ is $\bar{P}$. As M.K. Richter's proof ([8] Theorem 1) shows, any such $\succsim$ strongly rationalizes h. Q.E.D.

Remark 3: In fact, we have an even stronger conclusion from Lemma 6 in the sense that the asymmetric part $\succ$ of $\succsim$ is total.

Remark 4: It is easy to show that if $\succsim$ is transitive, then $\succsim$ is semitransitive and pseudotransitive. As an example in Bridges ([2], Section 3) shows, the converse is not true.

Proof of Theorem 1: Necessity of the SA: Lemmas 3 and 4. Sufficiency of the SA: Lemma 6 and Remark 4. Q.E.D.

Proof of Theorem 2: Necessity of the SA: Lemma 5. Sufficiency of the SA: Lemma 6 and Remark 4. Q.E.D.

For a characterization, we have

Theorem 3: Let h be a singleton-valued choice. Then the following conditions are equivalent.

(i) h satisfies the SA.

(ii) h can be rationalized by a preference $\succsim$ that is reflexive, total, and transitive, and whose asymmetric part $\succ$ is also total.

(iii) h can be rationalized by a preference $\succsim$ that is reflexive, total, and transitive.

(iv) h can be rationalized by a preference $\succsim$ that is reflexive, total, semitransitive, and pseudotransitive.

(v) h can be rationalized by a preference $\succsim$ that is reflexive, total, and semitransitive.

(vi) h can be rationalized by a preference $\succsim$ that is reflexive, total, and pseudotransitive.

Proof: (i) implies (ii) by Lemma 6 and Remark 3. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), a fortiori. (v) implies (i) by the necessity part of Theorem 1. So (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

And (ii) $\Rightarrow$ (vi), a fortiori. (vi) implies (i) by Lemma 5. So (i) $\Rightarrow$ (ii) $\Rightarrow$ (vi) $\Rightarrow$ (i). Q.E.D.

Remark 5: One implication of Theorem 3 is that even if we start with a reflexive, total, and semitransitive (or pseudotransitive) preference $\succsim$, there exists some nice reflexive, total, and transitive preference which rationalizes the same demand as $\succsim$. For the purpose of equilibrium existence theory, for example, this allows us to replace "difficult" preferences with ones that are easier to work with. For further ideas about this, see [7].

Remark 6: Note that Theorems 1, 2 and 3 do not depend on any topological properties of consumption set X, or any special properties of budget sets.

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