TWO NOTES ON HURWICZ'S TOPOLOGICAL LEMMA

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Introduction.

The object of these notes has to do with some general mathematical aspects of the theory of mechanisms. We start from a topological lemma stated by Professor L. Hurwicz on size inequality, which says the following:

Let S and T be topological spaces, with T having the similarity property[1]. Let $\emptyset : T \twoheadrightarrow S$ be a spot-threaded injective correspondence with a spot-domain[2] U. Then[3] $S \stackrel{F}{\geq} T$ if either of the following two conditions is satisfied:

(a) both S and T are Hausdorff spaces and T is locally compact; or

(b) the inverse function $\emptyset^{-1} : \emptyset(T) \to T$ is continuous on $\emptyset(U)$.

The spot threadedness assumption, as Hurwicz points out, was introduced and used, but not named, by Chander [1980, cor. 1.2]. The similarity property of the space T represent a possible generalization of lemmata 1 and 2 of Sato [1981] and Lemma 3, of Nayak [1982]. Hurwicz left open the question whether the space T, with the properties imposed on the lemma,

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1. A topological space T has the similarity property if every open set V has a subset V' which, in the relative topology, is homeomorphic to T.

2. The correspondence $\emptyset : T \twoheadrightarrow S$ is <u>spot-threaded with spot domain U</u> whenever U is an open subset of T and there exist some continuous function $f : U \to S$ such that $f(x) \in \emptyset(x)$ for all $x \in U$.

3. The notation $\stackrel{F}{\geq}$ refers to the Fréchet ordering. $S \stackrel{F}{\geq} F$ means that T can be inbedded in S.

could be a space which is not a finite-dimensional Euclidean space. In Section II of these notes we show an example of a topological space which satisfies all the assumptions on T and it is neither a finite dimensional Euclidean space nor can it be embedded in any of these spaces. That shows that Hurwicz' lemma is a true generalization of Sato and Nayak results quoted above. On the other hand we give sufficient conditions for a product space to have the similarity property, be locally compact and Hausdorff.

In Section I of this paper, we offer an extension of Hurwicz' lemma by introducing a weaker notion of the similarity property and by requiring the space T to have a particular locally compact subspace.

I. An extension of the Hurwicz Topological Lemma.

Let T be a topological space. Let U be a non-empty subspace of T.

Definition 1

We shall say that T has the *weak local similarity property at U* whenever there exist some $V \subset U$ which, in the topology of U, is homeomorphic to T.

Definition 2

We shall say that T has the *strong local similarity property at U* whenever every open subset $U' \subseteq U$ has a subset V' which, in the topology of U is homeomorphic to T.

Definition 3

We shall say that T has the *similarity property* whenever T has the strong local similarity property at T.

<u>Remark</u>: Consider the following statements.

(1) T has the weak local similarity property at some subspace U of T.

(2) T has the strong local similarity property at some subspace U of T.

(3) T has the similarity property.

It is clear that

(i) (3) $\Rightarrow$ (2) $\Rightarrow$ (1), and

(ii) (1) $\not\Rightarrow$ (2) $\not\Rightarrow$ (3) as the following examples show:

(a) (1) $\not\Rightarrow$ (2).

Let $T = T_1 \cup T_2$, where $T_1 = \{(x_1,x_2) \in R^2 : x_2 = 0\}$ and $T_2 = \{(y_1,y_2) \in R^2 : y_1 = 0\}$. Endow T with the topology inherited from R^2.

Let $U = \{t \in T : \|t\| \leq 1\}$. Then T has the weak local similarity property at U because the subset V of U, $V \equiv \{t' \in U : \|t\| < a < 1$, for some $a > 0\}$ is homeomorphic to T the homeomorphism being

$$\psi : V \to T \text{ , where } \psi(t') = \frac{t'}{a - \|t'\|} \text{ .}$$

However, T has not the strong local similarity property at any subspace. To show this,

let W be any subspace of T. We may assume that T contains some $P \neq (0,0)$, otherwise we are finished. Say $P \in T_2$. Let V be an open subset of W such that $P \in V$ and $T_1 \cap V = \emptyset$. Suppose there exists some $V' \subset V$ which is homeomorphic to T. Since T is connected, V' must be connected. But V' connected, $V' \subset V \subset T_2$ and T_2 homeomorphic to R, imply that V' must be an interval of R. However it is clear that an interval in R cannot be homeomorphic to T.

(b) (2) $\not\Rightarrow$ (3).

Let $T = T_1 \cup T_2$, where $T_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $T_2 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$. It is clear that T, as a subspace of $\mathbb{R}^2$, has the strong local similarity property at T_1 because any open subset of T_1 contains a subset which is homeomorphic to $\mathbb{R}^2$. On the other hand it is obvious that T does not have the similarity property.

<u>Topological Lemma on Size Inequality</u>.

Let $\phi : T \twoheadrightarrow S$ be a correspondence, where S and T are topological spaces, and let U be a non-empty subspace of T such that

(i) $\phi|_U : U \twoheadrightarrow S$ is an injective correspondence, and

(ii) there exists some continuous function $f : U \to S$ such that $f(t) \in \phi(t)$ for each $t \in U$. Then

$$S \overset{F}{\geq} T$$

if either of the following two conditions is satisfied:

(a) both S and T are Hausdorff spaces, U is locally compact and T has the strong local similarity property at U; or

(b) the inverse function $[\phi|_U]^{-1} : \phi(U) \to U$ is continuous and T has the weak local similarity property at U.

Proof: (follows Professor L. Hurwicz' proof).

<u>Step 1</u>: Let condition (a) hold. Let $V \subseteq U$ be a non-empty open subset of U. Since T is Hausdorff and U, as a subspace of T, is locally compact there exists some $V' \subseteq V$, V' open in U, such that the closure of V' in U, (denoted $cl|_U V$) is in V, and $cl|_U V'$ is a compact subset of U. By hypothesis (i) and (ii), $f|_{cl|_U V'} : cl|_U V' \longrightarrow f(cl|_U V')$ is a one-to-one onto continuous function from a compact space to a Hausdorff

space, hence a homeomorphism (Dugundji; p. 226, thrm 2.1(2)). Therefore $f|_{V'} : V' \to f(V')$ is an homeomorphism. Since T has the strong local similarity property at U, there exists some $T' \subset V'$ homeomorphic to T. Set $W' = f(T')$, then $f|_{T'} : T' \to W'$ is an homeomorphism. Hence $T \simeq T' \simeq W'$, and so

$$(*) \qquad W' \geq^F T' \geq^F T$$

because by definition of $\geq^F$, $A \simeq B$ implies $A \geq^F B$.

On the other hand, since $S' \geq W'$ we have

$$(**) \qquad S \geq^F W'$$

because the Frèchet ordering, $\geq^F$, is monotone with respect to set inclusion (in the relative topology). Since $\geq^F$ is transitive, the relations (*) and (**) imply

$$S \geq^F T .$$

<u>Step 2</u>: Let condition (b) hold. Since $[\emptyset|_U]^{-1} : \emptyset(U) \to U$ is continuous, by hypothesis (i) and (ii), it follows that $[\emptyset|_U]^{-1}\big|_{f(U)} : f(U) \to U$ is continuous. Since $f : U \to f(U)$ is continuous, one-to-one and onto, and $f \circ [\emptyset|_U]^{-1}\big|_{f(U)}$ is the identity function $i : f(U) \to f(U)$, we have that $f : U \to f(U)$ is an homeomorphism. Since T has the weak local similarity property at U, there exist some $T' \subset U$ which is homeomorphic to T. Hence $f|_{T'} : T' \to f(T')$ is an homeomorphism.

Proceding as in Step 1 we may conclude that

$$S \geq^F F .$$

q.e.d.

The following example shows that the above lemma is a true extension of Hurwicz' lemma (see (c)), concerning the local compactness and similarity property assumptions on T .

Let $T = [0,1] \cup Q$, where Q denotes all rationals, with the relative topology of the real line. It is clear that T has not the similarity property and it is not locally compact. However, the subspace $U = [0,1]$ is locally compact and T has the strong local similarity property at U , because any non-empty open subset of U , contains a subset which is homeomorphic to the real line.

On the other hand, with respect to the assumption (b) of the previous lemma we only require that T have the weak local similarity property at U instead of the stronger assumption used by Hurwicz (i.e., that T have the similarity property).

II General Results and some examples showing that the Hurwicz Topological Lemma goes beyond Euclidean spaces (with respect to T).

Lemma 1 is used in proving (Lemma 2) that a product of spaces having the similarity property has the similarity property. In turn, lemmata 2 and 3 are used to show (see proposition and example in pages 10 and 11) that the Hurwicz topological lemma goes beyond Euclidean spaces. We end up this section showing that (lemma 4) and open subspace of a topological space having the similarity property has also the similarity property.

Lemma 1

Let $\{X_\alpha\}_{\alpha\in J}$ and $\{Y_\alpha\}_{\alpha\in J}$ be two collections of topological spaces indexed by the same non-empty index set J .

If for each $\alpha \in J$, X_α is non-empty and homeomorphic to Y_α , then the product space $\prod\limits_{\alpha\in J} X_\alpha$ is homeomorphic to the product space $\prod\limits_{\alpha\in J} Y_\alpha$.

Proof:

For each $\alpha \in J$, Let $f_\alpha : X_\alpha \to Y_\alpha$ be the homeomorphism between the spaces X_α and Y_α . Denote by $x(\alpha)$ an element of X_α , and denote by $(x(\alpha))_{\alpha\in J}$, an element of the product space $\prod\limits_{\alpha\in J} X_\alpha$.

For each $\beta \in J$ define the function

$$f'_\beta : \prod_{\alpha\in J} X_\alpha \to Y_\beta \qquad \text{by}$$

$$f'_\beta((x(\alpha))_{\alpha\in J}) = f_\beta(x(\beta)) ,$$

then it is easy to see that f'_β , $\beta \in J$, is a continuous, open and onto map.

Define:

$$F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha \quad \text{as}$$

$$F((x(\alpha))_{\alpha \in J}) = (f'_\beta((x(\alpha))_{\alpha \in J}))_{\beta \in J} .$$

We claim that F is an homeomorphism. It is clear that F is a one-to-one and onto mapping. The mapping F is continuous because each of its coordinate functions is a continuous function (see Munkres; p. 115, Thm. 8.5). To finish the proof, let U be a basis element for the topology of $\prod_{\alpha \in J} X_\alpha$, then

$$U = \prod_{\alpha \in J} U_\alpha$$

where U_α is an open subset of X_α which equals X_α except for finitely many α's. Now, it is easy to see that

$$F(U) = \prod_{\beta \in J} f'_\beta(U_\beta)$$

and since f'_β, $\beta \in J$, is an onto open map, it follows that $F(U)$ is a basis element for the topology of $\prod_{\alpha \in J} Y_\alpha$.

q.e.d.

The converse of Lemma 1 is not true, as shown by the following two examples.

Example 1.

Let $A = \mathbb{R}^2$, $B = \mathbb{R}^6$, $C = \mathbb{R}^3$ and $D = \mathbb{R}^4$. Then $A \times B$ is homeomorphic with $C \times D$, however, A is not homeomorphic with C nor with D .

Example 2.

Let X be any topological space, and denote by Z_+ the set of positive integers. It can be shown that X^{Z+} is homeomorphic with $(X^{Z+})^{Z+}$, however it is not always true that X is homeomorphic with X^{Z+} .

Lemma 2.

Let $\mathcal{X} \equiv \prod_{\alpha \in J} X_\alpha$ be a product space, where J is any non-empty index set and for each $\alpha \in J$, X_α is a non-empty topological space having the similarity property. Then $\mathcal{X}$ has the similarity property.

Proof:

Let $U \subseteq \mathcal{X}$ be a non-empty open set. Then there exists some basis element $B \subseteq U$, $B = \prod_{\alpha \in J} U_\alpha$, where U_α is open in X_α for each α , and U_α equals X_α except for finitely many values of α . Since, for each $\alpha \in J$, X_α has the similarity property, there exist some $V_\alpha \subset U_\alpha$ which, in the relative topology, is homeomorphic to X_α .

Let $V = \prod_{\alpha \in J} V_\alpha$. Then $V \subseteq U$ and by lemma 1 it follows that V is homeomorphic with $\mathcal{X}$. Hence $\mathcal{X}$ has the similarity property.

q.e.d.

By using the same arguments as in the proof of Lemma 2, we can state the following two remarks.

Remark 1.

Let J be a non-empty index set. If for some $\alpha' \in J$, we have that the space $X_{\alpha'}$ has the weak local similarity property at some subspace $U_{\alpha'}$ then the product space $\prod_{\alpha \in J} X_\alpha$ has the weak local similarity property at some subspace U, namely, at $U = \prod_{\alpha \in J} U_\alpha$, where $U_\alpha = X_\alpha$, $\alpha \neq \alpha'$ and $U_\alpha = U_{\alpha'}$ for $\alpha \neq \alpha'$.

Remark 2.

Let J be a non-empty index set. If for each $\alpha \in J$, X_α has the strong local similarity property at some subspace U_α, then $X = \prod_{\alpha \in J} X_\alpha$ has the strong local similarity property at some subspace U, namely at $U = \prod_{\alpha \in J} U_\alpha$.

The converse of Lemma 2 is not true as the following example shows.

Example:

Let A be any topological space. Consider the product space A^{Z_+} where Z_+ is the set of positive integers. Then A^{Z_+} has the similarity property.

Proof:

Let U be an arbitrary open subset of A^{Z_+}. Then, by definition of the product topology U contains an open subset of the form $\prod_{n \in Z_+} V_n$ where for all $n \in Z_+$, V_n is non-empty and open in A, and $V_n \neq A$ only for finitely many n's. Therefore there exist some $n^* \in Z_+$ such that if $n \geq n^*$, then $V_n = A$. For each $n < n^*$ pick an element

$v(n) \in V_n$. Consider the subset of V , $V^* \subset V$, $V^* = \prod_{n\in Z} V^*_n$ where

$$V^*_n = \begin{cases} \{v(n)\} & \text{if } n < n^* \\ V_n & \text{if } n \geq n^* \end{cases}$$

$V^* \subset V \subset U$. V^* is homeomorphic to A^{Z+} . The homeomorphism $F : A^{Z+} \to V^*$

$$F((a(n)_{n\in Z_+})) = (f_m(a(n)_{n\in Z_+}))_{m\in Z_+}$$

where for each $m \in Z_+$ the coordinate function f_m , $f_m : A^{Z+} \to V^*_m$ is defined as follows

$$f_m((a(n))_{n\in Z}) = \begin{cases} v_m & \text{if } m < n^* \\ a(m - n^* + 1) & \text{if } m \geq n^* \end{cases}$$

The function F is one-to-one:

let $(a(n))_{n\in Z}$ and $(a'(n))_{n\in Z}$ be two distinct elements of A^{Z+} , i.e., there exists some $n' \in Z_+$, such that $a(n') \neq a'(n')$. Let $m' = n' + n^* - 1$ then the m'-th coordinate function will have different images for those two points in A^{Z+} .

The function F is onto:

Let $(v^*(n))_{n\in Z_+}$ be an element of V^* . Then the point $(a(n))_{n\in Z_+} \in A^{Z+}$ such that for all $n \in Z_+$, $a(n) = v^*(n + n^* - 1)$ has as its image the initially given point.

The function F is continuous:

Each coordinate function is either constant or a projection, therefore is continuous and therefore the function F defined from its coordinate functions is also continuous.

The inverse function of F , F^{-1} , is continuous:

Define $F^{-1} : V^* \to A^{Z+}$ as $F^{-1}((v^*(n))_{n\epsilon Z_+}) = g_m((v^*(n))_{n\epsilon Z+})_{m\epsilon Z+}$ where $g_m : V^* \to A$ is defined as $g_m((v^*(n))_{n\epsilon Z}) = v^*(m + n^* - 1)$.

Again since the coordinate function of F^{-1} are projections and therefore continuous, the function F^{-1} is also continuous.

Choosing A with no similarity property, i.e., A finite, establishes the desired result.

Taking $A = \{0,1\}$, it has been established that the Cantor set, homeomorphic to $\{0,1\}^{Z+}$, has the similarity property.

Note

A notationally more cumbersome but otherwise similar construction can be made to show that, for any space A the product space A^R has the similarity property.

Lemma 3

Let $X \equiv \prod_{\alpha\in J} X_\alpha$ be a product space, where J is an arbitrary non-empty index set and X_α is non-empty for each $\alpha \in J$.

X is locally compact if and only if each X_α is locally compact and X_α is compact for all but finitely many values of α.

Proof: (See Dugundji, pp. 239 - 240, thm. 6.5(4).)

As a direct application of the above lemmas, and using the fact that an arbitrary product of Hausdorff spaces is a Hausdorff space (Munkres, p. 197, Thm 2.2(a); Dugundji, p. 138, Thm 1.3(3)), we get the following:

Proposition

Let $\mathcal{X} = \prod_{\alpha\in J} X_\alpha$ be a product space, where J is an arbitrary non-empty index set and X_α is a non-empty topological space.

For each $\alpha \in J$, assume:

(a) X_α is a Hausdorff space,

(b) X_α has the similarity property, and

(c) X_α is locally compact and X_α is compact for all but finitely many values of α.

Then $\mathcal{X}$ is a locally compact Hausdorff space having the similarity property.

Proof:

$\mathcal{X}$ is locally compact by lemma 3, and it has the similarity property by lemma 2. It is Hausdorff by the theorem cited above.

q.e.d.

The following example shows (for any J) that the Hurwicz Topological Lemma applies not only to Euclidean spaces. When J is infinite, it applies to a space which is not even a subspace of a Euclidean space.

Example.

Let J be any arbitrary non-empty index set. For each $\alpha \in J$, let $X_\alpha = [0,1]$, as a subspace of the real line. Then the product space $\mathcal{X} = \prod_{\alpha \in J} X_\alpha$ has the following properties:

(a) It is a Hausdorff space because each X_α is Hausdorff. (See theorem 8.4, p. 115 in Munkres.)

(b) It is locally compact because for each $\alpha \in J$, X_α is a compact space and so, by the Tychonoff theorem, $\mathcal{X}$ is a compact space.

(c) It has the similarity property as a direct application of Lemma 2.

Lemma 4.

Let X be a topological space with the similarity property. Let U be a non-empty subspace of X. If U is open in X then U has the similarity property.

Proof:

Let U' be an open subset of U. Then U' is open in X. Since X has the similarity property, there exists some $V \subseteq U'$ which, in the relative topology of X, is homeomorphic to X. But the relative topology of V as a subspace of X is the same as the topology of V as a subspace of U. Let $f : X \to V$ be an homeomorphism, then $f|_U : U \to f(U)$ is an homeomorphism and $f(U) \subset V \subset U'$. Therefore U has the similarity property.

q.e.d.

REFERENCES

(1980) Chander, Parkash, "Dimension of Euclidean Message Spaces," Mimeo, ISI, New Delhi, Mimeo, undated, received November 1980.

(1966) Dugundji, James, Topology, Boston, Allyn and Bacon, 1966.

(1983) Hurwicz, Leonid, "On Informational Decentralization and Efficiency in Resource Allocation Mechanisms," typescript.

(1975) Munkres, James R., Topology, A First Course, Englewood Cliffs, New Jersey, Prentice-Hall, 1975.

(1982) Nayak, Jayendra, "The Informational Efficiency of the Walras Process in Pure Exchange Economies," Research Paper No. 20, Churchill College, University of Cambridge, undated, received October, 1982.

(1981) Sato, Fumitaka, "On the Informational Size of Message Spaces for Resource Allocation Processes in Economies with Public Goods," Journal of Economic Theory, 24, pp. 48 - 69, 1981.