

EXISTENCE AND FAIRNESS OF VALUE ALLOCATION
WITHOUT CONVEX PREFERENCES

by

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ABSTRACT

Shafer [8] proved that in a finite exchange economy value allocations exist, provided that each agent has convex, complete, transitive, compact and monotone preferences. However, if preferences are not convex, then value allocations may not exist. To remedy this difficulty we enlarge the set of value allocations introducing the concept of approximate value allocations, and show that in a finite exchange economy approximate value allocations exist, even if preferences are not convex, or compact, or monotone. This value existence result can be used to provide a very general value existence theorem for a sequence of finite economies. Further, we show that value allocations do discriminate in favor of or against any coalition of agents.

1. Introduction

Shapley [6] showed that in a finite exchange economy if the set of all attainable utility vectors (i.e., the Pareto set) is convex and compact, then cardinal¹ value allocations exist. It can easily be shown that the above result guarantees the existence of ordinal value allocations as well. However, Kannai and Mantel [4] showed that in a finite exchange economy with convex and monotone preferences for each agent, the Pareto set will not be convex for any utility function choice. Thus, as a consequence, value allocations may not exist. Shafer [8] however, assuming convex and monotone preferences for each agent and using a class of utility functions known as "minimum income functions,"² was able to prove the existence of ordinal value allocations even if the Pareto set was not convex.

It is the purpose of the present paper first to improve further the ordinal value allocations existence theorem by allowing not only for a nonconvex Pareto set but also for nonconvex and nonmonotone preferences for each agent. To this end we enlarge the set of value allocations by introducing the concept of approximate value allocations and show that in a finite exchange economy approximate value allocations exist. Second, we show that value allocations do not satisfy an equity criterion named coalitional fairness of allocations. Although our existence theorem was inspired from the elegant proof given in Shafer [8], we employ much weaker assumptions. In particular, we need not have either convex, or compact, or monotone preferences. The relaxation of these assumptions requires a proof of a novel type. However, we note that the recent nonstandard existence proof for a competitive

equilibrium by Khan-Rashid [3] and its translation by Anderson-Khan-Rashid [1] proved very useful in carrying out the arguments.

The cost that we pay for adopting very weak assumptions is that value allocations do not satisfy the requirement that the aggregate excess demand is zero. However, aggregate excess demand does have an upper bound. We should emphasize that it turns out that the bound depends on the norm of endowments, the square root of the number of agents in the economy, and a constant number of commodities. In fact, since we have no compact preferences the bound we derive is independent of preferences. Consequently, it can be easily computed.

Finally, it should be emphasized that the advantage of considering a finite exchange economy is that it provides a computable bound or error on aggregate excess demand. Consequently, we always know how far approximate value allocations are from value allocations. In contrast, if one considers "large" economies³ as in Shapley-Shubik [7], then the approximate notions are defined with respect to an arbitrary small positive number, i.e., ϵ (epsilon). Thus, all one could say in the context of a large but finite economy is that ϵ is a small positive number which tends to zero as the number of agents in the economy goes to infinity. Consequently, in a large but finite economy, approximate or ϵ -value allocations will tend to value allocations as the economy becomes infinite.⁴ In order to indicate the relationship between the fixed economy formulation and the sequential formulation, we will show how our value existence theorem can be used to provide a very general value existence result for a sequence of finite economies.

Section 2 of the paper presents the model and results; their proofs are concentrated in Section 3.

2. The Model and Results

2.1 Notation

\mathbb{R} denotes the set of real numbers.

\mathbb{R}^{ℓ} denotes the ℓ -fold Cartesian product of \mathbb{R} .

\mathbb{R}_+^{ℓ} denotes the positive orthant of \mathbb{R}^{ℓ} , i.e., $\mathbb{R}_+^{\ell} = \{x \in \mathbb{R}^{\ell} : x \geq 0\}$.

For any x, y in \mathbb{R}^{ℓ} , $x \geq y$ means $x_i \geq y_i$ for all $i=1,2,\dots,\ell$;
 $x > y$ means $x \geq y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for all i .

$e \in \mathbb{R}^{\ell}$ denotes $e = (1,1,\dots,1)$.

$|S|$ denotes the number of elements in the set S .

$\mathcal{P}(A)$ denotes the set of subsets of A .

For any $A \subset \mathbb{R}^{\ell}$, $\text{con } A$ denotes the convex hull of A .

\setminus denotes the set theoretic subtraction.

$$\|x\| = \sum_{i=1}^{\ell} |x_i| .$$

$S(x,r)$ denotes the open ball in \mathbb{R}_+^{ℓ} of radius r centered at x .

For any x, y in \mathbb{R}^{ℓ} , $x \cdot y$ denotes the inner product, i.e.,

$$x \cdot y = \sum_{i=1}^{\ell} x_i y_i .$$

2.2. Definitions

Let P denote the set of complete, transitive, continuous and locally nonsatiated binary relations \succsim on \mathbb{R}_+^l (the commodity space).

A utility function u for a binary relation \succsim on \mathbb{R}_+^l is a real valued function on \mathbb{R}_+^l such that $u(x) \geq u(y) \iff x \succsim y$.

A finite exchange economy E is a mapping of T into $P \times \mathbb{R}_+^l$, where T is a finite set of agents whose cardinality is n , i.e., $|T| = n$. Let \succsim_t be the projection of $E(t)$ onto P and $e(t)$ the projection of $E(t)$ onto \mathbb{R}_+^l . We interpret \succsim_t as the preference of agent t and $e(t)$ as his initial endowment.

A coalition is a subset of the set of agents.

A game with side payments $G = (T, v)$ consists of a finite set of agents T and a superadditive function v of $\mathcal{P}(T)$ into \mathbb{R} such that $v(\emptyset) = 0$.

The Shapley value of the game G assigns to each agent t a value which shows the marginal contribution made by that agent to all the coalitions of which he is a member, i.e., $s_t = \sum_{S \subset T} \frac{(|S|-1)! (|T|-|S|)!}{|T|!} [v(S) - v(S \setminus \{t\})]$. In other words the Shapley value of the same G assigned to an agent is given by summing up his dividends from the marginal value of each coalition in which he is a member.

Given a finite exchange economy E and a family $u = \{u_t\}_{t \in T}$ of utility functions representing the preferences $\{\succsim_t\}_{t \in T}$, define a game (T, v_u) by

$$v_u(S) = \max_{x(t)} \left\{ \begin{array}{l} \sum_{t \in S} u(x(t), t) : \sum_{t \in S} x(t) = \sum_{t \in S} e(t) \quad \text{and} \\ x(t) \in \mathbb{R}_+^l \quad \text{for all } t \in S \end{array} \right\} .$$

A function $x(t)$ of T into \mathbb{R}_+^l is said to be a value allocation for E if and only if:

(i)
$$\sum_{t \in T} x(t) = \sum_{t \in T} e(t)$$

(ii) there exists a family $u = \{u_t\}_{t \in T}$ of utility functions representing the preferences $\{r_t\}_{t \in T}$ such that for all $t \in T$, $u(x(t), t)$ is the Shapley value of the game (T, v_u) .

Hence, $x(t)$ is a value allocation for the finite exchange economy E if first, the aggregate excess demand is zero, and second, it yields to each agent a utility which is equal to his shares of the marginal value of each coalition in which he is a member.

2.3. Approximate Value Allocations

A price system p is a vector in \mathbb{R}^l such that $p \in S = \{x \in \mathbb{R}_+^l : \max x_i = 1, x_i > 0, i=1,2,\dots,l\}$. Let $M(x,p,t)$ be the minimum income function of agent t , (see McKenzie [5]) which allows him to achieve at the price vector p , a commodity combination at least as good as x , i.e., $M(x,p,t) = \min_{x' \in C_x} p \cdot x'$ where $C_x = \{x' \in \mathbb{R}_+^l : x' \succsim_t x\}$. It is known (see for instance McKenzie [5],⁵ or Shafer [8], p. 469) that $M(x,p,t)$ is continuous and represents a utility function for the preference r_t .

Let p be a price system, and consider the game $(T, v(p, \cdot))$ defined by
$$v(p, B) = \max_{x(t)} \left\{ \sum_{t \in B} M(x,p,t) : \sum_{t \in B} x(t) = \sum_{t \in B} e(t) \text{ and } x(t) \in \mathbb{R}_+^l \text{ for all } t \in B \right\}.$$

Denote by $Sh(p,t)$ the Shapley value of agent t of the above game.

A function $x(t)$ of T into \mathbb{R}_+^l is said to be an approximate value allocation for E if there exists price system p such that

(i) $M(x,p,t) = Sh(p,t)$ for all $t \in T$

(ii) $\sum_{i=1}^{\ell} \max \left\{ \sum_{t \in T} x_i(t) - \sum_{t \in T} e_i(t), 0 \right\} \leq (\ell+1) \sqrt{n} \max_{t \in T} \|e(t)\|$.

Condition (i) shows that there is a price system p at which if each trader will minimize his expenditure over his preferred set then his utility is equal to his Shapley value. Condition (ii) says that the aggregate excess demand is bounded above by a bound which depends on the norm of the initial endowments, the square root of the number of agents and a constant number of commodities. Note that the above concept of value allocation is different from the one used in Shafer [8] p. 429. In particular, due to non-convex preferences for all agents the aggregate excess demand is not zero as in Shafer [8], but does have an upper bound. However, if the bound or error $(\ell+1) \sqrt{n} \max_{t \in T} \|e(t)\|$ is zero then both notions coincide.

2.4 Coalitionally Fair Allocations

A function $x(t)$ of T into \mathbb{R}_+^{ℓ} is said to be an allocation if $\sum_{t \in T} x(t) = \sum_{t \in T} e(t)$. Let S_1, S_2 be two disjoint coalitions. An allocation $x(t)$ is coalitionally fair⁶ (c-fair) if there exist no $y(t)$ and $i=1,2$ such that for all $t \in S_i, y(t) \succ_t x(t)$ and $\sum_{t \in S_i} (y(t) - e(t)) = \left[\sum_{t \in S_j} (x(t) - e(t)) \right], j \neq i$. In other words, an allocation is said to be c-fair if no group of agents can redistribute among its members the net trade of any other group of agents and become better off. More simply, an allocation is c-fair if no coalition of agents envies the net trade of any other coalition of agents.

2.5 Results

Theorem 1: An approximate value allocation in E exists.

Theorem 2: A value allocation in E may not be c-fair.

Theorem 3: Let $E_n : T_n \rightarrow P \times \mathbb{R}_+^\ell$ be a sequence of finite economies.

If $|T_n| \rightarrow \infty$ and for any $S_n \subset T_n$ $\frac{|S_n|}{\sqrt{|T_n|}} \rightarrow 0 \Rightarrow \frac{\|\sum_{S_n} e_n(t)\|}{\sqrt{|T_n|}} \rightarrow 0$, then there exist $x_n : T_n \rightarrow \mathbb{R}_+^\ell$ and $p_n \in S$ such that:

$$(i) \quad M(x_n, p_n, t) = Sh(p_n, t) \quad \text{for all } t \in T_n.$$

$$(ii) \quad \frac{1}{|T_n|} \sum_{i=1}^{\ell} \max \left\{ \sum_{t \in T_n} x_n^i(t) - \sum_{t \in T_n} e_n^i(t), 0 \right\} < \frac{1}{\sqrt{|T_n|}} (\ell+1) \max_{t \in T_n} \|e_n(t)\| \rightarrow 0$$

Theorem 2 shows that a value allocation may discriminate in favor of or against some coalitions of agents. In other words, at a value allocation a group of agents can redistribute among its members the net trade of another group of agents and become better off.⁸

Theorem 3 shows the existence of approximate value allocation for a sequence of finite economies provided that initial endowments are integrable, i.e., no "small" group of agents can hold sufficiently large endowments.⁷ Conclusion (i) is the same as that in Theorem 1. Conclusion (ii) shows that as the number of agents goes to infinity, the average excess demand goes to zero. Consequently, the average demand approximately balances the average supply with the approximation getting better the larger the finite economy.

3. Proofs

Proof of Theorem 1: In order to prove Theorem 1 we compactify the excess demand space by a bound which depends on the number of agents in the economy and the initial endowments; and we let the price space vary with the number of

agents in the economy. It should be noted that since our bounds are chosen independently of the degree of nonconvexity of preferences, the approximating procedure will give rise to a computable error on aggregate excess demand.

Note first that, $M(x, p, t) = p \cdot x$ whenever p is normal to a plane of support of C_x . Moreover, by individual rationality,⁹ $Sh(p, t) \geq M(e(t), p, t) \geq 0$ for all $t \in T$.

Consider the set valued functions:

$$H(p, t) = \{x \in \mathbb{R}_+^\ell : p \cdot x \leq Sh(p, t)\}$$

$$D(p, t) = \{x \in \mathbb{R}_+^\ell : x \in H(p, t) \text{ and } x \text{ is maximal for } \sum_t \text{ in } H(p, t)\}.$$

Let,

$$e = \sum_{t \in T} e(t)$$

$$x = \sum_{t \in T} x(t)$$

$$D(p) = \sum_{t \in T} D(p, t)$$

$$\zeta(p) = D(p) - e$$

$$Z = \mathbb{R}_+^\ell - \{e\}$$

$$z = x - e.$$

Obviously, z is a point in $\zeta(p) \subset Z$.

Let,

$$Z' = \left\{ z \in Z : \|z\| \leq n \max_{t \in T} \|e(t)\| \right\}. \quad (1)$$

Define

$$\Delta = \left\{ p \in S : p_i \geq \frac{1}{\sqrt{n}}, \quad i=1, 2, \dots, \ell \right\}$$

$$\varphi(z) = \{p \in \Delta : p \cdot z = \max_{q \in \Delta} q \cdot z \text{ for all } q \in \Delta\}$$

$$\Theta(p, z) = \varphi(z) \times \text{con} \zeta(p) .$$

Thus, Θ is a set valued function from the set $\Delta \times Z'$ into itself. Standard arguments can be applied to show that the mapping Θ satisfies all the properties of the Kakutani fixed point theorem. Consequently, there exist $(p, z) \in \Theta(p, z)$, i.e., $z \in \text{con} \zeta(p)$ and for all $q \in \Delta$, $q \cdot z \leq p \cdot z \leq 0$. By the Corollary of the Shapley-Folkman theorem (Starr [9], p. 35) we can write:

$$z = \sum_{t \in T'} x(t) + \sum_{i=1}^{\ell} x'(t_i) - \sum_{t \in T} e(t) \text{ where } T' = T \setminus \{1, 2, \dots, \ell\} \text{ and}$$

$$x(t) \in D(p, t) \text{ for all } t \in T' \text{ and } x'(t_i) \notin D(p, t_i) \text{ for all } i=1, \dots, \ell .$$

Pick $x(t_i)$ so that $x(t_i) \in D(p, t_i)$ for all $i=1, 2, \dots, \ell$.

Then,

$$\begin{aligned} z(t_i) - z'(t_i) &= (x(t_i) - x'(t_i)) - (e(t_i) - e'(t_i)) \\ &= z(t_i) + e(t_i) - (z'(t_i) + e(t_i)) \\ &\leq z(t_i) + e(t_i) . \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=1}^{\ell} \max \{z_j(t_i) - z'_j(t_i), 0\} &\leq \sum_{j=1}^{\ell} (z_j(t_i) + e_j(t_i)) \\ &\leq \frac{p \cdot e(t_i)}{\min(p_1, \dots, p_{\ell})} \\ &\leq \sqrt{n} \|e(t_i)\| . \end{aligned} \quad (2)$$

Moreover,

$$q \cdot \left[\sum_{t \in T} x(t) - \sum_{t \in T} e(t) \right] = q \cdot \left[\sum_{i=1}^{\ell} x(t_i) - \sum_{i=1}^{\ell} e(t_i) \right] - q \cdot \left[\sum_{i=1}^{\ell} x'(t_i) - \sum_{i=1}^{\ell} e'(t_i) \right] + q \cdot z \quad (3)$$

From (2) and (3) it follows that for all $q \in \Delta$

$$q \cdot \left[\sum_{t \in T} x(t) - \sum_{t \in T} e(t) \right] \leq l \sqrt{n} \max_{t \in T} \| e(t) \| \quad (4)$$

Let

$$q^i = \begin{cases} 1 & \text{if } \left(\sum_{t \in T} x(t) - \sum_{t \in T} e(t) \right)^i > 0 \quad i=1, \dots, l \\ \frac{1}{\sqrt{n}} & \text{if } \left(\sum_{t \in T} x(t) - \sum_{t \in T} e(t) \right)^i \leq 0 \quad i=1, \dots, l \end{cases} \quad (5)$$

Substituting (5) into (4), we have

$$\begin{aligned} q \cdot \left[\sum_{t \in T} x(t) - \sum_{t \in T} e(t) \right] &= \sum_{\text{some } i} \left[\sum_{t \in T} x_i(t) - \sum_{t \in T} e_i(t) \right] \\ &\quad + \frac{1}{\sqrt{n}} \sum_{\text{some } i} \left[\sum_{t \in T} x_i(t) - \sum_{t \in T} e_i(t) \right] \\ &\leq l \sqrt{n} \max_{t \in T} \| e(t) \| \quad . \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^l \max \left\{ \sum_{t \in T} x_i(t) - \sum_{t \in T} e_i(t), 0 \right\} &\leq l \sqrt{n} \max_{t \in T} \| e(t) \| - \frac{1}{\sqrt{n}} \sum_{\text{some } i} \left[\sum_{t \in T} x_i(t) \right. \\ &\quad \left. - \sum_{t \in T} e_i(t) \right] \\ &= l \sqrt{n} \max_{t \in T} \| e(t) \| + \frac{1}{\sqrt{n}} \sum_{\text{some } i} \left[\left| \sum_{t \in T} x_i(t) \right. \right. \\ &\quad \left. \left. - \sum_{t \in T} e_i(t) \right| \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \ell \sqrt{n} \max_{t \in T} \| e(t) \| + \frac{1}{\sqrt{n}} \sum_{i=1}^{\ell} \left[\left| \sum_{t \in T} x_i(t) \right| \right. \\
 &\quad \left. - \left| \sum_{t \in T} e_i(t) \right| \right] \\
 &= \ell \sqrt{n} \max_{t \in T} \| e(t) \| + \frac{1}{\sqrt{n}} \left\| \sum_{t \in T} x(t) - \sum_{t \in T} e(t) \right\| \\
 &= \ell \sqrt{n} \max_{t \in T} \| e(t) \| + \frac{1}{\sqrt{n}} \| z \| \quad . \quad (6)
 \end{aligned}$$

Taking into account (1), (6) can be written as follows:

$$\sum_{i=1}^{\ell} \max_{t \in T} \left\{ \sum_{t \in T} x_i(t) - \sum_{t \in T} e_i(t), 0 \right\} \leq (\ell+1) \sqrt{n} \max_{t \in T} \| e(t) \| ,$$

which is condition (ii) of approximate value allocation for \bar{E} . Observe that since $x(t_i)$ is chosen arbitrarily to belong in the set $D(p, t_i)$ for all $i=1, 2, \dots, \ell$, then $x(t) \in D(p, t)$ for all $t \in T$. Since $x(t) \in D(p, t) \subset H(p, t)$ then $p \cdot x(t) \leq Sh(p, t)$. Suppose that there exists $r > 0$ such that $Sh(p, t) - p \cdot x(t) = r$. By local nonsatiation there exists x' in $S(x(t), r) \cap \mathbb{R}_+^{\ell}$ such that $x' \in H(p, t)$ and $x' \succ_t x(t)$, but this contradicts maximality of $x(t)$ in $H(p, t)$. Hence, $p \cdot x(t) = Sh(p, t) = M(x, p, t)$ which is condition (i) of the definition of approximate value allocation for \bar{E} . This completes the proof of the

Proof of Theorem 2: Consider an exchange economy with three agents and two commodities denoted by $T = \{1, 2, 3\}$ and x, y respectively. Their utility functions and their initial endowments are given as follows:

$$u_1(x_1, y_1) = \frac{1}{2} x_1 + \frac{1}{2} y_1, \quad e_1 = (0, 0)$$

$$u_2(x_2, y_2) = \left(\frac{1}{2} \sqrt{x_2} + \frac{1}{2} \sqrt{y_2} \right)^2, \quad e_2 = (0, 4)$$

$$u_3(x_3, y_3) = \left(\frac{1}{2} \sqrt{x_3} + \frac{1}{2} \sqrt{y_3} \right)^2, \quad e_3 = (4, 0)$$

A computation of the characteristic function v_u gives:

$$v_u(\{1\}) = 0$$

$$v_u(\{2\}) = v_u(\{3\}) = 1$$

$$v_u(\{1, 2\}) = 2$$

$$v_u(\{1, 3\}) = 2$$

$$v_u(\{2, 3\}) = v_u(\{1, 2, 3\}) = 4.$$

Moreover, we can compute the Shapley value of agent 1 and find that $Sh_1 = \frac{2}{6}$.

Note that by group rationality we have that $v(\{1, 2, 3\}) = \sum_{i \in T} Sh_i = 4$. Since agents 2 and 3 have identical utility functions, they are treated symmetrically

and thus we have that $Sh_2 = Sh_3 = \frac{11}{6}$. The value allocation is

$(x_1, y_1) = \left(\frac{2}{6}, \frac{2}{6} \right)$, $(x_2, y_2) = (x_3, y_3) = \left(\frac{11}{6}, \frac{11}{6} \right)$. However, the above value

allocation is not c-fair. In fact, note that for $S_1 = \{1\}$, $S_2 = \{2, 3\}$ the

assignment $(\bar{x}_1, \bar{y}_1) = (0, 0)$, $(\bar{x}_2, \bar{y}_2) = (\bar{x}_3, \bar{y}_3) = \left(\frac{13}{6}, \frac{13}{6} \right)$ makes each member

of coalition S_2 better off by obtaining the net trade of coalition S_1 , i.e.,

$u_i(\bar{x}_i, \bar{y}_i) = u_i\left(\frac{13}{6}, \frac{13}{6}\right) = \frac{13}{6} > u_i(x_i, y_i) = u_i\left(\frac{11}{6}, \frac{11}{6}\right) = \frac{11}{6}$ for $i \in S_2$ and

$(\bar{x}_2, \bar{y}_2) - (0, 4) + (\bar{x}_3, \bar{y}_3) - (4, 0) = (x_1, y_1) - (0, 0) = \left(\frac{2}{6}, \frac{2}{6} \right)$.

Proof of Theorem 3: Apply Theorem 1.

Footnotes

¹See [8] and its reference for a distinction between ordinal and cardinal value allocations.

²This will be defined in Section 2.3.

³i.e., sequences of finite economies.

⁴Alternatively we could say that as the number of agents in the economy goes to infinity, the aggregate excess demand tends to zero.

⁵McKenzie [5] in footnote 2 shows that $M(x,p,t)$ is concave in p and consequently, continuous in p .

⁶This concept of c-fair allocations was explored in [2] and [9] where we refer the reader to it's relationship to the literature on fairness.

⁷The integrability assumption can be weakened to $\frac{\max_{t \in I_n} \|e_n(t)\|}{\sqrt{|I_n|}} \rightarrow 0$. The latter assumption rules out dominant agents, i.e., monopolists or oligopolists, however, it allows for dominant classes of individuals.

⁸It should be remarked that the competitive equilibrium is c-fair (see [2]).

⁹Individual rationality and group rationality are two of the properties of the Shapley Value, i.e., $Sh(p,t) \geq M(e(t),p,t)$ and $\sum_{t \in I} Sh(p,t) = v(p,I) \leq p \cdot \sum_{t \in I} e(t)$, see also Shafer [8], p. 474.

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