LOCALLY STABLE PRICE MECHANISMS

by

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ABSTRACT

This paper develops necessary conditions for a price adjustment mechanism to achieve local stability at regular competitive equilibria. Two principal questions are: how closely must a locally stable mechanism be tailored to particular excess demand functions, and can any such mechanism be interpreted as a market adjustment process. In response to the first question, a variant of the (local) Newton method, termed the "orthogonal Newton method" is shown to require, in a dimensional sense, the minimal information about excess demand functions. The second question is answered in the negative by proving the nonexistence of any locally stable mechanism with the property that the price of any given commodity is not changed when its own market is in equilibrium. These and other results are obtained by using convergent price paths to generate a homotopy between the adjustment dictated by the mechanism and the actual direction of the equilibrium.

1. Introduction

The "Global Newton Method" introduced by Smale (1976) has stimulated renewed interest in price mechanisms which converge to competitive equilibrium. A previous era of research had ended with the famous example of Scarf (1960) showing that the tatonnement process can fail to be even locally stable at a unique equilibrium. This example created the presumption that any theory of the stability of competitive equilibrium must inevitably be confined to special classes of market excess demand functions. Although the Global Newton method has no apparent institutional interpretation, its generic convergence properties have reopened the question of whether there exist economically interesting price adjustment mechanisms which stabilize competitive equilibrium in general. The answer to this question requires a characterization of the stable mechanisms.

The object of the present paper is to provide some strong necessary conditions characterizing price mechanisms which achieve local stability. Before turning to the formal definitions and theorems, we first give an informal review of the results. Given an excess demand function f, a general price mechanism takes the form

(*) $\dot{p} = M(p,f)$.

Here we are taking the strictly positive orthant of R^n as the price space, with commodity $n+1$ as numeraire. At this level of generality, the second argument of M is the entire function f, not just the value of f at p. Thus (*) includes as special cases the tatonnement process

$\dot{p} = f(p)$,

and the (local) Newton method

$\dot{p} = -(Df(p))^{-1}f(p)$

(at least near regular equilibria so that $(Df(p))^{-1}$ exists), as well as other mechanisms which may involve higher derivatives or the behavior of f at prices other than p. A principal objective of this paper is to determine how much information about f is needed to ensure local stability.

At an equilibrium price p^*, the conventional sufficient condition for local stability is that all characteristic roots of the derivative $D_pM(p^*,f)$ have negative real parts. Under this stability condition, any sufficiently small perturbation of p away from p^* will result in a price path which remains near p^* and has p^* as its limit. Suppose that p^* is a regular equilibrium, that is, $Df(p^*)$ is nonsingular, and that instead of perturbing p we perturb f to an excess demand function f' which is near f in a C^1 topology. Then f' has an equilibrium $e(f')$ near p^*, and if the derivative D_pM is continuous in both arguments, all characteristic roots of $D_pM(e(f'),f')$ have negative real parts. Hence perturbing the demand function from f to f' results in the same path we would obtain by perturbing the price from $e(f')$ to p^* given the demand function f', that is, a path from p^* to $e(f')$. This observation suggests that instead of characterizing stable mechanisms by their response to price perturbations we can equivalently study their response to perturbations of the excess demand function. Let E' denote the set of demand functions f with $f(p^*) \neq 0$, and consider the following two functions on E' to $R^n - 0$ $(=\{x \in R^n: x \neq 0\})$:

a) $f \mapsto M(p^*,f)$; and

b) $f \mapsto e(f)-p^*$.

The function (a) describes the initial direction of movement dictated by the adjustment mechanism, and (b) gives the direction of the actual equilibrium. To characterize stable mechanisms we must relate the former function to the latter. It turns out that the price paths generated by a stable mechanism trace out a homotopy from (a) to (b). The homotopy equivalence of the adjustment mechanism and the equilibrium function is established in Theorem 2.10 below, and is the basis for all of the results in this paper. This homotopy does not require the characteristic root condition mentioned above, but relies on the fact of stability itself (a precise definition of local stability is given in

Definition 2.3 below).

To study the function $M(p^*,\cdot)$ we can additively decompose a demand function f into the vector of excess demands $f(p^*)$ and the function $f-f(p^*)$. Let $\mathcal{E}_{p^*}$ denote the set of excess demand functions f with $f(p^*)=0$ and $Df(p^*)$ nonsingular. Since $f-f(p^*) \in \mathcal{E}_{p^*}$ for all f with $Df(p^*)$ nonsingular, we can express $M(p^*,\cdot)$ as an adjustment function $A: R^n \times \mathcal{E}_{p^*} \to R^n$ with $M(p^*,f) = A(f(p^*),f-f(p^*))$. For example, the tatonnement process takes the form $A(f(p^*),f-f(p^*)) = f(p^*)$. Since the tatonnement process is not generally stable, we wish to know how much information about $f-f(p^*)$ must be built into the mechanism to ensure stability. For any f in $\mathcal{E}_{p^*}$, $A(\cdot,f): R^n - 0 \to R^n - 0$, and the homotopy equivalence mentioned above implies that the function $z \mapsto A(z,f)$ must have the same "degree" as the equilibrium function $z \mapsto e(f+z)$, where $f+z$ is the demand function $p \mapsto f(p)+z$. The degree of the equilibrium function is ± 1 as $\det(-Df(p^*)) \gtrless 0$, so the function $A(\cdot,f)$ must depend on at least the sign of the determinant of $Df(p^*)$. This result is stated in Theorem 3.3 below.

A much sharper result is obtained when we invoke the characteristic root condition for stability. Formally, the function A must be redefined for prices other than p^*, but suppressing this definition and the necessary smoothness assumptions for the purpose of smooth exposition, we obtain

$$D_p M(p^*,f) = D_z A(0,f) Df(p^*)$$

if $f(p^*)=0$. We will say that the mechanism is <u>hyperbolically locally stable</u> (HLS) if the characteristic roots of the right hand side all have negative real parts. For HLS mechanisms, Theorem 3.7 states that the functions $f \mapsto D_z A(0,f)$ and $f \mapsto -(Df(p^*))^{-1}$ are homotopy equivalent as functions on $\mathcal{E}_{p^*}$ to the space of nonsingular $n \times n$ matrices. For example, the (local) Newton method

$$M(p,f) = -(Df(p))^{-1}f(p)$$

has $D_pM(p,f) = -I$ if $f(p)=0$, where I is the identity matrix. In this case $D_zA(0,f) = -(Df(p^*))^{-1}$, so the two functions are identical.

As a characterization, homotopy equivalence is generally rather opaque, so it is natural to ask whether the above result can be strengthened to something more transparent. The answer to this question, which is yes and no, indicates why homotopy equivalence is a useful, perhaps even essential tool in the analysis of locally stable price mechanisms. First we will show that by modifying a mechanism constructed by Saari and Simon (1978, pp. 1123-1124) we can obtain an HLS mechanism which is less sensitive to f than the Newton method. Given f, apply the Gram-Schmidt process to the columns of $Df(p)$ and normalize the new columns to euclidean norm one, obtaining an orthonormal matrix $D^*f(p)$. Now use $D^*f(p)$ as if it were $Df(p)$ in the Newton method. That is, define $A^*(f(p^*), f-f(p^*)) = -D^*f(p^*)'f(p^*)$, where the prime denotes transpose. Then $D_zA^*(0,f)Df(p^*) = -D^*f(p^*)'Df(p^*)$, an upper triangular matrix with negative diagonal elements. We will call A^* the <u>orthogonal Newton method</u>. The Newton method maps $\mathcal{E}_{p^*}$ onto the space of nonsingular matrices, which is an open set of dimension n^2, while the orthogonal Newton method maps $\mathcal{E}_{p^*}$ onto a compact $n(n-1)/2$ dimensional manifold. However, the two manifolds are homotopy equivalent (the Gram-Schmidt process is a "strong deformation retraction") so the orthogonal Newton method does not contradict the above result. We would like to establish that among the class of HLS mechanisms, the orthogonal Newton method uses the least information about $f-f(p^*)$. The most transparent way to do this would be to show that for any HLS mechanism A there is a function h such that $D_zA^*(0,f) = h(D_zA(0,f))$ for all $f \in \mathcal{E}_{p^*}$. However, this sort of characterization is easily seen to be impossible. The characteristic root condition on $D_zA(0,f)Df(p^*)$ will remain satisfied for small changes in $Df(p^*)$ even if $D_zA(0,f)$ is held constant. Hence any HLS mechanism A may be modified so that $D_zA(0,\cdot)$ is constant on a C^1 open neighborhood of any given $f \in \mathcal{E}_{p^*}$. For this reason any characterization of locally stable mechanisms must be stated in terms of their global behavior on $\mathcal{E}_{p^*}$, and the tools of homotopy theory are well suited to this purpose.

The informational minimality of the orthogonal Newton method can be established in a coarser but still conventional form. Suppose we represent the derivative $D_zA(0,f)$ as being obtained by first projecting f to a summary statistic $\pi(f) \in X$, where X is a manifold, and then computing the above derivative as a continuous function on X. The smallest dimension needed for the manifold X to support this representation is a measure of the information required by $D_zA(0,\cdot)$. This measure, often generalized to permit nonmanifolds, is widely exploited in the literature on informationally efficient allocation mechanisms (e.g., Reiter (1977) and references therein). For A^* we can take $X=\mathcal{O}$; the orthogonal group, and $\pi(f) = -D^*f(p)'$, which is $D_zA^*(0,f)$ itself. Corollary 3.14 below states that any HLS mechanism requires a manifold X of dimension at least $n(n-1)/2$, the dimension of $\mathcal{O}$. Thus the orthogonal Newton method is in this sense informationally efficient.

Up to now we have required price mechanisms to stabilize the equilibrium $e(f)$ whenever $e(f)$ is a regular equilibrium near p^*. Since there is nothing special about the price p^*, this amounts to requiring every regular equilibrium to be locally stable, which may seem unreasonable. At the other extreme, probably the least that could be expected of a stable mechanism is that every regular demand function have at least one stable equilibrium. In particular, unique equilibria, if regular, should be locally stable. A mechanism which meets this requirement is said to be (locally) stable at unique equilibria, and an analogous modification is applied to hyperbolic local stability. For example, in the two commodity case $(n=1)$, $Df(p)<0$ at a unique regular equilibrium so the tatonnement process is HLS at unique equilibria if $n=1$. Surprisingly, the results obtained above under the stronger requirement apply almost directly to stability at unique equilibria. This is due to Proposition 4.2 below, which extends a result of Saari and Simon (1978, Lemma 5, p. 1113) to state that if $f(p)=0$ and $\det(-Df(p))>0$, then there is a demand function f° which agrees with f near p and has p as its unique equilibrium. Thus all of the above results carry over with $\mathcal{E}_{p*}$ replaced by $\mathcal{E}^{+}_{p*} = \{f \in \mathcal{E}_{p*}: \det(-Df(p^*))>0\}$ and $\mathcal{O}$ replaced by the rotation group $\mathcal{R}$, the set of orthonormal matrices with determinant $+1$. Of course the result that $A(\cdot,f)$ must be sensitive to the sign of $\det Df(p^*)$ is obviated; but the results for HLS mechanisms retain their strength because $\mathcal{O}$ is simply the disjoint union of $\mathcal{R}$ and its mirror image. In particular, the informational efficiency of the orthogonal Newton method is retained with no change in the minimum dimension.

Actual markets are not likely to adjust prices according to the Newton method, orthogonal or otherwise, so we still need to determine whether there exists a stable mechanism which can be interpreted as a market adjustment process. This is of course an ambiguous criterion, but it seems reasonable to require that if the excess demand for commodity j is zero then $\dot{p}_j=0$. Unfortunately

Theorem 5.2 states that no such mechanism can be locally stable at unique equilibria. This result does not require the characteristic root condition. In other words, a stable mechanism must use the price on any given market explicitly as a tool for equilibrating other markets. This suggests that unless restricted to special classes of excess demand functions, competitive markets can operate satisfactorily only if price adjustments are made by a central planning agency. On the other hand, Theorem 5.2 may rely too heavily on the dubious identification of commodities and markets. Perhaps different excess demand functions give rise to different market structures so that the class of natural market adjustment mechanisms differs across demand functions. Of course we cannot pursue the evolution of markets here, except to note this possibility as a qualification to Theorem 5.2.

The Newton and orthogonal Newton methods depend on $f-f(p^*)$ only through the derivative $Df(p^*)$. Thus the derivative contains enough information to ensure stability, so it is natural to investigate mechanisms of the form

$$\dot{p} = A(f(p), Df(p)).$$

We will call these generalized Newton methods. Saari and Simon (1978) have already studied such mechanisms for the purpose of determining whether any entries of the matrix $Df(p)$ can be ignored. An ignorable entry is a partial derivative which need not be evaluated in computing $\dot{p}$, so the maximum number of ignorable entries may have practical significance. Saari and Simon showed that if the mechanism is required to be locally stable at all regular equilibria then no entries are ignorable. They then considered mechanisms which, when started from a fixed open set of initial prices, would converge to some equilibrium for every regular demand function. They termed these effective price mechanisms. By considering demand functions with a unique equilibrium lying within the initial open set, they reduced the requirement to local stability at unique equilibria. In this case they exhibited a mechanism similar to the orthogonal Newton method which ignores a column of $Df(p)$. In the orthogonal Newton method, the last column of $D^*f(p)$ is determined by the first $n-1$ columns of $Df(p)$ and the sign of $\det Df(p)$. If p is the unique equilibrium, $\det(-Df(p))>0$ (e.g., Dierker (1972)), so the restriction to unique equilibria enables the orthogonal Newton method to ignore a column of $Df(p)$. Saari and Simon obtained several restrictions on the number and position of ignorable entries for effective price mechanisms, depending on the number of commodities. Their results were obtained by algebraic arguments showing that if $D_zA(0,Df(p))$ is nonsingular and too many entries of $Df(p)$ are ignored, a change in the ignored entries can cause $D_zA(0,Df(p))Df(p)$ to have a characteristic root with positive real part, preventing stability. Their assumptions concerning the nonsingularity of $D_zA(0,Df(p))$ are logically intermediate between our definition of local stability, which places no direct restrictions on the derivative, and hyperbolic local stability. (A more precise statement of their assumptions and results is given in Sections 6.5 and 6.7 below). Using the homotopy methods of the present paper, we also obtain in Corollary 6.6 the result that local stability at all regular equilibria precludes ignorable entries. Corollary 6.8 states that independently of the number of commodities, a mechanism which is HLS at unique equilibria cannot have two ignorable entries in different rows and columns of $Df(p)$. In other words, there are at most n ignorable entries, and all must be either in the same column or the same row. There is an HLS mechanism, specified in Section 3.6, which is analagous to the orthogonal Newton method and ignores the last row of

$Df(p)$ at unique equilibria, so these bounds are attainable. This conclusion subsumes all of the restrictions obtained by Saari and Simon for effective price mechanisms. However, as mentioned above, the characteristic root condition involved in hyperbolic local stability is somewhat stronger than the assumptions used by Saari and Simon.

Aside from these results, the principal purpose of this paper is to introduce a new technique of stability analysis. Modern economic theory has thrived on characterizations of mechanisms which achieve specified goals on domains of natural interest. I hope that the technique used here can be extended to other equilibrium concepts, and that stability will be among the hypotheses of future characterization theorems. Some more technical open questions are mentioned in Section 7 below.

2. General Price Mechanisms

2.1 Definitions: Let P be an open subset of R^n_{++}.[1] The space of market excess demand functions, denoted $\mathcal{E}$, is the space of C^1 functions on P to R^n with the topology of C^1 uniform convergence on compact subsets of P.[2] For each $p \in P$, let $\mathcal{E}_p = \{f \in \mathcal{E}: f(p) = 0 \text{ and } \det Df(p) \neq 0\}$.

2.2 Remarks: The open set P should be interpreted as small. In particular, if K is a compact cube in R^n_{++} and P is a neighborhood of K, then for any $f \in \mathcal{E}_p$ one can easily construct a C^1 function $f^\circ: R^n_{++} \to R^n$ satisfying

i) f° is bounded from below;

ii) $pf^\circ(p)$ is bounded from above;

iii) for any sequence $\{p^k\}_{k=1}^\infty \in R^n_{++}$ converging to some p° with $p^\circ_j=0$ for some $1\leq j\leq n$, $\|f^\circ(p^k)\| \to \infty$; and

iv) f° agrees with f on K.

Thus the function $(p_1,\ldots,p_n,1) \to (f^\circ(p),-pf^\circ(p))$ has the conventional properties of a market excess demand function with commodity $n+1$ as numeraire. Since we are only concerned with local stability, we can ignore the global properties (i-iii) in defining $\mathcal{E}$. Of course the extension f° may have additional equilibria, but this possibility will not matter until Section 4 below when we need to ensure the uniqueness of equilibrium.

2.3 Price Mechanisms: A price mechanism is a function $M: P \times \mathcal{E} \to R^n$ such that for each (p,f),

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1 $R^n_{++} = \{p \in R^n: p_j>0 \text{ for all } j\}$.

2 In the terminology of Hirsch (1976), p. 35, $\mathcal{E} = C^1_W(P,R^n)$.

i) $M(p,f)=0$ if an only if $f(p)=0$; and

ii) there is an open set $\mathcal{U} \subset P \times \mathcal{E}$ with $\{p\} \times \mathcal{E}_p \subset \mathcal{U}$ such that

a) M is continuous on $\mathcal{U}$; and

b) $M(\cdot,f)$ is C^1 on the open set $\{p': (p',f) \in \mathcal{U}\}$.

Given $f \in \mathcal{E}$ and $p^{\circ} \in P$, a price mechanism determines the ordinary differential equation

$$(*) \quad \dot{p} = M(p,f)$$
$$p(0)=p^{\circ}.$$

2.4 Remarks: For the Newton method mentioned in the introduction, $M(p,f) = -(Df(p))^{-1}f(p)$, which is only well-defined if $Df(p)$ is nonsingular. The Newton method can be extended to $P \times \mathcal{E}$ by specifying $M(p,f) = f(p)$ if $Df(p)$ is singular. This satisfies 2.3 (i) and (ii), and indicates why the regularity conditions in 2.3 (ii) are not imposed on all of $P \times \mathcal{E}$. Proposition 6.2 below shows that this awkwardness cannot be avoided.

2.5 Local Stability: Given a price mechanism M , let $D = \{(p^{\circ},f)$: 2.3(*) has a unique solution on the entire time domain $[0,\infty)\}$. Define the function $q: D \times [0,\infty) \to P$ by setting $q(p^{\circ},f;t)$ equal to the solution of (*) at time t. A price mechanism is locally stable if for each $p^* \in P$ and each neighborhood U^* of p^* , there is an open neighborhood $\mathcal{U}$ of $\{p^*\} \times \mathcal{E}_{p^*}$ such that for each $(p^{\circ},f) \in \mathcal{U}$,

i) $q(p^{\circ},f;t) \in U^*$ for all t ; and

ii) $\lim_{t \to \infty} q(p^{\circ},f;t)$ exists and is in $\{p: f \in \mathcal{E}_p\}$.

We will suppose without loss of generality that $\mathcal{U}$ is also small enough so that 2.3 (ii) is satisfied.

2.6 Remarks: Condition (ii) is usually termed asymptotic stability. Regarding the demand function f as fixed, condition (i) is usually termed Lyapunov stability, which together with asymptotic stability constitutes the conventional definition of local stability (e.g., Arrow and Hahn (1971, p. 279)). However, f is variable in the present model, so condition (i) also requires Lyapunov stability in response to small perturbations in the excess demand function. This is a natural requirement since a deviation from economic equilibrium is more likely to result from a change in the economic environment specified by f than from an exogenous change in prices. Also, as mentioned in the introduction, it is usually implied by the standard characteristic root condition (e.g., Proposition 6.3 below).

The two stability conditions imply that q converges continuously to equilibrium, so that we can add the endpoint ∞ to the time interval and regard q as a continuous function on $\mathcal{U} \times [0,\infty]$. This is established in Sections 2.7 and 2.8 below. The function q is used in Theorem 2.10 to establish, in a geometrically obvious way, the basic homotopy equivalence mentioned in the introduction.

2.7 Definition: Let $[0,\infty]$ denote the one-point compactification of $[0,\infty)$ (e.g., Kelley (1955, p. 150)). The sets of the form $(a,\infty) \cup \{\infty\}$ form a neighborhood base at ∞. In particular, the function $g: [0,1] \to [0,\infty]$ defined by

$$g(x) = \begin{cases} x/(1-x) & \text{if } x<1 \\ \infty & \text{if } x=1 \end{cases}$$

is a homeomorphism.

2.8 Lemma: Suppose that M is locally stable. Let $p^* \in P$, $f^* \in E_{p^*}$, and let U^* be a neighborhood of p^*. Let $\mathcal{U}$ be given by 2.5 above, and for each $(p^\circ,f) \in \mathcal{U}$, define $q(p^\circ,f;\infty) = \lim_{t \to \infty} q(p^\circ,f;t)$. Then q is continuous on $\mathcal{U} \times [0,\infty]$.

Proof: The continuity of q on $\mathcal{U} \times [0,\infty)$ follows from 2.3 (ii) and a standard result on the continuity of solutions to ordinary differential equations (e.g., Coddington and Levinson (1955), Theorem 7.1, p. 22). To prove continuity at $t=\infty$, let $\{p_n,f_n;t_n)\}_{n=1}^{\infty} \subset \mathcal{U} \times [0,\infty)$ converging to $(p^\circ,f;\infty)$ for some $(p^\circ,f) \in \mathcal{U}$, and let $p = q(p^\circ,f;\infty)$. Let $\varepsilon>0$, and using stability condition 2.5 (i), let $\mathcal{U}'$ be a neighborhood of (p,f), such that for each $(p',f') \in \mathcal{U}'$, $\|q(p',f';t) - p\| < \varepsilon$ for all t. By stability condition 2.5 (ii), which we have used already to define p, there is some $t^*>0$ such that $(q(p^\circ,f;t^*),f) \in \mathcal{U}'$. Since $(p_n,f_n) \to (p^\circ,f)$, we have $(q(p_n,f_n;t^*),f_n) \in \mathcal{U}'$ for sufficiently large n. Hence for large n, $\|q(p_n,f_n;t_n) - p\| = \|q[q(p_n,f_n;t^*),f_n;t-t^*] - p\| < \varepsilon$.

2.9 Definitions: If X is a topological space and $A \subset X$, let $X-A$ denote the set difference $\{x \in X: x \notin A\}$, topologized as a subset of X. If g and h are continuous functions on $X-A$ to $Y-B$, g and h are homotopic, written $g \simeq h$, if there is a continuous function $H: (X-A) \times [0,1] \to Y-B$ with $H(\cdot,0) = g$ and $H(\cdot,1) = h$.

Let $p^* \in P$, let U^* be a neighborhood of p^*, and let $\mathcal{U}$ be an open neighborhood of $\{p^*\} \times E_{p^*}$ given by 2.5 above, to remain fixed from now on. Let $E^* = \{f \in E: (p^*,f) \in \mathcal{U}\}$. Then E^* is an open neighborhood of E_{p*}. Define $q^*: E^* \to R^n$ by $q^*(f^*) = q(p^*,f,\infty) - p^*$. Then $q^*: E_{p*} \to \{0\}$ and $q: E^* - E_{p*} \to R^n - 0.$[1] Also $M(p^*,\cdot): E_{p*} \to \{0\}$ and $M(p^*,\cdot): E^* - E_{p*} \to R^n - 0$.

2.10 Theorem: The functions $q^*: E^* - E_{p*} \to R^n - 0$ and $M(p^*,\cdot): E^* - E_{p*} \to R^n - 0$ are homotopic.

Proof: Define the function $H: (E^* - E_{p*}) \times [0,\infty] \to R^n - 0$ by

$$H(f,\lambda) = \begin{cases} q(p^*,f;\lambda) - p^* & \text{if } \lambda \geq 1 \ ; \\ [q(p^*,f;\lambda) - p^*]/\lambda & \text{if } 0 \leq \lambda < 1 \ ; \text{ and} \\ M(p^*,f) & \text{if } \lambda = 0. \end{cases}$$

Then $H(\cdot,0) = M^*(p^*,\cdot)$ and $H(\cdot,\infty) = q^*$. The continuity of H follows from 2.3 (ii) and Proposition 2.9. Since $[0,\infty]$ is homeomorphic to $[0,1]$, the proof is complete.

2.11 Remarks: This homotopy equivalence is necessary but not sufficient for local stability. For example if $n=2$, let M^1 denote the Newton method and for each $-1 \leq \lambda \leq 1$, let $R(\lambda)$ denote the rotation matrix

$$R(\lambda) = \begin{bmatrix} \lambda & -(1-\lambda^2)^{\frac{1}{2}} \\ (1-\lambda^2)^{\frac{1}{2}} & \lambda \end{bmatrix}.$$

Then for each λ, the mechanism defined by $M^\lambda(p,f) = R(\lambda)\, M^1(p,f)$ is obviously homotopic to the Newton method, but $M^{-1} = -M^1$ is explosive and M^0 is cyclic.

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[1] In writing $R^n - \{0\}$ we will omit the brackets for notational simplicity.

3. Price Mechanisms at a Fixed Initial Condition

Having fixed the initial price at p^*, we can decompose a demand function f into the vector $f(p^*)$ and the function $f-f(p^*)$ to analyse how $M(p^*,\cdot)$ responds to these two components separately.

<u>3.1 Definitions</u>: Suppose that $A: R^n \times E_{p^*} \to R^n$ is a continuous function such that for each $f \in E^*$,

$$M(p,f) = A(f(p^*),f-f(p^*)).$$

By Definition 2.3 (i), for each $f \in E_{p^*}$, $A(\cdot,f): R^n-0 \to R^n-0$.

Let L denote the space of linear functions on R^n-0 to R^n-0, with the topology of uniform convergence on compact subsets of R^n-0. Each $\ell \in L$ is represented by a nonsingular $n\times n$ matrix, which can be viewed as a point in R^{n^2}. The topology of uniform convergence on compact subsets agrees with the topology on L as a subspace of R^{n^2}. Define the function $i: E_{p^*} \to L$ by setting $i(f)$ equal to the linear function determined by the matrix $-Df(p^*)^{-1}$. It is clear that i is continuous.

<u>3.2 Remarks</u>: The function $M(p^*,\cdot)$ is continuous on the open set E^*, and

for simplicity the above definition of A extends this domain of continuity to $R^n \times \mathcal{E}_{p*}$. That is, $M(p*,\cdot)$ is required to be continuous at all excess demand vectors $f(p*)$, though we maintain the restriction that $Df(p*)$ is nonsingular. None of the results below would be affected if we adopted the more tedious practice of keeping track of the proper domain of continuity in each proof, which will be necessary in applications of Theorem 2.10 in any case. Hence the above definition does not place a substantive restriction on the class of price mechanisms.

<u>3.3 Theorem</u>: Suppose that M is locally stable. Then for each $f \in \mathcal{E}_{p*}$, $A(\cdot,f)$ and $i(f)$ are homotopic as maps on R^n-0 to R^n-0. In particular, if $f,f' \in \mathcal{E}_{p*}$ with sign det $Df(p*) \neq$ sign det $Df'(p*)$ then $A(\cdot,f)$ and $A(\cdot,f')$ do not coincide on any neighborhood of 0 in R^n.

<u>Proof</u>: Let $f \in \mathcal{E}_{p*}$, and let U be closed ball about 0 in R^n such that $\{f+z\colon z \in U\} \subset \mathcal{E}*$, where $\mathcal{E}*$ is specified in 2.9 above, and $f+z$ denotes the function $p \mapsto f(p)+z$. By Theorem 2.10 above, there is a homotopy $H\colon (U-0) \times [0,1] \to R^n-0$ with $H(\cdot,0) = A(\cdot,f)$ and $H(z,1) = q*(f+z)$ for each $z \in U-0$, where $q*$ is defined in 2.9 above. Define: $H'\colon (U-0) \times [0,1] \to R^n-0$ by

$$H'(z,\lambda) = \begin{cases} q*(f+z) & \text{if } \lambda=1 \\ \lambda^{-1}q*(f+\lambda z) & \text{if } 0<\lambda<1 \\ -Df(p*)^{-1}z & \text{if } \lambda=0 \end{cases}, \text{ and}$$

and define $G\colon (U-0) \times [0,1] \to R^n-0$ by

$$G(z,\lambda) = \begin{cases} H(z,2\lambda) & \text{if } \lambda \leq 1/2 \text{ ; and} \\ H'(z,2-2\lambda) & \text{if } \lambda > 1/2. \end{cases}$$

Then G is a homotopy between $A(\cdot,f)$ and $i(f)$ as maps on $U-0$. Let ε denote the radius of U, and define A' and q' on R^n-0 to R^n-0 by

$$A'(z) = \begin{cases} A(z,f) & \text{if } z \in U-0 \text{ ; and} \\ A(\varepsilon\|z\|^{-1}z,f) & \text{otherwise,} \end{cases}$$

and

$$q'(z) = \begin{cases} -(Df(p^*))^{-1}z & \text{if } z \in U-0 \text{ ; and} \\ -(Df(p^*))^{-1}(\varepsilon\|z\|^{-1}z) & \text{otherwise.} \end{cases}$$

Modifying G in the same way gives a homotopy between A' and q'. The function $H'': (R^n-0) \times [0,1] \to R^n-0$ given by

$$H''(z,\lambda) = \begin{cases} A'(z) & \text{if } z \in U-0 \text{ ; and} \\ A(\lambda\varepsilon\|z\|^{-1}z + (1-\lambda)z,f) & \text{otherwise} \end{cases}$$

shows that $A(\cdot,f) \simeq A'$. Similarly, $q' \simeq i(f)$, which proves the first assertion.

If sign det $Df(p^*) \neq$ sign det $Df(p^*)$, then $i(f)$ and $i(f')$, which are linear maps, also have determinants of different sign by the definition of i. Therefore, for every neighborhood V of 0, $i(f)$ and $i(f')$ are not homotopic as maps on $V-0$ to R^n-0 (e.g., Milnor (1965), pp. 26-31). Hence $A(\cdot,f)|V \neq A(\cdot,f')|V$.

<u>3.4 Remarks</u>: The statement that $A(\cdot,f)$ is sensitive to the sign of

$\det Df(p^*)$ is a rather coarse implication of Theorem 2.10, although it yields a strong restriction on generalized Newton methods in Corollary 6.6 below. By using a little more homotopy theory, we will obtain sharper implications below (Theorem 5.2 and Proposition 6.2). However, by strengthening the stability requirement in a conventional way, we can obtain much stronger results directly. Given a price mechanism M and a demand function $f \in \mathcal{E}_{p^*}$, we can define a function $M' : P \times R^n \to R^n$ by $M'(p,z) = M(p, f+z-f(p))$. Then $M'(p^*,z) = A(z,f)$ for all z, and $M'(p,f(p)) = M(p,f)$ for all p. Also, $M'(p,0) = 0$ for all p. If we assume that M is sufficiently smooth so that M' is C^1 at $(p^*,0)$, we obtain $D_pM(p^*,f) = D_pM'(p^*,0) + D_zM'(p^*,0)Df(p^*) = $ $= D_zM'(p^*,0)Df(p^*) = D_zA(0,f)Df(p^*)$. Hence the conventional sufficient condition for local stability at p^* becomes the requirement that all characteristic roots of $D_zA(0,f)Df(p^*)$ have negative real parts. This motivates the following definition.

<u>3.5 Definition</u>: A price mechanism is hyperbolically locally stable (HLS) if

i) for each $f \in \mathcal{E}_{p^*}$, the function $z \mapsto A(z,f)$ is C^1;

ii) the function $f \mapsto D_zA(0,f)$ is continuous on $\mathcal{E}_{p^*}$; and

iii) for each $f \in \mathcal{E}_{p^*}$, all characteristic roots of the matrix $D_zA(0,f)Df(p^*)$ have negative real parts.

Note that formally, hyperbolic local stability is a property of A, that is, $M(p^*,\cdot)$. The behavior of M at prices other than p^* is not formally involved.

<u>3.6 The Orthogonal Newton Method</u>: The Newton method is an obvious HLS mechanism. We now define a more interesting HLS mechanism by modifying a mechanism

constructed by Saari and Simon (1978, pp. 1123-1124). If $Df(p)$ is non-singular, let $D^*f(p)$ denote the orthonormal matrix obtained by applying the Gram-Schmidt process to the columns of $Df(p)$. That is, the first column of $D^*f(p)$ is the first column $Df(p)$ normalized to euclidean norm one, the second column of $D^*f(p)$ is the normalized projection of the second column of $Df(p)$ onto the orthogonal complement of the first column; etc. Note for future reference that the last column of $D^*f(p)$ can be determined from the first $n-1$ columns of $Df(p)$ and the sign of $\det Df(p)$. Define $M^*(p,f) = -D^*f(p)'f(p)$, where prime denotes transpose. Then at p^*, we have $A^*: R^n \times \mathcal{E}_{p^*} \to R^n$ defined by $A^*(z,f) = -D^*f(p^*)'z$. Also $D_zA^*(z,f)Df(p^*) = -D^*f(p^*)'Df(p^*)$ and the right hand side is an upper-triangular matrix whose $j^{\underline{th}}$ diagonal element is $-\|c^j\|$, where c^j is the $j^{\underline{th}}$ column of $D^*f(p^*)$ before normalization. This implies that M^* is HLS. We will call M^* the orthogonal Newton method.

A similar HLS mechanism can be constructed by applying the Gram-Schmidt process to the rows of $Df(p)$, yielding an orthonormal matrix $D^\circ f(p)$. Define $A^\circ: R^n \times \mathcal{E}_{p^*} \to R^n$ by $A^\circ(z,f) = -D^\circ f(p^*)z$. Then $D_zA^\circ(0,f)Df(p^*) = -D^\circ f(p^*)'Df(p^*)$, so

$$Df(p^*)\ [D_zA(0,f)Df(p^*)]\ (Df(p^*))^{-1} = -Df(p^*)D^\circ f(p^*)' ,$$

which is a lower triangular matrix whose $j^{\underline{th}}$ diagonal element is $-\|r_j\|$, where r_j is the $j^{\underline{th}}$ row of $D^\circ f(p^*)$ before normalization. We also note that the last row of $D^\circ f(p)$ is determined by the first $n-1$ rows of $Df(p)$ and the sign of $\det Df(p)$.

<u>3.7 Theorem</u>: If A is HLS then $D_zA(0,\cdot)$ and i are homotopic as maps on $\mathcal{E}_{p^*}$ to L.

Proof: We will apply the previous homotopy technique to the "first variation" of the price adjustment equation at an equilibrium. Given $f \in E_{p*}$, consider the autonomous linear differential equation on the space of $n \times n$ matrices:

$$\dot{B} = D_zA(0,f)Df(p*)B + D_zA(0,f)$$

$$B(0) = 0 \quad \text{(the zero matrix)}.$$

Let $B(t,f)$ denote the solution at time t. Since all characteristic roots of $D_zA(0,f)Df(p*)$ have negative real parts, $B(t,f)$ converges to $(-Df(p*))^{-1} = i(f)$, and the convergence is continuous in f. That is, the function $B\colon [0,\infty] \times E_{p*} \to R^{n^2}$ obtained by setting $B(\infty,f) = i(f)$ is continuous. Define $H\colon [0,\infty] \times E_{p*} \to R^{n^2}$ by

$$H(t,f) = \begin{cases} B(t,f) & \text{if } t \geq 1 \ ; \\ t^{-1}B(t,f) & \text{if } 0<t<1 \ ; \text{ and} \\ D_zA(0,f) & \text{if } t=0. \end{cases}$$

Then H is continuous, $H(0,f) = D_zA(0,f)$, and $H(\infty,f) = i(f)$, so it only remains to show that $H(t,f) \in L$, that is, $H(t,f)$ is nonsingular for all $0<t<\infty$. It suffices to show that $B(t,f)$ is nonsingular for all $t>0$, so let c be a nonzero vector in R^n, and consider the differential equation on R^n given by

$$\dot{y} = D_zA(0,f)Df(p*)y + D_zA(0,f)c$$

$$y(0) = 0$$

with the solution $y(t)$. Again, this equation is stable and since it is autonomous, if $y(t)=0$ for some $t>0$, then $y(\cdot)\equiv 0$. However, the solution is given by $y(t) = B(t,f)c$, so $y(\infty) \neq 0$. Hence $B(t,f)c \neq 0$ for all

$t>0$, which proves the theorem.

3.8 Remarks: Theorem 3.7 is a much stronger statement about the dependence of M on $f-f(p)$ than is Theorem 3.3. Theorem 3.7 implies that up to homotopy, $D_zA(0,\cdot)$ is a 1-1 correspondence on $\mathcal{E}_{p*}$. This implication is established in Corollary 3.11 below. The next two sections give some preliminary definitions and results.

3.9 Definitions: Let $\mathcal{O}$ denote the subset of $\mathcal{L}$ consisting of orthonormal matrices. Then $\mathcal{O}$ is a smooth compact $n(n-1)/2$ -dimensional manifold consisting of the two diffeomorphic components $\{\ell \in \mathcal{O}\colon \det \ell = 1\}$ and $\{\ell \in \mathcal{O}\colon \det \ell = -1\}$ (e.g., Warner (1971), pp. 130-131).

Topological spaces Y and Z are homotopy equivalent, written $Y \simeq Z$, if there exist continuous functions $a\colon Y \to Z$ and $b\colon Z \to Y$ with $b \circ a \simeq 1_Y$ and $a \circ b \simeq 1_Z$, where 1_Y (respectively 1_Z) is the identity function on Y (respectively Z). It is easy to verify that $\simeq$ is an equivalence relation. The functions a and b are called homotopy equivalences. It is easy to verify that any map homotopic to a homotopy equivalence is itself a homotopy equivalence.

3.10 Proposition: Define $j\colon \mathcal{L} \to \mathcal{E}_{p*}$ by $j(\ell)(p) = -\ell^{-1}(p-p^*)$. Then $i \circ j = 1_{\mathcal{L}}$, and $j \circ i \simeq 1_{\mathcal{E}_{p*}}$. Let $g\colon \mathcal{L} \to \mathcal{O}$ denote the normalized Gram-Schmidt projection. Then g is a homotopy equivalence, so we have $\mathcal{E}_{p*} \simeq \mathcal{L} \simeq \mathcal{O}$.

Proof: Since $i(f) = (-Df(p^*))^{-1}$, it is immediate that $i \circ j = 1_{\mathcal{L}}$. For each $f \in \mathcal{E}_{p*}$, $j \circ i(f)$ is the function $p \mapsto Df(p^*)(p-p^*)$. Define $H\colon \mathcal{E}_{p*} \times [0,1] \to \mathcal{E}_{p*}$ by

$$H(\lambda,f)(p) = \begin{cases} f(p) & \text{if } \lambda=1 \ ; \\ \lambda^{-1}f(p^* + \lambda(p-p^*)) & \text{if } 0<\lambda<1 \ ; \text{ and} \\ Df(p^*)(p-p^*) & \text{if } \lambda=0. \end{cases}$$

Then H is the desired homotopy between $j \circ i$ and $1_{\mathcal{E}_{p^*}}$. If $a: \mathcal{O} \to \mathcal{L}$ is the inclusion map, we have $g \circ a = 1_{\mathcal{O}}$, and there is a geometrically obvious homotopy between $1_{\mathcal{L}}$ and $a \circ g$. Hence g is a homotopy equivalence, which completes the proof.

3.11 Corollary: If A is HLS then $D_zA(0,\cdot)$ is a homotopy equivalence. In particular, $j \circ D_zA(0,\cdot) \simeq 1_{\mathcal{E}_{p^*}}$.

Proof: By Theorem 3.7, $D_zA(0,\cdot) \simeq i$, and by Proposition 3.10, $j \circ i \simeq 1_{\mathcal{E}_{p^*}}$.

3.12 Remarks: A natural way to clarify the informational implication of this corollary is to ask how many parameters would be needed to paramaterize the knowledge of f required by $D_zA(0,f)$. This question is answered by Corollary 3.14 below, which also implies that the orthogonal Newton method is informationally minimal.

3.13 Definitions: Let X be a topological space, let $\pi: \mathcal{E}_{p^*} \to X$ be continuous, and let $B: R^n \times X \to R^n$ be a continuous function satisfying

i) for each $x \in X$, $B(\cdot,x)$ is C^1 ;

ii) the function $x \mapsto D_zB(0,x)$ is continuous; and

iii) $B(z,\pi(f)) = A(z,f)$ for all $(z,f) \in R^n \times \mathcal{E}_{p^*}$.

The space X is an m-dimensional manifold if X is Hausdorff and each $x \in X$

has an open neighborhood which is homeomorphic to R^m.

3.14 Corollary: If A is HLS and X is a manifold, then $\dim X \geq n(n-1)/2 = \dim \mathcal{O}$.

Proof: Let $m = n(n-1)/2$. Since $\mathcal{O}$ is a compact m-dimensional manifold with two components, $H_m(\mathcal{O},Z_2) = Z_2 \oplus Z_2$, where $H_m(\mathcal{O},Z_2)$ denotes the m^{th} homology group of $\mathcal{O}$ with coefficients in the integers mod 2 (Dold (1972), Corollary 3.4, p. 260). By Proposition 3.10, $H_m(E_{p^*},Z_2) = H_m(L,Z_2) = H_m(\mathcal{O},Z_2)$, and the map i induces an isomorphism on $H_m(E_{p^*},Z_2)$ to $H_m(L,Z_2)$. Let $X^\circ = \{x \in X: D_zB(0,x)$ is nonsingular$\}$. Then $\pi(E_{p^*}) \subset X^\circ$, and X° is an open subset of X, and thus a manifold of the same dimension. By Theorem 3.7, the abovementioned isomorphism factors through $H_m(X^\circ,Z_2)$ so $H_m(X^\circ,Z_2) \neq 0$. Hence by [Dold (1972), Proposition 3.3(a), p. 260] $\dim X \geq m$.

3.15 Remarks: For the orthogonal Newton method, we can take $X = \mathcal{O}$, $\pi(f) = -Df(p^*)'$, and $B(z,\ell) = \ell z$. Hence the orthogonal Newton method achieves the minimum dimension.

Corollary 3.11 above implies that all of the homotopy groups of $\mathcal{O}$ factor through the corresponding groups of X, so that X is homotopically at least as rich as $\mathcal{O}$. Corollary 3.16 below states that if π is sufficiently like a projection that π^{-1} admits a continuous selection, then X and $\mathcal{O}$ are homotopy equivalent.

3.16 Corollary: Suppose that A is HLS and there is a continuous function $\gamma: X \to E_{p^*}$ such that $\pi \circ \gamma = 1_X$. Then $X \simeq \mathcal{O}$.

Proof: Let B^* denote the function $D_z B(0,\cdot)$. Since $B^* \circ \pi = D_z A(0,\cdot)$, Corollary 3.11 implies that $j \circ B^* \circ \pi \simeq 1_{E_{p^*}}$. Hence $j \circ B^* \circ \pi \circ \gamma \simeq \gamma$, and since $\pi \circ \gamma = 1_X$, $j \circ B^* \simeq \gamma$. Therefore $\pi \circ (j \circ b^*) \simeq \pi \circ \gamma = 1_X$, so π and $j \circ B^*$ establish that $X \simeq E_{p^*}$. By Proposition 3.10, $E_{p^*} \simeq 0$, which completes the proof.

4. Stability at Unique Equilibria

Thus far we have required price mechanisms to be stable at p^* whenever p^* is a regular equilibrium. However, if $\det(-Df(p^*)) < 0$, we know from index considerations (e.g., Dierker (1972)) that if f is regular there must be at least two other equilibria. In this section we will only require stability at p^* if p^* is the unique equilibrum. This is equivalent to requiring stability when $\det(-Df(p^*)) > 0$, in the sense that if f satisfies the latter condition then there is a demand function f° which coincides with f near p^* and has p^* as its unique equilibrium. This result is stated in Proposition 4.2 below. A similar result for linear hyperbolic functions f was used in the same way by Saari and Simon (1978, Lemma 5, p. 1113). Since our results depend only on the behavior of demand functions near p^*, this criterion for uniqueness is sufficient.

<u>4.1 Definition</u>: Let $\mathcal{E}^+_{p^*} = \{f \in \mathcal{E}_{p^*}: \det(-Df(p^*)) > 0\}$. A price mechanism M is <u>locally stable at unique equilibria</u> if Definition 2.5 above is satisfied with $\mathcal{E}_{p^*}$ replaced by the subset $\mathcal{E}^+_{p^*}$. A mechanism M (more precisely A) is <u>HLS</u> at <u>unique equilibria</u> if Definition 3.5 above is satisfied with $\mathcal{E}_{p^*}$ replaced by $\mathcal{E}^+_{p^*}$. Let $\mathcal{L}^+ = \{\ell \in \mathcal{L}: \det \ell > 0\}$, and let $\mathcal{R} = \{\ell \in \mathcal{O}: \det \ell = 1\}$.

<u>4.2 Proposition</u>: Let $f \in \mathcal{E}^+_{p^*}$. Then there is an open neighborhood N of p^* and a C^1 function $f^\circ: R^n_{++} \to R^n$ satisfying

i) f° agrees with f on N;

ii) f° is bounded from below;

iii) $pf^{\circ}(p)$ is bounded from above;

iv) for any sequence $\{p^k\}_{k=1}^{\infty}$ in R^n_{++} converging to some $p^{\circ} \in \partial R^n_{++}$, $\lim \|f^{\circ}(p^k)\| = \infty$; and

v) $f^{\circ}(p) \neq 0$ if $p \neq p^*$.

Proof: Since $Df(p^*)$ is nonsingular, there is an ε-ball $B_{\varepsilon} \subset P$ centered at p^* with $\varepsilon < \min\{p_j : 1 \leq j \leq n\}/3$ and $f(p) \neq 0$ for all $p \in B_{2\varepsilon}$. Let $\sigma : B_{2\varepsilon} \to [0,1]$ be a C^1 function satisfying

a) $\sigma(p) = 1$ if $\|p^*-p\| \leq \varepsilon$;

b) $\sigma(p) = 0$ and $D\sigma(p) = 0$ if $\|p^*-p\| = 2\varepsilon$

Now define $f^*: B_{2\varepsilon} \to R^n$ by

$$f^*(p) = \begin{cases} f(p^* + \sigma(p)(p-p^*))/\sigma(p) & \text{if} \quad \sigma(p) \neq 0; \text{ and} \\ Df(p^*)(p-p^*) \quad \text{if} & \sigma(p) = 0. \end{cases}$$

Then f^* is a C^1 function on $B_{2\varepsilon}$ which agrees with f on B_{ε} and $f^*(p) \neq 0$ if $p \neq p^*$. Also, (b) implies that for each $p \in \partial B_{2\varepsilon}$, $f^*(p) = Df(p^*)(p-p^*)$ and $Df^*(p) = Df(p^*)$. Since $\det(-Df(p^*)) > 0$, sign det $Df(p^*) =$ sign det $(-I)$, where I is the identity matrix. Therefore there is a continuous function $\tau: [0,\varepsilon] \to L$ satisfying

c) $\tau(0) = Df(p^*)$ and $\tau(\varepsilon) = -I$

(Warner (1971), Theorem 3.68, p. 131). Since L is a smooth manifold (as an open subset of R^{n^2}) it is straightforward to smooth τ to a C^1 function satisfying, in addition to (c),

(d) $D\tau(0) = D\tau(\varepsilon) = 0$.

Now extend f^* to $B_{3\varepsilon}$ by defining $f^*(p) = \tau(\|p-p^*\|-2\varepsilon)(p-p^*)$ if $2\varepsilon \leq \|p-p^*\| \leq 3\varepsilon$. Then on $B_{3\varepsilon}$, f^* is a C^1 function satisfying $f^*(p) \neq 0$ if $p^* \neq 0$, and for all $p \in \partial B_{3\varepsilon}$, $f^*(p) = p^*-p$ and $Df^*(p) = -I$. Now extend f^* to R^n_{++} by setting $f^*(p) = p^*-p$ if $p \notin B_{3\varepsilon}$. For each $1 \leq j \leq n$, let $h_j: (-\infty, p^*_j) \to R$ be a C^1 strictly increasing function satisfying,

(e) h_j is bounded from below;

(f) $\lim_{x \to p^*_j} h_j(x) = \infty$; and

(g) $h_j(x) = x$ for all $x \in [-p^*_j/2, p^*_j/2]$.

Finally, define $f^\circ: R^n_{++} \to R^n$ by $f^\circ(p) = (h_j(f_j(p)))^n_{j=1}$. Assuming that ε was chosen small enough so that $|f_j(p)| \leq p^*_j/2$ for all $p \in B_\varepsilon$, $f^\circ(p) = f(p)$ for all $p \in B_\varepsilon$. Hence condition (i) of the Theorem is satisfied with $N = \text{int}\, B_\varepsilon$. Also, (e) yields condition (ii) and (f) yields condition (iv). Conditions (iii) and (iv) follow from the definition of f^* , (g), and the fact that each h_j is strictly increasing. Therefore f° is the desired extension.

<u>4.3 Remarks:</u> If an adjustment function A is restricted to $R^n \times E^+_{p^*}$, Theorem 3.3 is trivial. However, the results concerning hyperbolic local stability are much less affected. They are recorded with the necessary

modifications below. The proofs are identical to the corresponding proofs in Section 3.

4.4 Theorem: If A is HLS at unique equilibria then $D_zA(0,\cdot)$ and i are homotopic as maps on $\mathcal{E}^+_{p*}$ to $\mathcal{L}^+$.

4.5 Proposition: With the obvious domain restrictions, $i \circ j = 1_{\mathcal{L}^+}$, and $j \circ i \simeq 1_{\mathcal{E}^+_{p*}}$, so $\mathcal{E}^+_{p*} \simeq \mathcal{L}^+$. Also $\mathcal{L}^+ \simeq \mathcal{R}$.

4.6 Corollary: If A is HLS at unique equilibria then $D_zA(0,\cdot)$ is a homotopy equivalence. In particular $j \circ D_zA(0,\cdot) \simeq 1_{\mathcal{E}^+_{p*}}$.

4.7 Definition: The definition of B in 3.13 above is modified by replacing $\mathcal{E}_{p*}$ with $\mathcal{E}^+_{p*}$.

4.8 Corollary: Suppose that A is HLS at unique equilibria and that X is a manifold. Then $\dim X \geq n(n-1)/2 = \dim \mathcal{R}$.

Proof: Since $\mathcal{R}$ is a component of $\mathcal{O}$, $H_{n(n-1)/2}(\mathcal{R}; Z_2) = Z_2$, so the proof of Corollary 3.14 applies.

4.9 Corollary: Suppose that A is HLS at unique equilibria and there is a continuous function $\gamma: X \to \mathcal{E}^+_{p*}$ with $\pi \circ \gamma = 1_X$. Then $X \simeq \mathcal{R}$.

5. Market Mechanisms

This section applies Theorem 2.10 to show that except in the two commodity case there is no market mechanism which is locally stable at unique equilibria. A market mechanism is defined by the property that if the market for commodity j is in equilibrium then $\dot{p}_j = 0$.

5.1 Definition: A price mechanism M is a market mechanism if there exists a continuous function $A: R^n \times E_{p*} \to R^n$ such that for each $f \in E*$,

i) $M(p*,f) = A(f(p*), f-f(p*))$; and

ii) for each $1 \leq j \leq n$, $A_j(f(p*), f-f(p*)) = 0$ if $f_j(p*) = 0$, where A_j (resp. f_j) denotes the $j^{\underline{th}}$ coordinate of A (resp. f).

5.2 Theorem: If $n \geq 2$ there does not exist a market mechanism which is locally stable at unique equilibria.

Proof: Define the function $d: R \to E_{p*}$ by setting $d(\ell)$ equal to the demand function defined by $d(\ell)(p) = -\ell^{-1}(p-p*)$. Then d is continuous and since R is compact, $d(R)$ is compact. It follows easily that there is some $\varepsilon>0$ such that $\{f + (\varepsilon,0,\ldots,0): f \in d(R)\} \subset E*$. Let $\varepsilon^{\circ} = (\varepsilon,0,\ldots,0)$. Then for each $\ell \in R$, $q*(d(\ell) + \varepsilon^{\circ}) = \ell\varepsilon^{\circ} = \varepsilon\ell^1$, where ℓ^1 is the first column of ℓ. Let M be a market mechanism and suppose by way of contradiction that M is locally stable at unique equilibria. Then by Theorem 2.10, $A(\varepsilon^{\circ},d(\cdot))$ and $q*(d(\cdot) + \varepsilon^{\circ})$ are homotopic as maps on R to R^n-0. By Condition 5.1 (ii), $A(\varepsilon^{\circ},d(R))$ is a compact subset of $\{y \in R^n-0; y_j = 0 \text{ for all } j>1\}$, which is contractible in R^n-0 since $n \geq 2$. Therefore $A(\varepsilon^{\circ},d(\cdot))$ is homotopic to a constant function. However, the map $\ell \mapsto \ell^1$ is the projection in the standard fibre bundle representation of $\mathcal{R}$, and is not homotopic to a constant (e.g., Steenrod (1951), Corollary and subsequent remark, p. 54). Hence $q^*(d(\cdot) + \varepsilon^\circ)$ and $A(\varepsilon^\circ, d(\cdot))$ are not homotopic, and this contradiction completes the proof.

<u>5.3 Remarks</u>: The Theorem states that local stability at unique equilibria requires the adjustment $\dot{p}_j$ to depend directly on the excess demand on other markets. In fact, it must depend directly on the excess demand on all other markets. For example, suppose we partition the set of commodities $\{1,\ldots,n\}$ into disjoint nonempty subsets m_1, and m_2, and weaken the defining condition 5.1 (ii) to

(ii′) for some $j \in m_1$, $A_j(f(p^*), f-f(p)) = 0$ if $f_i(p^*) = 0$, for all $i \in m_1$.

Then the same method of proof shows this requirement to be inconsistent with local stability at unique equilibria.

6. Generalized Newton Methods

This section is devoted to mechanisms which determine $\dot{p}$ as a function of the excess demand vector $f(p)$ and the derivative $Df(p)$. The principal question, first posed by Saari and Simon (1978), is whether any of the entries of the matrix $Df(p)$ can be ignored. Before turning to this question we show in Proposition 6.2 below that such mechanisms, if locally stable at unique equilibria, cannot be continuously extended from the space of nonsingular matrices $Df(p)$ to the space of all matrices. Thus the somewhat awkward domain restriction in Definition 2.3 above is essential. Proposition 6.3 notes that if a generalized Newton method is HLS then it is locally stable, strengthening the motivation for hyperbolic local stability given in Section 3.4 above.

6.1 Definitions: A price mechanism M is a generalized Newton method[1] if there is a C^1 function $A\colon R^n \times L \to R^n$ satisfying

$$M(p,f) = A(f(p),\, Df(p))$$

for all $(p,f) \in P \times E$ with $Df(p)$ nonsingular. It follows that a generalized Newton method is HLS (resp. HLS at unique equilibria) if for every $\ell \in L$ (resp. $\ell \in L^+$), $D_zA(0,\ell)$ is nonsingular and all characteristic roots of $D_zA(0,\ell)\ell$ have negative real parts.

6.2 Proposition: Suppose there is a continuous function $\bar{A}\colon (R^n-0) \times R^{n^2} \to R^n-0$ which agrees with A on $(R^n-0) \times L$. Then M is not locally stable

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1 Saari and Simon use the term generalized Newton method for Smale's global Newton method.

at unique equilibria.

Proof: Let $\varepsilon > 0$ and let $\varepsilon^{o} = (\varepsilon,0,\ldots,0)$. Since R^{n^2} is contractible to 0, $A(\varepsilon^{o},\cdot)$ is homotopic to the constant function $\ell \mapsto A(\varepsilon^{o},0)$. However, exactly as in the proof of Theorem 5.2 above, Theorem 2.10 can be applied to show that this is inconsistent with stability at unique equilibria.

6.3 Proposition: If a generalized Newton method is HLS (resp. HLS at unique equilibria) then it is locally stable (resp. locally stable at unique equilibria).

Proof: This is a straightforward application of a standard result in the stability theory of ordinary differential equations (e.g., Coddington and Levinson (1955, Theorem 1.1, p. 314)).

6.4 Definition: An entry ij is ignorable if there is some neighborhood U of 0 in R^n such that for every $z \in U$, every $\ell \in L$, and every $\ell'_{ij} \in R$,

$$A(z,\ell) = A(z,\ell/\ell'_{ij})$$

if $\ell/\ell'_{ij} \in L$, where ℓ/ℓ'_{ij} is the matrix obtained by substituting ℓ'_{ij} for ℓ_{ij} in ℓ. Two entries ij and km are rc-distinct if $i \neq k$ and $j \neq m$.

6.5 Remarks: The definition of ignorable entry is due to Saari and Simon (they use the term ignorable coordinate). They define a price mechanism to be locally effective if at every regular equilibrium it is locally stable in the conventional sense (which does not require Lyaponov stability in response to perturbations of the demand function) and for some $\ell \in L$, $D_z A(0,\ell)$ is nonsingular. They show that a locally effective price mechanism admits no ignorable entries. In Corollary 6.6 we obtain the present analogue of their result as an application of Theorem 3.3 above.

6.6 Corollary: A locally stable generalized Newton method admits no ignorable entries.

Proof: Given any entry ij , choose $\ell \in L$ and $\ell'_{ij} \in R$ so that $\det \ell = -\det(\ell/\ell'_{ij})$. Then $\det i(\ell) = -\det i(\ell/\ell'_{ij})$, where $i(\ell) = -\ell^{-1}$. If M is locally stable, Theorem 3.3 above implies that $A(\cdot,\ell) \neq A(\cdot,\ell/\ell'_{ij})$ when restricted to any neighborhood of 0 in R^n. Hence ij is not ignorable.

6.7 Remarks: Saari and Simon define a mechanism to be effective if there is a fixed open set $V \subset P$ such that for every regular demand function f , all solutions of

$$\dot{p} = A(f(p),Df(p))$$

$$p(0) \in V$$

converge to an equilibrium of f, and $D_z A(0,Df(p))$ is nonsingular for some equilibrium p (p. 1103 and p. 1123). They obtain several restrictions on ignorable entries for effective price mechanisms (Theorem, pp. 1105-1106), which we paraphrase below:

i) there cannot be two rc-distinct ignorable entries if the corresponding entries of $D_z A(0,\ell)$ are not identically zero on L;

ii) if an entire column of entries is ignored, no other entries can be ignored; and

iii) if $n=2$ or 3, there cannot be two rc-distinct ignorable entries; and if $n=4$, there cannot be three ignorable entries two of which are rc-distinct.

They remark that the nonsingularity assumption on $D_z A(0,0f(p))$ can be weakened depending on the result (p. 1123).

If M is effective, it must be locally stable at p if $p \in V$ and f is a regular demand function having p as its unique equilibrium. Using a result similar to Proposition 4.2 above (Lemma 5, p. 1113), Saari and Simon infer that all characteristic roots of the nonsingular matrix

$$D_z A(0,\ell)\ell$$

must have nonpositive real parts whenever ℓ is a hyperbolic matrix with $\det(-\ell)>0$. If too many entries are ignored, they show that manipulating the ignored entries can create a characteristic root with a positive real part.

Corollary 6.8 below applies Corollary 4.6 to show that if the characteristic root condition is strengthened by requiring that the real parts be negative then rc-distinct entries cannot be ignored, a conclusion which implies (i-iii) above. This result was reported without proof in a preliminary announcement by Saari and Simon (1976). The Newton method A^* and its analogue A° defined in Section 3.6 above indicate that this is the strongest possible restriction on ignorable entries for mechanisms which are HLS at unique equilibria.

<u>6.8 Corollary:</u> A generalized Newton method which is HLS at unique equilibria

cannot ignore rc-distinct entries.

Proof: Suppose by way of contradiction that M is HLS at unique equilibria and has ignorable entries ij and km with $i \neq k$ and $j \neq m$. Then $n \geq 2$, and we will assume for the moment that $ij = 11$ and $km = 22$. Let $S^1 = \{(x,y) \in R^2: x^2 + y^2 = 1\}$, and define the function $\alpha: S^1 \to \mathcal{R}$ by

$$\alpha(x,y) = \left(\begin{array}{cc|c} x & -y & \\ & & 0 \\ y & x & \\ \hline & 0 & I_{n-2} \end{array}\right)$$

where I_{n-2} denotes the $(n-2) \times (n-2)$ identity matrix.

Define $\beta: S^1 \to \mathcal{L}^+$ by

$$\beta(x,y) = \left(\begin{array}{cc|c} 2 & -y & \\ & & 0 \\ y & 2 & \\ \hline & 0 & I_{n-2} \end{array}\right).$$

Since 11 and 22 are ignorable entries,

$$D_zA(0,\alpha(x,y)) = D_zA(0,\beta(x,y)) \quad \text{for all} \quad (x,y) \in S^1.$$

Define $H: S^1 \times [0,1] \to \mathcal{L}^+$ by

$$H(x,y;\lambda) = \left(\begin{array}{cc|c} 2-\lambda & -y(1-\lambda) & \\ & & 0 \\ y(1-\lambda) & 2-\lambda & \\ \hline & 0 & I_{n-2} \end{array}\right).$$

Then H is a homotopy between β and the constant function $(x,y) \to I_n$, so $D_zA(0,\alpha(\cdot)) = D_zA(0,\beta(\cdot)): S^1 \to \mathcal{L}$ is homotopic to the constant function

$(x,y) \mapsto D_z A(0, I_n)$. However, α generates the fundamental group of $\mathcal{R}$ [Whitehead (1942), p. 133], and thus the fundamental group of $\mathcal{L}^+$. Corollary 4.6 implies that the map $\ell \mapsto D_z A(0,\ell)$ induces an isomorphism on the fundamental group of $\mathcal{L}^+$, so $D_z A(0,\alpha(\cdot))$ cannot be homotopic to a constant map. This contradiction shows that 11 and 22 cannot be ignorable entries. For general ij and km, let $g\colon \mathcal{L}^+ \to \mathcal{L}^+$ be a linear isomorphism constructed by interchanging rows and columns so that entry ij goes to 11 and km goes to 22. Then in the above argument α, β, and H can be replaced by $g^{-1} \circ \alpha$, $g^{-1} \circ \beta$, and $H(g^{-1}(\cdot);\cdot)$ respectively, to complete the proof.

7. Remaining Problems

The above results give rise to several open questions. The first concerns the role of hyperbolic local stability. Several results, such as Theorem 3.7, cannot be stated without this condition, or at least the nonsingularity of $D_zA(0,f)$. However, I do not know whether Corollary 3.14 is true of general locally stable mechanisms, or whether Corollaries 4.8 and 6.8 are true for mechanisms which are locally stable at unique equilibria. Second, it may be possible to use the technique developed here to study mechanisms which are not locally stable for general regular demand functions. For example, it may be possible to obtain a unified homotopy-theoretic characterization of the demand functions for which the tatonnement process is locally stable. Third, it would of course be desirable to determine whether the above results can be duplicated for price mechanisms in discrete time. Finally, there is the natural question of whether the technique developed here can be extended to the study of global stability.

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