THE INCENTIVE IMPLICATIONS OF INCOMPLETE INSURANCE: THE MULTIPLICATIVE CASE

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ABSTRACT

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Suppose that a fair insurance policy is available to risk-averse economic agents contingent on only "observable" variables. The risk-averse agents will purchase incomplete insurance to maximize their expected utility. The first order condition is typically an equality of expected "marginal" utility, whether or not weighted by the level of income, conditional on observable states. It is interesting to know how the "levels" of utility are ranked among these states given the first order conditions. We will show that the ranking of the "level" of expected utility depends on the degree of risk aversion. Suppose an implicit contract model with severance payments. Workers are laid off with a fixed known probability. When a worker is laid off, he is paid severance payments and released for a search of new employment. Whether he is reemployed or self-employed or how much he is earning is unverifiable by the original employer, so that severance payments cannot be contingent on income after layoff. In this sense, severance payments are incomplete insurance.

Suppose that income after layoff is proportional to severance payments with the proportion being stochastic. The first order condition is given as an equality of an expected marginal utility of income after layoff weighted by that income to a marginal utility of wages for a retained worker weighted by the amount of severance payments. For example, when the relative risk aversion is more than one and constant, and the mean of proportion of yield on severance payments is less than one, the utility of the retained is larger than the expected utility of the laid off. Other cases are worked out, too. This claim is proved using the following theorem. Provided that the relative risk aversion is greater than one, the "level" of utility of sure income is greater (less) than the expected utility, iff the utility function is of increasing (decreasing, respectively) relative risk aversion. The result is reverse for the case that the degree of risk aversion is less than one. All assertions are rigorously proved.

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I. INTRODUCTION

There have been numerous contributions to economic theory of uncertainty and insurance. Despite well-developed theory in risk aversion in a complete market framework, economic behavior in a incomplete market case has not been investigated in a sufficient depth.

Suppose that a risk-averse agent faces different amounts of income for different states of nature. It is well-known that optimal risk-sharing between a risk-averse agent and a risk-neutral agent (insurance company) results in equalization of net income (income after premium and coverage) across different states of nature, given that all states of nature are verifiable, i.e. that markets are complete. Suppose now that a subset of states of nature are not verifiable individually, but only as a group of states, by an insurance (risk-neutral) agent. If uncertainty is additive to insurance coverage, then an optimal insurance policy is such that the expected <u>marginal</u> utility of net income over a group of unverifiable states of nature is equal to the <u>marginal</u> utility of the net (sure) income at a verifiable state of nature. If uncertainty is multiplicative, then an optimal insurance policy would equalize the expected <u>marginal</u> utility weighted by the multiplicative stochastic yield to the <u>marginal</u> utility of a sure income at a verifiable state. It is of a great interest to study a property of an optimal incomplete insurance policy. Especially it is important to compare the level of utility of a verifiable state to the level of expected utility of a group of unverifiable states, because moral hazard becomes an important problem in the case where the utility of the laid-off is larger than that of retained. Which level of (expected) utility is higher depends crucially upon the behavior of a degree of risk aversion. A preliminary result in this project, reported in Imai, Geanakoplos and Ito (1981), showed that in the case of additive uncertainty whether the absolute risk aversion is increasing or decreasing is a deciding factor. The purpose of this paper is to demonstrate that in the case of multiplicative uncertainty not only the relative risk aversion is increasing or decreasing but also the size of the degree of risk aversion decide which if the two levels of (expected) utility is large.

In this paper, a role of severance payments in a model of implicit contracts is studied as an example of incomplete insurance. Since severance payments usually does not depend on outcomes at alternative opportunities after layoff, they are considered at best incomplete insurance for layoff. Suppose that if a worker is not laid off, then he is paid at a sure (deterministic) wage. On the other hand, if a worker is laid off, severance payments are made to him. When a worker leaves the firm, he proceeds to search an opportunity at other firms. A worker with such a contract is covered against risk of layoff, but not against risks over alternative opportunities outside the originally-contracted firm. The first order condition is an equality between the expected marginal utility of the case of layoff and the marginal utility of the case of retention. The problem can be viewed as a two-stage lotteries with fair insurance available only for the first stage. Several results concerning comparative statics for this incomplete insurance are demonstrated. A crucial question, however, is whether an expected utility of a laid off worker is greater or less than that of a retained worker, given the optimal amount of severance payments. If the former is the case, there is an apparent problem of moral hazard among workers. In this paper, a probability of layoff is treated as fixed, and severance payments are not paid for voluntary quits. Hence, moral hazard is ruled out in our model, but expected to be explored in future research. The model and theorems in this paper has a general property of incomplete insurance, so that other interpretations and applications can be suggested.

II. THE MODEL

Let us suppose a firm announces to layoff randomly a certain proportion, (1-p), of its labor force. The rest are retained and paid the sure (deterministic) wage y. Let us assume that the probability of retention, p, is fixed and known. Each worker has an identical, monotone increasing, strictly-concave, and thrice-continuously-differentiable von Neumann-Morgenstern utility function, $u: R_+ \to R$. For all $y \in R_+$, $u'(y) > 0$, and $u''(y) < 0$. A laid-off worker proceed to search an alternative opportunity and may be rehired at another firm. The rehiring wage, $\tilde{y}$, is stochastic before the search. Its probability distribution is identical for and known to all workers. Suppose that the firm is risk-neutral and willing to offer fair insurance contingent on verifiable state of nature for its workers. (We may assume an outside insurance company or a labor union instead of the firm which arrange a fair insurance.) By fair insurance, we mean a contingent contract with an expected payoff being equal to zero. A decision of layoff or retention is observable and verifiable by both parties. Therefore, the firm agrees to arrange severance payments to a laid-off worker. However, fair insurance does not depend on a rehiring wage, because it can not be easily observable and verifiable by the original employer. Moreover, if the insurance were dependent on rehiring wage, it causes a moral hazard in worker's search in alternative employment. As Hall and Lilien (1979) observed, severance payments rarely depends on events outside the firm. We will investigate properties of an optimal severance payments as incomplete insurance of worker's net income.

Two specifications of stochastic alternative wages are possible. The first case is that a distribution of rehiring wage is independent of the size of severance payments. In other words, uncertainty is "additive" to severance payments. The net (stochastic) income of a laid-off worker can be written as $c + \tilde{w}$, where $\tilde{w}$ is a rehiring wage at another firm. This case has been analyzed in Imai, et.al. (1981). It was shown there that which level of utility is larger depends on whether the degree of risk aversion is increasing or decreasing. This case is applicable, if search process does not require a substantial monetary input, but rather a luck of finding an opportunity. However, if a laid-off worker has to use a part of severance payments to move and to search for a new employer, or to start up his own new business, then a distribution of rehiring wages may depend on severance payments. We specify in this paper in a way that the net income of a laid-off worker is proportional to severance payments made at the time of layoff, with that proportion being stochastic: $\tilde{y} = \tilde{r}c$. In other words, uncertainty is "multiplicative" to the severance payment.

An extensive form of our model is illustrated in Figure 1. For the first stage of uncertainty, fair insurance is available. This is shown in payments of c for the upper branch and of $(-(1-p)c/p)$ for the lower branch. The payment for the upper branch is interpreted as severance payments as insurance coverage for layoff, and the cost for the lower branch is interpreted as an insurance premium for the above-mentioned insurance. The net income for a retained worker is a sure wage minus the insurance premium, $y - (1-p)c/p$. If a worker is laid off, i.e., at the upper branch, he is exposed to the second-stage uncertainty which is not observable or verifiable by a risk-neutral agent (an insurance company). This net income, $\tilde{y}$, is proportional to severance payments. The level of optimal severance payments is determined by maximizing an ex ante (before the first-stage uncertainty is known) expected utility.

INSERT FIGURE 1 about here

A goal of this paper is to examine properties of optimal severance payments for proposed implicit contract models. One of the questions asked below is whether the level of utility in the retained worker, given an optimal severance payments, is greater than the level of expected utility of the laid-off worker before search. This question is posed by Geanakoplos and Ito (1982) for the case of additive uncertainty, and the question about the level of expected utility is answered in Imai, Geanakoplos and Ito (1981) without an extensive discussions. In the rest of this paper is to prove a theorem for the case of multiplicative uncertainty with extensive discussions on properties of optimal contracts in this environment.

III. RESULTS

Now let us investigate a case of multiplicative uncertainty. An expected utility before the layoff decision is $Z = pu(y-(1-p)c/p) + (1-p)Eu(\tilde{r}c)$. Maximizing Z with respect to severance payments c, we derive the following first order condition.

$$(3.1) \quad u'(y - qc) = E\tilde{r}u'(\tilde{r}c),$$

where $q = (1-p)/p$. Or, multiplying c on the both sides of the above equation,

$$(3.2) \quad cu'(y - qc) = E\tilde{r}cu'(\tilde{r}c).$$

It is easy to find a conditions on $u'(0)$ and on the range of r such that there exists c which satisfies (3.1). Given it exists, it is unique, because the left hand side of (3.1) is increasing in c and the right hand side of (3.1) is decreasing in c.

Now let us investigate comparative statics of the optimal level of severance payments. To that end, it is convenient to decompose a stochastic yield, $\tilde{r}$, into a mean and disturbances around the mean.

$$\tilde{r} = \bar{r} + \tilde{\theta}, \quad E\,\tilde{\theta} = 0.$$

Moreover, define a special case of a mean preserving spread as follows:

$$\tilde{\theta} = \begin{cases} \theta & \text{with probability } 1/2 \\ -\theta & \text{with probability } 1/2. \end{cases}$$

First, let us check an effect of an increase of the mean yield on the optimal level of severance payments. Using the implicit function theorem, as usual, and the first-order condition, we obtain the following expression.

$$(3.3) \qquad \frac{dc^*}{d\bar{r}} = \frac{E[u'(\tilde{r}c)(1-R_u(\tilde{r}c)]}{-\,q\,u''(y-qc) - E\tilde{r}u''(\tilde{r}c)}$$

Hence, we have:

if for all x, $R_u(x) \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} 1$, then $dc^*/d\bar{r} \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} 0$, respectively.

where R_u is the Arrow-Pratt relative risk aversion:

$$R_u(x) = -\,xu''(x)/u'(x).$$

Second, let us investigate an effect of a mean preserving spread on the optimal level of severance payments. Rewriting the first order condition for the special case of a mean preserving spread defined above,

$$(3.4) \qquad u'(y-qc) = [(\bar{r}+\theta)u'((\bar{r}+\theta)c) + (\bar{r}-\theta)u'(\bar{r}-\theta)c)]/2.$$

Using the implicit function theorem, we have the following:

$$(3.5) \qquad \frac{dc^*}{d\theta} = - \frac{u'((\bar{r}+\theta)c)(1-R_u((\bar{r}+\theta)+c)) + u'((\bar{r}-\theta)c)(1-R_u((\bar{r}-\theta)c)}{2qu''(y-qc) + (\bar{r}+\theta)^2 u''((\bar{r}+\theta)c) + (\bar{r}-\theta)^2 u''((r-\theta)c)}$$

Therefore, knowing $u'(x) > 0$, and $u''(x) < 0$, for all x;

$$\text{if for all } x,\ R_u(x) \begin{matrix} > \\ = \\ < \end{matrix} 0, \text{ then } dc^*/d\theta \begin{matrix} > \\ = \\ < \end{matrix} 1, \text{ respectively.}$$

Now we are interested in knowing whether the optimal severance payments exceeds an amount of the net (sure) income of the employed. Namely whether c^* or $(y-qc)$ is greater is a question. It may seem too large an amount of severance payments if it is larger than the net (sure) income. However, this is perfectly plausible, because severance payments are relevant only for the case of layoff with probability $(1-p)$ which may be very small. This question is not only interesting itself but also important for answering a crucial question whether the retained or the laid off has a higher (expected) utility given the optimal severance payments. Two steps are needed to answer this question. First, consider the case of certainty; and second add an effect of a mean preserving spread. Suppose that $\theta = 0$ is always the case. Then the first order condition is rewritten as

$$(3.6) \qquad c^*u'(y-qc^*) = \bar{r}c^*u'(\bar{r}c^*).$$

Recall that $g(x)=xu'(x)$ is increasing, constant, or decreasing, dependent upon the degree of relative risk aversion is greater than, equal to, or smaller than one, respectively. Therefore, defining $\bar{r}c = x$,

$$\frac{d(RHS(3.6))}{d(\bar{r}c)} \begin{matrix} < \\ = \\ > \end{matrix} 0, \quad \text{iff, for all } x,\ R_u(x) \begin{matrix} > \\ = \\ < \end{matrix} 1, \text{ respectively.}$$

Now suppose the case 1, where $R_u(x) > 1$, and $\bar{r} > 1$. By $\bar{r} > 1$, we know immediately $\bar{r}c > c$. By $R_u(x) > 1$, we know $g'(x) < 0$, which implies in relation to (3.6),

(3.7) $\bar{r}cu'(\bar{r}c) < cu'(c)$.

Replace the left hand side of (3.7) by the left hand side of (3.6):

$$cu'(y-(1-p)c/p) < cu'(c).$$

Dividing the both sides by $c > 0$,

$$u'(y-(1-p)c/p) < u'(c).$$

Since the marginal utility is a decreasing function,

$$(y-(1-p)c/p) > c,$$

that is,

(3.8) $py > c$.

Let us consider the second case: $R_u > 1$ and $\bar{r} < 1$. This implies

$$\bar{r}c < c.$$

Then all the inequalities in (3.7) through (3.8) become opposite.

In the third case: $R_u(x) < 1$ and $\bar{r} > 1$, it implies $\bar{r}c > c$. But now $g'(x) > 0$, so that all the inequalities in (3.7) through (3.8) become opposite, again.

As it may be easily guessed, in the fourth case:

$$R_u(x) < 1 \text{ and } \bar{r} > 1,$$

all the inequalities in (3.7) through (3.8) are holding true.

Lastly, if $R_u(x) = 1$, and/or $\bar{r} = 1$, then all the inequalities above become equalities. In summary, we have a classification of TABLE 1a, as for the question of whether severance payments is greater or smaller than net (sure) income.

Remember that the TABLE 1a is derived for the certainty case. Now let us consider the second effect, i.e., an effect of uncertainty. For this exercise, it is enough to recall the result for the mean preserving spread.

That is,

$$\frac{dc^*}{d\theta} \gtreqless 0, \quad \text{iff for all } x,\ R_u(x) \gtreqless 1, \text{ respectively.}$$

This uncertainty effect reinforces the case of certainty only in the case of $\bar{r} < 1$. If $\bar{r} = 1$, then $c^* > py$ for $R_u(x) > 1$, and vice versa. Note that for the case of $\bar{r} > 1$, the uncertainty effect and the case of certainty cancel each other, so that the total effect is ambiguous. The combined case is summarized in TABLE 1b.

INSERT TABLE 1a and 1b about here

By continuity of functions involved, we can show that $\bar{r}^*$ such that $c = py$ has the following properties. As uncertainty becomes small, i.e. $\theta \to 0$, $\bar{r}^*$ must converge to one. If $R_u(x) > 1$ for all x, and it becomes larger everywhere, then $\bar{r}^*$ becomes further away from one. By the same argument, if $R_u(x) < 1$, and it becomes smaller everywhere, then $\bar{r}^*$ becomes further away from one.

Finally, we are ready to answer a question whether a laid off worker is better or worse off in the expected utility sense than a retained worker in the multiplicative uncertainty case. As we have seen in answers to other questions, whether the degree of relative risk aversion is greater of smaller than one makes a crucial difference. It is also important to classify whether c^* is greater or smaller than py.

We start by the case where $c^* > py$, that is, $c^* > (y-qc)$. Recall that this is likely if $R_u(x) > 1$ and $\bar{r} < 1$, or if $0 < R_u(x) < 1$, and $\bar{r}$ is sufficiently larger than one. Multiplying $u'(.)$ on both sides of $c > (y-qc)$, we have

$$(3.9) \qquad c^* u'(y-qc^*) \; > \; (y-qc^*)u'(y-qc^*).$$

By substituting the first order condition, (3.2) into the left hand side of (3.9), it is shown that

$$(3.10) \qquad E\tilde{r}c^* u'(\tilde{r}c^*) \; > \; (y - qc^*)u'(y - qc^*).$$

Suppose that there exists $x^\dagger$ such that

$$(3.11) \qquad E\tilde{r}c^* u'(\tilde{r}c) = x^\dagger u'(x^\dagger).$$

From (3.11) and (3.10),

$$(3.12) \qquad x^\dagger u'(x^\dagger) \; > \; (y-qc^*)u'(y-qc^*).$$

Now we come to the point to differentiate sub-cases, depending on the size of the degree of relative risk aversion. Recall that if for all x, $R_u(x) > 1$, then $g'(x) < 0$. Then applying this relationship to (3.12),

$$x^\dagger < y - (1-p)c/p.$$

On the other hand, if for all x, $0 < R_u(x) < 1$, then $g'(x) > 0$, i.e.,

$$x^\dagger > y - (1-p)c/p.$$

In sum, we have the following relationships:

(3.13a) If for all x, $R_u > 1$, then $u(x^\dagger) < u(y-qc^*)$, and

(3.13b) If for all x, $0 < R_u(x) < 1$, then $u(x^\dagger) > u(y-qc^*)$.

We have derived several relationships between three levels of income. The retained received $(y - qc)$, and the laid-off received $\tilde{r}c^*$. We want to compare utility levels of these two levels of income. This can be done in two steps. Equations (3.13a) and (3.13b) give the relation between utility levels of $x^\dagger$ and $(y - qc)$. The missing link is the relationship between two (expected) utility levels of $x^\dagger$ and $\tilde{r}c^*$. For this purpose, we use the following theorem, which will be proved in the next section.

<u>THEOREM</u>

(3.14a) If for all x, $R_u(x) > 1$, then for any $\tilde{r}$,

$u(x^\dagger) \gtreqless Eu(\tilde{r}c^*)$, if and only if, for all x, $R'_u(x) \gtreqless 0$, respectively.

(3.14b) If for all x, $R_u(x) < 1$, then for all $\tilde{r}$,

$u(x^\dagger) \lesseqgtr Eu(\tilde{r}c^*)$, if and only if, for all x, $R'_u(x) \gtreqless 0$, respectively. ■

Combining (3.13a), (3.13b), (3.14a) and (3.14b), we have a relationship between $Eu(\tilde{r}c^*)$ and $u(y-qc^*)$:

(3.15a) Suppose for all x, $R_u(x) > 1$: Then

(i) if for all x, $R'_u(x) \leqslant 0$, then $Eu(\tilde{r}c^*) \leqslant u(x^T) < u(y-qc^*)$,

(ii) if for all x, $R'_u(x) > 0$, then it is ambiguous whether $Eu(\tilde{r}c^*)$ is greater or smaller than $u(y-qc^*)$, because both are greater than $u(x^T)$.

(3.15b) Suppose for all x, $0 < R_u(x) < 1$. Then

(i) if for all x, $R'_u(x) \geqslant 0$, then $Eu(\tilde{r}c^*) \geqslant u(x^T) > u(y-qc^*)$, and

(ii) if for all x, $R'_u(x) < 0$, then whether $Eu(\tilde{r}c^*)$ is greater of smaller then $u(y-qc^*)$ is ambiguous because both are smaller than $u(x^T)$.

For the case of $c^* < (y-(1-p)c/p)$, i.e., $c^* < py$, appropriate inequalities should be reversed, but a logic goes in the same way. Ambiguous cases occur in the case of increasing relative risk aversion: $R_u'(x) > 0$.

Lastly, if $c^* = py$, then $c^*u'(y-qc^*) = (y-qc^*)u'(y-qc^*)$.

Therefore, applying the first order condition,

$$E\tilde{r}c^{*}u'(\tilde{r}c^{*}) = (y-qc^{*})u'(.).$$

The above THEOREM, with condition $\tilde{x} = \tilde{r}c^{*}$, implies

$$x^{\dagger} = y - qc^{*}.$$

Therefore,

(3.16a) Suppose $R_{u}(x) > 1$. Then, for any $\tilde{r}$,

$$u(x^{\dagger}) \begin{matrix} > \\ = \\ < \end{matrix} Eu(\tilde{r}c^{*}), \text{ if and only if, for all } x, \; R'_{u}(x) \begin{matrix} > \\ = \\ < \end{matrix} 0, \text{ respectively;}$$

(3.16b) Suppose $0 < R_{u}(x) < 1$. Then, for any $\tilde{r}$,

$$u(x^{\dagger}) \begin{matrix} < \\ = \\ > \end{matrix} Eu(\tilde{r}c^{*}), \text{ if and only if, for all } x, \; R'_{u}(x) \begin{matrix} > \\ = \\ < \end{matrix} 0, \text{ respectively.}$$

Suppose that the degree of relative risk aversion is equal to one for all x, namely it is constant relative risk aversion: $R_{u}'(x) = 0$. Then $u(x^{\dagger})$ is equal to $Eu(\tilde{r}x^{*})$. Note that $x^{\dagger}$ is uniquely determined by the first order condition.

Now we are in position to summarize the results as TABLE 2.

INSERT TABLE 2 about here

Let us consider whether this multiplicative uncertainty case has a moral hazard problem like the one in the additive uncertainty case. First, we have to know a reasonable estimate of the degree and derivative of relative risk aversion. Although many economists agree that the absolute risk aversion is decreasing, such consensus is not reached for the relative risk aversion. The degree of risk aversion seems to be more than one, i.e. $R_u(x) > 1$ for all x, according to Grossman and Shiller (1981) and Friend and Blume (1975).[1] In Friend and Blume (1975) and Blume and Friend (1975), they claim that their findings are consistent with decreasing or constant relative risk aversion. Cohn, Lewellen, Lease, and Schlarbaum (1975) conclude the relative risk aversion decline as wealth increase across households. However, Siegel and Hoban (1982) have recently issued a caution on the above-mentioned studies. The last finding suggests that the relative risk aversion may be increasing in the less wealthy families, and decreasing in the more wealthy ones, where wealth includes human capital as well as financial assets. These studies are not conclusive not only because they differ in conclusions but also because they are cross-sectional studies instead of, say panel data analysis. But this is not a paper of empirical analysis. All it is desired here is a reasonable estimate of the degree and derivative of relative risk aversion. It seems to us that the degree of relative risk aversion is greater than one, and it may be decreasing or constant, or at least not strongly increasing. Suppose also that the mean of the stochastic yield on severance payments is larger than one, i.e., $\bar{r} > 1$. This assumption is very plausible if the laid of is allowed to keep money and not goes on to costly search. The last assumption implies, if $R_u(x)$ is larger than one not too much or disturbances around the mean of stochastic yield does not change the qualitative result too much, that $c^* < py$. After all, a combination of these assumptions derived from evidence point us to the case of lower right hand corner of the upper block in TABLE 2, the corner being enclosed by bold lines. Namely, with the optimal severance payments, the laid off is better off than the retained at the point of layoff and before search of rehiring: $Eu(\tilde{r}c) > u(y-qc)$.

IV. PROOF

In this section we give a proof of a theorem in the preceding section. First, we define the degree of absolute risk aversion of a utility function u, in the sense of Pratt (1964) and Arrow (1965): $A_u(x) = -u''(x)/u'(x)$. Recall that the relative risk aversion for utility function u is defined by $R_u(x) = -xu''(x)/u'(x)$. We would like to investigate different implications of utility functions with different degrees of risk aversion. Let us denote another von Neumann-Morgenstern utility function of class C^3 is denoted by v: $R_+ \to R$. For all $x \in R_+$, it is true that $v'(x) > 0$, but not necessarily the concavity. Now, let us define the certainty equivalence level of $\tilde{x}$ with respect to a utility function by $\underline{x}^u$: namely, $Eu(\tilde{x}) = u(\underline{x}^u)$. A relationship between $\underline{x}^u$ and a risk premium, π, defined in Pratt (1964) can be easily derived as follows: $\pi_u = \bar{x} - \underline{x}^u$, where $\bar{x}$ is the mathematical mean of $\tilde{x}$. Let us define a weighted marginal utility, g: $g(x) \equiv xu'(x)$. The "certainty equivalence" level of g with respect to $\tilde{x}$ is defined by $x^{\dagger}$: $x^T u'(x^T) = E\tilde{x}u(\tilde{x})$. Let us consider properties of g in relation to the degree of risk aversion:

(4.1) $$g'(x) = u'(x) + xu''(x)$$
$$= u'(x)\{1 - R_u(x)\}.$$

Hence the sign of $g'(x)$ is determined by whether the degree of absolute risk aversion is greater or smaller than one:

(4.2) For all x, $g'(x) \underset{<}{\overset{>}{=}} 0$, if and only if, for all x, $A_u(x) \underset{>}{\overset{<}{=}} 1$,

respectively. Its second derivative is expressed in terms of u and $R'_u(x)$:

(4.3) $$g''(x) = u''(x)\{1 - R_u(x)\} - u'(x)R'_u(x).$$

Now let us recall a theorem proved by Pratt, which states that a person with a more risk averse (in the sense of absolute risk aversion) utility function requires a greater risk premium, and vice versa. Since we denote in terms of "certainty equivalent" levels instead of risk premiums, Pratt's theorem [Pratt (1964; theorem 1, conditions (a) and (b))] is expressed as the following lemma. Then a theorem in the preceding section will be proved.

<u>LEMMA</u>

Consider two utility functions, u and v. For all x, $A_u(x) \lesseqgtr A_v(x)$, if and only if, for all probability distribution of $\tilde{x}$, $\underline{x}^u \gtreqless \underline{x}^v$, respectively.■

<u>PROOF OF THEOREM IN SECTION III</u>

First, consider the case (A): $0 < R_u(x) < 1$. Then the sign of $g'(x)$ is determined to be negative from (4.2). From $g'(x)$ defined by (4.1) and $g''(x)$ defined by (4.2), it is possible to define the absolute risk aversion for function g: $A_g(x) = -g''(x)/g'(x)$. Apply Pratt's theorem replacing v by g, we have the following relationship:

(4.4) For any $\tilde{x}$, $x^{\dagger} \lesseqgtr \underline{x}$, if and only if, for all x, $A_g(x) \gtreqless A_u(x)$,

respectively. Now check the difference in the degrees of absolute risk aversion in g and u.

$$A_g(x) - A_u(x) = -\frac{u''(1-R_u)-u'R'_u}{u'(1-R_u)} + \frac{u''}{u'} = \frac{R'_u(x)}{1-R_u(x)} .$$

Since we are considering the case that $R_u(x)$ is positive and less than one, the denominator is positive. Hence the sign of the difference in the degree of risk aversion has to agree with $R'_u(x)$:

(4.5) For all x, $A_g(x) \gtreqless A_u(x)$, if and only if, for all x, $R'_u(x) \gtreqless 0$, respectively. Since the utility function is monotone increasing, the following relationship is obvious.

$$(4.6) \quad x^{\dagger} \overset{<}{\underset{>}{=}} \underline{x}^{u}, \text{ if and only if } u(x^{\dagger}) \overset{<}{\underset{>}{=}} u(\underline{x}^{u}) \equiv Eu(\tilde{x}),$$

respectively. Recall that $x^{\dagger}$ is defined as the "certainty equivalent" level of weighted marginal utility. Combining (4.4), (4.5) and (4.6), we have that for any $\tilde{x}$,

$$u(x^{\dagger}) \overset{<}{\underset{>}{=}} Eu(\tilde{x}), \text{ if and only if, for all } x,\ R'_u(x) \overset{>}{\underset{<}{=}} 0,$$

respectively. This is precisely (3.16a). For the case of (B): $R_u > 1$, $g'(x)$ is negative. In order to use a lemma, we compare the degrees of absolute risk aversion of $u(x)$ and $(-g(x))$. However, noting that A_{-g} is equal to A_u, (4.4) is true. Since $1 - R_u(x) < 0$, (4.5) should be modified:

$$(4.5') \quad \text{For all } x,\ A_g(x) \overset{<}{\underset{>}{=}} A_u(x), \text{ if and only if, for all } x,\ R'_u(x) \overset{>}{\underset{<}{=}} 0,$$

respectively. combining (4.4), (4.5) and (4.6), (3.16b) is derived. ■

Two remarks are in order. First, if $R'_u(x)$ fluctuates between positive and negative, causing $R_u(x)$ becomes greater and smaller than one, over some range of x, then there is no unambiguous result obtained as for which is greater, $Eu(\tilde{x})$ or $u(x^{\dagger})$. Second, if $R_u(x) = 1$, for all x, i.e., implying $R'_u(x) = 0$, for all x, then $g'(x) = u'(x)(1 - R_u(x)) = 0$. Therefore, g is a constant function and $x^{\dagger}$ may not uniquely defined.

V. CONCLUDING REMARKS

Our theoretical model predicts that both additive and multiplicative uncertainty which is not verifiable by an insurance company results in severance payments so high that the expected utility of the laid off is higher than the sure utility of the retained, under reasonable assumptions on risk aversion. On the way to these results, we have proved a new theorem on incomplete insurance. There are two ways to interpret our counterintuitive results. First, severance payments cannot be so high in a real world in order to prevent an apparent moral hazard problem, which is assumed away in our model by a fixed p, and no severance payments to voluntary quits. If this is a case then it is of interest to investigate how severance payments have to be reduced to the second best solution with a moral hazard constraint. It is also important to think whether there is any institutional device to prevent a moral hazard, thus to increase severance payments toward the level of the first best solution. Second, severance payments defined in this model may be broadened to include human capital, which is increased through a tenure of employment before a layoff. Then it may be true that severance payments with human capital are indeed very large, so that the laid off is as well off, if not better off, as the retained. However, these questions are beyond a scope of this paper, and should be investigated soon.

Models considered in this paper imply general properties of incomplete insurance, not only for implicit contract models but also any other applications. For example, it is possible to interpret two branches of an extensive form of our model as "accident" and "no accident," or as "sick" and "healthy," where true costs including psychological hardships of accident or sickness are not observable and hence not insurable. These applications of incomplete insurance other than severance payments are highly desirable and on our agenda for future research. We believe that theorems proved in this paper will be relevant in other applications.

FOOTNOTES

1 Grossman and Shiller (1981) restricted to a class of utility functions with a constant risk aversion and measures the size of the degree of risk aversion. In that sense it is not consistent to combine their work with others which asserts decreasing or increasing relative risk aversion.

REFERENCE

Arrow, K.J., "The Theory of Risk Aversion," in Aspects of the Risk-Bearing, Yrjo Jahnssionin Saatio, Helsinki, 1965.

Blume, M.E., and I. Friend, "The Asset Structure of Individual Portfolios and Some Implications for Utility Functions," Journal of Finance vol.XXX, no.2, May (1975), 585-603.

Cohn, R.A., W.G. Lewellen, R.C. Lease, and G.G. Schlarbaum, "Individual Investor Risk Aversion and Investment Portfolio composition," Journal of Finance vol. XXX, no.2, May (1975), 605-620.

Friend, I. and M.E. Blume, "The Demand for Risky Assets," American Economic Review vol. 65, no.5, December (1975), 900-922.

Geanakoplos, J. and T. Ito, "On Implicit Contracts and Involuntary Unemployments," Discussion Paper No. 81-155R, Center for Economic Research, Department of Economics, University of Minnesota, June 1982.

Grossman, S. and R.J. Shiller, "The Determinants of the Variability of Stock Market Prices," American Economic Review vol. 71, no.2, May (1981), 222-227.

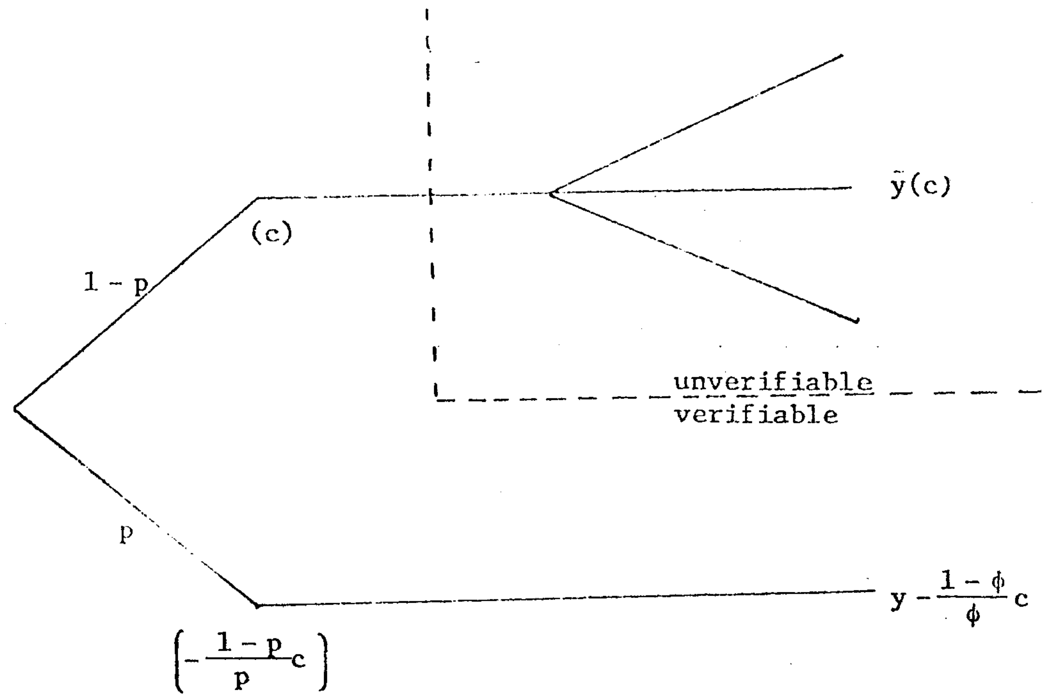
Hall, R.E. and D.M. Lilien, "Efficient Wage Bargains under Uncertain Supply and Demand," American Economic Review vol. 69, no.5, December (1979), 868-879.

Imai, H., J. Geanakoplos, and T. Ito, "Incomplete Insurance and Absolute Risk Aversion," Economics Letters vol. 8, (1981), 107-112.

Pratt, J.W. "Risk Aversion in the Small and in the Large," Econometrica vol. 32, no.1-2, January-April (1964), 112-136.

Ross, S.A., "Some Stronger Measures of Risk Aversion in the Small and the Large with Applications," Econometrica vol.49, no .3, May (1981), 621-638.

Siegel, F.W. and J.P. Hoban Jr., "Relative Risk Aversion Revisited," Review of Economics and Statistics vol. LXIV, no.3, August (1982), 481-487.



Model I [Additive Uncertainty]

$$\tilde{y}(c) = c + \tilde{w}$$

Model II [Multiplicative Uncertainty]

$$\tilde{y}(c) = \tilde{r}c$$

FIGURE 1

	$\bar{r} > 1$	$\bar{r} = 1$	$\bar{r} < 1$
For all x, $R_u(x) > 1$	$c^* > py$	$c^* = py$	$c^* < py$
For all x, $R_u(x) = 1$	$c^* = py$	$c^* = py$	$c^* = py$
For all x, $R_u(x) < 1$	$c^* < py$	$c^* = py$	$c^* > py$

Table 1a: Certainty Case $\theta = 0$

			$\bar{r} > 1$		
	$\bar{r} < 1$	$\bar{r} = 1$	$\theta >> 0$ $\bar{r} \simeq 1$	For some combination of θ and $\bar{r}$	$\theta \simeq 0$ $\bar{r} >> 1$
For all x, $R_u(x) > 1$	$c^* > py$	$c^* > py$	$c^* > py$	$c^* = py$	$c^* < py$
For all x, $R_u(x) = 1$	$c^* = py$	$c^* = py$	$c^* = py$	$c^* = py$	$c^* = py$
For all x, $R_u(x) < 1$	$c^* < py$	$c^* < py$	$c^* < py$	$c^* = py$	$c^* > py$

Table 1b: General Case $\theta > 0$

			for all x, $R_u'(x) > 0$	for all x, $R_u'(x) = 0$	for all x, $R_u'(x) < 0$
for all x, $R_u(x) > 1$	$\bar{r} < 1$ $\bar{r} = 1$	$c^* > py$	$Eu(\tilde{r}c^*) < u\left(y - \frac{1-p}{p}c^*\right)$		ambiguous
	$\bar{r} > 1$	$c^* = py$		$Eu(\cdot) = u(\cdot\cdot)$	
		$c^* < py$	ambiguous	$Eu(\tilde{r}c^*) > u\left(y - \frac{1-p}{p}c^*\right)$	
for all x, $0 < R_u(x) < 1$	$\bar{r} > 1$	$c^* > py$	$Eu(\tilde{r}c) > u\left(y - \frac{1-p}{p}c^*\right)$		ambiguous
		$c^* = py$		$Eu(\cdot) = u(\cdot\cdot)$	
	$\bar{r} = 1$ $\bar{r} < 1$	$c^* < py$	ambiguous	$Eu(\tilde{r}c^*) < u\left(y - \frac{1-p}{p}c^*\right)$	
for all x, $R_u(x) = 1$	for any $\bar{r}$	$c^* = py$	Not possible	$Eu(\tilde{r}c^*) = u\left(y - \frac{1-p}{p}c^*\right)$	Not possible

Table 2