MICROECONOMETRIC ANALYSIS OF KINK POINTS:

THE CONVEX CASE

by

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ABSTRACT

Consumer demand and production microeconomic models with kink points are analyzed. Kink points in demand or production schedules may arise due to binding nonnegative constraints, quantity rationing, production quotas or increasing block prices. The occurance of various kink points divides the schedule into various regimes with structural change. The switching conditions for the regimes can be determined by the comparison of market prices with some appropriate virtual prices. Several equivalent switching regime conditions are also derived in terms of demand functions. The proposed approach unifies the direct and dual approaches and suggests appropriate econometric models for convex budget sets in consumer demand and production economics.

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Lung-Fei Lee and Mark M. Pitt (*)

1. Introduction.

The kink points in the microeconomic analysis of consumer demand and production schedules may occur in many circumstances. Binding nonnegative constraints on consumption goods, inputs or outputs in production, create kink points on the boundary of the commodities space. Increasing block prices on demanded goods or decreasing block prices on produced outputs will imply convex budget set or convex production schedules, respectively. The vertices of the convex set are kink points. Increasing block prices on consumption goods can be regarded as a relatively weak form of quantity rationing on individual consumers. Strict quantity rationing on consumption for individual consumers or production quotas on individual producers impose upper bounds on the demand or production schedules on the rationed goods. For econometric analysis, the kink points are usually atoms in the probability space. The proper econometric specification of the problems will involve limited dependent variables; see, e.g., Tobin [1958], Amemiya [1974] and Maddala [1983]. Most of the econometric analyses of kink points are in the two goods models with a few exceptions. Gronau [1974] and Heckman [1974] analyze the labor supply problems of females. Burtless and Hausman [1978], Hausman [1979] and Wales and Woodland [1979] consider the labor supply on convex budget sets due to progressive taxation. Commodities such as water may also be priced with increasing block rates; see, Billings and Agthe [1980][1]. Demand systems with more than two goods and nonnegative binding constraints have been considered in Wales and Woodland [1983] and Lee and Pitt [1983].

The latter two articles have considered the multivariate models. The approach in Wales and Woodland [1983] is a direct approach with the specification of direct utility function. The probabilities of binding nonnegative constraints are determined by the Kuhn-Tucker inequalities conditions. The approach in Lee and Pitt [1983] specifies the notional demand equations or indirect utility function.[2] The observed demand for the consumed goods are related to the notional demand equations via virtual prices (Rothbarth [1941]). Virtual prices transform binding zero consumption quantities into nonbinding quantities, and provide a vigorous justification for structural shifts in the observed demand equations across different demand regimes. The conditions that determine the occurrence of the binding nonnegative constraints are provided in terms of notional demand equations. For the econometric analysis of kink points in the general situation, the above analysis needs to be extended and, in particular, the conditions that characterize the occurrence of different kink points and the different demand or production regimes need to be derived. In this article, a general analysis framework is introduced from which the regime conditions can be easily derived. Intuitive interpretation will be provided. The analysis extends the analysis in Lee and Pitt [1983] to the increasing block prices and quantity rationing cases and unifies the approach in Wales and Woodland [1983] and Lee and Pitt [1983] for demand analysis with binding nonnegative constraints.

This article is organized as follows. In Section 2, a general model of consumer demand on convex budget sets is specified. An approach is introduced in Section 3 to derive the switching regime conditions that characterize the occurrence of different regimes. Virtual prices are introduced from marginal utilities. In Section 4, we consider the various ways that virtual prices can be constructed. Virtual prices can be derived from direct utility function or demand equations. The conditional demand equations are shown to be related to the unconditional demand equations via the virtual prices. Alternate but equivalent regime criteria are derived in Section 5. In Section 6 we illustrate the analysis in a three-goods case. Econometric specification is considered. The linear expenditure system is analyzed in some detail and the likelihood function is derived in that section. The approach is extended to the analysis of production economics in Section 7. Final conclusions are in Section 8. Proofs of the theorems are collected in an appendix.

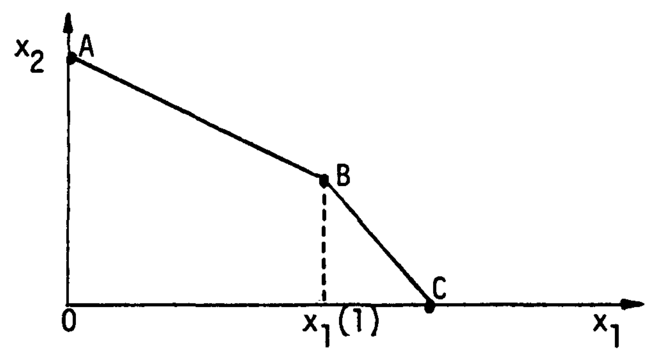
2. Consumer Demand on Convex Budget Sets.

Convex budget sets which differ from linear budget sets occur when there are binding nonnegative constraints, quantity rationing or increasing block prices. Consumers are assumed to be utility maximizers with given budget constraints.

Consider the two-goods case with increasing block prices for the commodity x_1. The marginal unit price is p_{11} if the demanded quantity of x_1 is less than or equal to $x_1(1)$, and the marginal unit price is p_{12}, $p_{12} > p_{11}$, if the quantity is greater than $x_1(1)$. The budget set for this case is illustrated in Figure 1.

FIGURE 1.

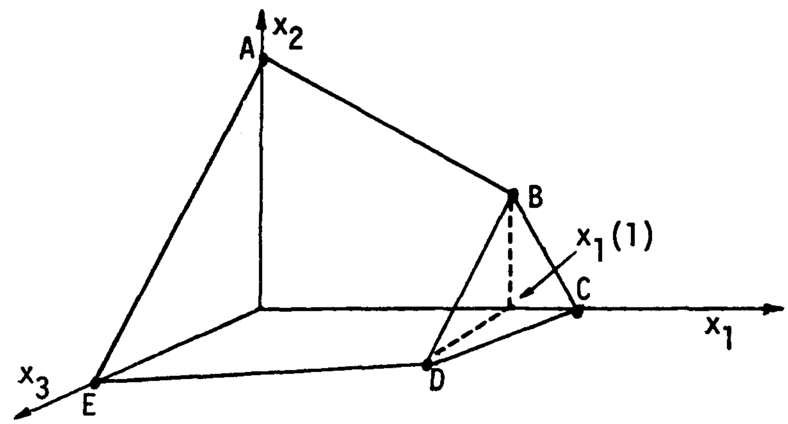
Two-goods case with increasing block price on x_1



With given income M, the extended budget line AB is $p_{11}x_1 + p_2x_2 = M$ and the budget line BC is $p_{12}x_1 + p_2x_2 = M + (p_{12} - p_{11})x_1(1)$. In addition to the nonnegative boundary points A and C, the point B is also a kink point. The quantity rationing case with quantity ceiling $x_1(1)$ can be regarded as a special case with $p_{12} = \infty$.[3] Figure 2 illustrates the three-goods case with increasing block prices for the commodity x_1.

FIGURE 2.

- Three-goods case with increasing block price on x_1



The budget plane ABDE is determined by $p_{11}x_1 + p_2x_2 + p_3x_3 = M$ and the budget plane BCD is based on $p_{12}x_1 + p_2x_2 + p_3x_3 = M + (p_{12} - p_{11})x_1(1)$ where $p_{12} > p_{11}$. The above cases have only two block rates. In general if there are I_1, $I_1 \geq 2$, different rates for x_1 which are changed at knots $x_1(1), x_1(2), \ldots, x_1(I_1-1)$, where $x_1(i) < x_1(i+1)$ for $i=1,\ldots,I_1-2$, the budget lines segments will be

$$p_{11}x_1 + p_2x_2 + p_3x_3 = M, \quad \text{if} \quad 0 \leq x_1 \leq x_1(1);$$

$$p_{12}x_1 + p_2x_2 + p_3x_3 = M + (p_{12}-p_{11})x_1(1), \quad \text{if} \quad x_1(1) < x_1 \leq x_1(2);$$

$$\vdots$$

$$p_{1I_1}x_1 + p_2x_2 + p_3x_3 = M + \Sigma_{\ell=1}^{I_1-1}(p_{1\ell+1}-p_{1\ell})x_1(\ell), \quad \text{if} \quad x_1(I_1-1) < x_1 .$$

where $p_{11} < p_{12} < \cdots < p_{1I_1}$.

For general formulation, each commodity may be subject to increasing block rates. For commodity j, assume that there are I_j, $I_j \geq 1$, different rates with $p_{j1} < p_{j2} < \ldots < p_{jI_j}$ and knots $x_j(1),\ldots,x_j(I_j-1)$. When $I_j=1$, it corresponds to the case of one rate. For notational convenience, the conventions $x_j(0) = 0$ and $x_j(I_j) = \infty$, are adopted. For the general case with m commodities, the budget segments are

$$p_{1i_1}x_1 + p_{2i_2}x_2 + \cdots + p_{mi_m}x_m = M_{i_1 i_2 \ldots i_m}$$

if $x_1(i_1-1) < x_1 \leq x_1(i_1),\ldots,x_m(i_m-1) < x_m \leq x_m(i_m)$, where $M_{i_1 i_2 \ldots i_m} = M + \Sigma_{j=1}^{m} \Sigma_{\ell=1}^{i_j-1} (p_{j\ell+1}-p_{j\ell})x_j(\ell)$. Let $U(x_1,\ldots,x_m)$ be the utility function which is continuously differentiable, increasing and strictly quasi-concave. The utility maximization problem is

$$\max_{x_1,\ldots,x_m} U(x_1,\ldots,x_m)$$

subject to

$$\begin{aligned} &\Sigma_{j=1}^{m} p_{ji_j} x_j \leq M_{i_1 i_2 \ldots i_m}, && i_j=1,\ldots,I_j; \\ &x_j \geq 0, && j=1,\ldots,m. \end{aligned} \tag{2.1}$$

where

$$M_{i_1 i_2 \ldots i_m} = M + \Sigma_{j=1}^{m} \Sigma_{\ell=1}^{i_j-1} (p_{j\ell+1}-p_{j\ell})x_j(\ell). \tag{2.2}$$

This formulation provides the general model for the consumer demand analysis in this article. As we have pointed out, the quantity rationing case with upper quantity limit $x_j(I_j)$ is a special case corresponding to infinite price $p_{jI_j} = \infty$.[4] Similar problems on production economics will be considered separately in a subsequent section.

3. Virtual Prices and Regime Criteria.

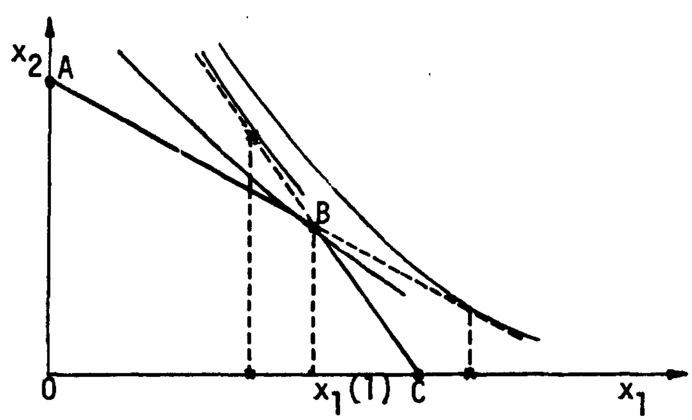
For econometric analysis, it is necessary to characterize the occurrence of the optimum solution at the different budget segments or the kink points with given explanatory variables. For the two-goods case, the conditions that characterize the different regimes are rather straightforward. Burtless and Hausman [1978] and Hausman [1979] have considered such cases. They specify a demand function for good 1

$$x_1^* = D_1(p_1, p_2, S)$$

as solution of the standard utility maximization problem, $\max_{x_1, x_2} \{U(x_1, x_2): p_1x_1 + p_2x_2 = S\}$. For the two-goods case in Figure 1, as specified in the previous section, it is clear that the optimum quantity x_1^* lies on the line segment AB if $D_1(p_{11}, p_2, M) < x_1(1)$, and the optimum quantity x_1^* lies on the line segment BC if $D_1(p_{12}, p_2, M_1) > x_1(1)$ where $M_1 = M + (p_{12} - p_{11})x_1(1)$. On the other hand, the optimum quantity x_1^* occurs at the kink point $x_1(1)$ if and only if $D_1(p_{11}, p_2, M) \geq x_1(1)$ and $D_1(p_{12}, p_2, M_1) \leq x_1(1)$, as predicted in Figure 3.

FIGURE 3.

Two-goods case: optimality at kink point



For the general m goods case, the generalization of these criteria is not obvious from the geometric view. One needs a systematic way to derive such criteria. The device we suggest is based on the Kuhn-Tucker conditions and the concept of virtual prices. Before we analyze the general problem (2.1), it is instructive to consider the two-goods case. Consider the regime that the optimum occurs at the kink point $\bar{x}$, $\bar{x} = (x_1(1),\bar{x}_2)$ where $\bar{x}_2 = (M - p_{11}x_1(1))/p_2$. Consider the utility maximization problem,

$$\max_{x_1,x_2} U(x_1,x_2)$$

$$\text{subject to} \quad \begin{aligned} p_{11}x_1 + p_2x_2 &\leq M; \\ p_{12}x_1 + p_2x_2 &\leq M_1 \end{aligned} \tag{3.1}$$

where $M_1 = M + (p_{12}-p_{11})x_1(1)$. The Lagrangean function is

$$L = U(x_1,x_2) + \lambda_1(M - p_{11}x_1 - p_2x_2) + \lambda_2(M_1 - p_{12}x_1 - p_2x_2).$$

The Kuhn-Tucker conditions that characterize the optimum regime at the kink point $\bar{x}$ are

$$\frac{\partial U(\bar{x})}{\partial x_1} - \lambda_1 p_{11} - \lambda_2 p_{12} = 0, \tag{3.2}$$

$$\frac{\partial U(\bar{x})}{\partial x_2} - (\lambda_1+\lambda_2)p_2 = 0, \tag{3.3}$$

$$p_{11}\bar{x}_1 + p_2\bar{x}_2 = M, \quad \lambda_1 \geq 0, \tag{3.4}$$

$$p_{12}\bar{x}_1 + p_2\bar{x}_2 = M_1, \quad \lambda_2 \geq 0. \tag{3.5}$$

These conditions are not directly useful as the regime criteria shall be expressed as inequalities. Define a variable ξ_1 as

$$\xi_1 = \frac{1}{\lambda_1+\lambda_2}\frac{\partial U(\bar{x})}{\partial x_1} \tag{3.6}$$

where λ_1 and λ_2 are solutions from (3.2)-(3.5). Equivalently,

$$\xi_1 = p_2\frac{\partial U(\bar{x})}{\partial x_1} / \frac{\partial U(\bar{x})}{\partial x_2}\,. \tag{3.6)'$$

It follows that equation (3.2) can be rewritten as

$$\xi_1 - \frac{\lambda_1}{\lambda_1+\lambda_2}p_{11} - \frac{\lambda_2}{\lambda_1+\lambda_2}p_{12} = 0. \tag{3.2)'$$

Equation (3.2)' implies that

$$\xi_1 - p_{11} = \frac{\lambda_2}{\lambda_1+\lambda_2}(p_{12}-p_{11}) \tag{3.7}$$

and

$$\xi_1 - p_{12} = \frac{\lambda_1}{\lambda_1+\lambda_2}(p_{11}-p_{12}). \tag{3.8}$$

As $p_{12} > p_{11}$, we have $\xi_1 \geq p_{11}$ and $p_{12} \geq \xi_1$. Thus when the optimum occurs at the kink point, it is necessary that $p_{12} \geq \xi_1 \geq p_{11}$. That these inequalities are sufficient conditions can be shown as follows. Define ω as

$$\omega = \frac{\partial U(\bar{x})}{\partial x_2} / p_2. \tag{3.9}$$

Since $p_{12} \geq \xi_1 \geq p_{11}$, where $\xi_1 = p_2\frac{\partial U(\bar{x})}{\partial x_1} / \frac{\partial U(\bar{x})}{\partial x_2} = \frac{\partial U(\bar{x})}{\partial x_1} / \omega$, there exists a $\mu \in [0,1]$ such that $\xi_1 = \mu p_{12} + (1-\mu)p_{11}$. Define ω_1 and ω_2 as

$$\omega_1 = \mu\omega \tag{3.10}$$

and

$$\omega_2 = (1-\mu)\omega. \tag{3.11}$$

Obviously, $\omega_1 \geq 0$, $\omega_2 \geq 0$ and $\omega = \omega_1+\omega_2$. Thus we have

$$\frac{\partial U(\bar{x})}{\partial x_1} - \omega_1 p_{12} - \omega_2 p_{11} = 0$$

$$\frac{\partial U(\bar{x})}{\partial x_2} - (\omega_1+\omega_2)p_2 = 0$$

$$p_{11}x_1(1) + p_2\bar{x}_2 = M, \quad \omega_1 \geq 0$$

$$p_{12}x_1(1) + p_2\bar{x}_2 = M_1, \quad \omega_2 \geq 0$$

which are the Kuhn-Tucker conditions which characterize that $\bar{x}$ is the optimum solution. Hence we conclude that the regime criteria that determine this regime are the inequalities

$$p_{11} \leq \xi_1 \leq p_{12}. \tag{3.12}$$

From the constructions in (3.9) and (3.6)', we have essentially that

$$\frac{\partial U(\bar{x})}{\partial x_2} - \omega p_2 = 0 \tag{3.13}$$

$$\frac{\partial U(\bar{x})}{\partial x_1} - \omega \xi_1 = 0. \tag{3.14}$$

Furthermore, define an "income" C as

$$C = M + (\xi_1 - p_{11})x_1(1). \tag{3.15}$$

Obviously,

$$\xi_1 x_1(1) + p_2\bar{x}_2 = C. \tag{3.16}$$

Thus the plane $\{(x_1,x_2)|\xi_1 x_1 + p_2 x_2 = C\}$ is tangent to the indifferent curve at the link point $\bar{x}$, which is point B in Figure 3, and supports the kink point as the optimal solution of utility maximization under the price vector (ξ_1,p_2) and income C. The price ξ_1 is known as the virtual price for good 1 at the quantity $x_1(1)$ and C is the corresponding virtual income in the quantity rationing literature, see, e.g., Rothbarth [1941], Neary and Roberts [1980] and Deaton [1981].

The virtual price is simply a shadow price. The kink point $x_1(1)$ is the demanded quantity, because the first block price p_{11} is relatively lower than ξ_1 at $x_1(1)$, and therefore the consumer buys as many quantities as permitted under p_{11}, but on the other hand, the second block price p_{12} is too high and therefore the consumer does not want to purchase any quantity under p_{12}. This provides an intuitive interpretation of the criteria inequalities in (3.12). From the geometric point of view in Figure 3, the criteria are on the comparisons of the tangent of the indifference curve at the point B with the slopes of the lines AB and BC. These criteria should be equivalent to the previous criteria based on the demand functions in Hausman [1979]. A proof for this equivalence will be provided in a later section. On the other hand, that the optimum occurs inside the line AB is characterized by $D_1(p_{11},p_2,M) < x_1(1)$. The condition $D_1(p_{12},p_2,M_1) > x_1(1)$ characterizes that the optimum occurs inside line BC.

With the construction of appropriate virtual prices, the comparison of these virtual prices with market prices can characterize the regime occurrence for the general m goods model in (2.1) and (2.2). As shown in the appendix, we have the following result.

<u>Theorem 1</u>. Let $x^* = (x_1^*,\ldots,x_m^*)$ be the demanded quantity vector. Consider the general regime in the form:

$$\begin{aligned}
&x_1^* = x_2^* = \ldots = x_{\ell_1-1}^* = 0;\\
&x_{\ell_1}^* = x_{\ell_1}(i_{\ell_1}^\circ), x_{\ell_1+1}^* = x_{\ell_1+1}(i_{\ell_1+1}^\circ),\ldots,x_{\ell_2-1}^* = x_{\ell_2-1}(i_{\ell_2-1}^\circ); \qquad (3.17)\\
&x_{\ell_2}(i_{\ell_2}^\circ-1) < x_{\ell_2}^* < x_{\ell_2}(i_{\ell_2}^\circ),\ldots,x_m(i_m^\circ-1) < x_m^* < x_m(i_m^\circ)
\end{aligned}$$

where $0 \le \ell_1 \le \ell_2 \le m$ and for some $i_{\ell_1}^\circ, i_{\ell_1+1}^\circ,\ldots,i_m^\circ$. The necessary and sufficient conditions that characterize this regime are

$$p_{j1} \geq \xi_j(x^*), \quad j=1,2,\ldots,\ell_1-1;$$

$$p_{ji_j^\circ} \leq \xi_j(x^*) \leq p_{j(i_j^\circ+1)}, \quad j=\ell_1,\ell_1+1,\ldots,\ell_2-1; \tag{3.18}$$

$$x_j(i_j^\circ-1) < x_j^* < x_j(i_j^\circ), \quad j=\ell_2,\ell_2+1,\ldots,m,$$

where $\xi_j(x^*)$ is the virtual price of good j at the optimum point x^*.

As shown in the subsequent section, the virtual prices $\xi_j(x^*)$ at x^* are functions of some observed prices, income and kink points:

$$\xi_j(x^*) = \xi_j(p_{\ell_2 i_{\ell_2}^\circ}, p_{\ell_2+1\, i_{\ell_2+1}^\circ}, \ldots, p_{mi_m^\circ}, M_{1\ldots1\, i_{\ell_1}^\circ \ldots i_{\ell_2-1}^\circ\, i_{\ell_2}^\circ \ldots i_m^\circ} - \Sigma_{j=\ell_1}^{\ell_2-1} p_{ji_j^\circ}\, x_j(i_j^\circ);\ 0,\ldots,0,x_{\ell_1}(i_{\ell_1}^\circ),\ldots,x_{\ell_2-1}(i_{\ell_2-1}^\circ)) \tag{3.19}$$

where the zeroes are the zero quantities of the first ℓ_1-1 goods. We note that

$$\begin{aligned} M_{1\ldots1 i_{\ell_1}^\circ \ldots i_{\ell_2-1}^\circ\, i_{\ell_2}^\circ \ldots i_m^\circ} &- \Sigma_{j=\ell_1}^{\ell_2-1} p_{ji_j^\circ}\, x_j(i_j^\circ) \\ &= M_{1\ldots1(i_{\ell_1}^\circ+1)\ldots(i_{\ell_2-1}^\circ+1) i_{\ell_2}^\circ \ldots i_m^\circ} - \Sigma_{j=\ell_1}^{\ell_2-1} p_{j(i_j^\circ+1)}\, x_j(i_j^\circ) \\ &= M_{1\ldots1(i_{\ell_1}^\circ+1) i_{\ell_1+1}^\circ \ldots i_{\ell_2-1}^\circ\, i_{\ell_2}^\circ \ldots i_m^\circ} - p_{\ell_1(i_{\ell_1}^\circ+1)}\, x_{\ell_1}(i_{\ell_1}^\circ) - \\ &\quad \Sigma_{j=\ell_1+1}^{\ell_2-1} p_{ji_j^\circ}\, x_j(i_j^\circ) \end{aligned} \tag{3.20}$$

etc.

4. Utility Function, Demand Equations and Virtual Prices.

Virtual prices can be constructed in principle from either direct utility function or indirect utility function. Consider the situation that ℓ goods are rationed at the quantities x_i°, $i=1,\ldots,\ell$. With a specified utility function $U(x_1,\ldots,x_m)$, a price vector p and income M, the constrained utility maximization problem is

$$\max_x U(x_1,\ldots,x_\ell,x_{\ell+1},\ldots,x_m)$$

$$\text{subject to} \quad x_i = x_i^\circ, \quad i=1,\ldots,\ell \tag{4.1}$$

$$\text{and} \quad p'x = M.$$

Implicitly, we always assume that $M > \Sigma_{i=1}^{\ell} p_i x_i^\circ$ so that the problem is well posed. The Lagrangean function is

$$L = U(x) + \lambda(M-p'x) + \Sigma_{i=1}^{\ell}\eta_i(x_i^\circ - x_i)$$

where λ and the η's are Lagrangean multipliers. The optimum solution x^* of (4.1) satisfies the first-order conditions:

$$\frac{\partial U(x^*)}{\partial x_i} - \lambda p_i - \eta_i = 0, \quad x_i^* = x_i^\circ, \quad i=1,\ldots,\ell \tag{4.2}$$

$$\frac{\partial U(x^*)}{\partial x_j} - \lambda p_j = 0, \qquad j=\ell+1,\ldots,m \tag{4.3}$$

$$p'x^* = M. \tag{4.4}$$

The optimum demanded quantities x_j^*, $j=\ell+1,\ldots,m$, for the unrationed goods can be solved from (4.3) and (4.4) conditional on the rationed goods at x_i°, $i=1,\ldots,\ell$;

$$x_j^* = D_j(p_{\ell+1},\ldots,p_m,M - \Sigma_{i=1}^{\ell} p_i x_i^\circ | x_1^\circ,\ldots,x_\ell^\circ) \qquad j=\ell+1,\ldots,m \tag{4.5}$$

which are the conditional demand equations for the unrationed goods. The virtual prices ξ_j, $j=1,\ldots,\ell$ for the rationed goods at $x_1^\circ,\ldots,x_\ell^\circ$ are

$$\xi_j(x^*) = \frac{1}{\lambda}\frac{\partial U(x^*)}{\partial x_j}$$

$$= p_m \frac{\partial U(x_1^\circ,\ldots,x_\ell^\circ,x_{\ell+1}^*,\ldots,x_m^*)}{\partial x_j} \bigg/ \frac{\partial U(x_1^\circ,\ldots,x_\ell^\circ,x_{\ell+1}^*,\ldots,x_m^*)}{\partial x_m} \qquad (4.6)$$

for $j=\ell+1,\ldots,m$. Substituting the conditional demand equations (4.5) into (4.6), the virtual prices can be written as functions of the observed prices for the unrationed goods income and the rationed quantities:

$$\xi_j(x^*) = \xi_j(p_{\ell+1},\ldots,p_m,M_R;x_1^\circ,\ldots,x_\ell^\circ) \qquad j=1,\ldots,\ell \qquad (4.7)$$

where M_R, $M_R = M - \Sigma_{i=1}^{\ell} p_i x_i^\circ$, is the remaining income for the unrationed goods. The virtual prices and the conditional demand equations as functions of p, M and the kink points provide the predictable conditions for the regime occurrence as the conditions (3.18). The complication in this direct approach is on the derivation of the conditional demand equations (4.5) from the first-order conditions.

Virtual prices for the rationed goods can also be derived from unconditional (notional) demand equations. The unconditional (notional) demand functions $D_i(p,M)$, $i=1,\ldots,m$, are solutions of the unconstrained utility maximization problem $\max\{U(x)|p'x = M\}$. With the rationed goods i at quantities x_i°, $i=1,2,\ldots,\ell$, the virtual prices which support the commodity vector $x^* = (x_1^\circ,\ldots,x_\ell^\circ,x_{\ell+1}^*,\ldots,x_m^*)$ as the unconstrained utility maximizer, will be characterized by the following demand relations:

$$x_i^\circ = D_i(\xi_1,\xi_2,\ldots,\xi_\ell,p_{\ell+1},\ldots,p_m,c), \quad i=1,\ldots,\ell; \qquad (4.8)$$

$$x_j^* = D_j(\xi_1,\xi_2,\ldots,\xi_\ell,p_{\ell+1},\ldots,p_m,c), \quad j=\ell+1,\ldots,m \qquad (4.9)$$

where

$$c = M + \Sigma_{i=1}^{\ell}(\xi_i - p_i)x_i^\circ \tag{4.10}$$

is known as the virtual income, and $\Sigma_{i=1}^{\ell}\xi_i x_i^\circ + \Sigma_{j=\ell+1}^{m} p_j x_j^* = c$ is the corresponding budget tangent plane. The virtual prices ξ_i for the rationed goods are solutions from the equations (4.8) and (4.10). The virtual prices for the unrationed goods are the observed prices (Neary and Roberts [1980]). This approach is useful for the specification of unconditional (or notional) demand functions or indirect utility function as the basic structure instead of a direct utility function. For the indirect utility approach, the unconditional demand equations can be derived from Roy's identity. The complication of this approach is to solve the demand equations (4.8) for the virtual prices ξ_i, $i=1,\ldots,\ell$.

The demanded quantities x_j^*, $j=\ell+1,\ldots,m$ for the unrationed goods satisfy the conditional demand equations (4.5). With the introduction of virtual prices ξ_i, $i=1,\ldots,\ell$ for the rationed goods, x_j^*, satisfy the unconditional demand equation,

$$x_j^* = D_j(\xi_1,\xi_2,\ldots,\xi_\ell,p_{\ell+1},\ldots,p_m,M_R + \Sigma_{i=1}^{\ell}\xi_i x_i^\circ), \quad j=\ell+1,\ldots,m \tag{4.11}$$

where $M_R = M - \Sigma_{i=1}^{\ell}p_i x_i^\circ$. Substituting the virtual price equations (4.7) into (4.11), it is the conditional demand equations (4.5). Thus the conditional demand equations can be derived from the unconditional demand equations via the virtual prices and vice-versa. As a function of prices $p_{\ell+1},\ldots,p_m$, remained income M_R and the rationed quantities x_i°, $i=1,\ldots,\ell$, the demand equations (4.11) can be interpreted as conditional demand equations conditional on x_i°, $i=1,\ldots,\ell$.

For the analysis of kink points in consumer demand, the kink points in each regime in (3.17) can be regarded as the rationed quantities of the commodities. The virtual prices for each regime can be derived as described in this section.

5. Alternative Regime Criteria.

The regime criteria (3.18) in Section 3 are expressed as inequalities in terms of virtual prices and conditional demand equations. The last section has pointed out that the conditional demand equations are unconditional demand equations evaluated at some appropriate virtual prices. In Burtless and Hausman [1978] for labor supply and Lee and Pitt [1983] for consumer demand with only nonnegative constraints, the regimes are completely determined by inequalities with unconditional demand equations. Explicitly, for the binding nonnegative constraints, we need to assume that the underlying utility function can be extended mathematically into the non-positive orthants and its extension processes, also the strict quasi-concavity and monotonicity properties. This assumption justifies that the unconditional demand functions can take the negative values and can be used to determine the occurrence of binding nonnegative constraints. We note that this extension is needed only for the presence of binding nonnegative constraints. In this section, we will show that the criteria (3.18) can be expressed completely in terms of unconditional demand equations or completely in terms of virtual prices.

Consider the quantity rationing problem (4.1). As before, let $x^* = (x_1^\circ,\ldots,x_\ell^\circ,x_{\ell+1}^*,\ldots,x_m^*)'$ be the optimal solution of (4.1), ξ_i, $i=1,\ldots,\ell$ and c be the corresponding virtual prices and virtual income. To simplify notations, denote $\bar{p} = (p_{\ell+1},\ldots,p_m)'$, $\bar{\xi} = (\xi_1,\ldots,\xi_\ell)'$ and $\bar{x}^\circ = (x_1^\circ,\ldots,x_\ell^\circ)'$. We note that $\bar{\xi} = \bar{\xi}(\bar{p},M_R;\bar{x}^\circ)$ is a vector valued function of $\bar{p}$, M_R and $\bar{x}^\circ$. Let

$$V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c) = U(x^*) \tag{5.1}$$

be the indirect utility function evaluated at the virtual prices and virtual income. The following theorem relates the marginal indirect utility with respect to the rationed quantities to virtual prices.

Theorem 2: $\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i$

$$= \frac{\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)}{\partial c} \cdot (\xi_i(\bar{p},M_R;\bar{x}^\circ) - p_i) \qquad i=1,\ldots,\ell.$$

Proof: $$\frac{\partial V(\bar{\xi},\bar{p},c)}{\partial x_i}$$

$$= \Sigma_{j=1}^{\ell} \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} + \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial c} \frac{\partial c}{\partial x_i}$$

$$= \Sigma_{j=1}^{\ell} \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} + \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial c} [(\xi_i - p_i) \quad \Sigma_{j=1}^{\ell} x_j^\circ \frac{\partial \xi_j}{\partial x_i}].$$

Roy's identity implies that $\frac{\partial V(\bar{\xi},\bar{p},c)}{\partial \xi_j} = -x_j^\circ \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial c}$. Hence it follows that

$$\frac{\partial V(\bar{\xi},\bar{p},c)}{\partial x_i} = - \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial c} \Sigma_{j=1}^{\ell} x_j^\circ \frac{\partial \xi_j}{\partial x_i} + \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial c} [(\xi_i - p_i) + \Sigma_{j=1}^{\ell} x_j^\circ \frac{\partial \xi_j}{\partial x_i}]$$

$$= \frac{\partial V(\bar{\xi},\bar{p},c)}{\partial c} \cdot (\xi_i - p_i).$$

Q.E.D.

This theorem implies that the marginal change in individual welfare as the level of rationed good is changed depends on the difference between its virtual price and its market price. Since the marginal individual utility of income is positive, the following corollary follows from Theorem 2.

Corollary 1: (i) $\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i > 0$ if and only if $\xi_i(\bar{p},M_R;\bar{x}^\circ) > p_i$;

(ii) $\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i < 0$ if and only if $\xi_i(\bar{p},M_R;\bar{x}^\circ) < p_i$,

and (iii) $\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i = 0$ if and only if $\xi_i(\bar{p},M_R;\bar{x}^\circ) = p_i$,

for $i=1,\ldots,\ell$.

The indirect utility function V in (5.1) as a function of the rationed goods preserves some concavity properties of the direct utility function as in the following lemma.

Lemma 1: The indirect utility function $V(\xi(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)$ is (strictly) quasi-concave with respect to any rationed good x_i, $i \in \{1,\ldots,\ell\}$ if the direct utility function $U(x)$ is (strictly) quasi-concave.

With this lemma, we can prove the following theorem which provides information about the location of the desirable (notational) demand quantities as compared with the location of the rationed quantities. The proofs of the lemma and this theorem are in the Appendix.

Theorem 3:

(i) $\partial V(\xi(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i \leq 0$ if and only if

$$D_i(p_i,\bar{p},M - \Sigma_{\substack{j=1\\ j\neq i}}^{\ell} p_j\ x_j^\circ | x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_\ell^\circ) \leq x_i^\circ,$$

and

(ii) $\partial V(\xi(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i \geq 0$ if and only if

$$D_i(p_i,\bar{p},M - \Sigma_{\substack{j=1\\ j\neq i}}^{\ell} p_j\ x_j^\circ | x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_\ell^\circ) \geq x_i^\circ,$$

for all $i=1,\ldots,\ell$.

Combining Theorem 3 and Corollary 1, we have immediately the following corollary.

Corollary 2:

(i) $\xi_i(\bar{p},M_R;\bar{x}^\circ) \leq p_i$ if and only if

$$D_i(p_i,\bar{p},M - \Sigma_{\substack{j=1\\ j\neq i}}^{\ell} p_j\ x_j^\circ | x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_\ell^\circ) \leq x_i^\circ,$$

and

(ii) $\xi_i(\bar{p},M_R;\bar{x}^\circ) \geq p_i$ if and only if

$$D_i(p_i,\bar{p},M - \Sigma_{\substack{j=1\\ j\neq i}}^{\ell} p_j x_j^\circ | x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_\ell^\circ) \geq x_i^\circ$$

for $i=1,\ldots,\ell$.

These results are intuitively appealing. At the rationed quantities, if the market price for good i is cheaper than the corresponding virtual price, the consumer would desire to purchase more than the rationed quantity of good i if the good i were unrationed; and vice-versa.

With these relations, the regime criteria in Theorem 1 can be rewritten as inequalities in terms of virtual prices or conditional demanded quantities. Thus there are different but equivalent ways to characterize the different regime occurrence.

6. An Econometric Specification--The Linear Expenditure System.

Consider an indirect utility function which has a special Gorman Polar form

$$V(p,M) = (M - \Sigma_{i=1}^{m} \beta_i p_i) \prod_{i=1}^{m} p_i^{-\alpha_i} \tag{6.1}$$

where $\alpha_i > 0$, $\Sigma_{i=1}^{m} \alpha_i = 1$. It can be shown that the corresponding utility function has a Klein-Rubin-Stone-Geary form

$$U(x) = \Sigma_{i=1}^{m} \alpha_i \, \ell n(x_i - \beta_i). \tag{6.2}$$

For the case that the nonnegative constraint of the good i is binding, it is necessary to assume that $\beta_i < 0$ so that (6.2) is a well-defined function. The (notional) demand functions can be derived from the Roy's identity which are

$$x_i = \beta_i + \alpha_i (M - \Sigma_{j=1}^{m} \beta_j p_j)/p_i, \qquad i=1,\dots,m.$$

To introduce stochastic elements into the model it is convenient to introduce additive errors into the expenditure equations,

$$p_i x_i = p_i \beta_i + \alpha_i (M - \Sigma_{j=1}^{m} \beta_j p_j) + \epsilon_i \qquad i=1,\dots,m \tag{6.3}$$

where ϵ_i has zero mean and may be correlated across different demand equations. It follows that $\Sigma_{i=1}^{m} \epsilon_i = 0$. This system has been used extensively in empirical works (see, e.g., Stone [1954]). It is attractive for our models because of its linear structures from which virtual prices can be derived analytically.[5/]

To demonstrate the econometric formulation of the likelihood function with this system and to illustrate the results derived in the previous section, let us consider the three-goods case in some detail. We assume that only the first good has two block prices. Figure 2 in Section 2 illustrates the budget set for this example. We have $M_{111}=M$ and $M_{211} = M + (p_{12} - p_{11}) x_1(1)$. Consider the situation that goods 1 and 2 are always consumed in some positive amounts but good 3 may or may not be purchased. There will be six regimes in total:

Regime 1: The demanded quantities lie on the interior of the plane ABDE.

Regime 2: The demanded quantities lie on the line BD (excluding point B).

Regime 3: The demanded quantities lie on the interior of the plane BCD.

Regime 4: The demanded quantities lie on the line AB (excluding point B).

Regime 5: The demanded quantities are at the point B.

Regime 6: The demanded quantities lie on the line BC (excluding point B).

Regime 1: $x_1^* < x_1(1)$ and $x_3^* > 0$. The demanded quantities satisfy the demand equations:

$$x_i^* = D_i(p_{11},p_2,p_3,M_{111}) \qquad i=1,2,3.$$

The regime occurrence is determined by the conditions

$$D_1(p_{11},p_2,p_3,M_{111}) < x_1(1) \quad \text{and} \quad D_3(p_{11},p_2,p_3,M_{111}) > 0.$$

Equivalent conditions can be derived with virtual prices. Let ξ_1 be the virtual price of good 1 at $x_1(1)$, which is determined from

$$x_1(1) = D_1(\xi_1,p_2,p_3,M_{111} + (\xi_1-p_{11})x_1(1)).$$

Furthermore, let $\xi_3^{(1)}$ be the virtual price of good 3 at 0, which is determined from $0 = D_3(p_{11},p_2,\xi_3^{(1)},M_{111})$. The equivalent regime criteria are

$$\xi_1(p_2,p_3,M_{111}-p_{11}x_1(1);x_1(1)) < p_{11},$$
$$\xi_3(p_{11},p_2,M_{111};x_3=0) > p_3.$$

For the linear expenditure system, the structures are as follows. The virtual price ξ_1 can be solved from the equation,

$$\xi_1 x_1(1) = \xi_1\beta_1 + a_1(M_{111} + (\xi_1 - p_{11})x_1(1) - \beta_1\xi_1 - \beta_2 p_2 - \beta_3 p_3) + \epsilon_1$$

which is

$$\xi_1 = \frac{a_1(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3) + \epsilon_1}{(1 - a_1)(x_1(1) - \beta_1)}. \tag{6.4}$$

The virtual price $\xi_3^{(1)}$ is derived from the equation,

$$0 = \xi_3^{(1)}\beta_3 + a_3(M_{111} - p_{11}\beta_1 - p_2\beta_2 - \xi_3^{(1)}\beta_3) + \epsilon_3$$

which is

$$\xi_3^{(1)} = -\frac{a_3(M_{111} - p_{11}\beta_1 - p_2\beta_2) + \epsilon_3}{(1 - a_3)\beta_3}. \tag{6.5}$$

The occurrence of this regime is determined by the following two conditions:

$$p_{11}(x_1(1) - \beta_1) - a_1(M_{111} - p_{11}\beta_1 - p_2\beta_2 - p_3\beta_3) > \epsilon_1,$$

$$p_3\beta_3 + a_3(M_{111} - p_{11}\beta_1 - p_2\beta_2 - p_3\beta_3) < \epsilon_3.$$

The demanded quantities are

$$x_i^* = \beta_i + a_i(M_{111} - \Sigma_{j=1}^3 p_j\beta_j)/p_i + \epsilon_i/p_i, \qquad i=1,2,3. \tag{6.6}$$

As $\Sigma_{i=1}^3 \epsilon_i = 0$ and one of the equations is redundant, the contribution of this regime to the likelihood function is the density $g_1(x^*)$,

$$g_1(x^*) = p_1 p_3\ f(p_1(x_1^* - \beta_1) - a_1(M_{111} - \Sigma_{j=1}^3 p_j\beta_j),\ p_3(x_3^* - \beta_3) - a_3(M_{111} - \Sigma_{j=1}^3 p_j\beta_j))$$

where $f(\epsilon_1, \epsilon_3)$ is the bivariate density function of ϵ_1 and ϵ_3.

<u>Regime 2</u>: $x_1^* = x_1(1)$ and $x_3^* > 0$. The demanded quantities x_2^* and x_3^* satisfy the equations:

$$x_i^* = D_i(\xi_1, p_2, p_3, M_{111} + (\xi_1 - p_{11})x_1(1)), \qquad i=2,3$$

where $\xi_1 = \xi_1(p_2, p_3, M_{111} - p_{11}x_1(1); x_1(1))$ is the virtual price of good 1 at $x_1(1)$. The regime criteria in terms of demand equations are

$$D_1(p_{12}, p_2, p_3, M_{211}) \leq x_1(1) \leq D_1(p_{11}, p_2, p_3, M_{111})$$

and $\quad D_3(\xi_1, p_2, p_3, M_{111} + (\xi_1 - p_{11})x_1(1)) > 0$. Let

$$\xi_{31} = \xi_3(p_2, M_{111} - p_{11}x_1(1); x_1(1), x_3=0)$$

be the virtual price of good 3 at $(x_1(1), x_3=0)$. The equivalent regime criteria are

$$p_{11} \leq \xi_1 \leq p_{12} \text{ and } \xi_{31} > p_3.$$

For the linear expenditure system, the regime conditions are

$$p_3\beta_3(1-\alpha_1) + \alpha_3(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3) + \alpha_3\epsilon_1 + (1-\alpha_1)\epsilon_3 > 0,$$

$$p_{11}(1-\alpha_1)(x_1(1)-\beta_1) - \alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3) \leq \epsilon_1$$

and $\quad \epsilon_1 \leq p_{12}(1-\alpha_1)(x_1(1)-\beta_1) - \alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3).$

The demanded quantities satisfy the following expenditure equations.

$$p_i(x_i^*-\beta_i)(1-\alpha_1) = \alpha_i(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3) + \alpha_i\epsilon_1 + (1-\alpha_1)\epsilon_i \qquad i=2,3.$$

Denote $\quad v_i = p_i(1-\alpha_1)(x_i^*-\beta_i) - \alpha_i(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3) \qquad i=2,3$

$$L_2 = p_{11}(1-\alpha_1)(x_1(1)-\beta_1) - \alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3) \text{ and}$$

$$U_2 = p_{12}(1-\alpha_1)(x_1(1)-\beta_1) - \alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2 - p_3\beta_3).$$

The contribution of this regime to the likelihood function is $g_2(x^*)$,

$$g_2(x^*) = p_3 \int_{L_2}^{U_2} f(\epsilon_1, \frac{v_3 - a_3\epsilon_1)}{1 - a_1}) d\epsilon_1 .$$

Regime 3: $x_1^* > x_1(1)$ and $x_3^* > 0$. The effective budget plane is $p_{12}x_1 + p_2x_2 + p_3x_3 = M_{211}$. The demand functions are $x_i^* = D_i(p_{12}, p_2, p_3, M_{211})$, $i=1,2,3$. The regime criteria are $D_1(p_{12}, p_2, p_3, M_{211}) > x_1(1)$ and $D_3(p_{12}, p_2, p_3, M_{211}) > 0$. Equivalent regime criteria are $p_{12} < \xi_1(p_2, p_3, M_{211} - p_{12}x_1(1); x_1(1))$ and $p_3 < \xi_3(p_{12}, p_2, M_{211}; x_3=0)$.

For the linear expenditure system, the demanded quantities satisfy the equations:

$$p_{12}x_1^* = p_{12}\beta_1 + a_1(M_{211} - p_{12}\beta_1 - p_2\beta_2 - p_3\beta_3) + \epsilon_1$$

$$p_ix_i^* = p_i\beta_i + a_i(M_{211} - p_{12}\beta_1 - p_2\beta_2 - p_3\beta_3) + \epsilon_i, \qquad i=2,3.$$

The contribution of this regime to the likelihood function is

$$g_3(x^*) = p_{12}p_3\ f(p_{12}(x_1^*-\beta_1) - a_1(M_{211} - p_{12}\beta_1 - p_2\beta_2 - p_3\beta_3),$$
$$p_3(x_3^*-\beta_3) - a_3(M_{211} - p_{12}\beta_1 - p_2\beta_2 - p_3\beta_3)).$$

Regime 4: $x_1^* < x_1(1)$ and $x_3^* = 0$. The demanded quantities are $x_i^* = D_i(p_{11}, p_2, \xi_3^{(1)}, M_{111})$, $i=1,2$ where $\xi_3^{(1)} = \xi_3(p_{11}, p_2, M_{111}; x_3=0)$ is the virtual price for good 3 at $x_3 = 0$ with the first budget plane. The regime criteria are $D_1(p_{11}, p_2, \xi_3^{(1)}, M_{111}) < x_1(1)$ and $\xi_3^{(1)} \leq p_3$. Let $\xi_{13} = \xi_1(p_2, M_{111} - p_{11}x_1(1); x_1(1), x_3=0)$ and $\xi_{31} = \xi_3(p_2, M_{111} - p_{11}x_1(1); x_1(1), x_3=0)$ be the virtual prices of goods 1 and 3, respectively, at $(x_1(1), x_3=0)$. Equivalent regime criteria are $\xi_3^{(1)} \leq p_3$ and $\xi_{13} < p_{11}$. In terms of demand equations, the conditions are $D_1(p_{11}, p_2, \xi_3, M_{111}) < x_1(1)$ and $D_3(p_{11}, p_2, p_3, M_{111}) \leq 0$.

For the linear expenditure, the demanded quantity x_1^* satisfies the expenditure equation

$$p_{11}x_1^* = p_{11}\beta_1 + \frac{\alpha_1}{1-\alpha_3}(M_{111} - p_{11}\beta_1 - p_2\beta_2) + \frac{\alpha_1}{1-\alpha_3}\epsilon_3 + \epsilon_1.$$

The conditions that determine the regime occurrence are

$$\alpha_1(M_{111} - p_{11}\beta_1 - p_2\beta_2) + \alpha_1\epsilon_3 + (1-\alpha_3)\epsilon_1 < (1-\alpha_3)p_{11}(x_1(1)-\beta_1),$$

$$p_3\beta_3 + \alpha_3(M_{111} - p_{11}\beta_1 - p_2\beta_2 - p_3\beta_3) + \epsilon_3 \le 0.$$

The contribution of this regime to the likelihood function is

$$g_4(x^*) = p_{11}\int_{-\infty}^{U_4} f(v_1 - \frac{\alpha_1}{1-\alpha_3}\epsilon_3, \epsilon_3)d\epsilon_3$$

where $v_1 = p_{11}(x_1^*-\beta_1) - \frac{\alpha_1}{1-\alpha_3}(M_{111} - p_{11}\beta_1 - p_2\beta_2)$ and

$$U_4 = -p_3\beta_3 - \alpha_3(M_{111} - p_{11}\beta_1 - p_2\beta_2 - p_3\beta_3).$$

Regime 5: $x_1^* = x_1(1)$ and $x_3^* = 0$.

The quantity x_2^* satisfies $x_2^* = D_2(\xi_{13}, p_2, \xi_{31}, M_{111} + (\xi_{13}-p_{11})x_1(1))$ where the virtual prices ξ_{13} and ξ_{31} are defined as in Regime 4. The regime criteria are $p_{11} \le \xi_{13} \le p_{12}$ and $\xi_{31} \le p_3$. The equivalent conditions in terms of demand equations are relatively complicated. Let $\xi_3^{(1)} = \xi_3(p_{11}, p_2, M_{111}; x_3=0)$ and $\xi_3^{(2)} = \xi_3(p_{12}, p_2, M_{211}; x_3=0)$ be the virtual prices of good 3 at $x_3 = 0$ corresponding to the first and second budget planes, respectively. The conditions are $D_1(p_{12}, p_2, \xi_3^{(2)}, M_{211}) \le x_1(1) \le D_1(p_{11}, p_2, \xi_3^{(1)}, M_{111})$ and $D_3(\xi_1, p_2, p_3, M_{111}-p_{11}x_1(1)) \le 0$.

For the linear expenditure system, the structures are as follows.

The virtual prices ξ_{13} and ξ_{31} for goods 1 and 3 at $(x_1(1), x_3=0)$ can be solved from the following linear equations system:

$$\xi_{13}x_1(1) = \xi_{13}\beta_1 + \alpha_1(M_{111} + (\xi_{13}-p_{11})x_1(1) - \xi_{13}\beta_1 - p_2\beta_2 - \xi_{31}\beta_3) + \epsilon_1$$

$$0 = \xi_{31}\beta_3 + \alpha_3(M_{111} + (\xi_{13}-p_{11})x_1(1) - \xi_{13}\beta_1 - p_2\beta_2 - \xi_{31}\beta_3) + \epsilon_3,$$

which are

$$\xi_{13} = \frac{\alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2) + (1-\alpha_3)\epsilon_1 + \alpha_1\epsilon_3}{(1 - \alpha_1 - \alpha_3)(x_1(1)-\beta_1)},$$

$$\xi_{31} = \frac{\alpha_3(M_{111} - p_{11}x_1(1) - p_2\beta_2) + \alpha_3\epsilon_1 + (1-\alpha_1)\epsilon_3}{(-\beta_3)(1 - \alpha_1 - \alpha_3)}.$$

The regime occurrence conditions are

$$(1-\alpha_1-\alpha_3)(x_1(1)-\beta_1)p_{11} \le \alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2) + (1-\alpha_3)\epsilon_1 + \alpha_1\epsilon_3$$
$$\le (1-\alpha_1-\alpha_3)(x_1(1)-\beta_1)p_{12}$$

and $$\alpha_3(M_{111} - p_{11}x_1(1) - p_2\beta_2) + \alpha_3\epsilon_1 + (1-\alpha_1)\epsilon_3 \le (-\beta_3)(1-\alpha_1-\alpha_3)p_3.$$

Denote

$$D_1 = (1-\alpha_1-\alpha_3)(x_1(1)-\beta_1)p_{11} - \alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2),$$

$$D_2 = (1-\alpha_1-\alpha_3)(x_1(1)-\beta_1)p_{12} - \alpha_1(M_{111} - p_{11}x_1(1) - p_2\beta_2),$$

and $$D_3 = (-\beta_3)(1-\alpha_1-\alpha_3)p_3 - \alpha_3(M_{111} - p_{11}x_1(1) - p_2\beta_2).$$

The contribution of this regime to the likelihood function is $g_5(x^*)$,

$$g_5(x^*) = \int_{-\infty}^{D_3}\int_{D_1}^{D_2} \frac{1}{1-\alpha_1-\alpha_3} f\left(\frac{(1-\alpha_1)u_1 - \alpha_1u_3}{1-\alpha_1-\alpha_3}, \frac{-\alpha_3u_1 + (1-\alpha_3)u_3)}{1-\alpha_1-\alpha_3}\right)du_1du_3.$$

<u>Regime 6</u>: $x_1^* > x_1(1)$ and $x_3^* = 0$.

The demanded quantities satisfy the equations:

$$x_i^* = D_i(p_{12},p_2,\xi_3^{(2)},M_{211}) \qquad i=1,2.$$

The regime criteria are $D_1(p_{12},p_2,\xi_3^{(2)},M_{211}) > x_1(1)$ and $\xi_3^{(2)} \leq p_3$. Equivalently, in terms of virtual prices, the conditions are $\xi_3^{(2)} \leq p_3$ and $\xi_1(p_2,M_{211}-p_{12}x_1(1);x_1(1),x_3=0) > p_{12}$. In terms of demand equations, the regime conditions are $D_1(p_{12},p_2,\xi_3^{(2)},M_{211}) > x_1(1)$ and $D_3(p_{12},p_2,p_3,M_{211}) \leq 0$.

For the linear expenditure system, the virtual price $\xi_3^{(2)}$ is

$$\xi_3^{(2)} = \frac{\alpha_3(M_{211} - p_{12}\beta_1 - p_2\beta_2) + \epsilon_3}{(-\beta_3)(1-\alpha_3)}.$$

The demanded quantity x_1^* satisfies the following expenditure equation,

$$p_{12}x_1^* = p_{12}\beta_1 + \frac{\alpha_1}{1-\alpha_3}(M_{211} - p_{12}\beta_1 - p_2\beta_2) + \frac{\alpha_1}{1-\alpha_3}\epsilon_3 + \epsilon_1.$$

The regime occurrence conditions are

$$p_{12}(x_1(1)-\beta_1) - \frac{\alpha_1}{1-\alpha_3}(M_{211} - p_{12}\beta_1 - p_2\beta_2) < \frac{\alpha_1}{1-\alpha_3}\epsilon_3 + \epsilon_1,$$

$$p_3\beta_3 + \alpha_3(M_{211} - p_{12}\beta_1 - p_2\beta_2 - p_3\beta_3) + \epsilon_3 \leq 0.$$

The contribution of this regime to the likelihood function is

$$g_6(x^*) = p_{12}\int_{-\infty}^{U} f(v_1 - \frac{\alpha_1}{1-\alpha_3}\epsilon_3,\epsilon_3)d\epsilon_3$$

where $v_1 = p_{12}(x_1^*-\beta_1) - \frac{\alpha_1}{1-\alpha_3}(M_{211} - p_{12}\beta_1 - p_2\beta_2)$ and

$$U = -p_3\beta_3 - \alpha_3(M_{211} - p_{12}\beta_1 - p_2\beta_2 - p_3\beta_3).$$

For each sample i, define the set of mutually exclusive regime indicators,

$$I_{ji} = 1, \text{ if } x_i^* \text{ is in the } j^{\underline{th}} \text{ regime},$$

$$= 0, \text{ otherwise}.$$

The likelihood function L_i for the sample i is therefore

$$L_i = \prod_{j=1}^{J} g_j(x_i^*)^{I_{ji}}$$

where J is the total number of regimes. For a random sample of size N, the likelihood function for the whole sample is

$$L = \prod_{i=1}^{N} \prod_{j=1}^{J} g_j(x_i^*)^{I_{ji}}.$$

7. Likelihood Function for a General Demand System.

The linear expenditure system is relatively simple in its functional form from which virtual prices can be derived analytically. However, one may not want to implement this system in certain empirical studies because of its restrictive theoretical properties, e.g., its linear Engel curve. The use of other general and flexible demand systems is feasible, even though virtual prices may not easily be derived, for the purpose of estimation.

Consider a general (notional) demand system with m goods

$$x_i = D_i(p_1,p_2,\ldots,p_m,M;\theta) + \epsilon_i, \qquad i=1,\ldots,m \tag{7.1}$$

where θ is a vector of unknown parameters and ϵ_i is the disturbance with zero mean. The budget constraint implies that $\Sigma_{i=1}^{m} p_i \epsilon_i = 0$. The disturbances ϵ_i, $i=1,\ldots,m$ are correlated and are heteroscedastic. Let $x^* = (x_1^*,\ldots,x_m^*)$ be the observed demand quantity vector. Without loss of generality, consider the general regime in Theorem 1. The criteria for the determination of this regime are the conditions (3.18). The virtual prices $\xi_1,\xi_2,\ldots,\xi_{\ell_2-1}$ are determined by the following equations:

$$\begin{aligned} 0 &= D_j(\xi_1,\xi_2,\ldots,\xi_{\ell_2-1},p_{\ell_2},\ldots,p_m,M_R + \Sigma_{k=\ell_1}^{\ell_2-1}\xi_k\, x_k(i_k^\circ);\theta) + \epsilon_j, \\ &\quad j=1,\ldots,\ell_1-1; \\ x_j(i_j^\circ) &= D_j(\xi_1,\xi_2,\ldots,\xi_{\ell_2-1},p_{\ell_2},\ldots,p_m,M_R + \Sigma_{k=\ell_1}^{\ell_2-1}\xi_k\, x_k(i_k^\circ);\theta) + \epsilon_j, \\ &\quad j=\ell_1,\ldots,\ell_2-1 \end{aligned} \tag{7.2}$$

where $M_R = M_{1\ldots 1 i_{\ell_1}^\circ \ldots i_{\ell_2-1}^\circ i_{\ell_2}^\circ \ldots i_m^\circ} - \Sigma_{k=\ell_1}^{\ell_2-1} p_k\, x_k(i_k^\circ)$.

The remaining demanded quantities are

$$x_j^* = D_j(\xi_1,\xi_2,\ldots,\xi_{\ell_2-1},p_{\ell_2},\ldots,p_m,M_R + \Sigma_{k=\ell_1}^{\ell_2-1}\xi_k x_k(i_k^\circ);\theta) + \epsilon_j,$$
$$j=\ell_2,\ell_2+1,\ldots,m. \tag{7.3}$$

These equations provide an implicit function from the disturbance vector $(\epsilon_1,\ldots,\epsilon_{m-1})$ to the vector $(\xi_1,\xi_2,\ldots,\xi_{\ell_2-1},x_{\ell_2}^*,x_{\ell_2+1}^*,\ldots,x_{m-1}^*)$. As $\Sigma_{i=1}^m p_i\epsilon_i = 0$, the equation x_m^* is functionally dependent on the other equations and is redundant. Given a joint density function for $(\epsilon_1,\ldots,\epsilon_{m-1})$, it implies a joint density function for $(\xi_1,\ldots,\xi_{\ell_2-1},x_{\ell_2}^*,x_{\ell_2+1}^*,\ldots,x_{m-1}^*)$ which can be derived in a straightforward manner as the Jacobian matrix can be easily derived from (7.2) and (7.3). Let $f(\xi_1,\ldots,\xi_{\ell_2-1},x_{\ell_2}^*,x_{\ell_2+1}^*,\ldots,x_{m-1}^*)$ denote the implied joint density function. It follows that the contribution of this regime to the likelihood function for an observation is

$$\int_{p_{\ell_2-1}i_{\ell_2-1}^\circ}^{p_{\ell_2-1}(i_{\ell_2-1}^\circ+1)} \cdots \int_{p_{\ell_1}i_{\ell_1}^\circ}^{p_{\ell_1}(i_{\ell_1}^\circ+1)} \int_{-\infty}^{p_{\ell_1-1,1}} \cdots \int_{-\infty}^{p_{11}} f(\xi_1,\xi_2,\ldots,\xi_{\ell_2-1},x_{\ell_2}^*, x_{\ell_2+1}^*,\ldots,x_{m-1}^*) \cdot d\xi_1 d\xi_2 \ldots d\xi_{\ell_2-1}.$$

For the estimation, the method of maximum likelihood can be used. However, the evaluation of the likelihood function may be cumbersome and expensive if there are integrals with dimensions more than two. It is an open question whether there were any computationally simple estimation methods that could be derived.

8. Production Analysis.

The above analyses are focused on the consumer demand models. They can be extended to the analyses of kink points in production economics. We will assume that the objective of the production unit is either profit maximization or cost minimization with given production technology. Kink points may occur because of binding nonnegative constraints on inputs or outputs in a multiple inputs or multiple outputs technology. Production quotas on outputs or quantity rationing on demand for inputs will create kink points. A relatively weak form of rationing is in the form of increasing block prices in inputs and a weak form of production quota is on the decreasing block prices in outputs.6/

Consider the profit maximization problem subject to quantities constraints:

$$\max_{x,q} p'q - r'x \tag{8.1}$$

$$\text{subject to} \quad F(q,x) = 0, \quad \bar{q} \geq q \geq 0, \quad \bar{x} \geq x \geq 0$$

where x and q are $k\times 1$ and $M\times 1$ vectors of inputs and outputs respectively, and $\bar{x}$ and $\bar{q}$ are the upper quantity limits. The technology function F is an increasing function of q's and a decreasing function of x's. Other standard regularity conditions on F such as differentiability and strict quasi-concavity will also be assumed. To characterize the occurrence of kink points, we can also use the concept of virtual prices. To illustrate the construction of virtual prices from production technology F, let us consider a rather simple regime with $x^* = (0,x_2^*,\ldots,x_K^*)'$ and $q^* = (\bar{q}_1,q_2^*,\ldots,q_M^*)'$ where the first input is not utilized and the first output is produced at the quota level. The Lagrangean function is

$$L = p'q - r'x + \lambda(0-F(q,x)) + \phi'q + \psi'x + \delta'(\bar{q}-q) + \omega'(\bar{x}-x)$$

where ϕ, ψ, δ and ω are vectors of Lagrangean multipliers. This regime is characterized by the following Kuhn-Tucker conditions:

$$-r_1 - \lambda \frac{\partial F(q^*,x^*)}{\partial x_1} + \psi_1 = 0, \quad \psi_1 \geq 0;$$

$$-r_i - \lambda \frac{\partial F(q^*,x^*)}{\partial x_i} = 0, \qquad i=2,\ldots,K;$$

$$p_1 - \lambda \frac{\partial F(q^*,x^*)}{\partial q_1} - \delta_1 = 0, \quad \delta_1 \geq 0;$$

$$p_j - \lambda \frac{\partial F(q^*,x^*)}{\partial q_j} = 0, \qquad j=2,\ldots,M.$$

$$F(q^*,x^*) = 0, \quad q^* \geq 0, \quad x^* \geq 0.$$

Define the virtual price ξ_{d1} for input 1 and virtual price ξ_{s1} for output 1 at $(x_1=0,\bar{q}_1)$ as

$$\xi_{d1} = -\lambda \frac{\partial F(q^*,x^*)}{\partial x_1}$$

and

$$\xi_{s1} = \lambda \frac{\partial F(q^*,x^*)}{\partial q_1}.$$

Since $\frac{\partial F(q^*,x^*)}{\partial x_1} < 0$ and $\frac{\partial F(q^*,x^*)}{\partial q_1} > 0$, ξ_{d1} and ξ_{s1} are strictly positive. It follows that $\psi_1 = r_1 - \xi_{d1}$ and $\delta_1 = p_1 - \xi_{s1}$. Therefore this regime is characterized by

$$r_1 \geq \xi_{d1}, \quad 0 < x_i^* < \bar{x}_i, \qquad i=2,\ldots,K$$

and $$p_1 \geq \xi_{s1}, \quad 0 < q_j^* < \bar{q}_j, \qquad j=2,\ldots,M.$$

Input 1 is not used because the market price is too high and output 1 is produced up to the quota limit because the sale price is good. This technique can be easily generalized to the other regimes.

The case of increasing block prices in inputs can be reformulated into the framework (8.1). Consider the simple case of a single output x and $q = f(x)$ is the production function. Suppose that the input x will be charged a low price r_1 if the purchased amount is not greater than $x_1(1)$; otherwise, a higher price r_2 will be charged to the amount in excess of $x_1(1)$. Hence the cost $c(x)$ is

$$\begin{aligned} c(x) &= r_1 x, &&\text{if } x \leq x_1(1); \\ &= r_1 x_1(1) + r_2(x - x_1(1)), &&\text{if } x > x_1(1). \end{aligned}$$

The problem $\max_x \{pq - c(x) | q = f(x),\ x \geq 0\}$ can be rewritten into an identical problem with two perfectly substituted inputs:

$$\max_{x_1, x_2}\ pq - r_1 x_1 - r_2 x_2$$
$$\text{subject to } q = f(x_1 + x_2),\quad 0 \leq x_1 \leq x_1(1),\quad x_2 \geq 0.$$

As the price of x_1 is cheaper than x_2, x_1 will always be purchased first. x_2 will be purchased only if x_1 has been purchased up to its upper limit quantity $x_1(1)$. $x_1(1)$ is a kink point in this model.

Similarly, the decreasing block prices in outputs can also be formulated in the framework (8.1). Consider a single output case that the output quantity q can be sold at price p_1 if the quantity is not greater than the quota amount $q(1)$; however, quantities in excess of $q(1)$ can only be sold at a lower price p_2. The revenue function will be

$$\begin{aligned} R(q) &= p_1 q &&\text{if } q \leq q(1); \\ &= p_1 q(1) + p_2(q - q(1)), &&\text{if } q > q(1). \end{aligned}$$

The profit maximization problem $\max_{x,q} \{R(q) - rx | q = f(x)\}$ can be rewritten identically as a model with two perfectly substituted outputs:

$$\max_{q_1,q_2,x} \; p_1q_1 + p_2q_2 - rx$$
$$\text{subject to} \quad q_1 + q_2 = f(x), \quad 0 \leq q_1 \leq q(1), \quad q_2 \geq 0.$$

The quantity $q(1)$ is a kink point in this model.

Instead of the direct approach with profit maximization with a specified production technology, one can also consider the dual approaches with the specification of profit or cost functions. For the dual approaches, the Shepard lemma or Hotelling-McFadden lemma (see, e.g., McFadden [1978]), will provide the (notional) input demand and output functions. Virtual prices for the kink points can be solved directly from those functions. The regime occurrence criteria can also be expressed completely as inequalities in terms of virtual prices or notional input demand and output functions. Results, that are analogous to Theorems 2 and 3, can be derived from restricted profit function. Consider the restricted profit maximization problem where some of the quantities of inputs and outputs are fixed:

$$\begin{aligned} &\max_{x,q} \; pq - rx \\ &\text{subject to} \quad F(q,x) \leq 0, \\ &\qquad\qquad\quad x_i = x_i^\circ, \qquad i=1,\ldots,K_0 \\ &\text{and} \qquad\quad\;\; q_j = q_j^\circ, \qquad j=1,\ldots,M_0 \end{aligned} \tag{8.2}$$

where $K_0 < K$ and $M_0 \leq M$. Suppose that the optimal variable inputs and outputs exist and are denoted as $x^*_{K_0+1},\ldots,x^*_K$ and $q^*_{M_0+1},\ldots,q^*_M$, the optimal value of profit as a function of input and output prices and the fixed quantities is a restricted profit function π_R:

$$\pi_R = \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)$$

where $\bar{x}^\circ = (x_1^\circ,\ldots,x_{K_0}^\circ)'$ and $\bar{q}^\circ = (q_1^\circ,\ldots,q_{M_0}^\circ)'$. Without restrictions on input and output quantities, the unconstrained profit maximization problem,

$$\max_{x,q} pq - rx \quad \text{subject to} \quad F(q,x) \leq 0 \tag{8.3}$$

will imply an unrestricted profit function π_u:

$$\pi_u = \Pi_u(p,r).$$

For the cases of binding nonnegative constraints, we need to assume that the function F can be mathematically extended into the negative regions such that the unrestricted (notional) optimal solution $(\tilde{x},\tilde{q})$ and the unrestricted profit function Π_R are well defined. Let $\bar{\xi}_d = (\xi_{d1},\ldots,\xi_{dK_0})'$ and $\bar{\xi}_s = (\xi_{s1},\ldots,\xi_{sM_0})'$ be the virtual price vectors for the constrained inputs and outputs at $(\bar{x}^\circ,\bar{q}^\circ)$. The restricted profit function is related to the unrestricted profit function via the virtual price vectors:

$$\begin{aligned}
&\Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ) \\
&= \Pi_u(\bar{\xi}_s,p_{M_0+1},\ldots,p_M,\bar{\xi}_d,r_{K_0+1},\ldots,r_K) + \Sigma_{j=1}^{M_0} (p_j-\xi_{sj})q_j^\circ - \\
&\quad \Sigma_{i=1}^{K_0}(r_i-\xi_{di})x_i^\circ.
\end{aligned} \tag{8.4}$$

The following result is an analogue to Theorem 1, which relates the marginal restricted profit function with respect to the rationed quantities to virtual prices.

<u>Theorem 4</u>: $\dfrac{\partial\Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)}{\partial x_i} = \xi_{di} - r_i, \qquad i=1,\ldots,K_0$

and $\dfrac{\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)}{\partial q_j} = p_j - \xi_{sj}, \qquad j=1,\ldots,M_0.$

Proof: From (8.4), we have

$$\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)/\partial x_i$$

$$= \Sigma_{j=1}^{M_0} \frac{\partial \Pi_u}{\partial \xi_{sj}} \frac{\partial \xi_{sj}}{\partial x_i} + \Sigma_{\ell=1}^{K_0} \frac{\partial \Pi_u}{\partial \xi_{d\ell}} \frac{\partial \xi_{d\ell}}{\partial x_i} - \Sigma_{j=1}^{M_0} \frac{\partial \xi_{sj}}{\partial x_i} q_j^\circ + \Sigma_{\ell=1}^{K_0} \frac{\partial \xi_{d\ell}}{\partial x_i} x_i^\circ + (\xi_{di} - r_i).$$

By the Hotelling-McFadden duality theorem, $\dfrac{\partial \Pi_u}{\partial \xi_{sj}} = q_j^\circ$ and $\dfrac{\partial \Pi_u}{\partial \xi_{d\ell}} = -x_\ell^\circ$ and hence it follows that

$$\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)/\partial x_i = \xi_{di} - r_i.$$

Similarly, we can prove that $\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)\partial q_j = p_j - \xi_{sj}$. Q.E.D.

Analogously to Lemma 1 in Section 5, we have

Lemma 2: The restricted profit function $\Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)$ is (strictly) quasi-concave with respect to any one of the rationed goods x_i or q_j, $i \in \{1,\ldots,K_0\}$ and $j \in \{1,\ldots,M_0\}$, if the production possibility set $\{(q,x)|F(q,x) \leq 0\}$ is (strictly) convex.

Denote the optimal variable inputs and outputs equations of the restricted profit maximization (8.2) as

$$x_i^* = D_i(p,r;x_1^\circ,\ldots,x_{K_0}^\circ,q_1^\circ,\ldots,q_{M_0}^\circ), \qquad i=K_0+1,\ldots,K$$

and

$$q_j^* = S_j(p,r;x_1^\circ,\ldots,x_{K_0}^\circ,q_1^\circ,\ldots,q_{M_0}^\circ), \qquad j=M_0+1,\ldots,M.$$

An analogous result to Theorem 3 is the following theorem.

Theorem 5: (i) $\dfrac{\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)}{\partial x_i} \leq 0$ if and only if

$$D_i(p,r;x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_{K_0}^\circ,\bar{q}^\circ) \leq x_i^\circ, \qquad i=1,\ldots,K_0$$

(ii) $$\frac{\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)}{\partial x_i} \geq 0 \text{ if and only if}$$

$$D_i(p,r;x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_{K_0}^\circ,\bar{q}^\circ) \geq x_i^\circ, \qquad i=1,\ldots,K_0$$

(iii) $$\frac{\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)}{\partial q_j} \leq 0 \text{ if and only if}$$

$$S_j(p,r;\bar{x}^\circ,q_1^\circ,\ldots,q_{j-1}^\circ,q_{j+1}^\circ,\ldots,q_{M_0}^\circ) \leq q_j^\circ, \qquad j=1,\ldots,M_0$$

and (iv) $$\frac{\partial \Pi_R(p,r;\bar{x}^\circ,\bar{q}^\circ)}{\partial q_j} \geq 0 \text{ if and only if}$$

$$S_j(p,r;\bar{x}^\circ,q_1^\circ,\ldots,q_{j-1}^\circ,q_{j+1}^\circ,\ldots,q_{M_0}^\circ) \geq q_j^\circ, \qquad j=1,\ldots,K_0.$$

The results in the above theorem are intuitively appealing. If the restricted profit is increasing as the restricted quantity of input is increased, the optimal input, if the restriction on this input is relaxed, must be more than the previous restricted quantity, and vice versa. Similarly, this is true for the production of outputs. The proofs of these results and the Lemma 2 are similar to the consumer demand analyses and are omitted here.

9. Conclusions.

This article has analyzed consumer demand and production economics with kink points in the demand or production schedules, which arise as binding nonnegative constraints, quantity rationing, increasing block prices for demanded goods or production quotas. The kink points are vertices in the convex budget sets and are atoms in econometric analyses. Various procedures are provided to characterize the occurrence of kink points so that the probabilities of the atoms can be derived from the basic economic structures. The basic structure can be either a specific utility function or indirect utility function for consumer demand analysis, and production technology or profit function for production analysis. Several equivalent sets of conditions are derived to characterize the occurrence of various kink points. The presence of kink points divides the demand schedule or production schedule into different regimes. The virtual prices provide the structural change of the schedules across regimes. By comparing various virtual prices with market prices, they can determine the regime where the demanded quantities or the produced quantities will lie on. Such comparisons are intuitively appealing as the virtual prices are reservation prices or shadow prices. Virtual prices can be derived either from direct utility function or from indirect utility function. For production analysis, they can be derived either from production technology or from profit function. Our analysis unifies the direct and dual approaches in consumption and production economics with kink points. Empirical application to consumer demand with nonnegative binding constraints can be found in Lee and Pitt [1983]. The analysis generalizes our previous work to the analysis of much more complicated quantity rationing and convex budget sets models.

Appendix: Proof of Theorems.

Proof of Theorem 1

The Lagrangean function for the general model (2.1) is

$$L = U(x_1,x_2,\ldots,x_m) + \Sigma_{i_1=1}^{I_1} \cdots \Sigma_{i_m=1}^{I_m} \lambda_{i_1 i_2\ldots i_m}(M_{i_1 i_2\ldots i_m} - \Sigma_{j=1}^{m} p_{ji_j}x_j)$$

$$+ \Sigma_{j=1}^{m} \psi_j x_j.$$

The Kuhn-Tucker conditions that characterize the regime in Theorem 1 are

$$\frac{\partial U(x^*)}{\partial x_j} - \Sigma_{i_j=1}^{I_j} (\Sigma_{i_1=1}^{I_1}\cdots\Sigma_{i_{j-1}=1}^{I_{j-1}} \Sigma_{i_{j+1}}^{I_{j+1}}\cdots\Sigma_{i_m=1}^{I_m} \lambda_{i_1\ldots\, i_{j-1}\, i_j\, i_{j+1}\ldots i_m})p_{ji_j}$$

$$+ \psi_j = 0,$$

for all $j=1,\ldots,m$;

$$\psi_j \geq 0,\quad j=1,\ldots,\ell_1-1;\quad \psi_{\ell_1} = \psi_{\ell_1+1} = \cdots = \psi_m = 0;$$

$$\Sigma_{j=\ell_1}^{\ell_2-1} p_{j(i^\circ_j+d_j)}x_j(i^\circ_j) + \Sigma_{j=\ell_2}^{m} p_{ji^\circ_j}\, x^*_j = M_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})}$$

$$i^\circ_{\ell_2}\ldots i^\circ_m;\qquad \lambda_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})i^\circ_{\ell_2}\ldots i^\circ_m} \geq 0$$

where the d's are dichotomous variables with values 0 or 1, and, for all remaining $(i_1,\ldots,i_m)$,

$$\Sigma_{j=\ell_1}^{\ell_2-1} p_{ji_j}\, x_j(i^\circ_j) + \Sigma_{j=\ell_2}^{m} p_{ji_j}\, x^*_j \leq M_{i_1\ldots i_m},\quad \lambda_{i_1\ldots i_m} = 0.$$

Hence,

$$\frac{\partial U(x^*)}{\partial x_j} - (\Sigma_{d_{\ell_1}=0}^{1}\cdots\Sigma_{d_{\ell_2-1}=0}^{1} \lambda_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})i^\circ_{\ell_2}\ldots i^\circ_m})p_{j1}$$

$$+ \psi_j = 0,\quad \psi_j \geq 0,$$

for $j=1,\ldots,\ell_1-1$;

$$\frac{\partial U(x^*)}{\partial x_j} - \Sigma^1_{d_j=0}(\Sigma^1_{d_{\ell_1}=0}\cdots\Sigma^1_{d_{j-1}=0}\,\Sigma^1_{d_{j+1}=0}\cdots\Sigma^1_{d_{\ell_2-1}=0}\,\lambda_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})i^\circ_{\ell_2}\ldots i^\circ_m})p_{j(i^\circ_j+d_j)} = 0$$

for $j=\ell_1,\ldots,\ell_2-1$, and

$$\frac{\partial U(x^*)}{\partial x_j} - (\Sigma^1_{d_{\ell_1}=0}\;\lambda_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})i^\circ_{\ell_2}\ldots i^\circ_m})p_{ji^\circ_j} = 0$$

for $j=\ell_2,\ldots,m$.

Define the virtual prices $\xi_j(x^*)$ at x^* for $j=1,\ldots,\ell_2-1$ as

$$\xi_j(x^*) = \frac{1}{\mu}\frac{\partial U(x^*)}{\partial x_j}$$

where $\mu = \Sigma^1_{d_{\ell_1}=0}\cdots\Sigma^1_{d_{\ell_2-1}=0}\,\lambda_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})i^\circ_{\ell_2}\ldots i^\circ_m}$. It follows that $\psi_j = \mu(p_{j1} - \xi_j(x^*))$ and hence $p_{j1} \geq \xi_j(x^*)$, for $j=1,\ldots,\ell_1-1$. Denote

$$\mu_{j0} = \Sigma^1_{d_{\ell_2-1}=0}\cdots\Sigma^1_{d_{j+1}=0}\,\Sigma^1_{d_{j-1}=0}\cdots\Sigma^1_{d_{\ell_1}=0}\,\lambda_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{j-1}+d_{j-1})i^\circ_j(i^\circ_{j+1}+d_{j+1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})\cdot i^\circ_{\ell_2}\ldots i^\circ_m}$$

and $\mu_{j1} = \Sigma^1_{d_{\ell_2-1}=0}\cdots\Sigma^1_{d_{j+1}=0}\,\Sigma^1_{d_{j-1}=0}\cdots\Sigma^1_{d_{\ell_1}=0}\,\lambda_{1\ldots1(i^\circ_{\ell_1}+d_{\ell_1})\ldots(i^\circ_{j-1}+d_{j-1})(i^\circ_j+1)(i^\circ_{j+1}+d_{j+1})\ldots(i^\circ_{\ell_2-1}+d_{\ell_2-1})i^\circ_{\ell_2}\ldots i^\circ_m}$

for $j=\ell_1,\ldots,\ell_2-1$. Hence $\mu[\xi_j(x^*) - (\frac{\mu_{j0}}{\mu}p_{ji^\circ_j} + \frac{\mu_{j1}}{\mu}p_{j(i^\circ_j+1)})] = 0$ and $\xi_j(x^*) = \frac{\mu_{j0}}{\mu}p_{ji^\circ_j} + \frac{\mu_{j1}}{\mu}p_{j(i^\circ_j+1)}$, $j=\ell_1,\ell_1+1,\ldots,\ell_2-1$. As $0 \leq \frac{\mu_{j0}}{\mu} \leq 1$ and $\frac{\mu_{j0}}{\mu} + \frac{\mu_{j1}}{\mu} = 1$, it follows that $p_{ji^\circ_j} \leq \xi_j(x^*) \leq p_{j(i^\circ_j+1)}$. Therefore, the conditions that will characterize this regime are

$$p_{j1} \geq \xi_j(x^*), \qquad j=1,\ldots,\ell_1-1;$$

$$p_{ji_j^\circ} \leq \xi_j(x^*) \leq p_{j(i_j^\circ+1)}, \qquad j=\ell_1,\ell_1+1,\ldots,\ell_2-1;$$

and $$x_j(i_j^\circ-1) < x_j^* < x_j(i_j^\circ), \qquad j=\ell_2,\ell_2+1,\ldots,m. \qquad \text{Q.E.D.}$$

Proof of Lemma 1

Without loss of generality, consider the rationed good 1. Consider $V(\xi(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)$ as a function of x_1° with other variables, namely x_i°, $i=2,\ldots,\ell;\bar{p}$ and M, as fixed arguments, and denote it as $\bar{V}(x_1^\circ)$. Let $x_1^\circ(0)$ and $x_1^\circ(1)$ be two different quantities of good 1, and suppose that $\bar{V}(x_1^\circ(1)) \geq \bar{V}(x_1^\circ(0))$. Consider any $\lambda \in (0,1)$ and $x_1^\circ(\lambda) = \lambda x_1^\circ(1) + (1-\lambda)x_1^\circ(0)$. As in (5.1), let $x_i^*(0)$ and $x_i^*(1)$, $i=\ell+1,\ldots,m$, be defined from the following relations:

$$\bar{V}(x_1^\circ(0)) = U(x_1^\circ(0),x_2^\circ,\ldots,x_\ell^\circ,x_{\ell+1}^*(0),\ldots,x_m^*(0))$$

$$\bar{V}(x_1^\circ(1)) = U(x_1^\circ(1),x_2^\circ,\ldots,x_\ell^\circ,x_{\ell+1}^*(1),\ldots,x_m^*(1)).$$

Define the convex combinations $x_i(\lambda) = \lambda x_i^*(1) + (1-\lambda)x_i^*(0)$, $i=\ell+1,\ldots,m$. By the quasi-concavity of U, $U(x_1^\circ(\lambda),x_2^\circ,\ldots,x_\ell^\circ,x_{\ell+1}(\lambda),\ldots,x_m(\lambda)) \geq \bar{V}(x_1^\circ(0))$. It follows that the optimality of the indirect utility function,

$$\bar{V}(x_1^\circ(\lambda)) \geq U(x_1^\circ(\lambda),x_2^\circ,\ldots,x_\ell^\circ,x_{\ell+1}(\lambda),\ldots,x_m(\lambda))$$

and hence $\bar{V}(x_1^\circ(\lambda)) \geq \bar{V}(x_1^\circ(0))$. If U is strictly quasi-concave, we have a strict inequality. Q.E.D.

Proof of Theorem 3

Let $\tilde{x}_i = D_i(p_i,\bar{p},M - \sum_{\substack{j=1 \\ j\neq i}}^{\ell} p_j x_j^\circ | x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_\ell^\circ)$ be the conditional (notional) demand quantities conditional on $x_1^\circ,\ldots,x_{i-1}^\circ,x_{i+1}^\circ,\ldots,x_\ell^\circ$. To prove the theorem, we will prove the following equivalent properties:

(i) $\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i = 0$ implies $\tilde{x}_i = x_i^\circ$;

(ii) $\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i > 0$ implies $\tilde{x}_i > x_i^\circ$;

and (iii) $\partial V(\bar{\xi}(\bar{p},M_R;\bar{x}^\circ),\bar{p},c)/\partial x_i < 0$ implies $\tilde{x}_i < x_i^\circ$.

Under the condition in (i), part (iii) in Corollary 1 implies that $\xi_i(\bar{p},M_R;\bar{x}^\circ) = p_i$ and hence $\tilde{x}_i = x_i^\circ$. To prove (ii), we use counter-arguments. Suppose that $\tilde{x}_i \leq x_i^\circ$. Under the condition in (ii), there exists quantity x_i^1 for good i such that $x_i^1 > x_i^\circ$ and $\bar{V}(x_i^1) > \bar{V}(x_i^\circ)$ where $\bar{V}$ was defined in the proof of Lemma 1. By the definition of $\tilde{x}_i$, $\bar{V}(\tilde{x}_i) = \max_x \{U(x)|p'x = M,\ x_j = x_j^\circ,\ j=1,\ldots,i-1,i+1,\ldots,\ell\}$. On the other hand, since $\bar{V}(x_i^1) = \max_x\{U(x)|p'x=M,\ x_j=x_j^\circ, j=1,\ldots,i-1,i+1,\ldots,\ell$ and $x_i=x_i^1\}$, $\bar{V}(\tilde{x}_i) \geq \bar{V}(x_i^1)$. As $\tilde{x}_i \leq x_i^\circ \leq x_i^1$ and $\bar{V}(x_i)$ is quasi-concave with respect to x_i, $\bar{V}(x_i^\circ) \geq \bar{V}(x_i^1)$, which is a contradiction. Property (iii) can be similarly proved. Q.E.D.

FOOTNOTES

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(1) On the other hand, commodities such as electricity may be priced with decreasing block rates which will create a concave budget set. The framework for the analysis of concave budget sets will involve the comparisons of indirect utilities over different regimes, and does not fit into our analysis in this article. The problems of concave budget sets will be considered elsewhere.

(2) Notional demand function is referred to the demand function derived from the utility maximization under the linear budget set. It is the regular demand function in the nonnegative commodities orthant but is extended mathematically into the negative commodity spaces.

(3) Our analysis can be generalized in a straightforward manner to incorporate quantity rationing with a fixed amount of quantity. This case is the main concern in the studies of Deaton [1981] and Blundell and Walker [1982].

(4) Our analysis can also consider the lower quantity limits which are positive instead of zeros with obvious modification.

(5) For some restrictive cases, more general functional form can be used. For example, for the analysis of quantity rationing with only one rationed good, a generalized Gorman indirect utility is used in Deaton [1981] and Blundell and Walker [1982]. For the model with only binding nonnegative constraints, the translog indirect utility function is considered in Lee and Pitt [1983].

(6) Two textbook examples on increasing block input prices in production can be found in Henderson and Quandt [1980]. One of the examples is on discontinuous labor contract for which the firm has to pay higher wage rates for overtime labor.

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