CONSUMER CHOICE OF INFORMATION
WITH A WEAK TOPOLOGY

by

Kevin D. Cotter

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Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

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1. Introduction

Some of the most interesting phenomena in the economics of uncertainty are those involving shifts in the private information of decision makers resulting from a change in their incentives. Some examples, which are discussed in the companion paper (Cotter, 1983), include the demise of the short-run Phillips curve in the 1970s, the seemingly inverted short-run correlation between announced money supply figures and interest rates, and the increase in the acquisition of second opinions regarding major medical expenses such as surgery.

To handle such endogenous shifts in information at a reasonable level of generality, a theory of the optimal choice of information is needed. Unfortunately, most models of information acquisition have used extreme symmetry assumptions regarding agents' characteristics and special parameterizations of the available choices. Consequently, it is difficult to extract many general principles from these models or to determine which assumptions lead to their findings. Some examples include Grossman and Stiglitz (1980), Verrecchia (1982), and Chan and Leland (1982).

To study information choice problems without explicitly computing the value of each choice or using symmetry arguments, some optimization result such as the Kuhn-Tucker theorems or the Maximum Theorem (Hildenbrand, 1974, p. 30) is needed. These require some mathematical structure such as a vector space or metric on the set of possible choices. Two different metrics on the space of all information (this is defined in Cotter, 1983) have been proposed. One was introduced by Boylan (1971) which will be called the Hausdorff metric in this paper. This is the metric used by Allen (1982) to study the effect of private information on consumer demand. The second metric is the pointwise convergence metric proposed by Cotter (1983, Section 2). To a large extent, either metric can be used to apply results such as the Maximum Theorem to choice problems of information, but they have substantially different mathematical properties which may affect their usefulness, which are discussed in Cotter (1983, Section 5). The topology generated by the pointwise convergence metric is weaker than that given by the Hausdorff metric. Hence any subset of the space of information which is compact with respect to the Hausdorff metric is also compact with respect to the pointwise convergence metric. Compactness is a key condition in optimization problems and fixed-point theorems. On the other hand, it is more difficult to show that a function defined on the space of information is continuous with respect to the pointwise convergence metric. An important question is whether the relationships between information and economic behavior is continuous with respect to either metric. If these relationships are continuous with respect to the Hausdorff metric but not the pointwise convergence metric (the reverse situation cannot occur) then the usefulness of the latter is diminished. On the other hand, if these relationships are continuous with respect to both metrics, then the pointwise convergence metric, with more compact sets and other convenient mathematical properties, should be used to study economic problems with information.

An appropriate metric of information should correspond to an economically motivated notion of similarity of information. In other words, two information fields should be close with respect to the metric if and only if they lead to similar economic behavior. The ideal metric is then the weakest one such that economic relationships are continuous with respect to the metric. If these relationships are continuous with respect to both the Hausdorff and the pointwise convergence metrics, then the pointwise convergence metric provides a more natural notion of similarity of information, since more pairs of information fields are close in that metric. The Hausdorff metric would fail to identify many of these pairs as being behaviorally close.

This paper has two purposes. First, the continuity properties of consumer demand demonstrated by Allen (1982, Sections 10 and 11) for the Hausdorff metric are shown to hold for the pointwise convergence metric. Second, these continuity results are applied to a consumer choice problem in which information as well as commodities are chosen.

The consumer problem with exogenous information is informally described in Section 2, and some notation is introduced. Section 3 provides a rigorous definition of conditional expected utility needed for the level of generality at which information is discussed in this paper. In Section 4, the model described in Section 2 is studied. The same continuity results obtained by Allen (1982, Sections 10 and 11) using the Hausdorff metric of information are demonstrated for the weaker pointwise convergence metric. In Section 5, information is allowed to be chosen by the consumer at a cost in commodities and observed prior to the choice for commodities. Using standard assumptions on the choice set of information-cost pairs, the demand correspondence for information and the resulting demand correspondence for goods are shown to be uppersemicontinuous. Problems arising from nonconvexities are discussed, and the model is extended to allow some prior information. Stochastic prices are considered in Section 6. In this case, the continuity results with exogenous information are unchanged provided the consumer does not infer from price. On the other hand, demand with information acquisition is poorly behaved in a very fundamental way.

The reader lacking a sound background in measure-theoretic probability and functional analysis is advised to skip Section 3, Propositions 4.1, 4.4, and 4.5, as well as most of the proofs. Results as well as some additional motivation found in Cotter (1983) are used freely in this paper.

Since a major purpose of this paper is to compare the metric in Cotter (1983) with the one used by Allen (1982), no attempt has been made to be as general as possible. Results on the effects of information about prices and product quality, as well as optimal choice of production of information, are forthcoming.

2. A Reader's Guide

Throughout this paper there are ℓ (finite) commodities, so the commodity space is R^{ℓ}. Define $R_{+}^{\ell} = \{x \in R^{\ell} | x_i \geqq 0$ for each $i\}$ and $R_{++}^{\ell} = \{x \in R^{\ell} | x_i \geqq 0$ for each $i\}$. Uncertainty is modelled as a probability space (Ω, F, μ) where Ω is the set of states of the world with generic element ω, the set (σ-field) of subsets (events) of Ω is F, and μ is a probability (measure) on (Ω, F). The resulting space of information is F^* with pointwise convergence metric ρ. See Cotter (1983, Sections 1 and 2) for definitions. The reader is advised to refer to that paper for properties of the space of information.

One major purpose of this paper is to compare the economic properties of the pointwise convergence metric of information with the Hausdorff metric. The latter was introduced by Boylan (1971) and applied to a consumer choice problem by Allen (1982, Section 10). To this end, the same problem is studied in Section 4. Only exogenous uncertainty about the consumer's utility of consumption for goods is considered. Examples of such uncertainty include the weather and (to a lesser extent, due to the availability of preventitive care) personal health. Uncertainty about prices, income, or product characteristics, as well as moral hazard, is excluded.

The consumer faces a utility function for goods which depends on the state of the world, written for the moment as $u(x,\omega)$. S/he has exogenously available information $B \in F^*$ which is used to help choose his/her optimal demand for goods. For fixed x and B, the utility function as a function of ω is a random variable, hence the conditional expectation $E[u(x,\cdot)|B]$ is a random variable. As x varies, the above expression defines a state-dependent utility function. The reader concerned with issues of measurability is invited to read Section 3 where a rigorous definition of conditional expected utility is provided.

The consumer observes the utility function in commodities $E[u(x,\cdot)|\mathcal{B}](\omega)$ corresponding to the true state ω, as well as the price vector for the commodities and his/her initial endowment, both assumed to be nonstochastic. S/he maximizes the observed utility function subject to the usual budget constraint. The resulting excess demand function depends on consumer characteristics (i.e., state-dependent utility, price, initial endowment, and information) and the state. For fixed consumer characteristics, excess demand as a function of ω is a random vector. Therefore, an alternate form of the excess demand function is a map from consumer characteristics to the vector space of integrable random vectors $L^1(R^\ell)$. The latter is a normed space with norm $\|f\| = \int_\Omega \|f(\omega)\| d\mu(\omega)$ where the norm inside the integral is the Euclidean norm (no confusion should result from using the same symbol for both norms). The domain of the excess demand function, which is the space of consumer characteristics, can be made a metric space in a natural way. The space of state-dependent utility functions has a metric τ which will be defined in the next section. Then the metric on the space of consumer characteristics is any one derived from the product topology, one example of which is $m\ (\mathcal{B},p,e,u),(\mathcal{B}',p',e',u') = \rho(\mathcal{B},\mathcal{B}') + \|p-p'\| + \|e-e'\| + \tau(u,u')$. In this (or any equivalent metric), two lists of consumer characteristics are close if each component is close.

Using the metrics on the domain and range defined above, the main result of Section 4 is that the excess demand function is continuous. This result is identical to the one obtained by Allen (1982) using the Hausdorff metric of information and the same hypotheses. Since the pointwise convergence metric is a weaker topology, the continuity result in this paper is stronger.

Propositions 4.2 and 4.6 must be interpreted with care. For a fixed state of the world, the excess demand function from consumer characteristics to ordinary vectors with the state fixed is not in general continuous. Continuity in the sense of the above results means that a slight change in consumer characteristics leads to a slight change in the excess demand vector for each state of the world except those in some event with small but positive probability. In other words, the probability of a large change is small.

A corollary to Proposition 4.6 is that the value of information is continuous. For a fixed state, the *ex post* utility of consumption can be computed when the consumer solves the above choice problem. In general, this is less than the maximum utility achievable when the state of the world is known in advance. The expected value of this indirect utility over all states is defined to be the value of the information possessed by the consumer. This result is used in Section 5 to discuss optimal choice of information. In that case choice problems reduce to maximizing the value of information function over some choice set.

For the remainder of this paper the state-dependent utility function will be written as U where $U(\omega)$ is the utility *function* in goods when the state is ω. Here $U(\omega)(x) \equiv u(x,\omega)$.

3. Utility and Expected Utility

Consider now a consumer facing uncertainty as given in the previous section. Assume that the "state of the world" $\omega \in \Omega$ matters to the consumer only through his/her utility of consumption. Let $\mathcal{U} = \{v: R_+^\ell \to R | v$ is continuous, strictly increasing, and strictly concave$\}$ be the space of utility functions, endowed with the topology of uniform convergence on compacta. Then $\mathcal{U}$ is a G_δ linear subspace of the separable Frechet space $C \equiv C(R_+^\ell, R)$ with metric d.

Let the state-dependent utility function be given by $U: \Omega \to \mathcal{U}$ which is Borel-measurable and satisfies the following condition, also Allen's (1982) Assumption 4.1.

Assumption 3.1: There exists $\Omega' \in F$ with $\mu(\Omega') = 1$ such that $U(\Omega')$ is relatively compact in $\mathcal{U}$.

In particular, Assumption 3.1 is satisfied if $u(x,\omega) \equiv U(\omega)(x)$ is jointly continuous, Ω is a locally compact Hausdorff space, and Ω' can be chosen to be relatively compact in Ω. In Example 2.2 of Cotter (1983), if utility is jointly continuous in temperature and consumption, Assumption 3.1 holds.

Some later results are stated in terms of the space of state-dependent utility functions. Define the latter to be $\mathcal{D} = \{U: \Omega \to \mathcal{U} | U$ is Borel-measurable and satisfies Assumption 3.1$\}$. Then $\mathcal{D} \subset L^1(\Omega, F, \mu) \equiv L^1(\mathcal{U})$. Give $\mathcal{D}$ the subspace norm topology. Thus $\mathcal{D}$ is a locally convex topological vector space with invariant metric $\tau(U,U') = \int_\Omega d(U(\omega),U'(\omega))d\mu(\omega)$.

Before discussing conditional expected utility, the latter must be defined carefully. Let $B \in F^*$. For each $x \in R_+^\ell$, $E[u(x,\cdot)|B] \in L^1(R)$ is defined up to a null set. One cannot simply claim that

$E[U|\mathcal{B}](\omega)(x) = E[u(x,\cdot)|\mathcal{B}](\omega)$ a.e. since the null sets can pile up over uncountably many x's, causing inequality over a large set. Defining a regular conditional probability as suggested by Kreps (1977) does not help in the case of sub-σ-fields (except in special cases) since the regular conditional probability defined depends on the value of x chosen, and the exceptional sets may be different for different x's. Using the integration theory of Frechet-valued functions in Rudin (1973, pp. 73-78) as suggested by Allen (1982) is inappropriate, since those integration results require conditions on the probability space and the state-dependent utility function that are stronger than those imposed in this paper.

The following proof gives a method of eliminating the null set problem and thereby constructing an expected utility function with the properties desired.

<u>Proposition 3.2</u>: If Assumption 3.1 holds, then $E[U|\mathcal{B}]$ is defined up to a null set. Further, $E[U|\mathcal{B}]$ is a measurable function from Ω to $\mathcal{U}$, and for each $x \in R^{\ell}_{+}$, $E[U|\mathcal{B}](\omega)(x) = E[u(x,\cdot)|\mathcal{B}](\omega)(x)$ a.e.

<u>Proof</u>: Let $\mathcal{C}^*$ be the topological dual of $\mathcal{C}$ with the weak* topology. By Assumption 3.1, $\sup_{\omega\in\Omega} d(U(\omega),0) < \bar{d} < \infty$. Define the following sets:

$$B = \{v \in \mathcal{C} | d(v,0) \leq \bar{d}\}$$

$$B^* = \{v^* \in \mathcal{C}^* |\ |v^*(v)| \leq 1 \text{ for every } v \in B\}.$$

By the Banach-Alaoglu theorem (Rudin, 1973, p. 66) B^* is weak* compact. Since $\mathcal{C}$ is separable, by a result of Rudin (1973, p. 77, Theorem 3.16), B^* is weak* metrizable hence there exists $\{v_n^*\}_{n=1}^{\infty} \subset B^*$ that is weak* dense in B^*. <u>Given</u> versions $\{E[v_n^*(U(\cdot))|\mathcal{B}]\}_{n=1}^{\infty}$, define $v_n^*(E[u|\mathcal{B}](\omega)) = E[v_n^*(U(\cdot))|\mathcal{B}](\omega)$ for each n and ω. Clearly $|v^*(U(\omega))| \leq 1$ for every $v^* \in B^*$, therefore

$$v^*(E[U|\mathcal{B}](\omega)) = \lim_{n\to\infty} v_n^*(E[U|\mathcal{B}](\omega)) = \lim_{n\to\infty} E[v_n^*(U(\cdot))|\mathcal{B}](\omega) =$$
$$E[v^*(U(\cdot))|\mathcal{B}](\omega)$$

the last statement following by definition of the weak* topology and bounded convergence. Now let $v^* \in C^*$. Then for some $r > 0$, $\frac{1}{r} v^* \in B^*$, so define

$$v^*(E[U|\mathcal{B}](\omega)) = E[v^*(U(\cdot))|\mathcal{B}](\omega)$$

for every $v^* \in C^*$. Since C^* separates points on C, this defines $E[U|\mathcal{B}](\omega)$ uniquely, and changing any version $E[V_n^*(U(\cdot))|\mathcal{B}]$ changes $E[U|\mathcal{B}]$ on a set of measure zero. Hence $E[U|\mathcal{B}]$ is well-defined up to null sets. Letting $v_x^*(U(\omega)) = u(x,\omega)$, $E[U|\mathcal{B}](\omega)(x) = E[u(x,\cdot)|\mathcal{B}](\omega)$ a.e. for every $x \in R_+^{\ell}$.

I now show that $E[U|\mathcal{B}](\omega) \in \mathcal{U}$ a.e. Letting $\lambda \in (0,1)$, $x_1, x_2 \in R_+^{\ell}$, $x_1 \neq x_2$, and all statements taken a.e.,

$$E[U|\mathcal{B}](\omega)(\lambda x_1 + (1-\lambda)x_2) = E[u(\lambda x_1 + (1-\lambda)x_2, \cdot)|\mathcal{B}](\omega)$$
$$< E[\lambda u(x_1,\cdot) + (1-\lambda)u(x_2,\cdot)|\mathcal{B}](\omega)$$
$$= \lambda E[u(x_1,\cdot)|\mathcal{B}](\omega) + (1-\lambda)E[u(x_2,\cdot)|\mathcal{B}](\omega)$$
$$= \lambda E[U|\mathcal{B}](\omega)(x_1) + (1-\lambda)E[U|\mathcal{B}](\omega)(x_2)$$

proving strict concavity. A similar argument can be used to prove strict monotonicity. This completes the proof. □

Below is a result (used but not proven by Allen (1982)) that will be used frequently in Section 4.

<u>Lemma 3.3</u>: Let $\{\mathcal{B}_n\}_{n=1}^{\infty} \subset \mathcal{F}^*$ be a sequence. Then for some $\Omega'' \in \mathcal{F}$, $\mu(\Omega'') = 1$, $\{E[U|\mathcal{B}](\omega) \mid n=1,2,\ldots, \omega \in \Omega''\}$ are equicontinuous. If $K \subseteq R_+^{\ell}$ is compact, they are uniformly equicontinuous on K.

<u>Proof</u>: Let $\varepsilon > 0$, $x \in R_+^{\ell}$. Then by Ascoli's theorem (Munkres (1975), p. 290) there exists $\delta(\varepsilon,x) > 0$ such that for every $x' \in R_+^{\ell}$ with

$\|x - x'\| < \delta(\varepsilon,x)$ and $\omega \in \Omega'$, $|u(x,\omega) - u(x',\omega)| < \varepsilon$. Hence for every n, for a.e. ω,

$$|E[u(x,\cdot)|\mathcal{B}_n](\omega) - E[u(x',\cdot)|\mathcal{B}_n](\omega)| \leq E[|u(x,\cdot) - u(x',\cdot)|\ |\mathcal{B}_n](\omega) < \varepsilon,$$

proving the first part of the lemma.

Now let $K \subset \mathcal{R}^\ell_+$ be compact. Let $\varepsilon > 0$. Then for each $x \in K$, there exists $\delta(\varepsilon,x) > 0$ such that for $x' \in B(x,\delta(\varepsilon,x))$, $x \in \mathcal{R}^\ell_+$, and for $\omega \in \Omega'$, $|u(x,\omega) - u(x',\omega)| < \varepsilon$. Then $\{B(x,\delta(\varepsilon,x))\}_{x \in K}$ is an open cover of K, so by the Lebesgue number lemma (Munkres (1975), p. 179) there exists $\delta > 0$ such that for $x,x' \in K$, $\|x - x'\| < \delta$, x and x' lie in the same ball of the above open cover, hence $|u(x,\omega) - u(x',\omega)| < \varepsilon$ for $\omega \in \Omega'$. As in the first part of the proof, for every n and a.e. ω, $|E[u(x,\cdot)|\mathcal{B}_n](\omega) - E[u(x',\cdot)|\mathcal{B}_n](\omega)| < \varepsilon$. This completes the proof.

If the consumption set is some convex subset of $\mathcal{R}^\ell$ rather than $\mathcal{R}^\ell_+$ then all of the results of this section continue to hold with obvious modifications. If some general consumption set is used later in this paper, the definitions of $\mathcal{U}$ and $\mathcal{D}$ are assumed to be modified accordingly.

The next major question is whether conditional expected utility is a continuous function of the information on which it depends. That will be addressed in the next section.

4. Continuity Properties of Information

In the previous section a function mapping information and state-dependent utility into conditional expected utility was defined. To obtain continuity of demand with respect to the information on which it depends, this function needs to be at least separately continuous. It is easily shown that it is continuous in U for fixed $\mathcal{B}$. The trick of the matter is to prove that the function is continuous in $\mathcal{B}$ for fixed U, or better yet, to show joint continuity.

Allen (1982, Theorems 10.1 and 10.8) showed joint continuity of the above function using the Hausdorff metric of information. Since the pointwise convergence metric is weaker (Cotter, 1983, Proposition 4.1), continuity in that metric is a stronger condition. The difficulty is that convergence in that metric is pointwise convergence of L^1 functions, whereas convergence of utility functions is uniform convergence on compacta. However, uniform equicontinuity of the set of conditional expected utility operators (cf. Cotter, 1983, Section 2) along with Assumption 3.1 (which Allen used) overcome this difficulty.

The following result demonstrates separate continuity of conditional expectation with respect to information; joint continuity is shown later.

<u>Proposition 4.1</u>: If $\mathcal{B}_n \to \mathcal{B}$ in $(\mathcal{F}^*, \mathcal{T}p)$ then $E[U|\mathcal{B}_n] \to E[U|\mathcal{B}]$ in $L^1(\mathcal{U})$ and in probability for fixed U.

<u>Proof</u>: (same as Allen's proof of Proposition 10.1). By Lemma 3.3, given $K \subset \mathbb{R}^\ell_+$ compact, for $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, x' \in K$, $\|x - x'\| < \delta$, and all n, for a.e. ω,

$$|E[u(x,\cdot)|\mathcal{B}_n](\omega) - E[u(x',\cdot)|\mathcal{B}_n](\omega)| < \varepsilon$$

$$|E[u(x,\cdot)|\mathcal{B}](\omega) - E[u(x',\cdot)|\mathcal{B}](\omega)| < \varepsilon$$

Pick $x_1, x_2, \ldots, x_n \in K$ such that $(B(x_i,\delta))_{i=1}^{n}$ cover K. Then for each i there exists $N(\varepsilon,i)$ such that for $n \geq N(\varepsilon,i)$

$$\int_\Omega |E[u(x_i,\cdot)|\mathcal{B}_n] - E[u(x_i,\cdot)|\mathcal{B}]|\,d\mu < \varepsilon.$$

Let $N(\varepsilon) = \max\{N(\varepsilon,1),\ldots,N(\varepsilon,n)\}$. Then for $n \geq N(\varepsilon)$,

$$\int_\Omega \sup_{x \in K} |E[u(x,\cdot)|\mathcal{B}_n] - E[u(x,\cdot)|\mathcal{B}]|\,d\mu$$

$$\leq \int_\Omega \sup_{x \in K} |E[u(x,\cdot)|\mathcal{B}_n] - E[u(x_i,\cdot)|\mathcal{B}_n]|\,d\mu$$

$$+ \int_\Omega |E[u(x_i,\cdot)|\mathcal{B}_n] - E[u(x_i,\cdot)|\mathcal{B}]|\,d\mu$$

$$+ \int_\Omega \sup_{x \in K} |E[u(x_i,\cdot)|\mathcal{B}] - E[u(x,\cdot)|\mathcal{B}]|\,d\mu < 3\varepsilon$$ (x_i is such that $\|x - x_i\| < \delta$),

therefore $E[U|\mathcal{B}_n] \to E[U|\mathcal{B}]$ in $L^1(\mathcal{U})$, and hence in probability. □

In addition to a state-dependent utility function U and information $\mathcal{B}$, suppose a consumer has an initial endowment $e \in \mathcal{R}_+^\ell$ and faces a price for goods $p \in \mathcal{R}_{++}^\ell$ both taken by the consumer as nonstochastic. If the true state of the world is ω, the consumer maximizes the observed conditional expected utility function $E[U|\mathcal{B}](\omega)\colon \mathcal{R}_+^\ell \to \mathcal{R}$ subject to the usual budget constraint $p\cdot x \leq p\cdot e$. For fixed U, e, p, and $\mathcal{B}$, the resulting excess demand can be traced out as a function of ω. This function is an integrable random variable, written as $\tilde{z}(\mathcal{B},p,e,U)(\omega)$. Thus as the other parameters vary, an excess demand function results:

$$\tilde{z}\colon\ F^* \times \mathcal{R}_{++}^\ell \times \mathcal{R}_+^\ell \times \mathcal{D} \to L^1(R^\ell)$$

Using Proposition 4.1 this is seen to be continuous in $\mathcal{B}$, e, and p for fixed U. Denote by z the usual excess demand function of p, e, and U.

<u>Definition:</u> Let X and Y be metric spaces and $F: X \twoheadrightarrow Y$ a correspondence. Then F is uppersemicontinuous if for every sequence $\{x_n\}$ in X with $x_n \to x$ and every sequence $\{y_n\}$ in Y with $y_n \quad F(x_n)$, there exists some subsequence $\{y_{n_j}\}$ with $y_{n_j} \to y$ and $y \quad F(x)$, and in addition, $F(x) \neq \emptyset$ for every x. Thus a correspondence is uppersemicontinuous if and only if it is nonempty-valued, compact-valued, and upperhemicontinuous (cf. Hildenbrand, 1974, pp. 21, 24). The correspondence F is lowersemicontinuous if for every sequence $\{x_n\}$ in X with $x_n \to x$ and every $y \quad F(x)$, there exists some sequence $\{y_n\}$ in Y with $y_n \to y$ and $y_n \quad F(x_n)$ for every n. A correspondence is continuous if it is both upper and lowersemicontinuous, and is equivalent to being compact-valued, nonempty-valued, and continuous in the sense of Hildenbrand (1974, p. 28).

Note that when the consumption set is R_+^ℓ, z is jointly continuous for all strictly positive p and nonnegative e. It is sufficient to show that the budget constraint correspondence $\gamma: R_{++}^\ell \times R_+^\ell \twoheadrightarrow R_+^\ell$ with $\gamma(p,e) = \{x \mid p \cdot x \leqq p \cdot e\}$ is continuous whenever $e = 0$. Lowersemicontinuity is immediate since $\gamma(p,0) = \{0\}$ and every budget set contains the origin. Let $\{(p_n, e_n)\}_n$ be a sequence converging to $(p,0)$ with $p >> 0$, and let x_n belong to $\gamma(p_n, e_n)$ for each n. Then $p_n \cdot x_n \to 0$ so since p is strictly positive, $x_n \to 0$. This proves uppersemicontinuity of γ. Hence excess demand is continuous for $e = 0$.

At least for the case of strictly positive prices and a nonnegative orthant consumption set, no minimum wealth problem arises.

<u>Proposition 4.2</u>: Suppose $\{(p_n, \mathcal{B}_n, e_n)\}_{n=1}^{\infty} \subset \mathbb{R}^{\ell}_{++} \times \mathcal{F}^* \times \mathbb{R}^{\ell}_{+}$ with $p_n \to p \in \mathbb{R}^{\ell}_{++}, \mathcal{B}_n \to \mathcal{B}$, $e_n \to e$. Then $\tilde{z}(p_n, \mathcal{B}_n, e_n, \mathcal{U}) \to \tilde{z}(p, \mathcal{B}, e, \mathcal{U})$ in probability and in $L^1(\Omega, \mathcal{F}, \mu; \mathbb{R}^{\ell})$.

<u>Proof</u>: By Lemma 3.3 and Ascoli's theorem (Munkres (1975, p. 290)), $\{E[U|\mathcal{B}_n](\omega)\}_{\substack{n=1\\ \omega\in\Omega''}}^{\infty} \cup \{E[U|\mathcal{B}](\omega)\}_{\omega\in\Omega''}$ is relatively compact for some $\Omega'' \in \mathcal{F}$ with $\mu(\Omega'') = 1$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that for every n satisfying $\|p_n - p\| < \delta$, $\|e_n - e\| < \delta$, and $v \in K$ satisfying $d(v, E[U|\mathcal{B}](\omega)) < \delta$ (d is a metric on $\mathcal{U}$) where $\omega \in \Omega''$, we have

$$\|z(p_n, v, e_n) - z(p, E[u|\mathcal{B}](\omega), e)\| < \varepsilon$$

(this is just uniform continuity of z on a compact set). For the same ε, there exists $A_\varepsilon \in \mathcal{F}$, $\mu(A_\varepsilon) \geq 1 - \varepsilon$, and N such that for $\omega \in A_\varepsilon$, $d(E[U|\mathcal{B}_N](\omega), E[U|\mathcal{B}](\omega)) < \delta$. Choosing the last δ to be smaller or n to be larger corresponding to the previous condition, we have that for $\omega \in A_\varepsilon$,

$$\|z(p_N, E[U|\mathcal{B}_N](\omega), e_N) - z(p, E[U|\mathcal{B}](\omega), e)\| < \varepsilon.$$

Then given $\varepsilon > 0$, there exists N such that

$$\mu[\|\tilde{z}(p_N, \mathcal{B}_N, e_N, \mathcal{U}) - \tilde{z}(p, \mathcal{B}, e, \mathcal{U})\| > \varepsilon] < \varepsilon$$

so $\tilde{z}(p_N, \mathcal{B}_N, e_N, \mathcal{U}) \to \tilde{z}(p, \mathcal{B}, e, \mathcal{U})$ in probability. Since excess demands lie in a compact set (the unions of all the budget sets), this proves L^1 convergence via bounded convergence. □

Define the value of information function as the expected utility of consumption. This is the definition used by Allen (1982, Section 11). Let $V: \mathcal{F}^* \times \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell}_{+} \times \mathcal{D} \to \mathbb{R}$ be $V(\mathcal{B}, p, e, U) = \int_\Omega U(\omega)(z(\mathcal{B}, p, e, U)(\omega) + e)d\mu(\omega)$.

Allen (1982, Theorem 11.3 and Corollary 11.4) showed that the value function is continuous in the Hausdorff metric of information.

<u>Proposition 4.3</u>: V is continuous in $\mathcal{B}$, p, and e for fixed U.

<u>Proof</u>: Let $\{(p_n, \mathcal{B}_n, e_n)\}_{n=1}^{\infty} \subset \mathbb{R}^{\ell}_{++} \times \mathcal{F}^* \times \mathbb{R}^{\ell}_{+}$ with $p_n \to p \in \mathbb{R}^{\ell}_{++}$ $\mathcal{B}_n \to \mathcal{B} \in \mathcal{F}^*$, $e_n \to e \in \mathbb{R}^{\ell}_{+}$. Let $z_n(\omega) = z_n(\mathcal{B}_n, p_n, e_n, U)(\omega)$ and $z(\omega) = z(\mathcal{B}, p, e, U)(\omega)$. Clearly $\{z_n(\omega) + e_n\}_{\substack{n=1 \\ \omega\in\Omega}}^{\infty} \cup \{z(\omega) + e\}_{\omega\in\Omega}$ lies in some compact set K (see above). Hence by Lemma 3.3, $\{u(\cdot,\omega)\}_{\omega\in\Omega''}$ is uniformly equicontinuous on K, so for $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - x'\| < \delta$, $x, x' \in K$, then $|u(x,\omega) - u(x',\omega)| < \varepsilon$ for every $\omega \in \Omega''$. Given such ε and δ, on a set $A_\varepsilon \in \mathcal{F}$ with $\mu(A_\varepsilon) \geq 1 - \varepsilon$, there exists n such that $|u(z_n(\omega) + e_n, \omega) - u(z(\omega) + e, \omega)| < \varepsilon$ hence $u(z_n(\omega) + e_n, \omega) \to u(z(\omega) + e, \omega)$ in probability. Since demands are bounded and $U(\Omega'')$ is compact, the above terms are bounded a.e., hence the integrals converge, proving the result. □

The next set of results demonstrate that all of the above functions are jointly continuous in utility as well as information, initial endowment, and price. The pointwise convergence topology has all of the continuity properties possessed by the Hausdorff topology that have been discussed so far by anyone.

<u>Proposition 4.4</u>: Let $\{U_n\}_{n=1}^{\infty} \subset \mathcal{D}$ be a sequence with $U_n \to U$ a.e., $U \in \mathcal{D}$. Suppose that for $K \subset \mathbb{R}^{\ell}_{+}$ compact, there exists $B(K) > 0$ such that for all $x \in K$ and all n, $|u_n(x,\omega)| \leq B(K)$ a.e. and $|u(x,\omega)| \leq B(K)$ a.e. Then if also $\mathcal{B}_n \to \mathcal{B}$, then $E[U_n|\mathcal{B}_n] \to E[U|\mathcal{B}]$ in $L^1(\Omega, \mathcal{F}, \mu; \mathcal{U})$ and in probability.

<u>Proof</u>: Let $K \subset \mathbb{R}^{\ell}_{+}$ be compact, $\varepsilon > 0$. Since $\limsup_{n\to\infty} \sup_{x\in K} |u_n(x,\omega) - u(x,\omega)| = 0$ for a.e. ω, by uniform boundedness there exists some

$N_1(\varepsilon) > 0$ such that for $n \geq N_1(\varepsilon)$ $\sup_{x\in K} \|u_n(x,\cdot) - u(x,\cdot)\| < \frac{\varepsilon}{3}$. Also, there exists $N_2(\varepsilon) > 0$ such that for $n \geq N_2(\varepsilon)$ $\sup_{x\in K} \|E[u(x,\cdot)|\mathcal{B}_n] - E[u(x,\cdot)|\mathcal{B}]\| < \frac{\varepsilon}{3}$. Then letting $n \geq \max\{N_1(\varepsilon), N_2(\varepsilon)\}$, $\sup_{x\in K} \|E[u_n(x,\cdot)|\mathcal{B}_n] - E[u_n(x,\cdot)|\mathcal{B}]\| \leq \sup_{x\in K} \|E[u_n(x,\cdot)|\mathcal{B}_n] - E[u(x,\cdot)|\mathcal{B}_n]\| + \sup_{x\in K} \|E[u(x,\cdot)|\mathcal{B}_n] - E[u(x,\cdot)|\mathcal{B}]\| + \sup_{x\in K} \|E[u(x,\cdot)|\mathcal{B}] - E[u_n(x,\cdot)|\mathcal{B}]\| < \frac{\varepsilon}{3} + 2\sup_{x\in K} \|u_n(x,\cdot) - u(x,\cdot)\| < \varepsilon$.

Let $n \geq \max\{N_1(\varepsilon), N_2(\varepsilon)\}$ be fixed. By Proposition 3.3, there exists $\delta(n) > 0$ such that for x, $x' \in K$ with $\|x - x'\| < \delta(n)$ and for a.e. ω,

$$|E[u_n(x,\cdot)|\mathcal{B}_n] - E[u_n(x',\cdot)|\mathcal{B}_n]| < \varepsilon$$
$$|E[u_n(x,\cdot)|\mathcal{B}] - E[u_n(x',\cdot)|\mathcal{B}]| < \varepsilon.$$

Choose $x_1^n, x_2^n, \ldots, x_M^n$ in K so that for $x \in K$, there exists x_i^n with $\|x - x_i^n\| < \delta(n)$. Then

$$\int_\Omega \sup_{x\in K} |E[u_n(x,\cdot)|\mathcal{B}_n](\omega) - E[u_n(x,\cdot)|\mathcal{B}](\omega)|\, d\mu(\omega)$$
$$\leq \int_\Omega \sup_{\substack{x\in K \\ \|x-x^n\| < \delta(n)}} |E[u_n(x,\cdot)|\mathcal{B}_n](\omega) - E[u_n(x_i,\cdot)|\mathcal{B}_n](\omega)|\, d\mu(\omega)$$
$$+ \int_\Omega \sup_i |E[u_n(x_i^n,\cdot)|\mathcal{B}_n](\omega) - E[u_n(x_i^n,\cdot)|\mathcal{B}](\omega)|\, d\mu(\omega)$$
$$+ \int_\Omega \sup_{\substack{x\in K \\ \|x - x_i^n\| < \delta(n)}} |E[u_n(x_i^n,\cdot)|\mathcal{B}](\omega) - E[u_n(x,\cdot)|\mathcal{B}](\omega)|\, d\mu(\omega)$$

The first and third terms are less than ε by the preceding paragraph, while the second term is less than ε by the remark in the first paragraph. Thus the entire quantity is less than 4ε.

Since $U_n \to U$ a.e., by uniform boundedness, there exists $N_3(\varepsilon)$ such that for $n \geq N_3(\varepsilon)$,

$$\begin{aligned}
&\int_\Omega \sup_{x\in K} |E[u_n(x,\cdot)|\mathcal{B}](\omega) - E[u(x,\cdot)|\mathcal{B}](\omega)|\,d\mu(\omega) \\
&\leq \int_\Omega \sup_{x\in K} |E[|u_n(x,\cdot) - u(x,\cdot)||\mathcal{B}](\omega)\,d\mu(\omega) \\
&< \varepsilon
\end{aligned}$$

so taking $n \geq \max\{N_1(\varepsilon), N_2(\varepsilon), N_3(\varepsilon)\}$,

$$\int_\Omega \sup_{x\in K} |E[u_n(x,\cdot)|\mathcal{B}_n](\omega) - E[u(x,\cdot)|\mathcal{B}](\omega)|\,d\mu(\omega) < 4\varepsilon$$

by application of the triangle inequality. This completes the proof. □

Corollary 4.5: Let $\{U_n\}_{n=1}^\infty \subset \mathcal{D}$ be a sequence such that $U_n \to U$ in $L^1(\mathcal{U})$. Then if $\mathcal{B}_n \to \mathcal{B}$, $E[U_n|\mathcal{B}_n] \to E[U|\mathcal{B}]$ in $L^1(\Omega,\mathcal{F},\mu;\mathcal{U})$ and in probability.

Proof: See the proof of Proposition 4.1. Note that no uniform boundedness condition is needed. □

The following results are proven in exactly the same way as Propositions 4.2 and 4.3.

Corollary 4.6: $\tilde{z}$ is jointly continuous in all its arguments.

Corollary 4.7: V is jointly continuous in all its arguments.

Later on the assumption of nonstochastic prices and initial endowments will be relaxed. The assumption that the consumer regard them as nonstochastic is essential, however. The problem is that if the consumer conditions expected utility on these variables, then the resulting demand function is not continuous. As explained in Section 2 of Cotter (1983), the map from random variables to information is not continuous.

The reader should note that the consumption set need not be R_+^ℓ in the above results. Any closed convex subset of R^ℓ bounded from below will do. In that case, all of the statements of this section hold for all (p,e) such that for some x in the consumption set, $p \cdot x < p \cdot e$. This formulation is appropriate to handle cases of time and effort as commodities.

Technical Remark: Since conditional expected utility is jointly continuous in utility and information, it is possible to define versions of $E[U|\mathcal{B}]$ on all of $\mathcal{D} \times F^*$ with a common exceptional set by defining versions on a countable dense subset of $\mathcal{D} \times F^*$ as in Proposition 3.2. This is useful in studying large economies under uncertainty with differential information.

5. A Consumer Choice Problem

The methods are now available to study information as a choice variable. Suppose the consumer has a choice problem of information. S/he observes p, e, and U, and faces a choice set $S(p,e) \subset F^* \times R^{\ell}_{+}$. Each choice $(\mathcal{B},y) \in S(p,e)$ is the acquisition of information $\mathcal{B}$ with the immediate payment of y which is subtracted from the initial endowment e. After making the choice, the conditional expected utility function corresponding to $\mathcal{B}$ and the unknown state ω is observed, and the consumer chooses excess demand as in the previous section. The choice problem of information then is

$$\max_{(\mathcal{B},y)\in S(p,e)} V(\mathcal{B},p,e-y,U).$$

Let the feasible information correspondence be $S: R^{\ell}_{++} \times R^{\ell}_{+} \twoheadrightarrow F^* \times R^{\ell}$. This correspondence describes the technical feasibility of information-cost pairs, and includes information choices due to expenditures of time and effort, provided they are listed as commodities. This correspondence should be regarded as a budget constraint, although information availability does not necessarily depend only on total income.

The main step in the classical proof of the uppersemicontinuity of demand with respect to price and initial endowment is the demonstration of continuity of the budget constraint correspondence (cf. Debreu, 1957, pp. 63-65). The continuity of preferences is then sufficient to apply the Maximum Theorem (Hildenbrand, 1974, pp. 29-30). In the present situation, however, one can do no better than to assume continuity of S a priori. Justifying such an assumption would require further understanding of the exchange and production of information.

Proposition 5.1: Suppose that S is continuous and for (p,e), there exists $(B,y) \in S(p,e)$ such that $y \leq e$. Then the information choice correspondence defined by $f: R^{\ell}_{++} \times R^{\ell}_{+} \twoheadrightarrow F^{*} \times R^{\ell}_{+}$ where $f(p,e) = \{(B,y) \mid V(B,p,e-y,U) \geq V(B',p,e-y',U)$ for every (B',y') in $S(p,e)\}$ is uppersemicontinuous. In addition, $\hat{V}(p,e)$, the resulting value function, is continuous.

Proof: Follows from Proposition 6.3 and Hildenbrand (1974, p. 30).

An analogous result can be proven for a general consumption set X. Care must be taken, however, to ensure that no minimum wealth problem arises. Though choices of minimum wealth will usually not be optimal, eliminating them from a choice correspondence may create a discontinuity in the correspondence.

Proposition 5.2: Let X be a convex closed subset of R^{ℓ} which is bounded from below. Define $M(p,e) = \{x \in X \mid p \cdot x' < p \cdot x$ for some x' in $X\}$. Suppose that $S: R^{\ell}_{++} \times X \twoheadrightarrow F^{*} \times R^{\ell}$ is continuous and $S(p,e)$ is contained in $M(p,e)$ for all (p,e). Then f is uppersemicontinuous and $\hat{V}$ is continuous.

Corollary 5.3: Suppose the hypotheses of either Propositions 5.1 or 5.2 holds. Then the excess demand correspondence $\hat{z}: F^{*} \times R^{\ell}_{++} \times R^{\ell}_{+} \times \mathcal{D} \twoheadrightarrow L^{1}(R^{\ell})$ (the third component of the domain may be replaced by X if necessary) defined by the composition of f with $\tilde{z}$ is uppersemicontinuous.

Proof: The composite of an uppersemicontinuous correspondence with a continuous function is uppersemicontinuous. See Hildenbrand (1974, p. 22).

Remark: Corollary 5.3 must be interpreted carefully with regard to individual states. It does not even claim that there exists any state ω such that the excess demand correspondence fixed at ω is uppersemicontinuous in (p,e). The above result takes excess demand as a map of prices and consumer characteristics into random vectors of excess demands. The statement of uppersemicontinuity is with respect to L^1 convergence. As is well known, the latter does not imply almost everywhere (a.e.) convergence.[1]

Suppose that the range of the information choice correspondence has a finite projection into F^*. Then uppersemicontinuity of demand for a.e. state of the world follows.

Proposition 5.4: Let $N \subset \Omega$ be a null set (i.e., $\mu(N) = 0$) such that for every $\omega \in N^c$, $U \in \mathcal{D}$, and $\mathcal{B} \in F^*$, $E[U|\mathcal{B}](\omega) \in \mathcal{U}$. Suppose that the hypotheses of Propositions 5.1 or 5.2 hold, and suppose that the range of S is such that its projection into F^* is finite. Then for every $\omega \in N^c$, the excess demand correspondence $\hat{z}_\omega$ mapping (p,e) into the set $\{z \in R^\ell \mid z = z(p,e-y,E[U|\mathcal{B}](\omega))$ for some $(\mathcal{B},y) \in f(p,e)\}$ is uppersemicontinuous for all (p,e) in the appropriate domain.

Proof: Suppose $(p_n,e_n) \to (p,e)$ in the appropriate domain. Let $z_n \in \hat{z}_\omega(p_n,e_n)$ for each n, so $z_n = z(p_n,e_n - y_n,E[U|\mathcal{B}_n](\omega))$ for some $(\mathcal{B}_n,y_n) \in f(p_n,e_n)$. By continuity of S and the fact that its range contains finitely many elements of F^*, there exists a subsequence $\{n_j\}_j$ with $\mathcal{B}_{n_j} = \mathcal{B}_0$ for some $\mathcal{B}_0$ and all j such that $y_{n_j} \to y_0$ with $(\mathcal{B}_0,y_0) \in S(p,e)$. Since z is continuous, z_{n_j} converges to $z(p,e - y,E[U|\mathcal{B}](\omega))$, completing the proof.

So far the consumer in these choice problems has possessed no information prior to the choice of an information-cost pair. When initial information is present, two additional problems arise. First, the subsequent information is combined with the initial information, which means that the combination must be continuous with respect to subsequent information. Second, since the consumer with initial information observes a different conditional expected utility function prior to choosing information for different states of the world, the choice of information will be state-dependent.

The problem of combining information was considered in Cotter (1983, Section 5). In light of Example 5.1 and Proposition 5.2 of that paper, it is clearly necessary that any initial information be a finite partition.

Let the consumer's initial information be A_0. If the true state of the world were ω the information choice problem would be:

$$\max_{(B,y)\in S(p,e)} E\left[u(\tilde{z}(B\vee A_0,p,e-y,U)(\cdot)+e-y,\cdot)\,|\,A_0\right](\omega)$$

where $\vee$ is the join operation defined in Cotter (1983, Section 5). Denote the resulting information choice set by $f(p,e,A_0,\omega)$. Since A_0 is a finite partition and hence the above expression is an integral over some subset of the state space, the proof of Proposition 4.3 plus Proposition 5.2 of Cotter (1983) verify the following result.

<u>Proposition 5.5</u>: Suppose the hypotheses of Propositions 5.1 or 5.2 hold. Then for any finite partition A_0 and fixed ω the correspondence $f(\cdot,\cdot,A_0,\omega)$ is uppersemicontinuous for (p,e) satisfying the hypotheses.

<u>Proof</u>: Let $\{A_1,A_2,\ldots,A_N\}$ be a disjoint partition of Ω determined by A_0. Let $\omega\in A_i$. Then the maximand is

$$\frac{1}{\mu(A_i)} \int_{A_i} u(z(\mathcal{B} \vee A_0, p, e-y, U)(\nu)+e-y,\nu)\, d\mu(\nu) \ .$$

By Proposition 5.2 of Cotter (1983), the map $\mathcal{B} \to \mathcal{B} \vee A_0$ is continuous. Hence the above expression is jointly continuous in $\mathcal{B}$ and y. Applying the Maximum Theorem completes the proof.

<u>Proposition 5.6</u>: Let $\hat{z}(p,e,A_0,U)(\omega)$ be the composition of $f(p,e,A_0,\omega)$ with $\tilde{z}$ (see p. 13). Then under the hypotheses of Propositions 5.1 or 5.2, $\hat{z}$ is uppersemicontinuous as a correspondence into $L^1(R^\ell)$ for fixed A_0.

<u>Proof</u>: Let $f_i(p,e) = f(p,e,A_0,\omega)$ for $\omega \in A_i$. By Proposition 5.5, f_i is uppersemicontinuous. Define $\hat{z}_i(p,e)$ to be f_i composed with $\tilde{z}$. Then $\hat{z}(p,e,A_0,U) = \sum_{i=1}^{N} \hat{z}_i(p,e) I_{A_i}$ where I_{A_i} is the indicator function of A_i. Let $p_n \to p$ and $e_n \to e$ all in the appropriate domain. Let $z_{in} \in \hat{z}_i(p_n,e_n)$ for each i and n. Then for fixed i, there exists a subsequence of $\{z_{ni}\}_{n,i}$ converging to some element of $\hat{z}_i(p,e)$ in $L^1(R^\ell)$ norm. Extracting a subsequence for each i yields the result.

<u>Proposition 5.7</u>: Suppose the hypotheses of Proposition 5.4 hold. Then for every $\omega \in N^c$, the excess demand correspondence $\hat{z}_{\omega,A_0}$ mapping (p,e) into the set $\{z \in R^\ell \mid z = z(p,e-y,E[U \mid \mathcal{B} \vee A_0](\omega))$ for some $(\mathcal{B},y) \in S(p,e)\}$ is uppersemicontinuous.

<u>Proof</u>: The excess demand function z is continuous in (p,e). Composing it with the correspondence in Proposition 5.4 completes the proof.

Note that the assumption that the consumer's initial information A_0 is a finite partition is crucial. Even if the join operation were continuous (which it is not; see Cotter, 1983, Example 5.1), the proofs of Propositions 5.5 and 5.6 would fail. In Proposition 5.5 the fact that $L^1(R^\ell)$

norm convergence implies almost everywhere convergence for constant random variables was used to obtain convergence on each piece of the partition. If information choice can differ between states in an arbitrary manner, no such result can be used. The problem in the proof of Proposition 5.6 is that the excess demand function $\tilde{z}$ is not continuous in state-dependent information. For arbitary initial information, the resulting information fed into $\tilde{z}$ is state-dependent, which creates a discontinuity as explained in the next section.

Since F^* has no convex structure (see Cotter, 1983, pp. 18 and 33) there exist no elementary conditions that will guarantee that excess demand in Corollary 5.3 is single-valued. In fact, the latter usually fails to even be convex-valued. The problem is that a consumer may be indifferent to two distinct information-cost pairs, each of which leads to distinct excess demands.

All of the above results hold when the Hausdorff topology of information is used, since the statements in Section 4 were proven by Allen (1982, Sections 10 and 11) for that topology. The requirement that the feasible information correspondence S is continuous is a stronger condition when continuity with respect to the Hausdorff metric is required. Thus the statements of this section are weakened.

Some objections may be raised to the requirement that the consumer must choose an expenditure for information in terms of a commodity bundle, rather than in terms of the value of that bundle. Since the price of commodities is known and fixed in advance (to the consumer) it really makes no difference to the consumer how the expenditure is made--the real variable is the value of that expenditure. The results of this section can be restated accordingly.

6. State-dependent prices

So far in this paper the price vector faced by the consumer has been assumed to be nonstochastic. When information is present, however, the excess demand depending on that information is state-dependent. Hence in an economy with information, a different market price for each state of the world must be chosen to clear markets. Unfortunately, Proposition 5.3 does not permit one to choose a state and find a market clearing price and set of excess demands for that state. That result does not imply that for any state the resulting excess demand at that state is uppersemicontinuous. Unless only finitely many choices of information are available to each consumer, computations must be taken in terms of state-contingent commodities (i.e., in $L^1(R^\ell)$) and hence the price space must be taken to be some subset of $L^1(R^\ell)$.

Stochastic prices and initial endowments create no new problems provided they are observed in advance of choice of demand, information is nonstochastic and exogenous, and the consumer does not infer from prices or initial endowments. Let the price and commodity spaces be $L^1(R^\ell)^2$ and define $z'\colon F^* \times L^1(R^\ell_{++}) \times L^1(R^\ell_+) \times \mathcal{D} \to L^1(R^\ell)$ to be $z'(B,p,e,U)(\omega) \equiv \tilde{z}(B,p(\omega),e(\omega),U)(\omega)$.

<u>Proposition 6.1</u>: z' is continuous.

<u>Proof</u>: This is a standard argument using Proposition 4.2 and Chebyshev's inequality. The details are omitted.

Serious problems do arise, however, when stochastic prices are combined with information acquisition. To simplify matters, assume that initial endowment remains nonstochastic. Suppose the consumer observes a state-dependent price (from which no inference is made) and faces the information acquisition problem stated in Section 5. Then the resulting excess demand correspondence given random vector p is $z^*(p,e,U) = \{z \in L^1(R^\ell) | z(\omega) \in \hat{z}(p(\omega),e,U)(\omega)$ for a.e. $\omega\}$. Proposition 5.3 does <u>not</u> imply that z^* is uppersemicontinuous. In face, if p is a constant random vector with (say) $p(\omega) = p_0$ for every ω then it is not even the case that $z^*(p,e,U)$ and $\hat{z}(p_0,e,U)$ are the same! The problem is that with the above definition of z^* the consumer is allowed to make two different actual choices of information at two different states ω and ω' even though $p(\omega) = p(\omega')$. To give the simplest possible example, suppose that at $p_0 \in R^\ell_{++}$ the consumer is indifferent at the information acquisition stage between, say, being informed and being uninformed. Let the corresponding random vectors in $L^1(R^\ell)$ be z_1 and z_2. Then $\hat{z}(p_0,e,U) = \{z_1,z_2\}$ while $z^*(p,e,U) = \{z \in L^1(R^\ell) | z = z_2 + (z_1 - z_2)I_E$ for some $E \in F\}$. Therefore z^* is not compact-valued. The definition of z^* permits random vectors in which the consumer is informed in some states and uninformed in others even though the consumer faces the same choice problem in all states.

In light of the above difficulties, one may believe that z^* should be defined so that for $z \in z^*(p,e,U)$, if $p(\omega) = p(\omega')$ the underlying choices of information are the same. If z^* is restricted in this way, however, then it cannot be uppersemicontinuous, as can be seen by taking a sequence of state-dependent prices converging to a constant price. In that case a number of excess demands "disappear" at the limit.

It is possible for z^* to be upperhemicontinuous in the sense of Hildenbrand (1974, p. 21) though such a result would be useless in the application of fixed-point theorems. Proving upperhemicontinuity directly would require that excess demand as given in Proposition 6.1 be continuous with respect to state-dependent information. The reason is that the optimal choice of information differs in different states of the world. Unfortunately, this does not appear to be the case in any reasonable topology of state-dependent information.

Other problems arise in attempting to demonstrate the existence of a Walrasian equilibrium with information acquisition, unless the set of choices of information is finite. As pointed out before, the excess demand correspondence $\hat{z}$ is not convex-valued. Convexifying the aggregate excess demand correspondence in an economy with a nonatomic space of agents (cf. Aumann, 1966; Hildenbrand, 1974) does not work for an infinite-dimensional commodity space. When the state space is infinite, the commodity space $L^1(R^\ell)$ is infinite-dimensional. Use of the Shapley-Folkman theorem to obtain approximate equilibria does not work does not work for infinite-dimensional spaces either. In addition, the price simplex in infinite-dimensional spaces is not compact in the norm topology.

In short, equilibrium analysis with information acquisition is not possible without assuming either a finite state space or finitely many choices of information. Finite probability spaces, as pointed out in Cotter (1983, p. 29) do not permit interesting specifications of information. It is straightforward to demonstrate existence of an exchange equilibrium in goods with information acquisition and uncountably many nonatomic agents provided that all agents have a finite choice set of information. The idea is to compute a market-clearing price for each state using Proposition 5.4, using the methods given in Aumann (1966). This result does not, however, use any results about the topology of information except for the measure-theoretic remark given at the end of Section 4.

Also, any equilibrium using state-dependent prices in which agents fail to infer from their observations of price is subject to serious objections on rationality as well as implementation grounds. To assume complete ignorance of any price-state relationships by consumers who at the same time are completely knowledgeable of other probabilities and information is at best dubious.

Finally, when prices are stochastic, then it does affect the choice of information problem whether the cost of information is given by consumption bundles or income, particularly when complete contingent markets are not permitted. In that case money may serve a role, which is consistent with the stylized fact that a monetary economy can be supported only if there are arbitrary barriers to trade.

7. Conclusion

In this paper the pointwise convergence metric of information as defined in Cotter (1983) has been applied to some problems of consumer choice under uncertainty. A consumer faces a state-dependent utility function, nonstochastic price and initial endowment, and initial information. The consumer maximizes his/her observed expected utility function conditional on the initial information subject to the observed budget constraint. The resulting excess demand function (as a map into random vectors) is jointly continuous in information, price, initial endowment, and utility. Further, the resulting value of information is continuous. As a result, consumer choice of an information-cost pair from a continuous correspondence prior to choice of demand for goods is well-behaved. The demand correspondence for information is uppersemicontinuous provided the consumer has no more than a finite partition for prior information. Unfortunately, that correspondence is not in general single-valued, meaning that the subsequent demand correspondence for goods is not convex-valued. When prices are stochastic, the demand function with exogenous information is still continuous, but now the demand correspondence with information acquisition is badly behaved.

The continuity results include all of those proven by Allen (1982, Sections 10 and 11) for the Hausdorff topology of information, though she does not consider choice of information. Since the pointwise convergence topology is weaker, it provides a more precise notion of similarity of information in terms of similarity of resulting demand. More study of the relationships between information and resulting economic behavior is needed to refine the economic interpretation of similarity of information as well as to demonstrate the power of an information metric.

The most important applications of an information metric are likely to be in models of endogenously determined information. Consequently, a wide variety of choice problems in the use and production of information require study. Some of these will be considered by this author in forthcoming work, particularly in the optimal production of information and information acquisition about product quality.

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