TRUNCATED EGALITARIAN AND MONOTONE PATH SOLUTIONS

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Abstract

Truncated Egalitarian and Monotone Path solutions are two families of generalizations of the Egalitarian solution which fail to satisfy WPO (Weak Pareto-Optimality) and SY (Symmetry) respectively. Any solution satisfying SY, IIA (Independence of Irrelevant Alternatives), CONT (Continuity), and MON (Monotonicity: everyone originally present should help out when new agents come in with no resources of their own) coincides with a Truncated Egalitarian solution if the number of agents n is greater than 2,but may permit utility substitutions if n=2. Similarly, any solution satisfying WPO, IIA, CONT and MON "essentially" coincides with a Monotone Path solution if n > 2,but may permit utility substitution if n=2.

Key words: Egalitarian solution

Truncated Egalitarian solution

Monotone Path solution

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1. Introduction An n-person problem is a subset of the n-dimensional euclidean space representing the utility allocations available to a group of n agents. An n-person solution defined on some class of admissible n-person problems is a method of associating to each problem in the class a unique feasible alternative, which is interpreted as a reasonable compromise among the agents' conflicting interests. Depending upon the context, a solution may be the rule followed by an impartial arbitrator, or may describe how the agents would resolve their problems on their own. The Egalitarian solution is the solution that selects the maximal feasible point at which the agents' utilities are equal.

An axiomatic characterization of the Egalitarian solution based on the possibility that the number of agents involved varies is proposed in Thomson (1981b). In that framework, a solution, by contrast to an n-person solution, is a method of solving any admissible problem that any group of agents may face. (The restriction of a solution to the problems involving a particular group of agents is called the component of the solution relative to that group.) The Egalitarian solution is the only one to satisfy Weak Pareto-Optimality (WPO): it is not possible to simultaneously increase the utilities of all the agents from the solution outcome, Symmetry (SY): if a problem is invariant under all permutations of the agents, they all get the same amount, Independence of Irrelevant Alternatives (IIA): if a problem is restricted by eliminating alternatives other than its solution outcome, then that point remains the solution outcome of the smaller problem, Continuity (CONT): small changes in problems lead to small changes in solution outcomes, and Monotonicity with respect to changes in the number of agents (MON): in order to accommodate the claims of more agents, sacrifices are made by all the agents originally present.

The purpose of the present paper is to investigate what additional solutions would be made possible by removing from this list of axioms WPO on the one hand, SY on the other.

Once a solution has been characterized, it is indeed natural to investigate how much freedom would be gained by sacrificing each of the axioms involved in the characterization and to find out what other axioms are compatible with the remaining ones. It is often the case that one would like to impose more axioms than can be jointly satisfied, and analyzing "trade-offs" among axioms is an important part of the axiomatic study of solutions. The best way to evaluate the "power" of an axiom is to determine whether removing it yields a "large" or a "limited" class of additional solutions -- for instance whether these solutions can be described in terms of a finite list of parameters or whether they constitute a complicated space -- and to what extent they share the same essential features. Of course, what is meant by "essential" is somewhat subjective. In the case of the Egalitarian solution though, one such feature is certainly that it permits no trade-off across the agents' utilities, i.e. it precludes arguments of the kind: agent i should give up α units of utility so as to permit agent j to gain β units of utility.

In addition, no axiom is ever fully convincing in all situations to which one would like to impose it. This is certainly the case of the two on which we will focus here. (What can be achieved by dropping the others is already known from our previous work and discussed at the end of this introduction.)

Let us first discuss weak Pareto-optimality. Although it does not seem restrictive to impose this requirement on an arbitrator's choice, it is not rare to observe non-arbitrated disputes to result in strictly dominated outcomes.

But what kind of violations of optimality will result? In our setting, it is natural to expect that they should be more and more severe as the number of agents involved grows, reflecting the greater difficulty of reaching a consensus in large groups. In some precise sense this expectation is confirmed by our analysis.

The case for dropping Symmetry is equally strong and involves interesting conceptual issues: the first motivation is that an arbitrator may want to solve a symmetric problem at a non-symmetric point in order to compensate for some inequities not explicitly described in the model. Similarly, an agent who is a "better bargainer" may be able to impose a non-symmetric outcome as compromise to a symmetric problem. Of course, it can be argued that such possibilities arise only if all relevant variables are not properly taken into account to start with. But identifying these variables and explicitly incorporating them into the model may not be an easy task. An alternative approach is simply to summarize their effect on representative "test problems" and deduce from this information, taken exogeneously, how the solution behaves everywhere. This requires that a "sufficient" family of test problems be discovered, sufficient in that it permit such an extension. A well-known illustration of this methodology is the characterization of the non-symmetric Nash solutions, each of which can be completely determined from the knowledge of how it solves a symmetric triangle. In that case, one test problem suffices. (See Roth (1979) for the complete argument.)

Another motivation for being interested in deleting the Symmetry axiom from the list of axioms characterizing the Egalitarian solution is that it is the only one of that list not to be invariant under ordinal transformations of the utility functions: if a problem is symmetric initially, it will not remain symmetric under such a transformation. Because we consider solutions defined on domains of convex problems and convexity is not a property that is invariant under all ordinal transformations, we may want to limit ourselves to concave transformations, but these still constitute a very large class under which the symmetry of a problem is typically not preserved. The only way to preserve symmetry is to subject all utilities to a common transformation, which implies a large degree of interpersonal comparability of utilities. Dropping the symmetry axiom considerably weakens the informational assumptions implicit in the analysis. The remaining axioms being all ordinal, it would be striking if they could be jointly satisfied only by solutions that are both invariant only under common transformations and in addition permit no utility substitutions. As it turns out, this is "almost" what happens.

We now turn to a summary of our results.

The existence of solutions satisfying the original list of axioms after the deletion of Weak Pareto-optimality on the one hand or Symmetry on the other, and permitting utility substitutions is easily demonstrated. An example of a solution satisfying SY, IIA, CONT and MON is the solution that selects the Nash outcome for problems involving two agents and the origin otherwise. An example of a solution satisfying WPO, IIA, CONT and MON is obtained as follows: first order the agents in some arbitrary way; then solve any problem faced by any group containing the group made up of the first two agents at the two-person Nash outcome of the intersection of the problem with the subspace corresponding to these two agents; finally, solve any problem faced by any other group at the alternative that is the most favorable to the agent with the highest rank in that ordering and yields zero utility to all the others.

These examples show that possibilities of utility substitutions arise whenever MON is made ineffectual by a violation of WPO or SY. The question is how extensive they can be.

Our main result is that possibilities of utility substitutions are in fact fairly limited and in particular never involve more than two agents in a non-trivial way. In establishing it, we are led to introducing two families of solutions generalizing the Egalitarian solution in two different directions.

We name the members of the first family Truncated Egalitarian solutions; they can be informally defined by describing how a typical Truncated Egalitarian solution solves a problem "expanding" away from the origin: the solution outcome of a "small" problem is its Egalitarian outcome but beyond a certain "size," it remains fixed at some maximal Egalitarian point. In addition, the common value of the coordinates of this maximal point for a given group is never greater than the corresponding value for any subgroup, and it is in this sense that the conjecture that violations of optimality should be more severe in small groups is confirmed. Each Truncated Egalitarian solution satisfies SY, IIA, CONT and MON and prohibits utility substitutions. As illustrated earlier, there are other solutions that satisfy these axioms but permit utility substitutions. But the possibilities of utility substitutions are limited to the two-person case. Of course, this is of significance because of the important role played by the two-person case both in the theory of bargaining but also in many concrete situations, labor-management conflicts being a prominent example.

The second family is the family of Monotone Path solutions, as discussed for instance in Thomson and Myerson (1980) and Myerson (1981): each of the components of such a solution is described by a continuous and monotone path in the utility space of the corresponding group of agents, starting at the origin, the solution outcome of a given problem involving that group being the intersection of its weak Pareto-optimal boundary with the path. In addition, the projection of the path relative to any group of agents onto the subspace pertaining to any one of its subgroups is the path relative to that subgroup. Each Monotone Path solution satisfies WPO, IIA, CONT and MON and prohibits utility substitutions. As above, there are other solutions satisfying the four axioms, and as above these other solutions permit non-trivial utility substitutions only for two-person groups. (We add the qualifier of "non-trivial" because a component pertaining to a group of cardinality greater than two could coincide with the component pertaining to a group of cardinality two that does happen to permit utility substitutions.) However, under two mild additional conditions limiting the extent to which an agent can be discriminated against, only Monotone Path solutions remain acceptable.

We close this introduction by briefly commenting on the deletion of each of the other three axioms, Independence of Irrelevant Alternatives, Continuity and Monotonicity. The deletion of the first of these axioms is studied in Thomson (1981a): it can in particular be replaced by Scale Invariance (which says that the solution is invariant under independent positive affine transformations of each of the agents' utilities) to yield a characterization of the Kalai-Smorodinsky solution (1975). The deletion of Continuity yields solutions that differ from the Egalitarian solution on a very small subclass of problems (specifically, the problems for which Weak Pareto-optimality is not equivalent to Pareto-optimality), and do not seem to offer much independent interest. The Continuity axiom is probably the most acceptable anyway. The Monotonicity axiom is the only one to impose constraints on how solutions should respond to population changes, and since the present paper is part of a long term research project specifically concerned with axiomatic theory with a variable population, its removal here would take us away from our main topic. It is clear however that a very large class of additional solutions would become acceptable.

The paper is organized as follows. Section 2 contains some preliminaries and the statement of the theorem that will be our point of departure. The removal of WPO from the list of axioms used in that theorem is studied in Section 3, and the removal of SY in Section 4.

2. Preliminaries The notations and definitions are as in Thomson (1981 a). There is a set $I = \{1,\ldots,n\}$ of potential agents. $\mathcal{P}$ is the class of subsets of I, with generic elements P,Q,... . The cardinality of the element P of $\mathcal{P}$ is denoted $|P|$. R_+^P is the cartesian product of $|P|$ copies of R_+ indexed by the members of P. Σ^P is the class of problems S that the group P may conceivably face. The following properties are required of S:

(i) S is convex, compact, and there exists x in S with $x > 0$.

(ii) S is comprehensive: for all x, y in R_+^P, if x belongs to S and $x \geq y$, then y also belongs to S.

A solution is a list $F = \{F^P, P \in \mathcal{P}\}$ where for each P in $\mathcal{P}$ and for each S in Σ^P, F^P associates to S a unique element $F^P(S)$ of S, interpreted as the recommended compromise for S. The Egalitarian solution $E = \{E^P, P \in \mathcal{P}\}$ is defined by setting $E^P(S)$ equal to the maximal point of S with equal coordinates.

We will use the following axioms:

Weak Pareto-optimality (WPO): For all P in $\mathcal{P}$, for all S in Σ^P, for all y in R^P, if $y > F^P(S)$, then y does not belong to S.

Symmetry (SY): For all P in $\mathcal{P}$, for all S in Σ^P, if for all one-to-one functions γ from P to P, $S = \{x' \in R_+^P \mid$ there exists x in S s.t. for all i in P, $x'_{\gamma(i)} = x_i\}$, then for all i, j in P, $F_i^P(S) = F_j^P(S)$.

Independence of Irrelevant Alternatives (IIA): For all P in $\mathcal{P}$, for all S, S' in Σ^P, if S contains S' and $F^P(S)$ belongs to S', then $F^P(S') = F^P(S)$.

Monotonicity (MON): For all P, Q in $\mathcal{P}$ with $P \subset Q$, for all S in Σ^P and T in Σ^Q, if $S = T \cap R^P$, then for all i in P, $F_i^P(S) \geq F_i^Q(T)$.

Continuity (CONT): For all P in $\mathcal{P}$, for all sequences $\{S^k\}$ of elements of Σ^P converging in the Hausdorff topology to some S in Σ^P, $F^P(S^k) \to F^P(S)$.

WPO says that there is no feasible way to make all agents better off than they are at the solution outcome, SY that if a problem is symmetric so should the solution outcome, IIA that if an outcome is judged best for some problem, it should still be judged best when competing against fewer alternatives, MON that everyone originally present should sacrifice when the rights of more agents have to be recognized and CONT that small changes in problems should be accompanied by small changes in solution outcomes.

We will take as point of departure the following characterization of the Egalitarian solution.[1/]

Theorem 1 (Thomson (1981b)). A solution satisfies WPO, SY, IIA, CONT and MON if and only if it is the Egalitarian solution.

[1/] For characterizations of the Egalitarian and related solutions involving a fixed number of agents, see Kalai (1977), Myerson (1977, 1981), Roth (1979) and Thomson and Myerson (1980).

3. Removing the axiom of Weak Pareto-optimality We investigate here what solutions other than the Egalitarian solution satisfy the list of axioms of Theorem 1 after the deletion of Weak Pareto-optimality.

The trivial solution $0 = \{0^P, P \in \mathcal{P}\}$ where for each P in $\mathcal{P}$, 0^P associates to every S in Σ^P the origin of R^P_+ obviously satisfies SY, IIA, CONT and MON. More generally, let α in R_+ be given and for each P in $\mathcal{P}$ and S in Σ^P, choose the solution outcome of S to be equal to $E^P(S)$ if αe_P does not belong to S and equal to αe_P otherwise. Any solution obtained in this way also satisfies the four axioms. These solutions can be further generalized by letting α depend on P provided that the α that pertains to each group be never greater than the α that pertains to any subgroup. This leads us to the following definition.

Given a list $\alpha = \{\alpha^P, P \in \mathcal{P}\}$ of non-negative and possibly infinite real numbers such that for all P, Q in $\mathcal{P}$ with $P \subset Q$, $\alpha^P \geqq \alpha^Q$, the associated Truncated Egalitarian solution (or TE solution) E_α is defined by the property that for all P in $\mathcal{P}$ and for all S in Σ^P, $E^P_\alpha(S) = E^P(S)$ if $\alpha^P e_P$ does not belong to S and $E^P_\alpha(S) = \alpha^P e_P$ otherwise.

All TE solutions satisfy SY, IIA, CONT and MON. The Egalitarian solution itself is a particular member of the family obtained by choosing $\alpha^P = \infty$ for all P. The distinguishing features of the TE solutions have a natural interpretation. The fact that a TE solution is fully responsive to expansions of opportunities over some initial range and then totally unresponsive can be interpreted to mean that negotiations are likely to lead to a satisfactory agreement if not too much is at stake and to stall if the problem at hand becomes really important. The monotonicity of the α^P means that it is relatively more difficult to reach a satisfactory agreement in large groups than in small groups.

We prove below that any solution satisfying the four axioms coincides with some TE colution E_α except perhaps when $|P| = 2$ for problems S such that $\alpha^Q e_P \leqq E^P(S) \leqq \alpha^P e_P$ for all Q in $\mathcal{P}$ with $Q \supset P$. In that range utility substitutions are possible. As a simple example, consider the solution F defined by setting $F^P = 0^P$ if $|P| \geqq 3$ and $F^P = N^P$ otherwise (N^P is the $|P|$-person Nash solution, which selects for all P in $\mathcal{P}$ and for all S in Σ^P the maximizer of $\{\prod_{i \in P} x_i \mid x \in S\}$). Note that the choice of F^P for $|P| \geqq 3$ essentially deprives MON of all power.

It is worthwhile comparing the components E_α^P described above to the solutions of "proportional character" discovered by Roth (1981) in analogous circumstances. Roth was interested in determining what solutions other than the n-person Egalitarian solution would satisfy a list of axioms previously shown to characterize that solution but from which the optimality axiom had been removed. The new solutions include a one parameter family F_k^n indexed by k in $[0,1]$: given some n-person problem S, $F_k^n(S)$ is obtained by scaling down the Egalitarian outcome $E^n(S)$ by the factor k, i.e. $F_k^n(S) = kE^n(S)$.

The ways in which the E_α^P and the F_k^n fail to reach optimality are therefore interestingly different. The solution outcomes recommended by all elements of both families are always in the direction of the Egalitarian outcome. They coincide with it or with some fixed point for the elements of the first family. They fail to reach it by a constant factor for the elements of the second family.

We now turn to a characterization of all solutions satisfying SY, IIA, MON and CONT.

Lemma 1. If a solution $F = \{F^P, P \in \mathcal{P}\}$ satisfies IIA and CONT, then for all P in $\mathcal{P}$ there exists a point $\bar{x}$ in R_+^P of coordinates possibly infinite such that for all S in Σ^P, $F^P(S) = \bar{x}$ if $\bar{x}$ belongs to S and $F^P(S)$ is weakly Pareto-optimal for S otherwise.

Proof: Suppose that for some P in $\mathcal{P}$ and for some $\bar{S}$ in Σ^P, $F^P(\bar{S})$ is not weakly Pareto-optimal for $\bar{S}$. Set $\bar{x} \equiv F^P(\bar{S})$. Let now S in Σ^P be given and $x \equiv F^P(S)$.

(i) First we show that if $\bar{x}$ belongs to the relative interior of S, $x = \bar{x}$. We first prove this in the case when x belongs to $\bar{S}$ and then we prove that x belongs to $\bar{S}$.

Supposing then that x belongs to $\bar{S}$, we define S' in Σ^P by $S' \equiv \text{cch}\{\bar{x} + \varepsilon e_P, x\}$ where $\varepsilon > 0$ is chosen sufficiently small to guarantee that $\bar{x} + \varepsilon e_P$ belongs to $\bar{S} \cap S$ (this is possible since $\bar{x}$ belongs to the relative interior of $\bar{S} \cap S$). We observe that $\bar{S}$ contains S' and that $F^P(\bar{S})$ belongs to S'. By IIA, $F^P(S') = F^P(\bar{S}) = \bar{x}$. Also, S contains S' and $F^P(S)$ belongs to S'. By IIA, $F^P(S') = F^P(S) = x$. Therefore $x = \bar{x}$.

We prove that x belongs to $\bar{S}$ by contradiction. We define a continuous function $S(\cdot)$: $[0,1] \to \Sigma^P$ such that $S(0) = S$, $S(1) = \bar{S}$ and $\bar{x}$ belongs to the relative interior of $S(t)$ for all t. For any t in $[0,1]$, if $F^P(S(t))$ belongs to $\bar{S}$, then $F^P(S(t)) = \bar{x}$, by the above paragraph. The path of $F^P(S(t))$ is continuous by CONT; it starts outside of $\bar{S}$, ends at $\bar{x}$ but cannot be in $\bar{S}$ unless it coincides with $\bar{x}$. This is impossible since $\bar{x}$ is in the relative interior of $\bar{S}$.

(ii) Next, we show that if $\bar{x}$ does not belong to the relative interior of S, x is weakly Pareto-optimal for S. Suppose not, and let S' in Σ^P containing both $\bar{S}$ and S be given. Since both $\bar{x}$ and x belong to the relative interior of S', by (i) applied twice, we conclude that $F^P(S') = \bar{x}$ and that $F^P(S') = x$ Therefore $x = \bar{x}$, in contradiction with the assumption that $\bar{x}$ does not belong to the interior of S and that x is not weakly Pareto-optimal for S.

To take care of the possibility that there is no $\bar{S}$ in Σ^P such that $F^P(\bar{S})$ is not weakly Pareto-optimal for $\bar{S}$, it suffices to choose $\bar{x}$ with some infinite coordinates. Q.E.D.

On the basis of Lemma 1, we conclude that to each solution F satisfying SY as well as IIA and CONT can be associated a list $\alpha = \{\alpha^P, P \in \mathcal{P}\}$ where each α^P is a non-negative and possibly infinite real number, such that for all P in $\mathcal{P}$ and for all S in Σ^P, $F^P(S) = \alpha^P e_P$ if this point belongs to S, and $F^P(S)$ is weakly Pareto-optimal for S otherwise.

Note that so far we have made no use of MON. Next, we have

Lemma 2. If a solution $F = \{F^P, P \in \mathcal{P}\}$ satisfies SY, IIA, CONT and MON, its associated sequence α is such that for all P, Q in $\mathcal{P}$ with $P \subset Q$, $\alpha^P \geqq \alpha^Q$.

Proof: Suppose by way of contradiction that for some P, Q in $\mathcal{P}$ with $P \subset Q$, $\alpha^Q > \alpha^P$. Let T in Σ^Q be defined by $T \equiv \mathrm{cch}\{\alpha^Q e_Q\}$. It follows from the definition of the sequence α that $F^Q(T) = \alpha^Q e_Q$, and since $\alpha^P e_P$ belongs to $S \equiv T \cap R^P = \mathrm{cch}\{\alpha^Q e_P\}$ that $F^P(S) = \alpha^P e_P$. But this is in contradiction with MON. Q.E.D.

Lemma 3. If a solution $F = \{F^P, P \in \mathcal{P}\}$ satisfies SY, IIA, CONT and MON, then for all P, Q in $\mathcal{P}$ with $|P| = 2$ and $P \subset Q$, and for all S in Σ^P such that $\alpha^Q e_P$ belongs to S, $F^P(S) \geqq \alpha^Q e_P$.

Proof: Let T in Σ^Q be given such that T contains $\alpha^Q e_Q$ and $S = T \cap R^P$. By definition of the sequence α, $F^Q(T) = \alpha^Q e_Q$, and by MON, $F^P(S) \geqq \alpha^Q e_P$

Q.E.D.

Given P in $\mathcal{P}$ with $|P| = 2$, let $\bar{\alpha}^P \equiv \max\{\alpha^Q \mid Q \in \mathcal{P}, P \subset Q\}$. (In this definition we could impose the requirement that $|Q| = 3$; this is in view of Lemma 2.

Lemma 4. If a solution $F = \{F^P, P \in \mathcal{P}\}$ satisfies SY, IIA, CONT and MON, then for all P in $\mathcal{P}$ with $|P| = 2$ and for all S in Σ^P such that $\bar{\alpha}^P e_P$ does not belong to S, $F^P(S) \geqq E^P(S)$.

Proof: Let a in R_+ be such that $E^P(S) = ae_P$, and let us assume that $\max\{\sum_{i \in P} x_i \mid x \in S\} \leqq 3a$. Let Q in $\mathcal{P}$ with $|Q| = 3$ be such that $P \subset Q$ and that $\alpha^Q > a$ (that such a Q exists follows from that fact that $a < \bar{\alpha}^P$ and the definition of $\bar{\alpha}^P$). Let T in Σ^Q be defined by $T \equiv \{x \in R^Q_+ \mid \sum_{i \in Q} x_i \leqq 3a\}$. Since

$\alpha^Q e_Q$ does not belong to T, $F^Q(T)$ is weakly Pareto-optimal for T. This, in addition to SY implies that $F^Q(T) = ae_Q$. Also S is contained in $T \cap R^P$. Let then T' in Σ^Q be defined by $T' \equiv \text{cch}\{S, ae_Q\}$. We have that T contains T' and that $F^Q(T)$ belongs to T'. By IIA, $F^Q(T') = F^Q(T)$. Since $S = T' \cap R^P$, we conclude by MON that $F^Q_P(T') = ae_P = E^P(S) \leqq F^P(S)$, the desired conclusion. If the inequality $\max\{\sum_{i \in P} x_i \mid x_i \in S\} \leqq 3a$ were not satisfied, we would conclude by applying CONT (the argument would be as in Lemma 1).

Lemma 5. If a solution $F = \{F^P, P \in \mathcal{P}\}$ satisfies SY, IIA, CONT and MON, then for all P in $\mathcal{P}$ with $|P| \geqq 3$ and for all S in Σ^P such that $\alpha^P e_P$ does not belong to S, $F^P(S) \geqq E^P(S)$.

Proof: Let Q in $\mathcal{P}$ be given with $|Q| \geqq 3$ and suppose by way of contradiction, that for some T in Σ^Q such that $\alpha^Q e_Q$ does not belong to T, it is not the case that $F^Q(T) \geqq E^Q(T)$. This means that for some i in Q, $\varepsilon \equiv E^Q_i(T) - F^Q_i(T) > 0$. Since $\alpha^Q e_Q$ does not belong to T, we conclude by Lemma 1 that $F^Q(T)$ is weakly Pareto-optimal for T; therefore, for some j in Q, $F^Q_j(T) - E^Q_j(T) \geqq 0$. Let then $y \equiv (1-\varepsilon/2)E^Q(T)$, T' in Σ^Q be defined by $T' \equiv \text{cch}\{y, F^Q(T)\}$, $P \equiv \{i,j\}$, and $S \equiv T' \cap R^P$. We have that $S = \text{cch}\{y_P, F^Q_P(T)\}$ and $E^P(S) = y_P$. Since $\alpha^P e_P$ does not belong to S, which results from the fact that $E^P_i(S) = (1-\varepsilon/2)\, E^Q_i(T) < E^Q_i(T) \leqq \alpha^Q \leqq \alpha^P$, it follows from Lemma 3 that $F^P(S) \geqq E^P(S) = y_P$. But this point is in fact Pareto-optimal for S, and therefore $F^P(S) = y_P$.

On the other hand, we have that T contains T' and that $F^Q(T)$ belongs to T'. By IIA, $F^Q(T') = F^Q(T)$. We then apply MON by comparing what agent j gets in T' and $S = T' \cap R^P$. This yields $F^Q_j(T) \leqq F^P_j(S) = E^Q_j(T) - \varepsilon < F^Q_j(T)$, a contradiction. Q.E.D.

All of the above results are collected in the following theorem.

Theorem 2: A solution $F = \{F^P, P \in \mathcal{P}\}$ satisfies SY, IIA, CONT and MON if and only if

(i) it coincides with a Truncated Egalitarian solution E_α except perhaps when $|P| = 2$ on the subclass Σ_α^P of Σ^P of problems S such that $\alpha^P e_P \geqq E^P(S) \geqq \bar{\alpha}^P e_P$ where $\bar{\alpha}^P \equiv \max\{\alpha^Q \mid Q \in \mathcal{P},\ P \subset Q\}$.

(ii) for each P in $\mathcal{P}$ with $|P| = 2$, it coincides on Σ_α^P with a solution $\tilde{F}^P$ satisfying SY, IIA and CONT such that $\tilde{F}^P(S) \geqq \bar{\alpha}^P e_P$ for all S in Σ_α^P.

(iii) for all i in I, $\alpha^i \geqq \sup\{x_i \mid x = F^P(S),\ P \in \mathcal{P},\ i \in P,\ S \in \Sigma^P\}$.

Proof: Necessity follows directly from Lemmas 1-5. We will simply observe that (iii) is imposed to ensure that MON is satisfied when groups of cardinalities 1 and 2 are compared. Sufficiency is straightforward. Q.E.D.

The following remarks should help clarifying the significance of Theorem 2.

(i) Note that the solutions described in Theorem 2 do not satisfy the axiom of Anonymity (AN) i.e. their components pertaining to two groups of the same cardinality may differ. In order for Anonymity to hold, one should impose the additional requirements that $\alpha^P = \alpha^{P'}$ whenever $|P| = |P'|$, and that the two-person solutions mentioned in (ii) be identical.

(ii) As a simple corollary of Theorem 2, we obtain a characterization of all solutions satisfying SY, IIA, CONT and MON as well as the following axiom of Scale Invariance.

Scale Invariance (S.INV): For all P in $\mathcal{P}$, for all S, S' in Σ^P, for all λ in R_{++}^P, if $S' = \{x' \in R^P \mid$ there exists $x \in S$ s.t. for all $i \in P$, $x_i' = \lambda_i x_i\}$ then for all i in P, $F_i^P(S') = \lambda_i F_i^P(S)$.

Corollary 1: A solution F satisfies SY, IIA, CONT, MON and S.INV if and only if either F = 0 or for all P in $\mathcal{P}$ with $|P| \geqq 3$, $F^P = 0^P$ and otherwise $F^P = N^P$.

Proof: It follows directly from Theorem 2 that for all P in $\mathcal{P}$ with $|P| \geqq 3$, $F^P = 0^P$. If $|P| = 2$, we apply a theorem of Roth (1977) which says that SY, IIA and S.INV are simultaneously satisfied only by the trivial solution 0^P or by the Nash solution N^P. A straightforward appeal to (iii) of Theorem 2 permits us to conclude the argument for $|P| = 1$.

(iii) A significantly weaker condition than Scale Invariance is Homogeneity (HOM): it says that the solution outcome should be invariant under common positive linear transformations of the utilities. (The λ_i appearing in the statement of S.INV should be identical.) We have directly

Corollary 2: A solution F satisfies AN, IIA, CONT, MON and HOM if and only if either $F = 0$, or for all P in $\mathcal{P}$ with $|P| \geqq 3$, $F^P = 0^P$ and otherwise $F^P = N^P$, or there exists some $k \geqq 4$ such that for all P in $\mathcal{P}$ with $|P| \geqq k$, $F^P = 0^P$ and otherwise $F^P = E^P$.

Proof: We simply note that if for some P in $\mathcal{P}$ with $|P| = 3$ and some S in Σ^P, $F^P(S) \geq 0$, then $F^P = E^P$. Then by Theorem 2, we have that $F^P = E^P$ for all P in $\mathcal{P}$ with $|P| \leqq 2$.

(iv) Utility substitutions are impossible as soon as for all P in $\mathcal{P}$ with $|P| = 2$, $\bar{\alpha}^P = \alpha^P$.

(v) We conclude this section with an example of a non-trivial solution of the form described in Theorem 2. (See Figure 1.)

Example 1. For all P in $\mathcal{P}$ with $|P| \geqq 2$, let $\alpha^P \equiv 1/|P|$ and for all P in $\mathcal{P}$ with $|P| \geqq 3$, and S in Σ^P let $F^P(S) \equiv \alpha^P e_P$ if this point belongs to S and $F^P(S) \equiv E^P(S)$ otherwise. For all P in $\mathcal{P}$ with $|P| \geqq 2$, and S in Σ^P, let $F^P(S) \equiv (1/2)e_P$ if this point belongs to S, $E^P(S)$ if $(1/3)e_P$ does not belong to S and $\operatorname{argmax}\{x_i x_j \mid x \in S;\ 4x_i - x_j \leqq 1;\ -x_i + 4x_j \leqq 1\}$ for all other S in Σ^P, where i and j designate the two members of P. (The shaded area of Figure 1 represents the points of S that satisfy the constraints.)

The solution so defined satisfies SY (in fact AN), IIA, MON and CONT and presents all the essential characteristics of any solution satisfying these four axioms.

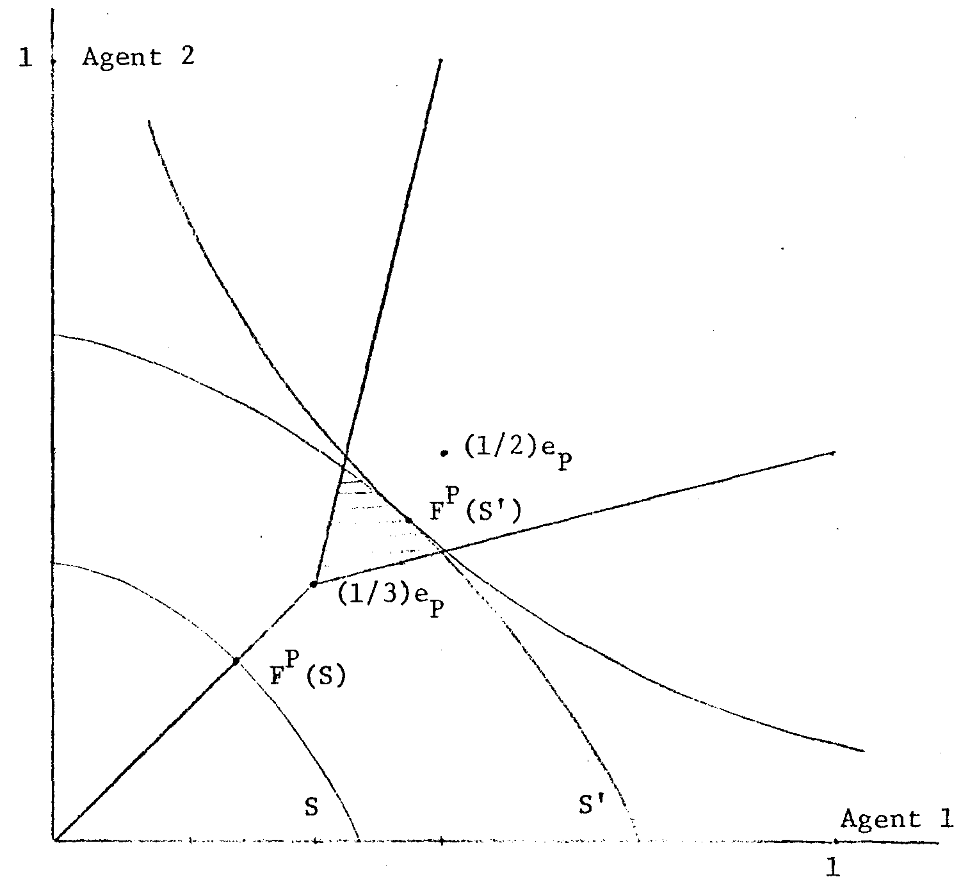

Figure 1

4. Removing the axiom of Symmetry.

4-1 Another characterization of Monotone Path solutions

We investigate here what solutions other than the Egalitarian solution satisfy the list of axioms of Theorem 1 after the deletion of Symmetry.

We start out with the description of a family of solutions that constitute a natural generalization of the Egalitarian solution and satisfy WPO, IIA, MON and CONT. We refer to them as Monotone Path solutions.

A monotone path in R_+^P is the range G^P of a function $\phi^P = \{\phi_i^P \mid i \in P\}$ from R_+ to R_+^P where each ϕ_i^P is continuous, increasing and onto R_+. A monotone path solution on Σ^P is a function from Σ^P to R_+^P such that there exists a monotone path G^P in R_+^P with the property that for all S in Σ^P, the solution outcome of S be the intersection of G^P with the weak Pareto-optimal boundary of S. With a slight abuse of notation we will denote such a solution by G^P.

A Monotone Path solution or MP solution is a solution $G = \{G^P, P \in \mathcal{P}\}$ where for each P in $\mathcal{P}$, G^P is a monotone path solution on Σ^P and furthermore

(1) For all P, Q in $\mathcal{P}$ with $P \subset Q$, $G^P = G_P^Q$ (G_P^Q is the projection of G^Q on R_+^P).

Note that Condition (1) implies that a solution obtained by arbitrarily choosing each F^P to be a monotone path solution on Σ^P is not an MP solution, and that if F is an MP solution then for all i and for all P, P' containing i, ϕ_i^P may be chosen equal to $\phi_i^{P'}$.

Monotone path solutions on alternative classes of n-person problems are discussed in Thomson and Myerson (1980) and Myerson (1981). The proportional solutions discussed by Kalai (1977) are monotone path solutions obtained by taking the ϕ_i^P to be proportional to each other. The Egalitarian solution is the MP solution obtained by choosing the ϕ_i^P to be identical functions. Like all the proportional solutions, the MP solutions prohibit utility substitutions.

It is easy to see that all MP solutions satisfy WPO, IIA, MON and CONT, as claimed earlier. It is shown below that there are other solutions with these four properties. However, the MP solutions are the only ones to satisfy the four properties as well as two mild additional conditions whose effect is to limit the extent to which a solution can discriminate against an individual. If either one of these properties is not met, other solutions exist that permit utility substitutions, but utility substitutions remain limited in any case; as in Section 3, they essentially concern only two-person components. If neither one of these properties is met, the possibilities of substitutions, although they remain limited to two-person components, expand significantly for that case. In fact, the two-person components cannot be given a simple characterization anymore and we will then simply illustrate the range of possibilities by giving examples.

We now formally introduce the two conditions mentioned above. The first one is familiar. It is in particular used by Myerson (1977) and Roth (1977).

Strong Individual Rationality (SIR): For all P in $\mathcal{P}$ and for all S in Σ^P, $F^P(S) > 0$.

This says that all agents gain something from their cooperation. This condition ensures the willing participation of all of them.

Unbounded Payoffs (U): For all P in $\mathcal{P}$, for all i in P and for all M in R, there exists $t > 0$ such that $F_i^P(\{x \in R_+^P \mid \sum_{i \in P} x_i \leqq t\}) \geqq M$.

The most an individual can be discriminated against is when he is always assigned utility zero, independently of the problem at hand. Condition U prevents this and a bit more, as it requires that the utility that each individual obtains on the class of simplicial problems should not be bounded.

We now have

<u>Theorem 3</u>: A solution satisfies WPO, IIA, MON and CONT as well as SIR and U if and only if it is a Monotone Path solution.

<u>Proof</u>: Let Q in $\mathcal{P}$ with $|Q| = 3$ and $t > 0$ be given and let T in Σ^Q be defined by $T \equiv \{y \in R^Q_+ \mid \sum_{i \in Q} x_i \leq t\}$. By SIR, we conclude that for each $t > 0$, $y \equiv F^Q(T(t)) > 0$. Let P in $\mathcal{P}$ with $P \subset Q$ and $|P| = 2$ be given and let $x \equiv y_P$. We have that $\sum_{i \in P} x_i < t$. Let now T' in Σ^Q be such that T contains T', y be in T' and x be Pareto-optimal for $S' \equiv T' \cap R^P$. By IIA, $F^Q(T') = y$, and by MON, $F^P(S') = x$. This equality is true for any S' obtained in this way and since x belongs to the relative interior of $T \cap R^P$, CONT and IIA imply that for all S in Σ^P admitting of x as weakly Pareto-optimal point, $F^P(S) = x$.

Let now t vary in $]0,\infty[$ and let G^P be the path traced out by x. By CONT, G^P is a continuous path. To show that G^P is also monotone, suppose by way of contradiction that there exist x and x' in G^P with $x \neq x'$ such that neither $x > x'$ nor $x' > x$. Then both x and x' are weakly Pareto-optimal for S in Σ^P defined by $S \equiv \text{cch}\{x, x'\}$ and by the conclusion of the above paragraph applied twice, we have that $F^P(S) = x$ and $F^P(S) = x'$, a contradiction. By U, we also conclude that both coordinates of x go to ∞ as t goes to ∞. By this argument, we know that each F^P with $|P| = 2$ is a monotone path solution on Σ^P.

Next we show that the two-person paths are consistent in the sense that they can be obtained from the projection of higher order paths. To see this

let Q in $\mathcal{P}$ be given with $|Q| = 3$. W.l.o.g. we assume that $Q = \{1,2,3\}$. Let $P \equiv \{1,2\}$, $P^1 \equiv \{2,3\}$ and $P^2 \equiv \{1,3\}$. Now, let (a_1,a_2) be a point of G^P. By U, there exists $a_3 > 0$ such that (a_2,a_3) belongs to G^{P^1}. We claim that (a_1,a_2,a_3) belongs to G^Q and (a_1,a_3) to G^{P^2}. To see this, let T in Σ^Q be defined by $T \equiv \text{cch}\{(a_1,a_2,a_3)\}$. To show that (a_1,a_2,a_3) belongs to G^Q, we first observe that by feasibility $F^Q(T) \leqq (a_1,a_2,a_3)$.
Suppose then, by way of contradiction, that the inequality is strict for one coordinate. W.l.o.g. we can assume that $F_1^Q(T) < a_1$. By WPO, equality holds for at least one of the other coordinates. Again w.l.o.g. we can assume that $F_2^Q(T) = a_2$. Let then T' in Σ^Q be defined by $T' \equiv T \cap \{x \in R_+^Q \mid x_1 + x_2 \leqq F_1^Q(T) + F_2^Q(T)$. We have that T contains T' and $F^Q(T)$ belongs to T'. By IIA, $F^Q(T') = F^Q(T)$. Also, by the first paragraph, we have that $F^P(T' \cap R^P)$ is the intersection of G^P with the weak Pareto-optimal boundary of $T' \cap R^P$. This intersection is strictly dominated by (a_1,a_2). Then we have $a_2 = F_2^Q(T') > F_2^Q(T' \cap R^P)$ in contradiction with MON. This argument shows that $F^Q(T) = (a_1,a_2,a_3)$ and applying MON again that $F^Q_{P^2}(T) \leqq F^{P^2}(T \cap R^{P^2}) = F^{P^2}(\text{cch}\{(a_1,a_3)\})$ i.e. that $F^{P^2}(\text{cch}\{(a_1,a_3)\}) = (a_1,a_3)$, so that (a_1,a_3) is a point of G^{P^2}.

This argument can be easily generalized to show that for all Q in $\mathcal{P}$ there exists a monotone path G^Q such that for all P in $\mathcal{P}$ with $P \subset Q$ and $|P| = 2$, $G_P^Q = G^P$. By an argument similar to the one presented in the previous paragraph, we would then prove that for all T in Σ^Q, $F^Q(T) = G^Q(T)$.

QED

In order to further illustrate Theorem 3, we present examples showing how relaxing <u>either</u> SIR or U generates new solutions that can exhibit utility substitutions. We indicate the form that these utility substitutions can take. Finally we show that relaxing both SIR and U permits yet additional utility substitutions.

To facilitate the description of these results, we introduce the Dictatorial solutions: Given P in $\mathcal{P}$ and i in P, D^i is the solution on Σ^P selecting the alternative that gives maximal utility to agent i and zero utility to all others.

4-2 Relaxing the Unboundedness condition

First, we offer a slight generalization of the MP solutions. For each P in $\mathcal{P}$ we weaken the requirement on the list $\{\phi_i^P,\ i \in P\}$ defining G^P that each ϕ_i^P be onto R_+ to the requirement that at least for one i in P, ϕ_i^P be onto R_+. This guarantees the unboundedness of G^P and the existence of an intersection of G^P with the weak Pareto-optimal boundary of any S in Σ^P. Then we define an MP^1 solution as a list $G = \{G^P,\ P \in \mathcal{P}\}$, where for each P in $\mathcal{P}$, G^P is so generalized and the consistency condition (1) is replaced by

(1)' For all P, Q in $\mathcal{P}$ with $P \subset Q$, $G^P \supset G_P^Q$.

An example of a now admissible two-person component is the component $F^{\{2,3\}}$ of the solution depicted in Figure 2. That example was constructed to show that the relaxation of the requirements on the ϕ_i^P introduces possibilities of utility substitutions.

Example 2. $I = \{1,2,3\}$. $F^I \equiv G^I$ is a monotone path in R^I approximating $[0, e_I] \cup \{e_I + \lambda e_3 \mid \lambda > 0\}$. $F^{\{2,3\}} \equiv G^I_{\{2,3\}}$ and $F^{\{1,3\}} \equiv G^I_{\{1,3\}}$. Given S in $\Sigma^{\{1,2\}}$, $F^{\{1,2\}}(S) \equiv E^{\{1,2\}}(S)$ if $e_{\{1,2\}}$ does not belong to intS and $\operatorname{argmax}\{x_1x_2 \mid x \in S,\ x_1 + \varepsilon x_2 \geq 1+\varepsilon,\ \varepsilon x_1+x_2 \geq 1+\varepsilon\}$ for $\varepsilon < 0$ otherwise. (The path $[0,e_I] \cup \{e_I + \lambda e_3 \mid \lambda > 0\}$ is only weakly increasing and therefore does not define a monotone path solution in Σ^I, which is why we take an "approximation" to it. Note that we choose $\varepsilon < 0$ to guarantee CONT).

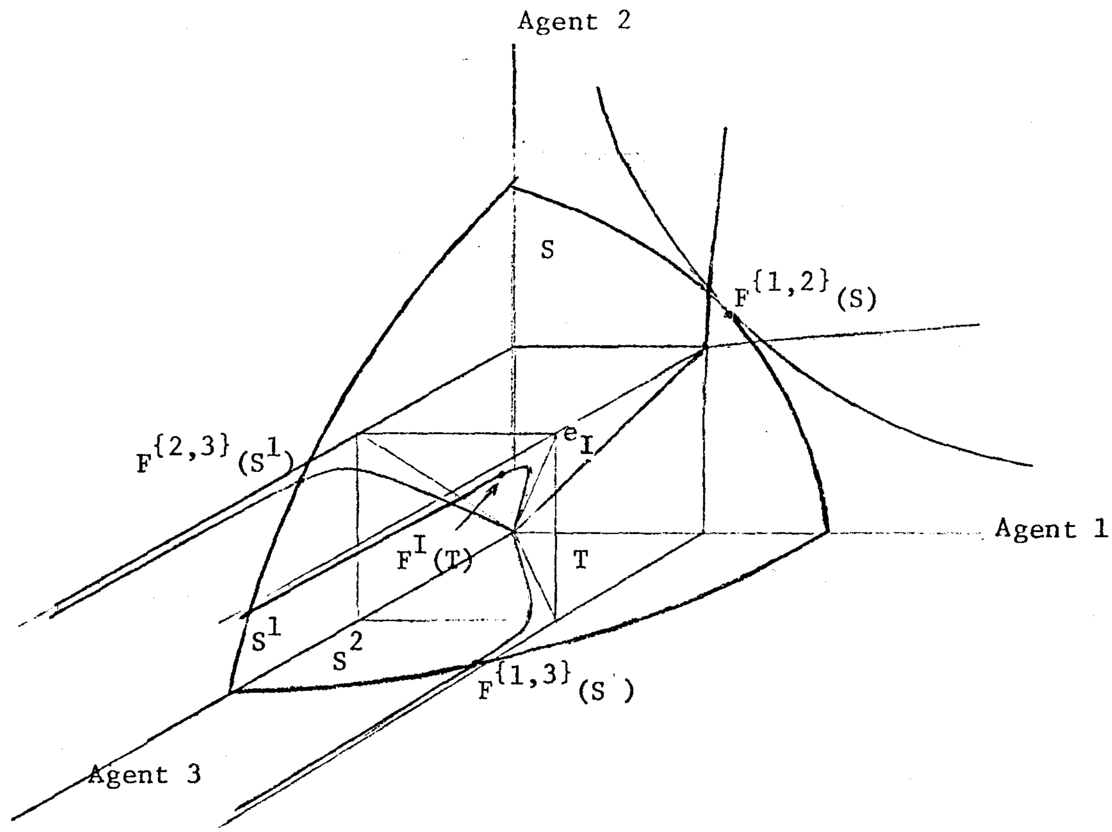

Figure 2

In no three-person problem are agents 1 and 2 ever allocated more than 1 so that the constraints imposed by MON on the pair F^I, $F^{\{1,2\}}$ are automatically satisfied whenever $F^{\{1,2\}}(S)$ can be chosen to dominate $e_{\{1,2\}}$.

By generalizing this observation, we obtain the following characterization theorem.

Theorem 4: A solution F satisfies WPO, IIA, MON, CONT and SIR if and only if there is an MP^1 solution with which it coincides except perhaps for one P in $\mathcal{P}$ with $|P| = 2$ and for all S in Σ^P not containing $\bar{\beta}^P \equiv \sup\{x_P \mid x \in G^I\}$

$F^P(S) = \tilde{F}^P(S)$ where $\tilde{F}^P$ is a solution defined on that subclass of problems, satisfying WPO, IIA and CONT and $\tilde{F}^P(S) \geqq \bar{\beta}^P$ for all S in the subclass.

Proof: It is obtained by a simple adaptation of the proof of Theorem 3.

4-3 Relaxing the Strong Individual Rationality condition

First, we offer a second trivial generalization of the MP solutions by permitting each component path to be initially non-decreasing and then increasing (instead of being increasing throughout) i.e. to initially lie in a subspace of smaller dimensionality than the space to which the component applies, and to successively enter subspaces of greater and greater cardinalities. Such a solution will be referred to as an MP^2 solution. Formally, this is achieved by letting each ϕ_i^P be 0 on some non-degenerate interval $[0,\gamma_i]$ (but still increasing on $[\gamma_i,\infty[$). An example of a now admissible two-person component is the component $F^{\{1,3\}}$ of the solution depicted on Figure 3.

That example exhibits all the essential features exhibited by solutions satisfying WPO, IIA, MON, CONT and U.

Example 3. $I = \{1,2,3\}$. Given S in $\Sigma^{\{1,2\}}$, $F^{\{1,2\}}(S) \equiv \operatorname{argmax}\{x_1x_2 \mid x \in S, x \leqq e_{\{1,2\}}\}$ if $e_{\{1,2\}}$ does not belong to S and $E^{\{1,2\}}(S)$ otherwise. $F^{\{2,3\}} = G^{\{2,3\}} \equiv [0,e_2] \cap \{e_2 + \lambda e_{\{2,3\}} \mid \lambda > 0\}$. $F^{\{1,3\}} = G^{\{1,3\}} \equiv [0,e_1] \cap \{e_1 + \lambda e_{\{1,3\}} \mid \lambda > 0\}$. Given S in Σ^I, $F^I(S) \equiv F^{\{1,2\}}(S \cap R^{\{1,2\}})$ if WPO(S) $\{e_{\{1,2\}} + \lambda e_I \mid \lambda > 0\} = \phi$, and that point of intersection otherwise.

What is interesting about this example is that for problems "large" enough, the solution coincides with an MP solution. Only one two-person component permits utility substitutions in some domain of the form $[0,\gamma_1] \times [0,\gamma_2]$, and there it coincides with the three-person component. The next theorem shows that this is the general situation in the three-person case and that if there are more than three agents, it remains true

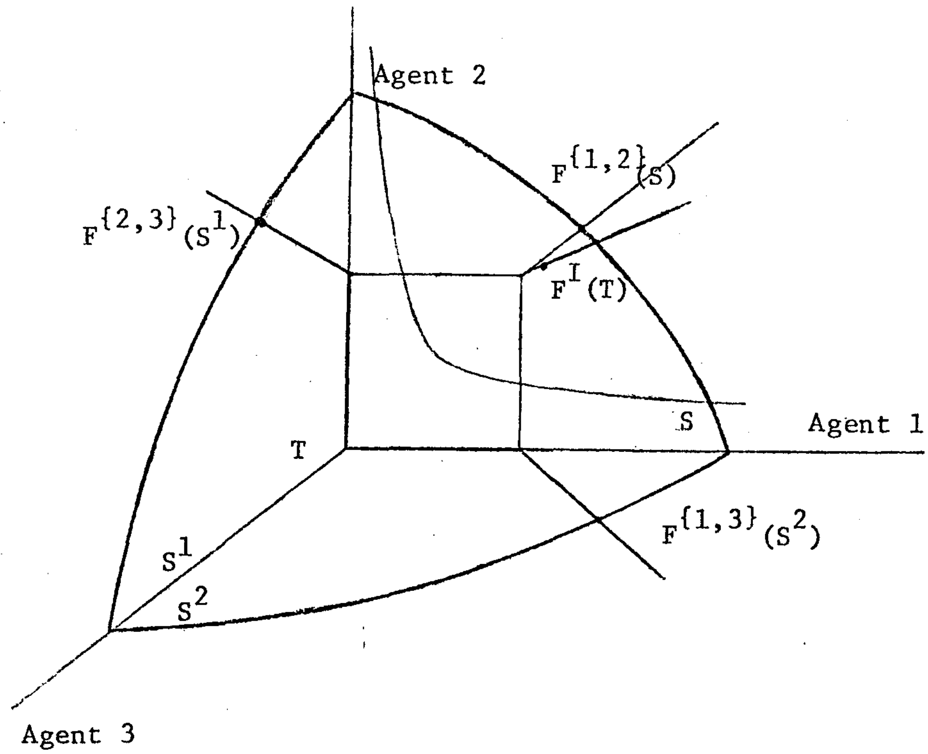

Figure 3

that only one two-person component can permit utility substitutions. Otherwise the solution coincides with an MP^2 solution.

Theorem 5. A solution F satisfies WPO, IIA, MON, CONT and U if and only if there is an MP^2 solution with which it coincides except that, if we denote by γ the maximal point of G^I that belongs to a two-dimensional subspace and P the corresponding two person group, then for all Q in $\mathcal{P}$ with $G \supset P$ and for all T in Σ^Q not containing γ_Q, $F^Q(T) = \tilde{F}^P(T \cap R^P)$ where $\tilde{F}^P$ is a solution defined on that subclass satisfying WPO, IIA, CONT and $\tilde{F}^P(S) \leqq \gamma_P$ for all S in the subclass.

Proof: It is obtained by a simple adaptation of the proof of Theorem 3. QED

This theorem says that SIR plays a very minor role indeed in preventing utility substitutions.

4-4 Relaxing both the Unboundedness and the Strong Individual Rationality Conditions

We saw that relaxing U yields solutions that permit utility substitutions for "large" problems and relaxing SIR yields solutions that permit utility substitutions for "small" problems. When both conditions are relaxed, solutions are obtained that combine both features but other possibilities arise. Because the solutions that satisfy WPO, IIA, MON and CONT do not constitute a simple family, we will not attempt a characterization and we will instead provide illustrating examples.

First we have

Example 4. Lexicographic dictatorial solutions: The agents are ordered in some arbitrary way. Then given any P in $\mathcal{P}$ and any S in Σ^P, $F^P(S)$ is taken to be the point of S that yields maximal utility to the member of P with the highest rank in the ordering and zero utility to all the others.

It is for these solutions that the violations of SY are the worst. A more general class of solutions satisfying the same four axioms and containing the dictatorial solutions can be obtained from the MP^2 solutions by permitting the component path relative to any Q to totally lie in a subspace R_+^P of R_+^Q.

Example 5. (See Figure 4.) $I = \{1,2,3\}$. $F^I = F^{\{1,2\}} = N^{\{1,2\}}$; $F^{\{2,3\}} = D^2$; $F^{\{1,3\}} = D^1$. (We write $F^2 = F^{\{1,2\}}$ to mean $F^I(T) = F^{\{1,2\}}(T \cap R^{\{1,2\}})$ for all T in Σ^I with a slight abuse of notation.)

In that example, SIR is violated in the worst way for one agent (agent 3) since he is totally ignored by all components relative to groups containing him. Utility substitutions are permitted over the whole of $R^{\{1,2\}}$ to solve any problem involving both agents 1 and 2.

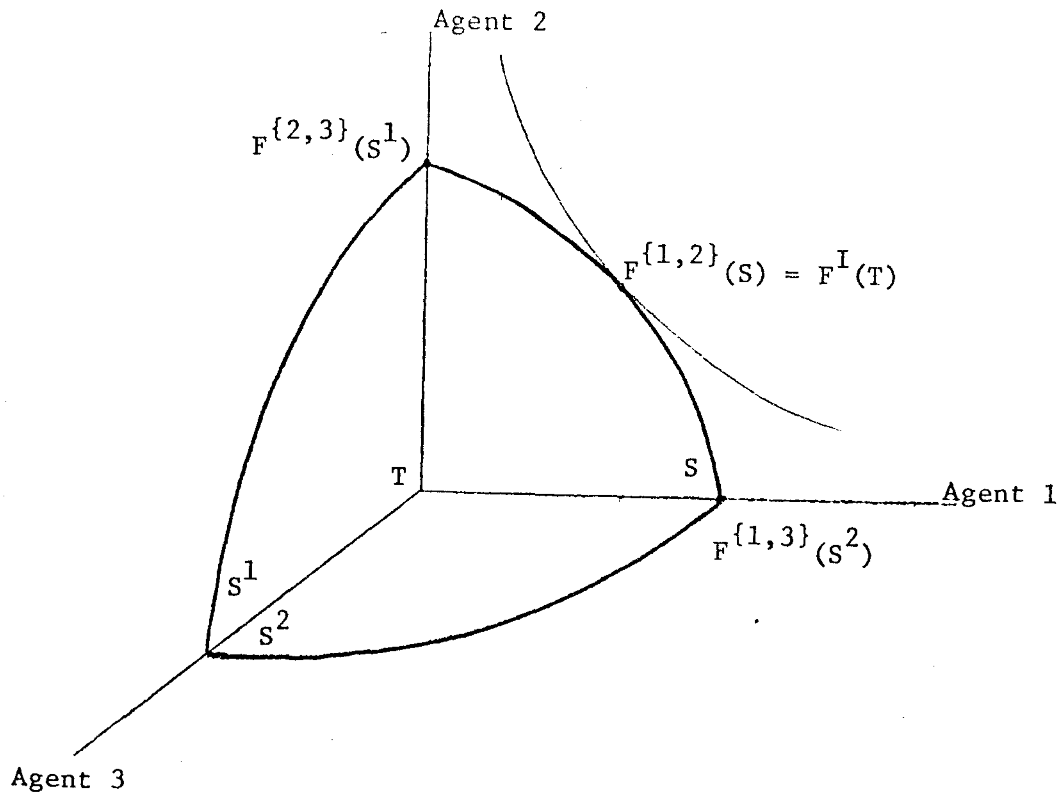

Figure 4

$I = \{1,2,3\}$

In the next example, it is the opposite. One agent (agent 3) always gets his preferred alternative. The components of the solution relative to any group containing him coincide with D^3. Utility substitutions remain possible over the whole of $R^{\{1,2\}}$ to solve any two-person problem involving both agents 1 and 2.

Example 6. (See Figure 5.) $I = \{1,2,3\}$. $F^I = F^{\{2,3\}} = F^{\{1,3\}} = D^3$; $F^{\{1,2\}} = N^{\{1,2\}}$.

The next example shows that the components of a solution relative to two different two-person groups may both permit utility substitutions provided the domains over which this possibility arises satisfy certain compatibility conditions; in our example, $F^{\{1,2\}}$ coincides with $N^{\{1,2\}}$ in $R \times [0,1]$ while $F^{\{2,3\}}$ coincides with $N^{\{2,3\}}$ in a subset of $[1,\infty[\times R_+$.

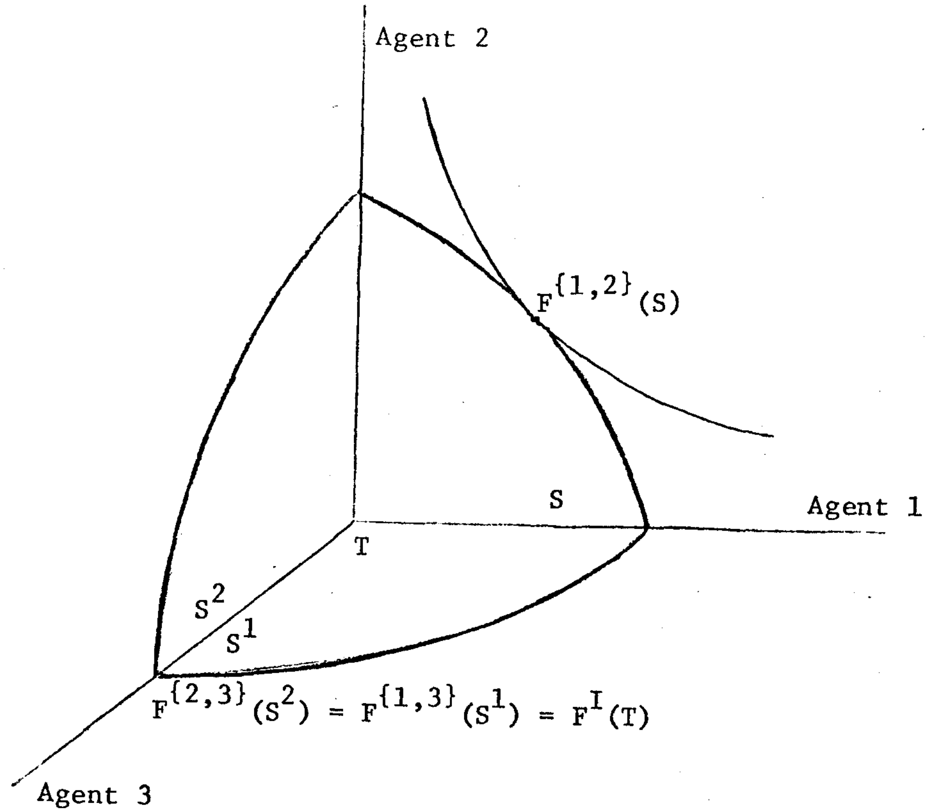

Figure 5

Example 7. (See Figure 6.) $I = \{1,2,3\}$. Given S in $\Sigma^{\{1,2\}}$, $F^{\{1,2\}}(S)$ $\text{argmax}\{x_1x_2 \mid x \in S,\ x_2 \leq 1\}$. Given S in $\Sigma^{\{2,3\}}$,

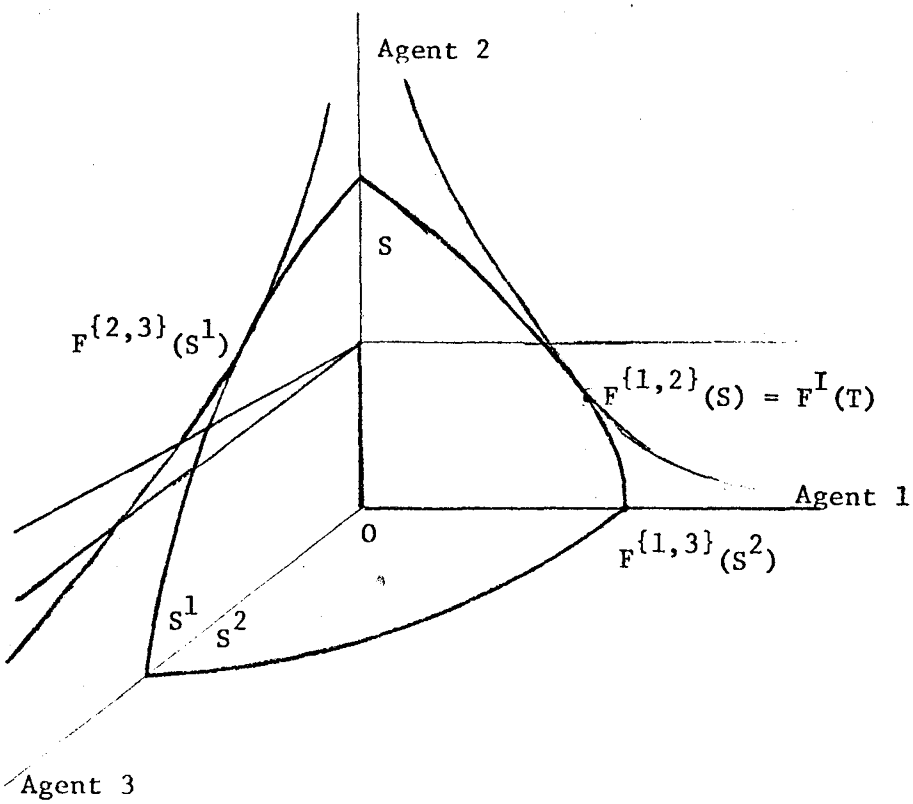

Figure 6

$F^{\{2,3\}}(S) \equiv D^2(S)$ if $\max\{x_2 \mid x \in S\} < 1$ and $\text{argmax}\{x_2 x_3 \mid x \in S, x_2+\varepsilon x_3 \geqq 1+\varepsilon\}$ where $\varepsilon < 0$ otherwise. $F^{\{1,3\}} \equiv D^1$ and $F^I \equiv F^{\{1,2\}}$.

The final example shows that for a given component the domain over which utility substitutions are possible may be made up of two separate subdomains; in that example $F^{\{1,2\}}$ coincides with $N^{\{1,2\}}$ in $[0,1] \times [0,1]$ and with $N^{\{1,2\}}$ in a subset of $[3/2,\infty[\times [3/2,\infty[$.

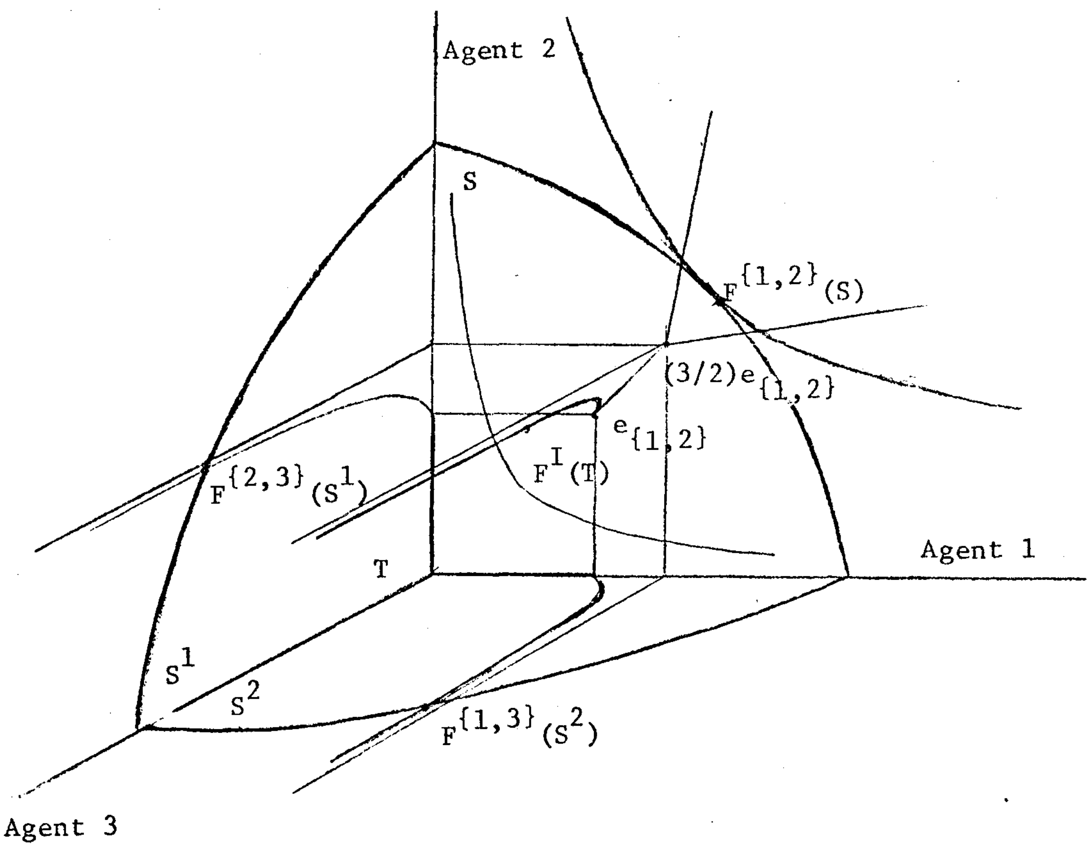

Figure 7

Example 8. (See Figure 7.) $I = \{1,2,3\}$. Given S in $\Sigma^{\{1,2\}}$, $F^{\{1,2\}}(S) \equiv$ $\text{argmax}\{x_1 x_2 \mid x \in S, x \leqq e_{\{1,2\}}\}$ if $e_{\{1,2\}}$ does not belong to S, $\text{argmax}\{x_1 x_2 \mid x \in S, x_1+\varepsilon x_2 \geqq 1+\varepsilon, \varepsilon x_1+x_2 \geqq 1+\varepsilon\}$ where $\varepsilon < 0$ if $(3/2)e_{\{1,2\}}$ belongs to S and $E^{\{1,2\}}(S)$ in all other cases. Given T in Σ^I, $F^I(T) \equiv F^{\{1,2\}}(T \cap R^{\{1,2\}})$ if $e_{\{1,2\}}$ does not belong to T and F^I coincides with an MP^2 solution G^I containing $[0,(1/2)e_{\{1,2\}}]$ and approximating $[(1/2)e_{\{1,2\}}, (3/2)e_{\{1,2\}}] \cup \{(3/2)e_{\{1,2\}} + \lambda e_3 \mid \lambda > 0\}$. $F^{\{1,3\}} \equiv G^I_{\{1,3\}}$ and $F^{\{2,3\}} \equiv G^I_{\{2,3\}}$.

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