SPECIFICATION AND ESTIMATION OF DEMAND SYSTEMS WITH LIMITED DEPENDENT VARIABLES

by

Lung-Fei Lee and Mark M. Pitt*

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

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ABSTRACT

Econometric models for consumer demand systems with zero demand quantities are specified which consider observed demands as the result of optimal behavior. Our approach utilizes the concept of virtual prices to transform binding zero quantities into nonbinding quantities and to provide a rigorous justification for structural shifts in the observed (positive) demand equations across demand regimes. Switching conditions, which determine the occurrence probabilities of different demand regimes, are provided in terms of notional (latent) demand equations. This approach can estimate demand relationships derived from either direct or indirect utility functions. The econometric model is used to estimate a system of demand equations based on a household sample survey from Indonesia.

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1. Introduction.

A number of recent studies have used household level data to estimate demand relationships [Pitt (1983a, 1983b), Deaton and Irish (1982), Strauss (1982), and Pitt and Rosenzweig (1983)]. Many of these studies have made use of household budget surveys from developing countries. The poor infrastructure of these countries results in market separation sufficient to provide the price variability required to estimate demand relationships at the level of the household from a single cross-section. These micro data sets offer a number of important advantages. For example, demographic variables such as family size and age composition are major determinants of household budget patterns but their effects are not easily measured with aggregated data. Using budget surveys from Bangladesh, Indonesia, and Sierra Leone, respectively, Pitt (1983a, 1983b), and Strauss were able to incorporate demographic and other household specific variables into their analyses. Strauss' data also enabled him to estimate an economic model of household-firm behavior for farm households. A particularly rich sample survey from Indonesia enabled Pitt and Rosenzweig to study the relationship between food prices, food consumption, health and the efficiency of household farm production. It is obvious that these kinds of issues are best analyzed with micro data sets. Unfortunately, many studies using micro data have suffered from the lack of an unrestrictive and theoretically consistent approach to dealing with a common attribute of these data sets, the non- (or otherwise bounded) consumption of goods. As micro data sets become increasingly

available, it is important that this econometric problem be resolved so that the interpretation of results is unclouded by econometric inconsistency.

Household budget data, which typically contains information on the consumption of very disaggregated commodities over relatively short periods, often contain a significant proportion of observations for which expenditure on one or more goods is zero. Such samples contain limited dependent variables in that certain demand values occur with finite positive probabilities. The estimation of demand systems with limited dependent variables differs from the well-known limited dependent variable models of Tobin (1958) and Amemiya (1974) in that demand systems involve complex structural interactions and cross-equation restrictions.

As is well known, systems of demand equations can be derived directly by maximizing a utility function subject to a budget constraint, or indirectly from a cost or indirect utility function with Roy's identity. Furthermore, demand functions that add up, are homogeneous of degree zero, and have symmetric, negative definite compensated price responses are integrable into a theoretically consistent preference ordering (Hurwicz and Uzawa (1971)). Moreover, the stochastic specification of demand systems with limited dependent variables must also be compatible with demand theory. This issue has been addressed in a recent article by Wales and Woodland (1983) where a system of demand equations with limited dependent variables was derived by maximizing a random utility function subject to a budget constraint. In this direct utility function approach, the Kuhn-Tucker inequalities determine the limited demand quantities. In this article, we propose a more useful approach based only upon the "notional" demand equations which does not require specification of the underlying utility function. This is a great convenience since it is easier to think of specifications for demand equations (or cost and indirect utility functions) than for

direct utility functions and also because it circumvents the frequent intractability of the Kuhn-Tucker conditions.[1/] In particular, our approach permits the specification of systems of demand equations with limited dependent variables derived from popular flexible functional forms for the indirect utility function, such as the translog. Contradicting the claims of Wales and Woodland (p. 273) that an indirect utility approach is inappropriate for dealing with non-negativitiy constraints, we will show that such an approach is not only possible but more useful than the direct utility approach. Our approach is in the tradition of the theory of consumer demand under rationing set forth in Tobin and Houthakker (1950-51), Pollak (1969, 1971), Howard (1977), Neary and Roberts (1980), and Deaton (1981), and utilizes the concept of the "virtual price" originated by Rothbarth (1941).

This article is organized as follows. In Section 2, we formally present the consumers' problem with binding non-negativity constraints and relate it to the consumers' problem under rationing. In Section 3, we set forth rules which discriminate among all possible consumption regimes. Section 4 provides the stochastic version of the demand equations and the likelihood function. In Section 5, our results are compared with the existing literature on limited dependent variables. In Section 6 we apply the econometric model to estimate a system of demand equations based on household survey data from Indonesia. Conclusions are set forth in the last section.

2. Demand Systems With Zero Demands.

As in traditional demand theory, assume that there exists a set of functional relationships which relate the demand for goods as functions of prices v,

$$(2.1) \qquad q_i = D_i(v) \qquad i=1,\ldots,K$$

which are defined on $v > 0$,[2/] where K is the number of goods and v is a vector of prices normalized by total expenditure M, $v = p/M$. The equations (2.1) are meaningful demand functions only if they are consistent with maximizing a well-behaved utility function subject to a budget constraint $py = M$ and the nonnegativity constraints $y \geq 0$. The existence of such demand equations are guaranteed by some regularity assumptions on the utility function $U(y)$, e.g., $U(y)$ is strictly quasi-concave, increasing and continuously differentiable. When $q_i \geq 0$ for all i, it corresponds to the solution of the traditional problem

$$(2.2) \qquad \max_y \{U(y) | v'y = 1, \quad y \geq 0\}$$

However, if some elements of the vector q in (2.1) are negative, the corresponding nonnegativity constrained problem (2.2) will have a boundary solution. If the utility function $U(y)$ possesses the <u>interior property</u> that the utility for a commodity combination in which one or more quantities is zero is lower than for any combination in which all quantities are positive, then the demand equations for all goods will be positive for all positive price vectors v. Zero consumption of certain goods can occur with behavioral interpretation only if the utility function does not possess this interior property.[3/] In our analysis, we will rule out these utility functions and assume, furthermore, that the utility function is not only well-defined on the nonnegative commodity space

with behavioral meaning but can also be extended mathematically into regions with negative quantities such that the equations (2.1) are the unique solution of the following problem:

$$(2.3) \qquad \max_{y} \{U(y) | vy = 1\}.$$

The equations (2.1), which are the solutions to the utility maximization problem <u>without</u> nonnegativity constraints (2.3), are referred to as <u>notional demand equations</u>. This conceptualization is convenient and operationally relevant in that most flexible demand systems which do not correspond to utility functions with the interior property have negative notional demands when some nonnegativity constraints are binding.

The notional demands q_i are latent variables. For a given vector of normalized prices v, the observed demand quantities vector x is the (nonnegative) solution to (2.2). If some of the x_i are zero, the vector x may not necessarily equal $D(v) = [D_1(v),\ldots,D_K(v)]'$. However, there exists a vector of normalized prices v^* such that $x = D(v^*)$. These prices v^* are known as <u>virtual prices</u> (Rothbarth (1941)). Neary and Roberts have shown that if the preference function is strictly quasi-concave, continuous and strictly monotonic, any allocation can be supported with virtual prices. Strict monotonicity also implies that support prices will always be strictly positive. Furthermore, the virtual prices corresponding to demands which are positive are the actual market prices themselves, that is, if $x_i > 0$, then $v_i^* = v_i$.[4/]

As each good may be consumed at some price vector but not at others, there are many possible patterns of consumed and non-consumed goods. Each pattern constitutes a single <u>demand regime</u>, and for each regime there exists a unique vector of virtual prices which support the observed demand vector x. Consider, for example, the regime with zero quantities demanded for the first L goods

and positive quantities for the remaining K-L goods. The vector of normalized virtual prices corresponding to this regime is characterized by $v_i^* = v_i$ for $i = L + 1,\ldots,K$, and the observed demand equations are

$$(2.4) \qquad x_i = D_i(v_1^*,\ldots,v_L^*,v_{L+1},\ldots,v_K), \qquad i = L + 1,\ldots,K$$

$$(2.5) \qquad 0 = D_i(v_1^*,\ldots,v_L^*,v_{L+1},\ldots,v_K), \qquad i = 1,\ldots,L.$$

The virtual prices $v_1^*,\ldots,v_L^*$ can be solved as functions of the observed prices $v_{L+1},\ldots,v_K$, i.e.,

$$(2.6) \qquad v_i^* = \xi_i(v_{L+1},\ldots,v_K), \qquad i = 1,\ldots,L.$$

Hence, the observed demand quantities (2.4) can be written as

$$(2.7) \qquad x_i = D_i(\xi_1(v_{L+1},\ldots,v_K),\ldots,\xi_L(v_{L+1},\ldots,v_K),v_{L+1},\ldots,v_K).$$

The demand equations (2.7) are conditional (restricted) demand equations, conditional on the first L goods having zero demands. For each regime, the observed set of conditional demand equations (2.7) depends on the notional demand equations (2.1) The virtual prices provide the link between the conditional and the notional demand equations. These virtual prices replace the actual prices of those goods which are not consumed and hence do not appear in the demand equations of the remaining goods. The equations (2.7) are the estimable equations, and the unknown parameters of the utility function U(y) or the corresponding notional demand equations (2.4) and (2.5) can be identified by estimating this conditional demand system.[5/]

3. Criteria for Discriminating Among Demand Regimes.

In the traditional consumer problem (2.2), the determination of the demand regime, that is, which goods are consumed and which are not, is simply reflected by the corresponding Kuhn-Tucker conditions. However, an explicit statement of these conditions requires knowledge of the direct utility function. The problem with this Kuhn-Tucker approach is that for many popular demand systems the underlying utility function does not have a closed-form expression, hence econometric implementation is problematical. Below we will demonstrate that alternative criteria for discriminating among demand regimes can be derived solely from the notional demand equations (2.1). These regime switching conditions will be used to determine regime occurrence probabilities for use in the econometric modeling discussed in Section 4 of this article.

In the simple case of an interior solution, that is, all demand quantities are positive, it is clear that the notional demands must obey

$$D_i(v_i,\ldots,v_K) > 0 \quad \text{for all} \quad i = 1,\ldots,K.$$

Without loss of generality, consider the regime for which the demanded quantities of the first L goods are zero and the quantities of the remaining $K-L$ goods are positive. Denote $\bar{v} = (v_{L+1},\ldots,v_K)$. This regime is characterized by the conditions

$$\begin{aligned}
&D_1(v_1,\xi_{12}(v_1,\bar{v}),\ldots,\xi_{1L}(v_1,\bar{v}),\bar{v}) \le 0\\
&D_2(\xi_{21}(v_2,\bar{v}),v_2,\xi_{23}(v_2,\bar{v}),\ldots,\xi_{2L}(v_2,\bar{v}),\bar{v}) \le 0\\
&\vdots\\
&D_L(\xi_{L1}(v_L,\bar{v}),\ldots,\xi_{LL-1}(v_L,\bar{v}),v_L,\bar{v}) \le 0\\
&D_i(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v}) > 0, \qquad i = L+1,\ldots,K
\end{aligned} \tag{3.1}$$

where $\xi_j(\bar{v})$, $j = 1,\ldots,L$ are the solutions to (2.5) (as in (2.6)) and the virtual prices $\xi_{12}(v_1,\bar{v}),\ldots,\xi_{1L}(v_1,\bar{v})$ are the solutions to the equations

$$(3.2) \qquad 0 = D_i(v_1,v_2^*,v_3^*,\ldots,v_L^*,\bar{v}) \qquad i = 2,3,\ldots,L$$

and the other ξ's are similarly defined. In order to simplify notation, henceforth the arguments of the functions ξ will be omitted. The demand equations (3.1) are <u>conditional notional demand equations</u>. For example, the first equation $D_1(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v})$ is the notional demand for good 1 conditional on the zero demands for the goods $2,3,\ldots,L$. The second equation, $D_2(\xi_{21},v_2,\xi_{23},\ldots,\xi_{2L},\bar{v})$ is the notional demand for good 2 conditional on goods $1,3,4,\ldots,L$ all having zero demands. This conceptualization of conditional demand is essentially the same as that set out in the rationing literature, except that the "ration" above is always zero.

For the simple two-good case, the regime having observed demands $x = (0,x_2)$, $x_2 > 0$, is characterized by the condition $D_1(v_1,v_2) \leq 0$. The condition $D_2(\xi_1,v_2) > 0$ always holds because $D_1(\xi_1,v_2) = 0$, and $v_1D_1(\xi_1,v_2) + v_2D_2(\xi_1,v_2) = 1$ and therefore $D_2(\xi_1,v_2) = 1/v_2$. Figure 1 clearly demonstrates that this single condition is sufficient and necessary for the consumption pattern $x = (0,\frac{1}{v_2})$.

The role of conditional notional demands in determining the demand regime is best illustrated geometrically in the three-good case. Figure 2 illustrates the consumers' problem in this instance. The budget constraint is the triangular region (simplex) bounded by A^*, B^*, and C^*. The hyperplane defined by these three points is the <u>budget hyperplane</u>. The line A_3B_3, which contains the segment A^*B^*, is the locus of points in the budget hyperplane at which the demand for good 3 is zero, i.e., $y_3 = 0$. Similarly, $y_2 = 0$ along B_2C_2 and $y_1 = 0$ along A_1C_1.

FIGURE 1

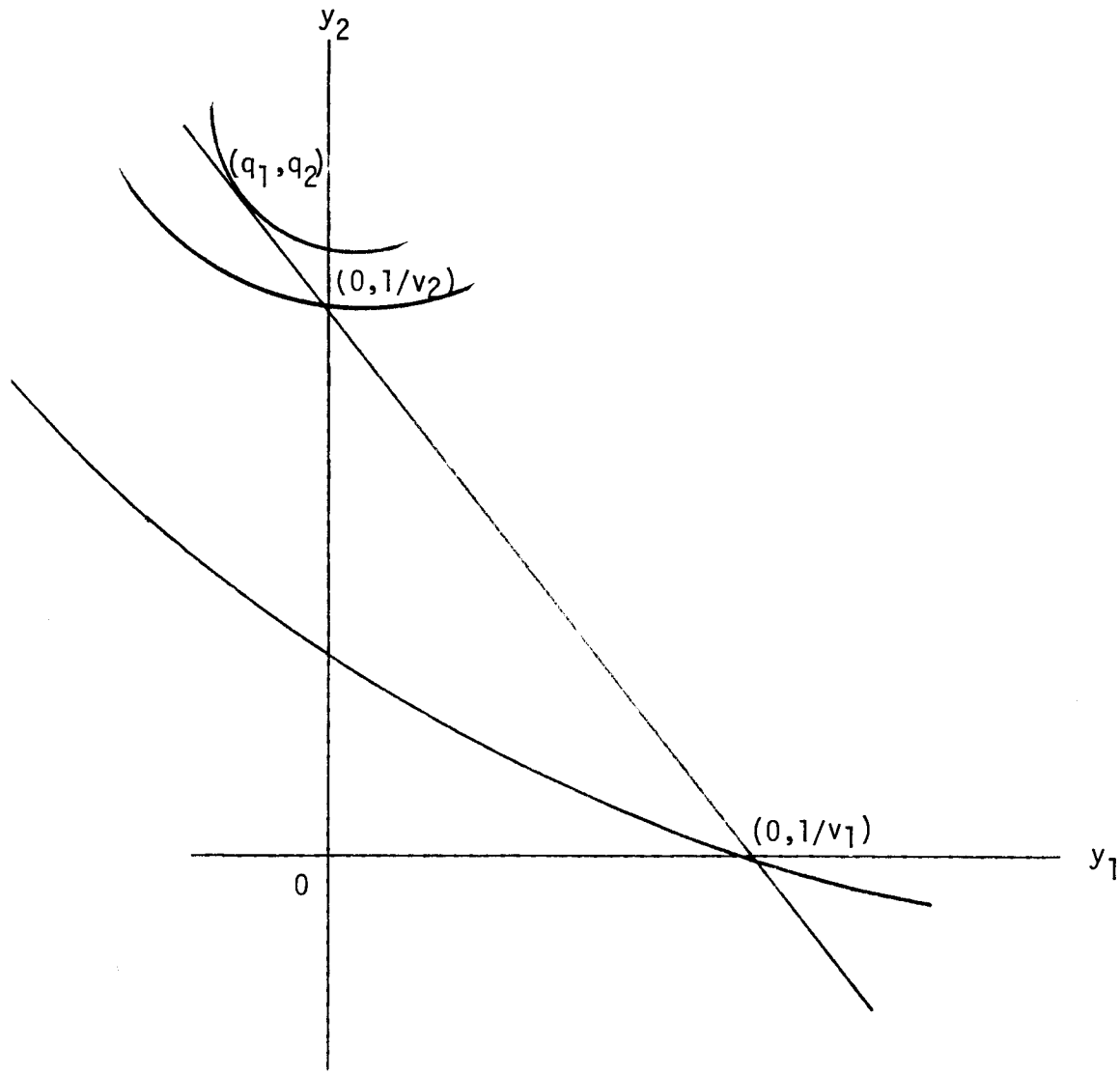


FIGURE 2

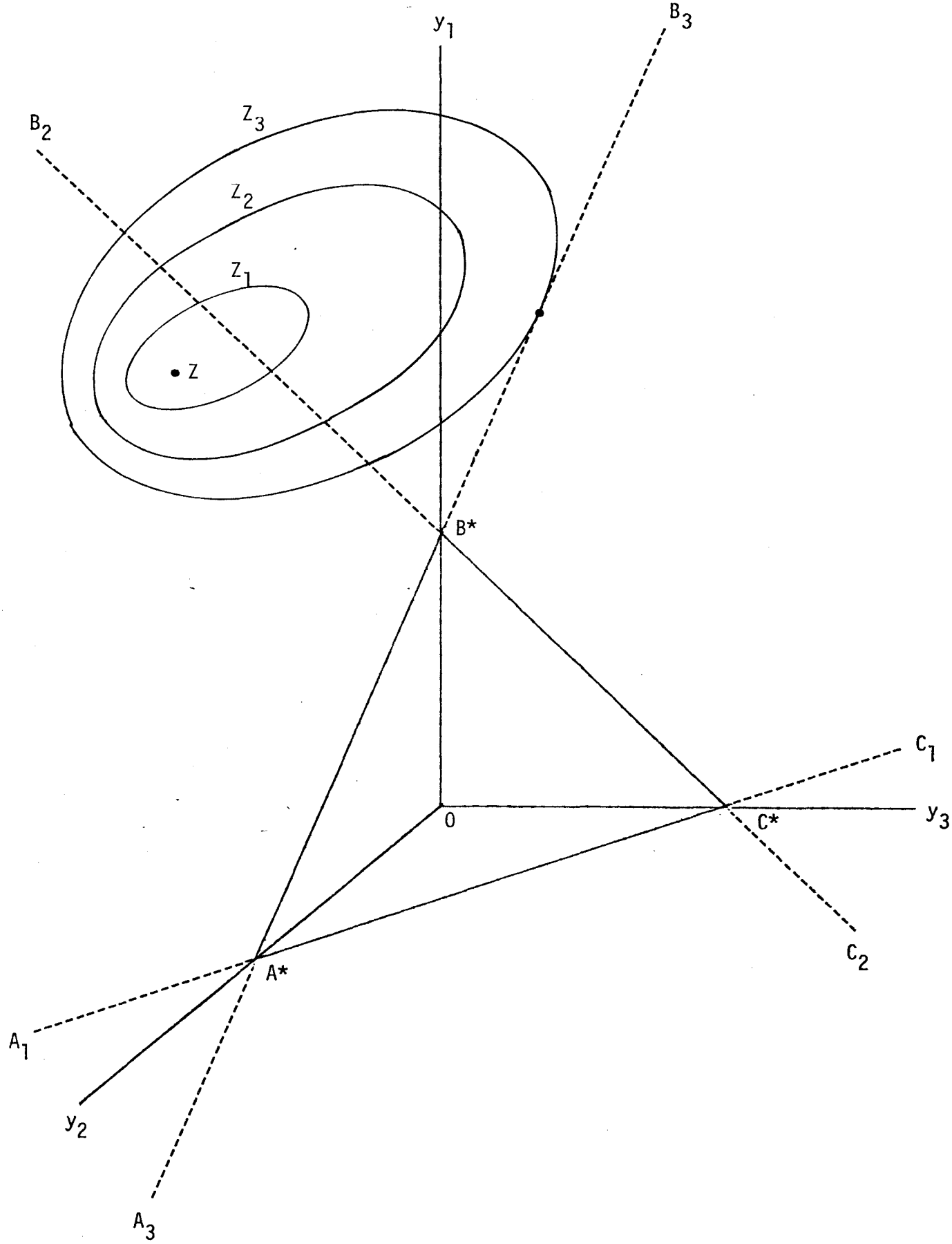
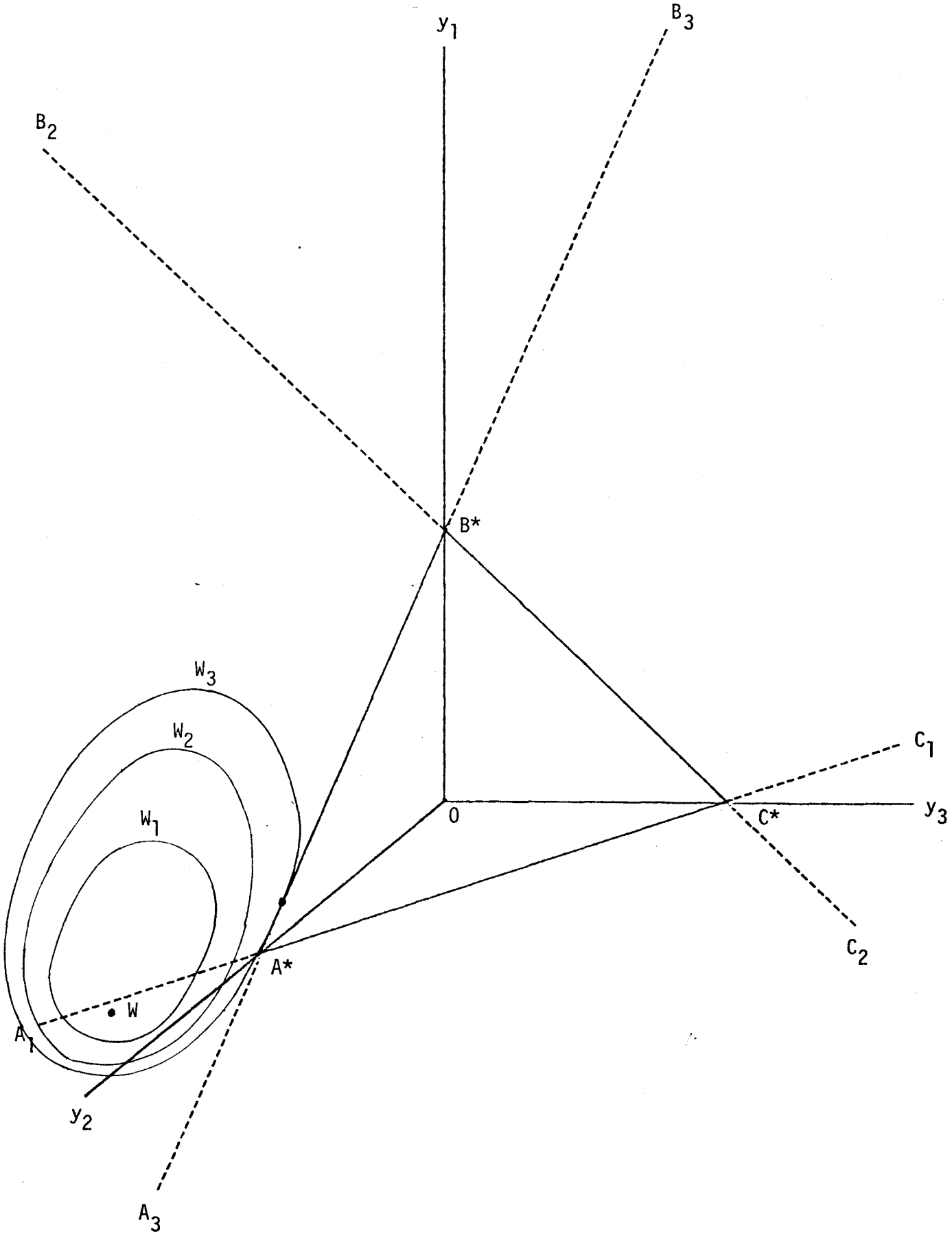


FIGURE 3



It is obvious that an interior solution requires that an indifference surface be tangent to the budget hyperplane within the region $A^*B^*C^*$. An observed demand will be zero only if an indifference surface is tangent to the budget hyperplane outside of the interior of $A^*B^*C^*$. Such a tangency occurs at the point Z having notional demands $q_1 > 0$, $q_2 > 0$, and $q_3 < 0$. Define <u>indifference contours</u> as the loci of points providing identical levels of utility consistent with the budget hyperplane. Z_1, Z_2, and Z_3 are such contours associated with point Z. It is clear that these contours are convex and never cross since they are merely cross-sections of the unrestricted indifference surfaces.

Equations (3.1) state that the demand regime $(y_1 > 0,\ y_2 > 0,\ y_3 = 0)$ is uniquely characterized by the conditions

$$
\begin{aligned}
&D_1(v_1,v_2,\xi_3) > 0 \\
(3.3)\qquad &D_2(v_1,v_2,\xi_3) > 0 \\
&D_3(v_1,v_2,v_3) \leq 0,
\end{aligned}
$$

that is, the demands for goods 1 and 2 conditional on the nonconsumption of good 3 must both be positive and the notional demand for good 3 is nonpositive. Geometrically, this means checking where the tangency of the notional indifference contours (such as Z_3) occurs along the <u>conditional budget line</u>. Since the line A_3B_3 represents $q_3 = y_3 = 0$, it is therefore the conditional (on $y_3 = 0$) budget line. Note that the contour Z_3 is tangent to this conditional budget line above point B^* where the conditional notional demand for good 2 is negative, thus violating (3.3). Therefore, even though the unconditional notional demand for good 2 is positive, its conditional demand is negative and therefore point Z is not consistent with the demand regime $(y_1 > 0,\ y_2 > 0,\ y_3 = 0)$.

This is not the case for a point such as W in Figure 3, having notional demands $(q_1 < 0,\ q_2 > 0,\ q_3 < 0)$. Its conditional notional demands, given by the tangency of the indifference contour W_3 and the conditional budget line, correspond to the conditions (3.3) and thus to the specified demand regime.

This analysis is readily extended to models with more than three goods. A mathematical proof is provided as an appendix. Intuitively, evaluating conditional notional demand functions to determine which of a set of nonnegativity constraints are binding should be equivalent to evaluating the Kuhn-Tucker conditions for the direct utility function underlying those demand functions. This important conjecture is also established in the appendix.

4. Econometric Model Specification.

In order to estimate the notional demand equations, a functional form needs to be specified which includes a finite number of unknown parameters plus stochastic components. These stochastic components reflect random preferences or other unexplained factors. Let θ be the vector of unknown parameters and ϵ the vector of random components. The stochastic notional demand equations are

$$(4.1) \qquad q_i = D_i(v;\theta,\epsilon).$$

These demand equations can be derived either by maximizing the direct utility function subject to the budget constraint as in (2.3) or from an indirect utility function through Roy's identity. Let $H(v;\theta,\epsilon)$ be an indirect utility function defined as

$$(4.2) \qquad H(v;\theta,\epsilon) = \max_q \{U(q;\theta,\epsilon) | vq = 1\}.$$

Applying Roy's identity, the notional demand equations are

$$(4.3) \qquad q_i = \frac{\partial H(v;\theta,\epsilon)}{\partial v_i} \Big/ \Sigma_{j=1}^{K} v_j \frac{\partial H(v;\theta,\epsilon)}{\partial v_j} \qquad i = 1,\ldots,K.$$

To derive the observed demand equations (2.7) and the regime switching, replace the actual prices v in the notional demand equations (2.1) with the relevant virtual price. These virtual prices are the solutions to the equations (4.3) with q_i set equal to zero if its notional value is negative. In the analysis of quantity rationing, Deaton (1981) has noted that it may be difficult to analytically derive virtual price functions from most flexible function forms for the indirect utility function. In subsequent work on labor supply and commodity demands, Deaton and Muellbauer (1981), and Blundell and Walker (1982) find a generalization of the Gorman polar form of the indirect utility function

to be particularly convenient (albeit restrictive) in the derivation of virtual prices. This results from the asymmetric manner in which leisure is specified compared to commodity demands, and because leisure is the only rationed good. Our problem is somewhat different. On the one hand we consider a model in which there are many commodities whose demands may be restricted, which complicates matters. On the other hand, all of the restricted demands in our model are zero rather than positive as in the rationing literature. With zero restricted demands, the derivation of virtual prices is considerably simplified as the denominator in Roy's identity (4.3) drops out of the virtual price functions. If demands for the first L goods are zero, the virtual prices ξ_i are solved from the equations

$$(4.4) \qquad 0 = \partial H(\xi_1,\ldots,\xi_L,\bar{v};\theta,\epsilon)/\partial v_i \qquad i = 1,\ldots,L$$

and the remaining (positive) demands are

$$(4.5) \qquad x_i = \frac{\partial H(\xi_1,\ldots,\xi_L,\bar{v};\theta,\epsilon)}{\partial v_i} \Big/ \Sigma_{j=1}^{K} v_j \frac{\partial H(\xi_1,\ldots,\xi_L,\bar{v};\theta,\epsilon)}{\partial v_j}, \qquad i = L+1,\ldots,K.$$

Since $\Sigma_{j=1}^{K} v_j \frac{\partial H(v;\theta,\epsilon)}{\partial v_j} = \frac{-\partial H(v;\theta,\epsilon)}{\partial M}$ is always negative, in order to observe this pattern of demand the following must hold

$$(4.6) \qquad \begin{aligned} \omega_1 &\equiv \frac{\partial H(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v};\theta,\epsilon)}{\partial v_1} \geq 0 \\ &\vdots \\ \omega_L &\equiv \frac{\partial H(\xi_{L1},\ldots,\xi_{LL-1},v_L,\bar{v};\theta,\epsilon)}{\partial v_L} \geq 0 \end{aligned}$$

$$\text{and} \quad \frac{\partial H(\xi_1,\ldots,\xi_L,\bar{v};\theta,\epsilon)}{\partial v_i} < 0 \quad \text{for} \quad i = L+1,\ldots,K.$$

Let $f(\omega_1,\ldots,\omega_L,x_{L+1},\ldots,x_K)$ be the joint density function of

$(\omega_1,\ldots,\omega_L,x_{L+1},\ldots,x_K)$. The contribution of this consumption pattern to the likelihood function is

$$(4.7)\qquad \ell_c(x;\theta) = \int_0^\infty \cdots \int_0^\infty f(\omega_1,\ldots,\omega_L,x_{L+1},\ldots,x_{K-1})d\omega_1,\ldots,d\omega_L.$$

Note that the demand equation for x_K can be dropped from the likelihood function due to the linear dependence $\Sigma_{i=L+1}^{K} v_i x_i = 1$. The probability that this regime occurs is

$$(4.8)\qquad P_r(c) = \int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty f(\omega_1,\ldots,\omega_L,x_{L+1},\ldots,x_K)d\omega_1,\ldots,d\omega_L\right)dx_{L+1},\ldots,dx_K$$

for $L < K - 1$. For $L = K - 1$, $Pr(c) = \int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty \cdots \int_0^\infty f(\omega_1,\ldots,\omega_L, x_{L+1},\ldots,x_{K-1})d\omega_1,\ldots,d\omega_L\right)dx_{L+1},\ldots,dx_{K-1}$. Let $I_i(c)$ be a dichotomous indicator such that $I_i(c) = 1$ if the observed consumption-pattern for individual i is the demand regime c, zero otherwise. The likelihood function for a sample with N observations is

$$(4.9)\qquad L = \prod_{i=1}^{N} \prod_{c} [\ell_c(x_i;\theta)]^{I_i(c)}.$$

The parameter vector θ can be estimated by the method of maximum likelihood.

As an illustration of the notional demand approach, consider the translog indirect utility function of Christensen, Jorgenson and Lau (1975),

$$(4.10)\qquad H(v;\theta,\epsilon) = \sum_{i=1}^{K} \alpha_i \ln v_i + \frac{1}{2}\Sigma_{i=1}^{K}\Sigma_{j=1}^{K}\beta_{ij}\ln v_i \ln v_j + \Sigma_{i=1}^{K}\epsilon_1 \ln v_i$$

where ϵ is a K-dimensional vector of normal variables $N(0,\Sigma)$.[6/] normalization, it is convenient to set $\sum_{i=1}^{K} \alpha_i = -1$ and $\sum_{i=1}^{K} \epsilon_i = 0$.[7/] notional share equations derived from Roy's identity are

$$(4.11)\qquad v_i q_i = \frac{\alpha_i + \Sigma_{j=1}^{K}\beta_{ij}\ln v_j + \epsilon_i}{D} \qquad i = 1,\ldots,K$$

where $D = -1 + \Sigma_{i=1}^{K}\Sigma_{j=1}^{K}\beta_{ij}\ln v_j$. Consider the regime for which the quantity demanded for one of the goods is zero and positive for all others, i.e., $x_1 = 0$, $x_2 > 0,\ldots,x_K > 0$. The virtual price ξ_1 as a function of $v_2,\ldots,v_K$, is

$$\ln\xi_1 = -(\alpha_1 + \Sigma_{j=2}^{K}\beta_{1j}\ln v_j + \epsilon_1)/\beta_{11}.$$

The remaining positive share equations are

$$(4.12)\qquad v_i x_i = \frac{\alpha_i - \alpha_1\frac{\beta_{i1}}{\beta_{11}} + \Sigma_{j=2}^{K}(\beta_{ij} - \beta_{1j}\frac{\beta_{i1}}{\beta_{11}})\ln v_j + \epsilon_i - \frac{\beta_{i1}}{\beta_{11}}\epsilon_1}{\Sigma_{j=2}^{K}(\beta_{\cdot j} - \beta_{\cdot 1}\frac{\beta_{1j}}{\beta_{11}})\ln v_j - (1 + \frac{\alpha_1}{\beta_{11}}\beta_{\cdot 1}) - \frac{\beta_{\cdot 1}}{\beta_{11}}\epsilon_1}, \qquad i = 2,\ldots,K$$

where $\beta_{\cdot j} = \Sigma_{i=1}^{K}\beta_{ij}$. Note from the above equations that ϵ_i can be expressed as functions of x_i and ϵ_1. The switching conditions for this demand regime are

$$\epsilon_1 \geq -(\alpha_1 + \Sigma_{j=1}^{K}\beta_{1j}\ln v_j)$$

and $x_i > 0$, $\quad i = 2,\ldots,K$.

Let $f(\epsilon_1)$ be the density function of ϵ_1 and $g(\epsilon_2,\ldots,\epsilon_{K-1}|\epsilon_1)$ be the conditional density function, conditional on ϵ_1. The Jacobian transformation $J_1(x,\epsilon_1)$ from $(\epsilon_2,\ldots,\epsilon_{K-1})$ to $(x_2,\ldots,x_{K-1})$, which can be derived from (4.12), is a function of x and ϵ_1. The likelihood function for this demand regime is

$$\int_{-(\alpha_1 + \Sigma_{j=1}^{K}\beta_{1j}\ln v_j)}^{\infty} g(\epsilon_2,\ldots,\epsilon_{K-1}|\epsilon_1)f(\epsilon_1)d\epsilon_1$$

where $\epsilon_i, i=2,\ldots,\epsilon_{K-1}$ are functions of x and ϵ_1 from (4.12). Now consider the demand regime in which the demands for the first two commodities are zero and all remaining demands are positive. The virtual prices ξ_1 and ξ_2 as functions of $v_3\ldots,v_K$ are

$$\begin{pmatrix} \ln\xi_1 \\ \ln\xi_2 \end{pmatrix} = -B^{-1}\begin{pmatrix} \alpha_1 + \Sigma_{j=3}^{K}\beta_{1j}\ln v_j \\ \alpha_2 + \Sigma_{j=3}^{K}\beta_{2j}\ln v_j \end{pmatrix} - B^{-1}\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

where $B = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$. The remaining positive shares are

$$(4.13) \qquad v_i x_i = \frac{\alpha_i + \beta_{i1}\ln\xi_1 + \beta_{i2}\ln\xi_2 + \Sigma_{j=3}^{K}\beta_{ij}\ln v_j + \epsilon_i}{-1 + \beta_{\cdot 1}\ln\xi_1 + \beta_{\cdot 2}\ln\xi_2 + \Sigma_{j=3}^{K}\beta_{\cdot j}\ln v_j}, \qquad i = 3,\ldots,K.$$

The $\epsilon_i, i=3,\ldots,K$, can be expressed (from (4.13)) as functions of x, ϵ_1 and ϵ_2. The regime switching conditions are as follows. The virtual price $\xi_{12}(v_1, v_3, \ldots, v_K)$ is

$$\ln\xi_{12} = -(\alpha_2 + \Sigma_{\substack{j=1 \\ j\neq 2}}^{K}\beta_{2j}\ln v_j + \epsilon_2)/\beta_{22}$$

and the first condition is

$$\epsilon_1 - \frac{\beta_{12}}{\beta_{22}}\epsilon_2 \geq s_1$$

where

$$s_1 = \frac{\beta_{12}}{\beta_{22}}(\alpha_2 + \Sigma_{\substack{j=1 \\ j\neq 2}}^{K}\beta_{2j}\ln v_j) - \alpha_1 - \Sigma_{\substack{j=1 \\ j\neq 2}}^{K}\beta_{1j}\ln v_j.$$

Furthermore, the virtual price $\xi_{21}(v_2, v_3, \ldots, v_K)$ is

$$\ln\xi_{21} = -(\alpha_1 + \Sigma_{j=2}^{K}\beta_{1j}\ln v_j + \epsilon_1)/\beta_{11}$$

and the second condition is

$$\epsilon_2 - \frac{\beta_{21}}{\beta_{11}}\epsilon_1 \geq s_2$$

where

$$s_2 = \frac{\beta_{21}}{\beta_{11}}(\alpha_1 + \Sigma_{j=2}^{K}\beta_{1j}\ln v_j) - \alpha_2 - \Sigma_{j=2}^{K}\beta_{2j}\ln v_j.$$

Let $\eta_1 = \epsilon_1 - \frac{\beta_{12}}{\beta_{22}}\epsilon_2$ and $\eta_2 = \epsilon_2 - \frac{\beta_{21}}{\beta_{11}}\epsilon_1$. Let $g(\epsilon_3,\ldots,\epsilon_{K-1}|\eta_1,\eta_2)$ be the conditional density function of $(\epsilon_3,\ldots,\epsilon_{K-1})$, conditional on η_1 and η_2 and $f(\eta_1,\eta_2)$ be the marginal density of η_1 and η_2. The likelihood function for this regime is

$$\int_{s_2}^{\infty}\int_{s_1}^{\infty} g(\epsilon_3,\ldots,\epsilon_{K-1}|\eta_1,\eta_2)f(\eta_1,\eta_2)d\eta_1 d\eta_2.$$

The likelihood function for other regimes can similarly be derived.

For some cases, we may specify the utility function or the demand equations instead of the indirect utility function. Consider the demand equations derived from the quadratic utility function and analyzed in Wegge (1968)

$$(4.14) \qquad \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_{K-1} \end{pmatrix} = \alpha^{-1}\gamma$$

and $q_K = \frac{1}{v_K}(1 - v_1q_1 - \ldots - v_{K-1}q_{K-1})$ where α is a $(K-1) \times (K-1)$ matrix and γ is a $(K-1)$ vector. The $(i,j)^{\underline{th}}$ entry of α is

$$\alpha_{ij} = a_{ij}v_K^2 - a_{iK}v_jv_K - (a_{kj}v_iv_K - a_{KK}v_iv_j)$$

and the $i^{\underline{th}}$ element of γ is

$$\gamma_i = v_i(a_{K0}v_K + a_{KK}) - a_{i0}v_K^2 - a_{iK}v_K + (v_i - v_K)v_K\epsilon_i$$

where $(\epsilon_1,\ldots,\epsilon_K)$ are random terms $N(0,\Sigma)$ and a_{i0}, a_{ij} are unknown parameters. The quadratic utility function corresponding to these demand equations is

$$(4.15) \qquad U(y) = \Sigma_{i=1}^{K} a_{i0}y_i + \frac{1}{2}\Sigma_{i=1}^{K}\Sigma_{j=1}^{K} a_{ij}y_iy_j + \Sigma_{i=1}^{K}\epsilon_i y_i$$

where $a_{ij} = a_{ji}$. A simple normalization that can be adopted for parameter identification is $\mathrm{var}(\epsilon_1) = 1$. Wales and Woodland (1983) have derived the likelihood function for this demand system with limited dependent variables from the Kuhn-Tucker conditions. Below, we derive the likelihood using the virtual price approach. It is important to note that a_{ij} is a function depending on v_i, v_j and v_K only, and not on the other prices, and that γ_i is a function of v_i and v_K only. Consider the demand regime where $x_1 = 0$ and all other x_i's are positive. Evaluating a and γ at the virtual price vector $(\xi_1, v_2, \ldots, v_K)$, we have

$$a|_{v_1 = \xi_1} \begin{pmatrix} 0 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{K-1} \end{pmatrix} = \gamma|_{v_1 = \xi_1}.$$

This implies that the positive demand equations are

$$(4.16) \qquad \begin{pmatrix} x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{K-1} \end{pmatrix} = \begin{pmatrix} a_{22}, \ldots, a_{2,K-1} \\ \vdots \\ a_{K-1,2}, \ldots, a_{K-1,K-1} \end{pmatrix}^{-1} \begin{pmatrix} \gamma_2 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_{K-1} \end{pmatrix}$$

and the first regime condition is

$$(4.17) \qquad (a^{-1})_{1.}\gamma \leq 0$$

where $(a^{-1})_{1.}$ denotes the first row of a^{-1}. Denote $\omega_1 = (a^{-1})_{1.}\gamma$. The likelihood function for this regime is

$$\int_{-\infty}^{0} g(\gamma_2, \ldots, \gamma_{K-1} | \omega_1) f(\omega_1) d\omega_1 \left| \det \begin{pmatrix} a_{22}, \ldots, a_{2K-1} \\ \vdots \qquad\qquad \vdots \\ a_{K-1,2}, \ldots, a_{K-1,K-1} \end{pmatrix} \right| .$$

Now consider the demand regime $x_1 = 0$, $x_2 = 0$ and $x_i > 0$, $i = 3,\ldots,K$. At the virtual prices $\xi_1(v_3,\ldots,v_K)$ and $\xi_2(v_3,\ldots,v_K)$,

$$a\Big|_{\substack{v_1 = \xi_1 \\ v_2 = \xi_2}} \begin{Bmatrix} 0 \\ 0 \\ x_3 \\ \vdots \\ x_{K-1} \end{Bmatrix} = \gamma\Big|_{\substack{v_1 = \xi_1 \\ v_2 = \xi_2}}$$

which implies that

$$(4.18) \qquad \begin{Bmatrix} x_3 \\ \vdots \\ x_{K-1} \end{Bmatrix} = \begin{Bmatrix} a_{33},\ldots,a_{3K-1} \\ \vdots \qquad\qquad \vdots \\ a_{K-1,3},\ldots,a_{K-1,K-1} \end{Bmatrix}^{-1} \begin{Bmatrix} \gamma_3 \\ \vdots \\ \gamma_{K-1} \end{Bmatrix}.$$

Since, at the virtual price $\xi_{12}(v_1,v_3,\ldots,v_K)$,

$$(4.19) \qquad a\Big|_{v_2 = \xi_{12}} \begin{Bmatrix} x_1 \\ 0 \\ x_3 \\ \vdots \\ x_{K-1} \end{Bmatrix} = \gamma\Big|_{v_2 = \xi_{12}},$$

the solution of x_1 from (4.19) provides the first regime switching condition, which is

$$(4.20) \qquad \begin{Bmatrix} a_{11} & a_{13},\ldots,a_{1K-1} \\ a_{31} & a_{33},\ldots,a_{3K-1} \\ \vdots & \vdots \qquad\qquad \vdots \\ a_{K-11} & a_{K-1,3},\ldots,a_{K-1,K-1} \end{Bmatrix}^{-1}_{1.} \begin{Bmatrix} \gamma_1 \\ \gamma_3 \\ \vdots \\ \gamma_{K-1} \end{Bmatrix} \leq 0.$$

Similarly, the second regime switching condition is

$$(4.21)\qquad \begin{pmatrix} a_{22},\ldots,a_{2K-1} \\ \vdots \\ a_{K-12},\ldots,a_{K-1,K-1} \end{pmatrix}^{-1} 1. \begin{pmatrix} \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{K-1} \end{pmatrix} \leq 0$$

Denote the random variates on the left hand sides of (4.20) and (4.21) by ω_1 and ω_2 respectively. The log likelihood function for this regime is

$$\int_{-\infty}^{0}\int_{-\infty}^{0} g(\gamma_3,\ldots,\gamma_{K-1}|\omega_1,\omega_2)f(\omega_1,\omega_2)d\omega_1 d\omega_2 \left|\det \begin{pmatrix} a_{33},\ldots,a_{3K-1} \\ \vdots \qquad\qquad \vdots \\ a_{K-1,3},\ldots,a_{K-1K-1} \end{pmatrix}\right|$$

where $\gamma_3,\ldots,\gamma_{K-1}$ are functions of x from (4.18). The likelihood function for other regimes can be similarly derived. Note that the explicit expressions for the virtual prices cannot be derived analytically in the quadratic case. Nevertheless, this example has demonstrated that it may not be necessary to explicitly solve for the virtual prices in order to derive the observed demand equations and the regime conditions.

5. Relationship of This Approach With Tobit Models.

The econometric model of consumer demand with limited dependent variables set out above is not the familiar Tobit model (Tobin (1958)) nor the simultaneous equations Tobit model of Amemiya (1974). However, there are both structural differences and similarities among these models.

Consider the two-good case. The notional demand equations are

$$(5.1) \qquad q_1 = D_1(v_1, v_2; \theta, \epsilon)$$

and

$$(5.2) \qquad q_2 = D_2(v_1, v_2; \theta, \epsilon).$$

These two equations are functionally dependent because they satisfy the budget constraint

$$(5.3) \qquad v_1 q_1 + v_2 q_2 = 1.$$

Thus there are at most three different regimes for the observed quantities $x = (x_1, x_2)$.

Regime 1: Both $x_1 > 0$, $x_2 > 0$.

The positive demand equations are $x_1 = D_1(v_1, v_2; \theta, \epsilon)$ and the conditions are $D_1(v_1, v_2; \theta, \epsilon) > 0$ and $D_2(v_1, v_2; \theta, \epsilon) > 0$.

Regime 2: $x_1 = 0$, $x_2 > 0$.

The positive demand equation is $x_2 = D_2(\xi_1, v_2; \theta, \epsilon) = \frac{1}{v_2}$ and the regime condition is $D_1(v_1, v_2; \theta, \epsilon) \leq 0$.

Regime 3: $x_1 > 0$, $x_2 = 0$.

The positive demand equation is $x_1 = D_1(v_1, \xi_2; \theta, \epsilon) = \frac{1}{v_1}$ and the regime condition is $D_2(v_1, v_2; \theta, \epsilon) \leq 0$.

Because of the budget constraint (5.3), the condition $D_2(v_1, v_2; \theta, \epsilon) \leq 0$ is equivalent to the condition $D_1(v_1, v_2; \theta, \epsilon) \geq \frac{1}{v_1}$ and the condition $D_1(v_1, v_2; \theta, \epsilon) \leq 0$ is equivalent to the condition $D_2(v_1, v_2; \theta, \epsilon) \geq \frac{1}{v_2}$. Combining the three regimes

$$
\begin{aligned}
x_1 &= \frac{1}{v_1} && \text{if } D_1(v_1, v_2; \theta, \epsilon) \geq \frac{1}{v_1} \\
&= D_1(v_1, v_2; \theta, \epsilon) && \text{if } \frac{1}{v_1} > D_1(v_1, v_2; \theta, \epsilon) > 0 \qquad (5.4) \\
&= 0 && \text{if } D_1(v_1, v_2; \theta, \epsilon) \leq 0
\end{aligned}
$$

which is, in effect, the two-limit Tobit model of Rosett and Nelson (1975). If good 2 was always consumed in positive amounts, we have a standard Tobit model. With appropriate separability assumptions, the K goods case can be reduced to a set of independent two-limit Tobit equations. This approach significantly reduces the computational burden but such strong restrictions on preferences may not be realistic. In the unrestricted case, consumer demand equations are more closely related to the multivariate and simultaneous equations Tobit models of Amemiya (1974). As an example, consider the case where one of the commodities is always consumed, say $x_K > 0$. Following Wales and Woodland (1983), the Kuhn-Tucker conditions for utility maximization are

$$
\begin{aligned}
v_K U_i(x; \theta, \epsilon) - v_i U_K(x; \theta, \epsilon) &\leq 0 \qquad i = 1, \ldots, K-1 \\
v'x &= 1
\end{aligned}
$$

and

$$
x_i(v_K U_i(x; \theta, \epsilon) - v_i U_K(x; \theta, \epsilon)) = 0 \qquad i = 1, \ldots, K-1.
$$

After substituting $x_K = (1 - \Sigma_{i=1}^{K-1} v_i x_i)/v_K$ into the remaining equations, these conditions can be rewritten as

(5.5) $\quad F_i(x_1,\ldots,x_{K-1};\theta,\epsilon) \geq 0 \qquad i = 1,\ldots,K-1$

where $F_i(x_1,\ldots,x_{K-1};\theta,\epsilon) = v_i U_K(x;\theta,\epsilon) - v_K U_i(x;\theta,\epsilon)$, and

(5.6) $\quad x_i = 0$ if $F_i(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_{K-1};\theta,\epsilon) > 0$

and

(5.7) $\quad F_i(x_1,\ldots,x_{K-1};\theta,\epsilon) = 0$, otherwise.

This system, which relates the dependent variables $x_1,\ldots,x_{K-1}$ in a direct interactive way, is similar to the formulation of simultaneous equations with limited dependent variables of Amemiya even though the equations in the latter model are usually expressed in linear form. Furthermore, the former system may not have the conventional meaning of simultaneity. Our approach differs from that approach in that we specify the "reduced form" equations first and incorporate the structural differences of different regimes through the use of virtual prices. In general, the observed system is a switching multivariate equations model where the regime criteria consist of multivariate inequality rules.

6. An Illustration: Analyzing Energy Price Policy in Indonesia.

In this section we discuss the application of the econometric model set out above to estimate a system of demand equations derived from the quadratic utility function (4.15). The data used in the estimation are taken from the 1976 National Socio-Economic Survey of Indonesia (SUSENAS 1976) carried out by the Biro Pusat Statistik (Central Bureau of Statistics). This survey provides detailed information on the expenditure pattern of 51,816 households. The survey was conducted in three subrounds--January-April, May-August, and September-December, with approximately one-third of the sample surveyed in each subround.

The purpose of this exercise is not only to illustrate our approach to estimating demand systems with limited dependent variables, but also to analyze the change in welfare among consumers resulting from increases in the price of energy in Indonesia. This is an important issue in the Indonesian government's deliberations on reducing the large subsidies provided consumers of energy. The measure of welfare gain and loss used is the compensating variation associated with any price change. We will only briefly sketch the compensating variation results and their policy implications here as they are discussed at length in Pitt and Lee (1983).

Five expenditure categories are distinguished in this analysis. These categories, aggregates of the more than 150 individual consumption items provided by the SUSENAS survey, are food and beverages, apparel, fuel, housing and other nondurables. Price indices for the five expenditure aggregates are constructed from price data derived directly from the 1976 SUSENAS tapes, from retail prices used in the construction of cost of living indices specific to 39 cities, from other household surveys and from other market price information collected by various Indonesian government agencies. The poor infrastructural and market separation of island Indonesia result in substantial price variation in cross-section. The construction of each price index as well as the other data used in

TABLE 1

Parameter Estimates of the
Quadratic Utility Function*

	Maximum Likelihood Estimate	Asymptotic t-ratio
c_{10}	.2482	1.57
c_{11}	.1551	7.07
c_{12}	.1769	1.24
c_{13}	.0259	0.20
c_{20}	.0502	0.16
c_{21}	-.0577	-1.07
c_{22}	-.0420	-0.11
c_{23}	-.0042	-0.01
c_{30}	.0703	0.49
c_{31}	.0697	3.32
c_{32}	.0922	0.85
c_{33}	.0165	0.17
c_{40}	.0949	0.70
c_{41}	.0569	2.58
c_{42}	.0649	0.57
c_{43}	.0123	0.11
c_{50}	-.1034	-1.37
c_{51}	-.0671	-5.68
c_{52}	-.0534	-0.75
c_{53}	-.0283	-0.38
a_{11}	-.0142	-32.27
a_{21}	.0439	22.43
a_{31}	-.0069	-17.51
a_{41}	.0126	7.86
a_{51}	.0132	31.92

...Continued

	Maximum Likelihood Estimate	Asymptotic t-ratio
a_{22}	.0384	1.65
a_{32}	-.0087	-5.52
a_{42}	-.1646	-14.53
a_{52}	-.0824	-22.93
a_{33}	.0059	11.07
a_{43}	-.0029	-4.00
a_{53}	-.0002	-.057
a_{44}	-.1990	-15.53
a_{54}	.0131	5.90
a_{55}	-.0178	-22.25
$\sigma 2$	.6434	3.80
$\sigma 3$	.8345	12.69
$\sigma 4$	1.0095	15.02
$\sigma 5$	.4766	20.25
$\rho 21$	-.1153	-1.15
$\rho 31$	-.0150	-0.29
$\rho 41$	-.1755	-2.74
$\rho 51$	-.3336	-5.97
$\rho 32$	-.0216	-0.11
$\rho 42$	-.0146	-0.06
$\rho 52$	.0250	0.12
$\rho 43$	.0046	0.06
$\rho 53$	-.0136	-0.18
$\rho 54$	.0299	0.42

*For the parameters c_{ij}, i refers to goods and has the values: i=1, food; i=2, apparel; i=3, housing; i=4, fuel; i=5, other nondurables. The values of j are: j=0, intercept; j=1, household size; j=2, Jan-April; j=3, May-August. The parameters a_{ij}, variances σ_i and correlation coefficients ρ_{ij} have subscripts which refer only to goods as in i above.

this analysis is described in Pitt and Lee.

The parameters of the quadratic utility function were estimated using a sample of 767 households randomly drawn from the 1976 SUSENAS survey. Zero levels of consumption were often observed for both the clothing and housing expenditure categories. The zero levels of housing expenditure represent an imputed market valuation of housing of zero. This would correspond to housing of such simple construction or poor quality that no market exists for its sale, construction or rental.

The presence of two commodities for which zero demand is observed means that there are four demand regimes (as defined in Section 2): a regime where all five goods are consumed, two regimes in which either clothing or marketable housing is not consumed, and a regime in which both clothing and marketable housing are not consumed. There are households in the sample corresponding to all four regimes.

In order to allow for the effects of season and household composition on demands, the parameters a_{io} of the utility function (4.15) were assumed to be linear functions of seasonal dummy variables and the size of the households (SIZE), as follows:

$$(6.1) \qquad a_{io} = c_{io} + c_{i1}\,\text{SIZE} + c_{i2}\,\text{Jan-April} + c_{i3}\,\text{May-August}, \qquad i = 1,5$$

The maximum likelihood estimates of the parameters of the quadratic utility function were obtained using the quadratic hill-climbing methods of Goldfeld and Quandt (1972) and are presented in Table 1. The normalization adopted for parameter identification is var $(\epsilon_1) = 1$. The asymptotic t-ratios reported in the table suggest that the seasonal component of demands (c_{i2}, c_{i3}) is not significant. However, household size (c_{i1}) is significant in the relations (6.1) for the cases of food, housing, fuel and other nondurables. The positive signs of the coefficients c_{11}, c_{31}, c_{41} mean that the marginal utility of food, housing and fuel

are higher the larger is household size given an allocation of the five commodities. All but two of the quadratic terms of the utility function (a_{ij}) are statistically different from zero at the .01 level of significance. All of the variance σ_i are significantly greater than zero as expected and some of the correlation coefficients of the stochastic terms are also statistically significant.

Table 2 presents compensating variation estimates by income class for urban and rural households in Indonesia. For Indonesia as a whole, the compensating variation associated with a 25% increase in the energy price in 1976 is Rp 162 (US$0.39) per month per household or .795% of average household expenditure. While the upper income class has the largest absolute compensating variation, the middle income class has the largest proportionate compensating variation. The lowest income class has both the lowest absolute and proportional welfare loss, as measured by compensating variation.

The patterns across income classes for urban and rural households separately are rather different. Compensating variation as a percentage of base (actual) expenditures declines with income among urban household but increases with income among rural households. As Pitt and Lee demonstrated, there is also substantial inter-regional variation in these results. As there were 26,648,000 households in Indonesia in 1976, a suitably distributed transfer of Rp 51,804 million (US$124.8 million) would have been sufficient to make Indonesian households indifferent between the actual level of energy prices and a level of energy prices 25% higher, assuming that there were no other changes in prices and income. This transfer seems small indeed compared to the economic subsidies provided petroleum fuels. In fiscal 1981/82, this subsidy was Rp 2.8 trillion (US$4.4 billion or Rp 1.50 trillion in 1976 prices).

Finally, in Table 3 the demand effects of compensated fuel price increases of 25% and 50% are presented. As is required by demand theory, fuel demand falls in

TABLE 2

Compensating Variations for a 25 Percent Increase in Fuel Prices (Rupiah per month per household)

	Income Group	Average Household Size	Share of Households in Class	Base Expenditures (Rp)	Compensated Expenditures (Rp)	Compensating Variation (Rp)	Percent C.V.
Urban Households	Lower	6.09	3.68	16090.	16231.	141.	.877
	Middle	5.43	7.89	27257.	27453.	196.	.720
	Upper	4.54	5.88	52948.	53276.	327.	.618
	Average	5.27	17.44	33560.	33789.	229.	.682
Rural Households	Lower	5.15	44.35	11972.	12061.	90.	.748
	Middle	4.45	29.49	21186.	21371.	186.	.877
	Upper	3.61	8.71	34224.	34543.	319.	.931
	Average	4.74	82.56	17612.	17760.	148.	.841
Total Households	Lower	5.22	48.03	12287.	12380.	94.	.761
	Middle	4.66	37.38	22467.	22655.	188.	.837
	Upper	3.98	14.59	41767.	42089.	322.	.771
	Average	4.83	100.00	20393.	20555.	162.	.795

TABLE 3

Effects of Compensated Fuel Price Increases on Demand

(percent change)

	Compensated 25 Percent Fuel Price Increase			Compensated 50 Percent Fuel Price Increase		
	Urban	Rural	Total	Urban	Rural	Total
Fuel	-2.200[a]	-6.436	-5.359	-4.360[b]	-12.632	-10.529
Food	+0.167	+0.518	+0.448	+0.344	+1.057	+0.915
Apparel	-1.694	-3.694	-3.313	-3.341	-7.234	-6.491
Housing	-0.068	+0.587	+0.403	-0.118	+1.267	+0.878
Other Nondurables	+1.518	+7.472	+5.562	+3.049	+14.950	+11.313

a) These values can be converted into compensated arc elasticities through division by 25.

b) These values can be converted into compensated arc elasticities through division by 50.

response to a compensated increase in its price. The arc price elasticity of demand for fuel corresponding to a compensated 25% increase in fuels price is -.214. It is only slightly less in absolute value for a 50% price increase.

The table also demonstrates that for the total of rural and urban households, food, housing and other nondurables are substitutes for fuel while apparel is a complement. Other nondurables and apparel consumption are the proportionately most sensitive to a fuel price increase. There is some difference between rural and urban households. Rural household demand for all five goods is more sensitive to a change in the fuel price than are urban households. In particular, rural household fuel elasticities are triple those of urban households.

7. Conclusions.

In this article, we have considered the specification and estimation of models of consumer demand where the demanded quantities for certain commodities are zero for some consumers. The models specified recognize that the observed demands are the result of optimal choice. Our approach generalizes the previous econometric work of Wales and Woodland (1983) in that demand relationships derived from either direct and indirect utility functions can be estimated. Our approach utilizes the concept of virtual prices originated in the quantity rationing literature. Virtual prices transform binding zero consumption quantities into nonbinding quantities, and provide a rigorous justification for structural shifts in the observed (positive) demand equations across demand regimes. Switching conditions, which determine the occurrence probabilities of different demand regimes, are provided in terms of notional (latent) demand equations. The relationship between this approach and the conventional single equation and simultaneous equation Tobit models is also considered.

We have used the approach described in this article to estimate a complete system of demand equations using a household budget survey for Indonesia. A demand system for five commodities was estimated with a sample of 767 households. The computational cost of estimating this demand system, in which there were at most two nonconsumed goods for each household, was quite moderate. However, because the econometric model is highly nonlinear and the censoring problem is multivariate in nature, computational difficulty and cost may increase rapidly with the number of conconsumed commodities.

Finally, although this article has concentrated on the anlysis of consumer demand, the approach can also be applied to the analysis of production at the level of the firm where the derived demand for certain inputs may be zero. An

example of this problem is found in Pitt (1983c), where an industrial census from Indonesia is used to estimate a multi-input production structure. In the second stage of a two-stage cost minimization problem, firms choose among a set of alternative fuels to meet an energy requirement determined in the first stage. As these fuels are close substitutes, most firms in the sample do not consume all of them. However, as estimation of production structures with binding nonnegativity constraints differs somewhat from the estimation of consumer demand systems, its discussion will appear elsewhere.

Footnotes

1. This point is observed in the approach of Burtless and Hausman (1978) in studying the effects of taxation on labor supply.

2. In this article, we use the convention that $v > 0$ means $v_i > 0$ for all i, and $v \geq 0$ means $v_i \geq 0$ for all i.

3. Survey sampling errors, such as reporting errors, may introduce zero quantities in observed samples. Such measurement error problems will not be considered in this article. A relatively simple model with reporting errors is in Deaton and Irish (1982).

4. Sometimes only the components of p^* which are subject to rationing are referred to as virtual prices. In this article, virtual prices are those price vectors which support boundary points of the simplex $\{y | y \geq 0, \ v'y = 1\}$.

5. Browning (1983) has shown that the unconditional cost function can theoretically be recovered from a conditional cost function. The necessary conditions for the conditional cost function are also sufficient for the recovery of the unconditional cost function when the rationed quantities are positive. Our approach starts with the unconditional functions. Identification in this article refers to parameter identification given functional forms for the unconditional functions.

6. One can also specify other distributions if they are of interest. Normality is of interest because of its additive property.

7. It is necessary to specify $\Sigma_{i=1}^{K} \epsilon_i = 0$, since, for the homogeneous case $\Sigma_{i=1}^{K} \beta_{ij} = 0$ so that $D = -1$ in the share equations (4.11) and the sum of the shares is unity.

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Appendix: Proof that the Conditions (3.1) Uniquely Characterize Demand Regimes

Inductive argument will be used to prove that the conditions (3.1) are valid conditions for the determination of the corresponding demand regime. First, consider the simple two-goods case. From the budget constraint $v'x = 1$, it is clear that one of the goods must be consumed. Consider the regime in which demand for the first good is zero and positive for the second, i.e., $x_1 = 0$ and $x_2 > 0$. The solution of the unconstrained problem (2.3) gives the notional demand equations

$$\text{(A.1)} \qquad q_i = D_i(v_1,v_2) \qquad i = 1,2.$$

If both q_1 and q_2 are positive, the constrained problem (2.2) will have an interior solution $x_1 > 0$ and $x_2 > 0$, which is a contradiction. Because of the constraint $v_1q_1 + v_2q_2 = 1$ and $v > 0$, only one of the components of $q = (q_1,q_2)$ can be negative or zero. As $U(y)$ is quasi-concave, the set

$$S = \{y \mid v'y = 1, U(y) > U(\tfrac{1}{v_1},0)\}$$

is convex and therefore necessarily a connected set. Since $U(q_1,q_2) > U(0,\frac{1}{v_2})$ and $U(0,\frac{1}{v_2}) > U(\frac{1}{v_1},0)$ (as $(x_1,x_2) = (0,\frac{1}{v_2})$), it follows that q_1 must either be negative or zero. This situation is clearly illustrated in Figure 1. Therefore $x_1 = 0$ is characterized by $D_1(v_1,v_2) \leq 0$, and since $x_2 = D_2(\xi_1(v_2),v_2)$, $x_2 > 0$ is characterized by $D_2(\xi_1(v_2),v_2) > 0$.

Now consider the K goods case where $x_1 = 0$ but $x_i > 0$ for $i = 2,\ldots,K$. Given the consumption pattern $x_1 = 0$, $x_i > 0$ for all $i = 2,\ldots,K$, we want to show that the conditions (3.1) are necessary. By the construction of the virtual price vector for $x_1 = 0$, $x_i > 0$, $i = 2,\ldots,K$,

$$x_i = D_i(\xi_1(v_2,\ldots,v_K),v_2,\ldots,v_K) \qquad i = 2,\ldots,K.$$

Hence $D_i(\xi_1, v_2, \ldots, v_K)$ for $i = 2, \ldots, K$ are necessarily positive. It remains to be shown that $D_1(v_1, v_2, \ldots, v_K) \leq 0$. Define the function $V(y_1, r)$ as

$$\text{(A.2)} \qquad V(y_1, r) = \max_{y_2, \ldots, y_K} \{U(y_1, y_2, \ldots, y_K) \mid v_2 y_2 + \ldots + v_K y_K = r\}$$

As noted by Neary and Roberts (1980), the function $V(y_1, r)$ will be quasi-concave on (y_1, r) when the utility function U is quasi-concave. With this construction, it is obvious that the optimal problem in (2.3) can be reduced to a two-commodity problem

$$\text{(A.3)} \qquad \begin{aligned} &\max_{y_1, \ldots, y_K} \{U(y_1, y_2, \ldots, y_K) \mid v_1 y_1 + v_2 y_2 + \ldots + v_K y_K = 1\} \\ &\quad = \max_{y_1, r} \{V(y_1, r) \mid v_1 y_1 + r = 1\}. \end{aligned}$$

Let (q_1, r^*) be the optimal solution of the latter problem in (A.3). It follows that

$$\text{(A.4)} \qquad \begin{aligned} q_1 &= D_1^*(v_1, 1) \\ &= D_1(v_1, v_2, \ldots, v_K) \end{aligned}$$

and

$$\text{(A.5)} \qquad r^* = D_2^*(v_2, 1)$$

where D_1^* and D_2^* denote the solution (q_1, r) as a function of the normalized price vector $(v_1, 1)$ where 1 is the normalized price for the aggregate good r. To prove that $q_1 < 0$ is necessary, it is sufficient to prove that $(0,1)$ is the optimal solution of the constrained problem

$$\text{(A.6)} \qquad \max_{y_1, r} \{V(y_1, r) \mid v_1 y_1 + r = 1, \quad y_1 \geq 0, \quad r \geq 0\}.$$

For all $(y_1, r) \geq 0$ and $v_1 y_1 + r = 1$, we have

(A.7) $\quad U(0,x_2,\ldots,x_K) \geq V(y_1,r)$.

If this were not so, there would exist a vector (y_1,r) and a vector $\bar{y} = (y_1,y_2,\ldots,y_K)$ such that $v_2y_2 + \ldots + v_Ky_K = r$, $v_1y_1 + r = 1$ and $U(\bar{y}) > U(0,x_2,\ldots,x_K)$. It follows that there exists a small λ, $1 \geq \lambda > 0$ and a vector z, $z = \lambda x + (1-\lambda)\bar{y}$, such that $v'z = 1$, $z \geq 0$ and $U(z) > U(x)$ by the strictly quasi-concavity of U. As this would be a contradiction, the inequality in (A.7) must hold. By the construction of V from (A.2), it follows that $V(0,1) \geq U(0,x_2,\ldots,x_K)$. Hence $(0,1)$ is the optimal solution of (A.6). It follows from the two-goods case that it is necessary that $D_1^*(v_1,1) \leq 0$, i.e., $D_1(v_1,v_2,\ldots,v_K) \leq 0$. Furthermore, that these conditions are sufficient to determine the demand regime $x_1 = 0$, $x_i > 0$ for $i = 2,\ldots,K$, can be shown as follows. We note that the solution of the optimum problem

(A.8) $$\max_{y_1,r} \{V(y_1,r) \mid v_1y_1 + r = 1\}$$

is given in (A.4) and (A.5). The condition $D_1(v_1,v_2,\ldots,v_K) \leq 0$ implies that the optimal solution of the constrained problem in (A.6) for the two-good case is $(0,1)$ since $D_1^*(v_1,1) \leq 0$. Hence it follows from (A.6) and (A.2)

(A.9) $$\begin{aligned} V(0,1) &= \max_{y_1,r} \{V(y_1,r) \mid v_1 y_1 + r = 1,\ y_1 \geq 0,\ r \geq 0\} \\ &\geq \max_{y_1,\ldots,y_K} \{U(y_1,y_2,\ldots,y_K) \mid v_1y_1 + v_2y_2 + \ldots + v_Ky_K = 1,\ y \geq 0\}. \end{aligned}$$

By construction (as in (A.2)), we have

(A.10) $$V(0,1) = \max_{y_2,\ldots,y_K} \{U(0,y_2,\ldots,y_K) \mid v_2y_2 + \ldots + v_Ky_K = 1\}.$$

The virtual price vector $(\xi_1(v_2,\ldots,v_K),v_2,\ldots,v_K)$, supports the quantities $(0,D_2(\xi_1,v_2,\ldots,v_K),\ldots,D_K(\xi_1,v_2,\ldots,v_K))$ and

$$U(0,D_2(\xi_1,v_2,\ldots,v_K),\ldots,D_K(\xi_1,v_2,\ldots,v_K))$$
$$(A.11) \qquad = \max_{y_1,\ldots,y_K} \{U(y_1,\ldots,y_K)|\xi_1 y_1 + v_2 y_2 + \ldots + v_K y_K = 1\}.$$

Since

$$\max_{y_1,\ldots,y_K} \{U(y_1,\ldots,y_K)|\xi_1 y_1 + v_2 y_2 + \ldots + v_K y_K = 1\}$$
$$\geq \max_{y_2,\ldots,y_K} \{U(0,y_2,\ldots,y_K)|v_2 y_2 + \ldots + v_K y_K = 1\},$$

it follows from this inequality and the relations in (A.9), (A.10) and (A.11) that

$$U(0,D_2(\xi_1,v_2,\ldots,v_K),\ldots,D_K(\xi_1,v_2,\ldots,v_K))$$
$$\geq \max_{y_1,\ldots,y_K} \{U(y_1,\ldots,y_K)|v_1 y_1 + v_2 y_2 + \ldots + v_K y_K = 1, \quad y \geq 0\}.$$

But since $D_i(\xi_1,v_2,\ldots,v_K) > 0$, $i = 2,\ldots,K$ and $\sum_{i=2}^{K} v_i D_i(\xi_1,v_2,\ldots,v_K) = 1$, it follows that

$$U(0,D_2(\xi_1,v_2,\ldots,v_K),\ldots,D_K(\xi_1,v_2,\ldots,v_K))$$
$$= \max_{y_1,\ldots,y_K} \{U(y_1,\ldots,y_K)|v_1 y_1 + v_2 y_2 + \ldots + v_K y_K = 1, \quad y \geq 0\}.$$

Therefore the solution vector $x = (x_1,\ldots,x_K)'$ of the constrained optimal problem (2.3) is simply $(0,D_2(\xi_1,v_2,\ldots,v_K),\ldots,D_K(\xi_1,v_2,\ldots,v_K))$ and the demand regime is $x_1 = 0$, $x_i > 0$ for all $i = 2,\ldots,K$.

Consider the more general demand regime where the quantities of the first L goods are zero and all others are positive. We want to show that the conditions in (3.1) characterize this regime. To simplify the notation, denote $\bar{v} = (v_{L+1},\ldots,v_K)$, $\bar{y} = (y_{L+1},\ldots,y_K)$ and $\bar{x} = (x_{L+1},\ldots,x_K)$, which are subvectors of the corresponding vectors v, y and x. That the conditions (3.1) are necessary for the regime $x_1 = 0,\ldots,x_L = 0$, $\bar{x} > 0$ can be proven as follows. The virtual price vector $(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v})$ supports the demand

quantities and therefore $x_i = D_i(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v})$, $i = L+1,\ldots,K$. It then follows that

$$D_i(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v}) > 0 \qquad i = L+1,\ldots,K.$$

To prove that the remaining conditions of (3.1) are necessary, it is sufficient to show that the first condition in (3.1) holds because the other conditions can be established in an analogous fashion. Define a utility function in the commodity space $(y_1,\bar{y})$ as

$$\text{(A.12)} \qquad V_1(y_1,\bar{y}) = U(y_1,0,\ldots,0,\bar{y})$$

and consider the corresponding constrained optimization problem,

$$\text{(A.13)} \qquad \max_{y_1,\bar{y}} \{V_1(y_1,\bar{y}) \mid v_1y_1 + \bar{v}'\bar{y} = 1, \quad y_1 \geq 0, \quad \bar{y} \geq 0\}$$

and the unconstrained optimization problem,

$$\text{(A.14)} \qquad \max_{y_1,\bar{y}} \{V_1(y_1,\bar{y}) \mid v_1y_1 + \bar{v}'\bar{y} = 1\}.$$

Since $x = (0,\ldots,0,\bar{x})$ is the solution of the general constrained problem in (2.2), it follows that

$$U(0,\ldots,0,\bar{x}) \geq \max_{y_1,y} \{V_1(y_1,\bar{y}) \mid v_1y_1 + \bar{v}'\bar{y} = 1, \quad y_1 \geq 0, \quad \bar{y} \geq 0\}.$$

However, note that the subvector $(0,\bar{x})$ satisfies the constraints (A.13), so therefore

$$\text{(A.15)} \qquad V_1(0,\bar{x}) = \max_{y_1,\bar{y}} \{V_1(y_1,\bar{y}) \mid v_1y_1 + \bar{v}'\bar{y} = 1, \quad y_1 \geq 0, \quad \bar{y} \geq 0\}$$

and $(0,\bar{x})$ is the solution to the problem (A.13). Denote the notional demand functions of the problem (A.14) as $D_{1i}(v_1,\bar{v})$, $i = 1,L+1,\ldots,K$. By construction (as in (3.2)) at the virtual prices $\xi_{1i}, i = 2,\ldots,L$,

$$\max_y \{U(y) | v_1 y_1 + \xi_{12} y_2 + \ldots + \xi_{1L} y_L + \bar{v}'\bar{y} = 1\}$$

$$= U(D_1(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}),0,\ldots,0,D_{L+1}(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}),\ldots,$$

(A.16) $$D_K(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}))$$

$$= V_1(D_1(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}),D_{L+1}(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}),\ldots,$$

$$D_K(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v})).$$

As

$$\max_y \{U(y) | v_1 y_1 + \xi_{12} y_2 + \ldots + \xi_{1L} y_L + \bar{v}'\bar{y} = 1\}$$

$$\geq \max_{y_1,\bar{y}} \{V_1(y_1,\bar{y}) | v_1 y_1 + \bar{v}'\bar{y} = 1\}$$

and $V_1 D_1(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}) + \Sigma_{i=L+1}^K v_i D_i(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}) = 1$, it follows that

(A.17) $$V_1(D_1(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}),D_{L+1}(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}),\ldots,D_K(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v})) = \max_{y_1,\bar{y}} \{V_1(y_1,\bar{y}) | v_1 y_1 + \bar{v}'\bar{y} = 1\}.$$

Therefore,

(A.18) $$D_{1i}(v_1,\bar{v}) = D_i(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}) \qquad i = 1,L+1,\ldots,K.$$

As shown in the previous paragraph, the demand regime $(0,\bar{x})$, $\bar{x} > 0$, for the K-L+1 commodity space, implies that $D_{11}(v_1,\bar{v}) \leq 0$. It then follows that $D_i(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v}) \leq 0$, and hence the conditions are necessary for the demand regime $x = (0,\ldots,0,\bar{x})$, $\bar{x} > 0$. That these conditions are also sufficient can be established as follows. First, it can be shown that for the reduced dimension problem (A.13), the conditions

(A.19) $$D_1(v_1,\xi_{12}(v_1,\bar{v}),\ldots,\xi_{1L}(v_1,\bar{v}),\bar{v}) \leq 0$$
$$D_i(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v}) > 0 \qquad i = L+1,\ldots,K$$

will characterize the demand regime $(0,q_{L+1},\ldots,q_K)$ with $q_i > 0$, $i = L+1,\ldots,K$ and furthermore the optimal solution for (A.13) is the vector

(A.20) $\quad (0,D_{L+1}(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v}),\ldots,D_K(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v}))$.

By induction, we know that the consumption pattern $(0,q_{L+1},\ldots,q_K)$, $q_i > 0$, $i = L+1,\ldots,K$ for the problem (A.13) is characterized by the conditions:

$$D_{11}(v_1,\bar{v}) \leq 0$$
$$D_{1i}(\xi_{11}(\bar{v}),\bar{v}) > 0 \qquad i = L+1,\ldots,K$$

where $(\xi_{11}(\bar{v}),\bar{v})$ is the virtual price vector for the reduced dimension commodities system. To prove that the statement in (A.19) holds, we will show that

$$D_{1i}(\xi_{11},\bar{v}) = D_i(\xi_1,\ldots,\xi_L,\bar{v}) \qquad i = L+1,\ldots,K.$$

The relation $D_{11}(v_1,\bar{v}) = D_1(v_1,\xi_{12},\ldots,\xi_{1L},\bar{v})$ has already been demonstrated in (A.18). By the construction of ξ_{11},

$$V_1(0,D_{1L+1}(\xi_{11},\bar{v}),\ldots,D_{1K}(\xi_{11},\bar{v}))$$
$$= \max_{y_1,\bar{y}}\{V_1(y_1,\bar{y})|\xi_{11}y_1 + \bar{v}'\bar{y} = 1\}.$$

But,

$$\max_{y_1,\bar{y}}\{V_1(y_1,\bar{y})|\xi_{11}y_1 + \bar{v}'\bar{y} = 1\}$$
$$\geq \max_{\bar{y}}\{V_1(0,\bar{y})|\bar{v}'\bar{y} = 1\}$$
$$\geq V_1(0,D_{1L+1}(\xi_{11},\bar{v}),\ldots,D_{1K}(\xi_{11},\bar{v}))$$

and therefore

(A.21) $\quad \max_{y_1,\bar{y}}\{V_1(y_1,\bar{y})|\xi_{11}y_1 + \bar{v}'\bar{y} = 1\} = \max_{\bar{y}}\{V_1(0,\bar{y})|\bar{v}'\bar{y} = 1\}$.

Furthermore, since

$$V_1(0,D_{L+1}(\xi_1,\ldots,\xi_L,\bar{v}),\ldots,D_K(\xi_1,\ldots,\xi_L,\bar{v}))$$

$$= U(0,0,\ldots,0,D_{L+1}(\xi_1,\ldots,\xi_L,\bar{v}),\ldots,D_K(\xi_1,\ldots,\xi_L,\bar{v}))$$
$$= \max_y \{U(y)|\xi_1 y_1 + \ldots + \xi_L y_L + \bar{v}'\bar{y} = 1\}$$
$$\geq \max_{\bar{y}} \{U(0,\ldots,0,\bar{y})|\bar{v}'\bar{y} = 1\}$$

and

$$\max_{\bar{y}} \{U(0,\ldots,0,\bar{y})|\bar{v}'\bar{y} = 1\}$$
$$= \max_{\bar{y}} \{V_1(0,\bar{y})|\bar{v}'\bar{y} = 1\}$$
$$\geq V_1(0,D_{L+1}(\xi_1,\ldots,\xi_L,\bar{v}),\ldots,D_K(\xi_1,\ldots,\xi_L,\bar{v})),$$

we have

$$\text{(A.22)} \qquad V_1(0,D_{L+1}(\xi_1,\ldots,\xi_L,\bar{v}),\ldots,D_K(\xi_1,\ldots,\xi_L,\bar{v})) = \max_{\bar{y}} \{V_1(0,\bar{y})|\bar{v}'\bar{y} = 1\}.$$

It follows from (A.21) and (A.22) that the vector $(0,D_{L+1}(\xi_1,\ldots,\xi_L,\bar{v}),\ldots,D_K(\xi_1,\ldots,\xi_L,\bar{v}))$ is the solution to the problem (A.14) and therefore

$$D_{1i}(\xi_{11},\bar{v}) = D_i(\xi_1(\bar{v})),\ldots,\xi_L(\bar{v}),\bar{v}) \qquad i = L+1,\ldots,K.$$

This proves the statement (A.19). With similar arguments, we can characterize all the reduced dimension commodities spaces of the form $\{y_\ell,y_{L+1},\ldots,y_K\}$, $\ell = 1,\ldots,L$. Also note that the solutions for all of these reduced dimension problems have the form $(0,x_{L+1},\ldots,x_K)$ where

$$x_i = D_i(\xi_1(\bar{v}),\ldots,\xi_L(\bar{v}),\bar{v}) \qquad i = L+1,\ldots,K.$$

Finally, it remains to be shown that the K-dimensional vector $x = (0,\ldots,0,x_{L+1},\ldots,x_K)$ is the optimum solution of the general constrained problem in (2.2). For each of the reduced dimension problems, the solution $(0,x_{L+1},\ldots,x_K)$ is characterized by a set of Kuhn-Tucker conditions. For the problem with utility

function V_i, $i \in \{1,\ldots,L\}$, the Kuhn-Tucker conditions are

$$\frac{\partial V_i(0,x_{L+1},\ldots,x_K)}{\partial y_1} - \frac{\partial V_i(0,x_{L+1},\ldots,x_K)}{\partial y_K}\frac{v_i}{v_K} < 0$$

$$\text{(A.23)} \qquad \frac{\partial V_i(0,x_{L+1},\ldots,x_K)}{\partial y_j} - \frac{\partial V_i(0,x_{L+1},\ldots,x_K)}{\partial y_K}\frac{v_j}{v_k} = 0, \qquad j = L+1,\ldots,K-1$$

$$v_{L+1}x_{L+1} + \ldots + v_K x_K = 1, \quad x_i > 0 \quad \text{for all} \quad i = L+1,\ldots,K-1.$$

Since $V_i(y_i,y_{L+1},\ldots,y_K) = U(0,\ldots,0,y_i,0,\ldots,0,y_{L+1},\ldots,y_K)$ for each i, $i = 1,\ldots,L$,

$$\frac{\partial V_i(0,x_{L+1},\ldots,x_K)}{\partial y_j} = \frac{\partial U(0,\ldots,0,x_{L+1},\ldots,x_K)}{\partial y_j}, \qquad j = i,L+1,\ldots,K.$$

The combination of the L sets of conditions in (A.23) forms the complete set of Kuhn-Tucker conditions for the general problem (2.2) and therefore $x = (0,\ldots,0, x_{L+1},\ldots,x_K)$ is the optimal solution. Thus we conclude that the conditions (3.1) are sufficient to imply the desired regime.

As all other demand regimes are merely rearrangements in the order of goods so that the quantities of the first L goods are zeroes and the remaining ones are positive, the notional demand functions can completely characterize all possible consumption patterns.