THE INFORMATIONAL REQUIREMENTS OF LOCAL STABILITY IN DECENTRALIZED ALLOCATION MECHANISMS

by

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Abstract

This paper presents a general model of continuous time decentralized adjustment processes. Each agent chooses a message in response to his private characteristics and the state of the mechanism, and the messages of all agents determine an adjustment in the state. The first result is that, under a local stability condition, the paths from initial states to stationary states describe a homotopy equivalence between the map from environments to their equilibrium states and the map from environments to adjustments at a given initial state. This equivalence is subsequently applied to adjustment processes for exchange environments with the competitive message space as the state space. It is shown that the local stability of competitive equilibria requires each trader's message space to be large enough to permit the communication of the derivative of his demand function as well as his demand. Pareto optimal trades can be achieved as locally stable equilibria of an adjustment process in which the states are trades, and traders communicate only their normalized utility gradients. Hence the competitive equilibrium concept appears to lose its well-known informational efficiency property if local stability is required.

1. Introduction

An informationally decentralized allocation mechanism as modelled in the seminal paper of Hurwicz (1960) is an iterative communication process which generates the information needed to implement the allocations dictated by a given performance correspondence. The model is explicitly dynamic, and subsequent papers have described mechanisms which converge to Pareto optimal allocations for large classes of environments (e.g. Hurwicz, Radner and Reiter (1975a) and (1975b), and Mitsui (1981)). However, the theory introduced by Hurwicz is perhaps better known for its achievements in measuring the minimal amount of communication needed to implement a given performance correspondence (e.g. see Reiter (1977), Jordan (1979) and the references therein). Except for an example by Reiter (1979), the latter issue has only been studied in static versions of the model, where the problem of implementing a given performance correspondence formally reduces to the problem of checking, in a decentralized way, whether a prospective allocation is consistent with it. The object of the present paper is to place the minimal communication problem in a dynamic context and to study the informational requirements of local stability, in addition to implementation in the static sense.

Section 2 below provides a general model of continuous-time decentralized adjustment processes. The model is motivated by an extensive discussion at the beginning of that section, so the remainder of this introduction will be confined to a list of topics and results. Theorem 2.10 provides the principal analytical tool of the paper by establishing a homotopy equivalence between the initial direction of movement dictated by the adjustment process and the direction of the eventual equilibrium.

Although the statement of this result and its subsequent use are somewhat technical, it is proved by an obvious use of the paths generated by the adjustment process itself. This result is applied in Theorem 2.14 to obtain general conditions ensuring that the required dimension of the message space is at least as large as the number of variables being adjusted. This is a natural conclusion, but one which admits natural counter examples, as shown in Section 4. Another application is made in Section 3 to show that any adjustment process for exchange environments which is locally stable at regular Walrasian equilibria must use messages large enough for each trader to communicate his excess demand and the derivative of his excess demand function at the current price. In Section 4 we define a Pareto improving trade adjustment process in which each trader communicates his demand price at the current trade. This process is shown by Theorem 4.2 to be informationally efficient and by Proposition 4.5 to be locally stable at every Pareto optimal allocation. The results of Section 3 and 4, taken together, seem to indicate that the well-known informational efficiency of the competitive allocation mechanism is limited to the static model.

2. Decentralized Adjustment Processes

We first give an informal sketch of some developments in informational decentralization theory in order to motivate the model introduced in this paper. In somewhat condensed form, the model of a decentalized allocation mechanism pioneered by Hurwicz (1960) can be described as follows. Each agent has a set of characteristics E^i, known only to himself, and the set of environments is the product $E = \Pi_i E^i$. A performance correspondence $\pi\colon E \twoheadrightarrow Y$ associates with each environment $e = (e^i)_{i=1}^N$ a subset $\pi(e)$ of the set Y of possible decisions or allocations. A decentralized allocation mechanism consists of a communication process by which agents communicate enough information about their private characteristics to enable an allocation in $\pi(e)$ to be selected. Agent i has a set of possible messages M^i, and he transmits messages according to a message response function $f^i\colon E^i \times M \to M^i$, where $M = \Pi_i M^i$. That is, agent i with the characteristic e^i will transmit the message $m_{t+1}^i = f^i(e^i;m_t)$ in response to the joint-message $m_t = (m_t^1,\ldots,m_t^N)$. If we define the function $f\colon E \times M \to M$ by $f(e,m) = (f^i(e^i,m))_i$, we can describe the communication process by the first-order difference equation

$$(*)\quad m_{t+1} - m_t = f(e,m_t) - m_t.$$

A continuous time analogue can be constructed by modelling the message response as a continuous adjustment, $\dot{m}^i = f^i(e^i,m)$, yielding the ordinary differential equation

$$(*')\quad \dot{m} = f(e,m).$$

Returning to the discrete time case, we can define the equilibrium message correspondence $\mu\colon E \twoheadrightarrow M$ which associates with each environment $e \in E$ the set of stationary messages $\mu(e) = \{m \in M\colon f(e,m) = m,$ that is, $f^i(e^i,m) = m^i$ for all $i\}$. For the continuous time version, $\mu(e) = \{m \in M\colon f(e,m) = 0\}$. Each message is associated with an allocation according to an outcome function $g\colon M \to Y$, so the set of equilibrium allocations for an environment e is the set $g(\mu(e))$. The allocation mechanism is said to realize or achieve the performance correspondence π if

R) $g(\mu(e)) = \pi(e)$ for each $e \in E$.

For some purposes a condition weaker than (R) is appropriate. For example, if each agent's characteristics include preferences on the set of allocations, and π is the Pareto correspondence, the allocation mechanism is "nonwasteful" if

R') $\mu(e) \neq \emptyset$ and $g(\mu(e)) \subset \pi(e)$ for each $e \in E$.

These realization criteria involve only the equilibrium messages and their associated allocations. In order to interpret the response functions as describing an adjustment process as well as a set of equilibria, we need to impose some form of stability. Local stability can be stated informally as

LS) for each $e \in E$, every solution of (*) (resp. (*')) with initial condition near $\mu(e)$ converges to some message in $\mu(e)$.

A principal focus of informational decentalization theory since its inception has been the study of the amount of communication needed to realize a given performance correspondence. A mechanism which minimizes communication is said to be informationally efficient. The size of the message space M, in terms of dimension or some generalization thereof, is the most commonly used measure of the amount of communication. The best known achievement in this area is the proof that the competitive mechanism is informationally efficient relative to nonwasteful mechanisms for classical exchange environments. Various versions of this result, its analogue concerning the Lindahl mechanism for environments with public goods, and other results on informational efficiency relative to mechanisms satisfying (R) or (R') have appeared in numerous papers. However, as far as I am aware, only Reiter (1979) has studied informational efficiency relative to both (R) and (LS). The addition of a stability requirement cannot reduce the necessary amount of communication, and one would naturally conjecture that in some cases communication must increase. This conjecture is verified by Reiter (1979), for an example of a space of environments and a performance function among the class of abstract decision problems studied in Hurwicz, Reiter and Saari (1980). The additional communication required for stability is also the focus of the present paper. However, we will modify the model described above in order to accomodate developments which have occurred since Hurwicz's original formulation.

In a seminal paper on informational efficiency, Mount and Reiter (1974) introduced an elegant technique for abstracting the equilibrium behavior of a decentralized allocation mechanism. From the above formulation, define for each i the correspondence $\mu^i: E^i \twoheadrightarrow M$ by $\mu^i(e^i) = \{m \in M: m^i = f^i(e^i,m)\}$. That is, $\mu^i(e^i)$ is the set of message

N-tuples such that agent i with characteristic e^i is in equilibrium. Then $\mu(e) = \cap_{i=1}^{N}\mu^i(e^i)$, so the influence of the response functions on the equilibrium message correspondence is captured entirely by the correspondences μ^i. Mount and Reiter pursued this abstraction a step further by condensing the response functions to the equilibrium correspondence μ, together with an analytical condition on μ ensuring that μ is the intersection of individual correspondences μ^i. The "crossing condition" can be written

×) for each $e, e' \in E$ and each i

$$\mu(e) \cap \mu(e') = \mu(e^1,\ldots,e^{i-1},e'^i,e^{i+1},\ldots,e^N) \cap \mu(e'^1,\ldots,e'^{i-1},e^i, e'^{i+1},\ldots,e'^N).$$

A set theoretic argument (Mount and Reiter (1974), Lemma 5, p. 171) shows that (×) is equivalent to the existence of correspondences $\mu^i\colon E^i \twoheadrightarrow M$ for each i with $\mu(e) = \cap_{i=1}^{N}\mu^i(e^i)$ for all $e \in E$. A correspondence $\mu\colon E \twoheadrightarrow M$ which satisfies (×) was termed a "coordinate correspondence". Hence the study of the equilibrium behavior of decentralized allocation mechanisms was reduced to the study of coordinate correspondences. It has since become more or less conventional to define a decentralized allocation mechanism as a coordinate correspondence $\mu\colon E \twoheadrightarrow M$ together with an outcome function $g\colon M \to Y$, and, again following Mount and Reiter, even to omit the requirement that M be a product of individual message spaces M^i. Indeed, to represent the competitive message space as a product of individual message spaces (as in Hurwicz (1972) for example) requires response functions which are not symmetric across traders.

Although this abstraction has streamlined the study of informational efficiency relative to (R) alone, it precludes even the statement of stability conditions. In order to recover stability analysis without unravelling the Mount-Reiter abstraction we will introduce a generalization of the original model. In the continuous time version of the original response function framework, a joint message m can be interpreted as a state of the system. Each agent adjusts his coordinate of the state variable according to the rule $\dot{m}^i = f^i(e^i,m)$, independently of the adjustments made by others. However, if the message space is not assumed to be a product of individual message spaces, different agents cannot make independent adjustments. For this reason, we will suppose instead that agent i chooses a "control" message c^i, and that the joint-control $c = (c^i)_{i=1}^N$ together with the state m determines an adjustment $\dot{m}$ according to an adjustment rule: $\dot{m} = \alpha(c,m)$. This of course includes the original model because if $M = \Pi_i M^i$ we can set $c^i = \dot{m}^i$ and $\alpha(c,m) = c$. In this generalization the response function of agent i, $f^i\colon E^i \times M \to C^i$, takes values in a set C^i of control messages, and we have added an adjustment function $\alpha\colon (c,m) \mapsto \dot{m}$ which is independent of e. The new analogue of (*') is

(*") $\dot{m} = \alpha(f(e,m),m)$.

This model of a dynamic adjustment process permits the stationary message correspondence $\mu\colon E \twoheadrightarrow M$ to take on the full generality of the Mount-Reiter formulation. We will use this model to study the following question: Given a correspondence $\mu\colon E \twoheadrightarrow M$, what can be said about the class of adjustment processes which have the equilibrium message correspondence μ and satisfy a local stability condition. In particular, what can be said about the dimension of the control message space C?

Since we will focus on the dimension of C, it may be appropriate to comment here on the usefulness of measuring communication by the dimension of a message space. Under some natural regularity assumptions, the dimension of the message space can be interpreted as the number of real variables which must be communicated. In the informational efficiency literature, most proofs which establish a lower bound for this number do so by putting messages in one to one correspondence with a cannonical set of economic variables. For example, any nonwasteful mechanism for exchange environments must communicate the equilibrium allocation and the hyperplane which supports all traders' upper contour sets at that allocation. The competitive mechanism is informationally efficient because it communicates only these variables. Hence the study of informational efficiency in terms of minimal message space dimension involves the qualitative selection of economic variables to be communicated. However, unless this selection of variables can be shown to be uniquely determined, as in the case of the competitive allocation mechanism (see Jordan (1979)), we must generally be satisfied with counting them.

We now define the model of adjustment processes formally.

<u>2.1 Definitions</u>: There are N agents, indexed by the superscript i, $1 \leq i \leq N$. For each i, let E^i denote the set of <u>characteristics</u> for agent i. Each E^i is assumed to be a topological space. The Cartesian product $E = \Pi_i E^i$ is the space of <u>environments</u>, with generic element

$e = (e^i)_i$.[1/]

An <u>adjustment process</u> is defined as follows. Let M denote the set of <u>state messages</u>, and for each i, let C^i denote the set of <u>control messages</u> for agent i and let $f^i: E^i \times M \to C^i$. The <u>response function</u> f^i describes the rule used by agent i to select a control message depending on his own characteristic and the current state message. Let $C = \Pi_i C^i$ with generic element $c = (c^i)_i$. We will assume that M and C^i, for each i, are open subsets of finite dimensional Euclidean spaces. Let $\alpha: C \times M \to R^n$, where $n = \dim M$. The function α is the <u>adjustment function</u>, and the adjustment process is denoted $((f^i,C^i)_i,\alpha)$. The functions f^i, $1 \leq i \leq N$, together with the adjustment function α determine the autonomous ordinary differential equation

$$(1) \quad \dot{m} = \alpha(f^1(e^1,m),\ldots,f^N(e^N,m);m)$$

for each environment $(e^i)_i$. Let the function $F: E \times M \to R^n$ denote the right hand side of (1). The <u>equilibrium message correspondence</u> $\mu: E \twoheadrightarrow M$ is defined by $\mu(e) = \{m \in M: F(e,m) = 0\}$.

<u>2.2 Remarks</u>: Despite the decentralized nature of the adjustment process, the equilibrium correspondence μ need not be a coordinate correspondence. Indeed, consider any correspondence $\mu^0: E \twoheadrightarrow M$, and let $\psi: E \to M$ be a selection from μ^0. Define an adjustment process as follows. For each i, let $C^i = E^i$, let $f^i(e^i,m) = e^i$, and let

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[1] The notation $(x^i)_i$ denotes the N-tuple $(x^i)_{i=1}^N$.

$$\alpha(e,m) = \begin{cases} 0 & \text{if} \quad m \in \mu^0(e); \text{ and} \\ \psi(e) - m & \text{otherwise.} \end{cases}$$

The equilibrium correspondence of this adjustment process is clearly μ^0. The differential equation (1) in this case becomes

$$(1') \quad \dot{m} = F(e,m) = \begin{cases} 0 & \text{if} \quad m \in \mu^0(e); \text{ and} \\ \psi(e) - m & \text{otherwise.} \end{cases}$$

If M is convex and $\mu^0(e)$ is a closed subset of M for each $e \in E$, this adjustment process also satisfies a stability condition. That is, for each e and any initial condition m^0, there is a continuous piecewise differentiable function $m^*\colon [0,\infty) \to M$ with $m^*(0) = m^0$, $\dot{m}^*(t) = F(e,m^*(t))$ for almost every t, and $\lim_{t\to\infty} m^*(t)$ exists and lies in $\mu^0(e)$ (m^* traverses the interval $[m^0,\psi(e))$ via the equation $m^*(t) = \psi(e) + [\exp(-t)](m^0 - \psi(e))$ but stops if it hits an element of $\mu^0(e)$ enroute).

In general we can define an augmented equilibrium correspondence $\tilde{\mu}\colon E \twoheadrightarrow C \times M$ by $\tilde{\mu}(e) = \{(c,m)\colon c^i = f^i(e^i,m)$ for each i and $\alpha(c,m) = 0\}$. Then $\tilde{\mu}$ is a coordinate correspondence, with $\tilde{\mu}^i$ defined by $\tilde{\mu}^i(e^i) = \{(c,m)\colon c^i = f^i(e^i,m)$ and $\alpha(c,m) = 0\}$. With this definition, the augmented equilibrium message space is $\tilde{\mu}(E)$. We will concentrate here on the size of C, but it may also be useful to investigate the size of $\tilde{\mu}(E)$.[2/]

The fact that the equilibrium correspondence need not be a coordinate correspondence significantly broadens the scope of the present model. In

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[2] I am indebted to Professor M. Richter for this suggestion.

particular, given a performance correspondence $\pi: E \twoheadrightarrow Y$ where Y is Euclidean, we can set $M = Y$ and $\mu = \pi$, so that the adjustment process adjusts the outcome directly. To illustrate this flexibility we will apply the model to the problem of finding the joint-maximum of several well-behaved functions. Let $Y = R^K$ and for each i, let E^i denote the set of C^2 functions $e^i: Y \to R$ satisfying

i) $D^2e^i(y)$ is negative definite for each $y \in Y$; and

ii) $De^i(y) = 0$ for some $y \in Y$.

Define the single-valued performance correspondence $\pi: E \to Y$ by letting $\pi(e)$ maximize $\Sigma_i e^i$ on Y. We first construct an adjustment process whose equilibrium correspondence is π. For each i, let $C^i = R^K$ and define $f^i: E^i \times Y \to C^i$ by $f^i(e^i,y) = De^i(y)$. Define $\alpha: C \times Y \to R^K$ by $\alpha(c,y) = \Sigma_i c^i$. In this case, equation (1) becomes

$$(1'') \quad \dot{y} = \Sigma_i De^i(y)$$

so we have described a decentralized gradient process. Assumptions (i) and (ii) ensure that the equilibrium correspondence is π and that the process is globally stable.

We now describe a very similar adjustment process with a coordinate equilibrium correspondence. Let $M = \{(y,q^1,\ldots,q^N) \in Y \times R^{KN}: \Sigma_{i=1}^N q^i = 0\}$, and for each i, define $\mu^i: E^i \twoheadrightarrow M$ by $\mu^i(e^i) = \{(y,q^1,\ldots,q^N) \in M: De^i(y) = q^i\}$. Define $\mu: E \twoheadrightarrow M$ by $\mu(e) = \bigcap_{i=1}^N \mu^i(e^i)$, so that μ is obviously a coordinate correspondence. Define the outcome function $g: M \to Y$ by $g(y,(q^i)_i) = y$. Then for each $e \in E$, $\mu(e)$ is a singleton and $g(\mu(e)) = \pi(e)$. Moreover, an argument which is conventional in the informational efficiency literature shows that any message space M' which admits a well-behaved[3/] coordinate correspondence $\mu': E \twoheadrightarrow M'$ and an outcome function $g': M' \to Y$ with $g' \circ \mu' = \pi$ must have at least dimension $KN = \dim M$. For each i, again let $C^i = R^K$ and define $f'^i: E^i \times M \to C^i$ by $f'^i(e^i;y,(q^j)_j) = De^i(y)$. Let $p: R^{KN} \to \{(q^i)_i \in R^{KN}: \Sigma_i q^i = 0\}$ be the orthogonal projection, and define $\alpha': C \times M \to R^{K(N+1)}$ by $\alpha'(c;y,(q^i)_i) = (\dot{y},(\dot{q}^i)_i)$ where

$$\dot{y} = \Sigma_i c^i; \text{ and}$$

$$(\hat{q}^i)_i = p((c^i)_i) - (q^i)_i.$$

This adjustment process yields the differential equation

$$(1''') \quad \begin{aligned} \dot{y} &= \Sigma_i De^i(y) \\ (\dot{q}^i)_i &= p((De^i(y))_i) - (q^i)_i. \end{aligned}$$

Hence the equilibrium correspondence is μ and the process is again globally stable.

In studying the informational requirements of stability we will focus on the dimension of the space C of control messages. For the above problem of finding a joint maximum, both adjustment processes use the same space $C = R^{KN}$. For the first process, in which the state message space is the outcome space Y, one can conclude from equilibrium considerations alone that C has the smallest possible dimension. To show this, for each i let E^{0i} denote the subset of E^i consisting of functions e^i

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[3] I.e. μ' is "locally threaded" (admits continuous local selections) with respect to a C^2 topology on E.

of the form $e^i(y) = -\Sigma_{j=1}^K (y_j - a_j^i)^2$, where $a_j^i \in R$ for each j. Then E^{0i} can be identified with R^K, with generic element a^i. Let $E^0 = \Pi_i E^{0i}$, with generic element $a = (a^i)_i$. On E^0, π is given by $\pi(a) = N^{-1}\Sigma_i a^i$. Fix $y \in Y$, and consider a response function $f^i(\cdot,y)\colon E^{0i} \to C^i$. We will show that this function must be 1-1. Let $i = 1$ and let $a^1, a'^1 \in E^{01}$ with $a^1 \neq a'^1$. For each $i > 1$, let $a^i \in E^{0i}$ with $\Sigma_{i>1} a^i = Ny - a^1$, so that $\pi(a^1, a^2, \ldots, a^N) = y$. Then $\alpha(c,y) = 0$, where $c^i = f^i(a^i,y)$ for each i. Since $a'^1 \neq a^1$, $\pi(a'^1, a^2, \ldots, a^N) \neq y$, so we must have $\alpha(c'^1, c^2, \ldots, c^N; y) \neq 0$, where $c'^1 = f^1(a'^1,y)$. Hence $c'^1 \neq c^1$, so $f^1(\cdot,y)$ is 1-1 on E^{01}. Similarly $f^i(\cdot,y)$ is 1-1 on E^{0i} for each $i > 1$. Therefore, for each i, if C^i is Euclidean and f^i is continuous on $E^{0i} \times Y$, $\dim C^i \geq K$.

For the second adjustment process described above for the joint-maximization problem, equilibrium considerations alone do not imply that C has minimal dimension. For example, for each i let $C'^i = R$ and define $f''^i\colon E^i \times M \to C'^i$ by $f''^i(e^i; y, (q^j)_j) = \| De^i(y) - q^i \|^2$, where $\| \cdot \|^2$ denotes the square of the Euclidean norm. Define $\alpha''\colon C' \times M \to R^{K(N+1)}$ by $\alpha''(c; y, (q^i)_i) = (\dot{y}, (\dot{q}^i)_i)$ where

$$\dot{y} = (\| c \|^2, 0, \ldots, 0) \in R^K; \text{ and}$$

$$\dot{q}^i = 0 \text{ for each } i.$$

Then the equilibrium message correspondence is again μ, but $\dim C' = N$. However, this adjustment process is clearly unstable. In Section 2.15 below we will apply Theorem 2.14 to conclude that local stability requires $\dim C \geq KN$. Hence in this case, increasing the state message space from Y to M does not reduce the size of the control message space needed to ensure local stability.

We now proceed with the formal study of locally stable adjustment processes. Section 2.3 below imposes Lipschitz conditions on the response functions and the adjustment function so that solutions to the differential equation (1) exist and are locally unique. Section 2.4 gives the definition of local stability. An implication of local stability is given in Proposition 2.5. The definition and this implication are discussed in Section 2.6.

<u>2.3 Lipschitz Conditions</u>: For each i, we assume that f^i is continuous and for each $e^i \in E^i$, the function $f^i(e^i,\cdot): M \to C^i$ is locally Lipschitzian. That is, for any $m^0 \in M$ there is a neighborhood M' of m^0 and a constant $K > 0$ such that for any $m, m' \in M'$, $\| f^i(e^i,m) - f^i(e^i,m')\| \leq K\| m - m'\|$. The adjustment function α is also assumed to be locally Lipschitzian (in the arguments c and m jointly).

<u>2.4 Definitions</u>: For each $(e^0,m^0) \in E \times M$, consider the differential equation with initial condition

$$(2) \quad \dot{m} = F(e^0,m)$$

$$m(0) = m^0.$$

Let $D = \{(e^0,m^0) \in E \times M$: (2) has a solution on the entire time domain $[0,\infty)\}$. Define $m^*: D \times [0,\infty) \to M$ by setting $m^*(e^0,m^0;t)$ equal to the solution of (2) at time t. The adjustment process is <u>locally stable</u> if for each $e \in E$, each $m \in \mu(e)$, and each neighborhood V of m, there are neighborhoods U_1 of e and U_2 of m with $U_2 \subset V$, $U_1 \times U_2 \subset D$, and a function $\mu^*: U_1 \times U_2 \to M$ satisfying, for each $(e^0,m^0) \in U_1 \times U_2$,

a) $m^*(e^0,m^0;t) \in V$ for all t; and

b) $\lim_{t\to\infty} m^*(e^0,m^0;t) = \mu^*(e^0,m^0)$, and $\mu^*(e^0,m^0) \in \mu(e^0)$.

<u>2.5 Proposition</u>: Let $e \in E$ and $m \in \mu(e)$, let V be a neighborhood of m, and suppose that the adjustment process is locally stable. Then in the notation of Definition 2.4, the neighborhoods U_1 and U_2 can be chosen so that

i) m^* is continuous;

ii) for any sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$, $\{m^*(\cdot;t_n)\}_{n=1}^{\infty}$ converges uniformly to μ^*; and

iii) μ^* is continuous.

<u>Proof</u>: Statement (i) follows from the Lipschitz condition imposed in 2.3 above and a standard result on the continuity of solutions to ordinary differential equations (e.g. Coddington and Levinson (1955), Theorem 7.1, p. 22). To prove (ii), let U_1 and U_2 be given by Definition 2.4 and let U_1' and U_2' be smaller neighborhoods of e and m respectively with compact closure. To prove (ii) on $U_1' \times U_2'$ it suffices to show that for any sequence $\{(e_n^0,m_n^0)\}_{n=1}^{\infty}$ in $U_1' \times U_2'$ converging to some $(e^0,m^0) \in U_1 \times U_2$ and any sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$, $m^*(e_n,m_n;t_n) \to \mu^*(e^0,m^0)$. Applying the definition of local stability to e^0 and $m^{00} = \mu^*(e^0,m^0)$, let V' be a neighborhood of m^{00} and let U_1'' be a neighborhood of e^0 and U_2'' a neighborhood of m^{00} such that

if $e' \in U_1''$ and $m' \in U_2''$ then $m^*(e',m';t) \in V'$ for all t. By 2.4(b), there is some t^0 such that $m^*(e^0,m^0;t^0) \in U_2''$, and by (i) there is some n^0 such that for all $n > n^0$, $e_n^0 \in U_1''$ and $m^*(e_n,m_n,t^0) \in U_2''$. For each $n > n^0$, let $m_n^0 = m^*(e_n,m_n,t^0)$. Then for $t_n > t^0$, $m^*(e_n,m_n,t_n) = m^*(e_n,m_n^0;t_n - t^0)$ so for $n > n^0$ and $t_n > t^0$, $m^*(e_n,m_n,t_n) \in V'$ by 2.4(a). Since V' is arbitrary, this proves (ii), and (iii) follows from (i) and (ii).

2.6 Remarks: The two parts to the definition of local stability, 2.4(a) and (b), are both natural stability requirements but have somewhat different interpretations. Condition 2.4(a) states that a small perturbation will not cause the system to wander too far away from equilibrium. This has a "structural stability" implication which we will amplify below. Condition 2.4(b) is an "asymptotic stability" condition, stating that the system will eventually converge to equilibrium. If the environment e is fixed, so the set U_1 becomes the singleton $\{e\}$, 2.4(a) and (b) constitute the conventional definition of local asymptotic stability for the differential equation (1) (e.g. Coddington and Levinson (1955, p. 314)). We have extended this definition by allowing perburbations of the environment as well as the state message. Indeed, in most economic contexts, disequilibria may be more likely to result from changes in the environment than from perturbations of the state variables.

Of course, by insisting on stability under small changes in the environment, we have implicitly imposed a structural stability condition on the equilibrium correspondence μ. This condition is represented explicitly in Proposition 2.5 as the continuity of μ^*. In particular, given e^0 and $m^0 \in \mu(e^0)$, the function $\mu^*(\cdot,m^0)$ is a local continuous selection from μ passing through m^0 at e^0. Hence μ is lower semi-continuous and "locally threaded". For this reason, when we apply the model to the competitive equilibrium correspondence in Section 3 below, we will have to restrict attention to regular equilibria. However, it should be emphasized that local stability does not imply local uniqueness. In fact we will establish in Section 4 below the local stability of an adjustment process on exchange environments whose equilibrium correspondence is the Pareto correspondence.

To establish that the equilibria of an autonomous differential equation are locally stable it is conventional to differentiate the right hand side at an equilibrium and check that the characteristic roots of the matrix representing the derivative all have negative real parts. This criterion is sufficient but of course not necessary. The following result shows that if E is Euclidean and F is C^1 in both arguments then the same criterion ensures local stability according to our definition.

<u>2.7 Proposition</u>: Suppose that

i) E is an open subset of a finite dimensional euclidean space;

ii) F is C^1; and

iii) for each $e \in E$ and each $m \in \mu(e)$, all eigenvalues of the matrix $D_m F(e,m)$ have negative real parts.

Then the adjustment process is locally stable. Moreover, for each $e \in E$ and $m \in \mu(e)$, there are neighborhoods U_1 of e and U_2 of m and a C^1 function $\mu^0: U_1 \to U_2$ such that if $e' \in U_1$ and $m' \in \mu(e') \cap U_2$ then $m' = \mu^0(e')$.

Proof: The second assertion follows from the implicit function theorem applied to the equation

$$F(e,m) = 0$$

which implicitly defines μ. The continuity of μ^0 together with a standard proof of local asymptotic stability of solutions to ordinary differential equations (e.g. Coddington and Levinson (1955), pp. 314-315) establishes local stability.

2.8 Remarks: Given a correspondence $\mu: E \twoheadrightarrow M$, what can be said of the class of locally stable adjustment process with equilibrium correspondence μ? Suppose that μ is single-valued. Then the most transparent adjustment process is defined by $C^i = E^i$ and $f^i(e^i,m) = e^i$ for each i, and $\alpha(e,m) = \mu(e) - m$. In this case $F(e,m) = \mu(e) - m$, so the adjustment $F(e,m)$ is always in the direction of $\mu(e)$. Surprisingly, there is a powerful generalization of this relation between $F(e,m)$ and $\mu(e)$ which applies to all locally stable adjustment processes.

By way of illustration suppose that E is a Euclidean space with $\dim E = \dim M$ and that μ is 1-1 and continuous on E. Let $e^0 \in E$ and let $E - e^0 = \{e \in E: e \neq e^0\}$. By translating the origin of M we can assume that $\mu(e^0) = 0$. Since μ is 1-1, μ maps $E - e^0$ to $R^n - 0$, where $R^n - 0 = \{m \in R^n: m \neq 0\}$. Now consider the function $m^*(\cdot,0;1)$ on $E - e^0$. Since μ is 1-1 and (2) is an autonomous ODE, $m^*(\cdot,0;1)$: $E - e^0 \to R^n - 0$. Moreover, the definition of local stability implies that by letting t pass from 1 to ∞ we can describe a homotopy between $m^*(\cdot,0;1)$ and μ on $E - e^0$ to $R^n - 0$. Now consider the function

$H: (E - e^0) \times [0,1] \to R^n - 0$ defined by

$$H(e,\lambda) = \begin{cases} \lambda^{-1} m^*(e,0;\lambda) & \text{if } \lambda > 0; \text{ and} \\ F(e,0) & \text{if } \lambda = 0. \end{cases}$$

This describes a homotopy between $m^*(\cdot,0;1)$ and $F(\cdot,0)$, so μ and $F(\cdot,0)$ are homotopic as maps on $E - e^0$ to $R^n - 0$.

To picture the implications of this fact, suppose that $n = 2$ and trace a circular path about e^0 in $E - e^0$. Applying μ to this path gives a path about 0 in $R^n - 0$ which is the homeomorphic image of a circle. Then applying $F(\cdot,0)$ must yield a path about 0 in the same "direction" as the path given by μ (although $F(\cdot,0)$ need not be 1-1). This is depicted in Figure 1 below. In particular, if $F(\cdot,0)$ and μ are C^1 and the derivatives $D\mu(e^0)$ and $D_e F(e^0,0)$ are nonsingular, then the determinants of the two derivatives must have the same sign.

The general statement of this homotopy equivalence is given in Theorem 2.10 below, and is the principal analytical tool of this paper.

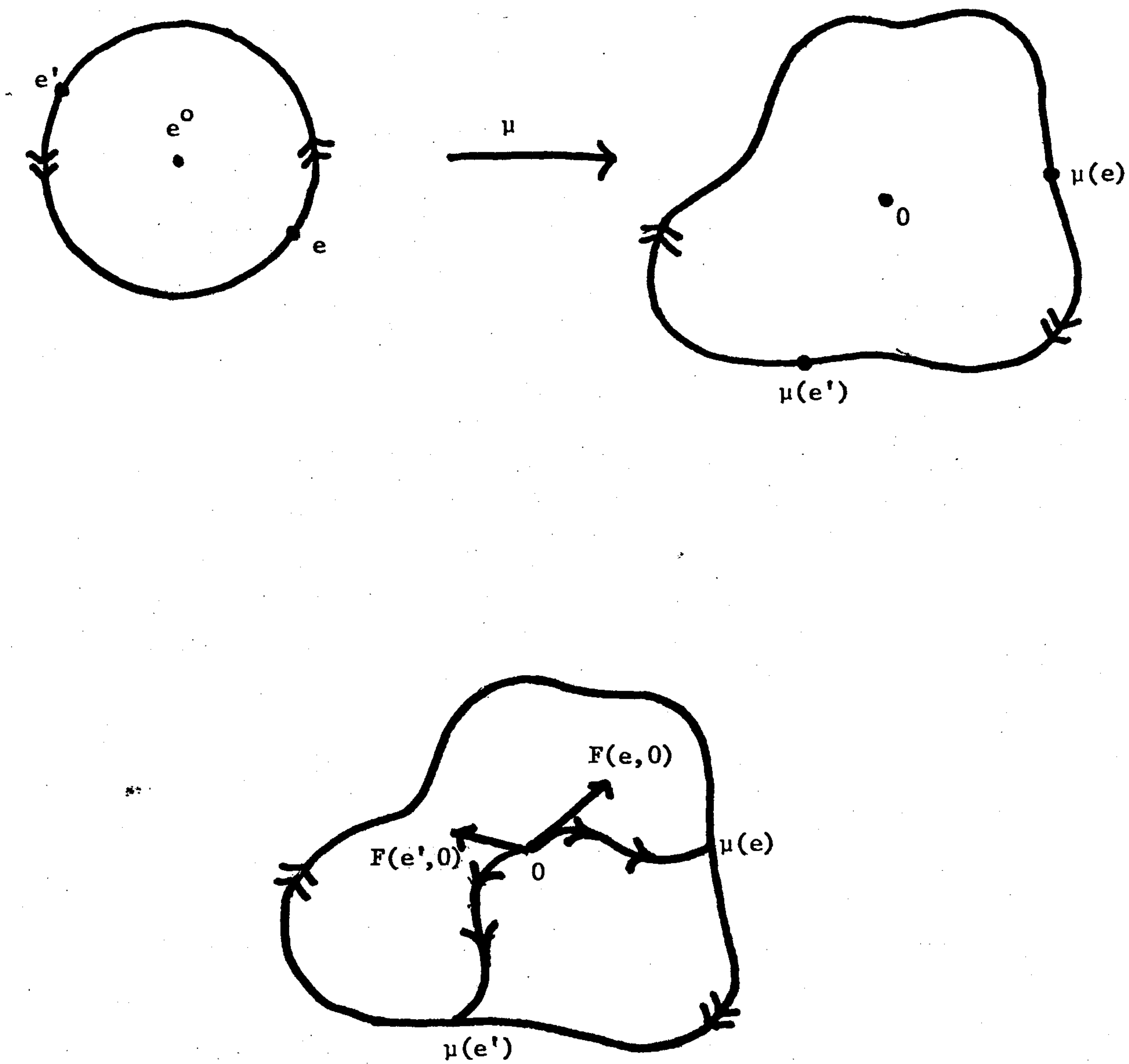

Figure 1[4/]

[4/]The double arrow ≫ denotes orientation.

<u>2.9 Definitions</u>: Let X and X' be topological spaces and let $h: X \to X'$ be continuous. If $A \subset X$ and $A' \subset X'$, h is a <u>map of pairs</u>, written $h: (X,A) \to (X',A')$ if $h(A) \subset A'$. A <u>homotopy</u> between two maps of pairs h and h' is a continuous function $H: X \times [0,1] \to X'$ with $H(\cdot,0) = h$, $H(\cdot,1) = h'$, and $H(A,\lambda) \subset A'$ for each $\lambda \in [0,1]$.

If $A \subset X$, $X - A$ denotes the set $\{x \in X: x \notin A\}$. Similarly, $R^n - 0 = \{x \in R^n: x \neq 0\}$.

<u>Theorem 2.10</u>: Suppose that the adjustment process is locally stable, let $e^0 \in E$, let $m^0 \in \mu(e^0)$, and let $U = U_1 \times U_2$ be a neighborhood of (e^0,m^0) in $E \times M$ given Proposition 2.5 above. Let $\mu^*: U \to M$ be given by 2.4 and define graph $\mu^* \equiv \{(e,m) \in U: m = \mu^*(e,m)\}$. Define the function $\mu': (U,U - \text{graph } \mu^*) \to (R^n,R^n - 0)$ by $\mu'(e,m) = \mu^*(e,m) - m$, where $n = \dim M$. Then the maps
$\mu': (U,U - \text{graph } \mu^*) \to (R^n,R^n - 0)$ and
$F: (U,U - \text{graph } \mu^*) \to (R^n,R^n - 0)$ are homotopic.

<u>Proof</u>: Define the function $G: U \times [0,1] \to R^n$ by

$$G(e,m;\lambda) = \begin{cases} \mu'(e,m) & \text{if } \lambda = 0 \\ m^*(e,m;\lambda^{-1}) - m & \text{if } \lambda > 0. \end{cases}$$

Then by the definition of Proposition 2.5 (ii), G is a homotopy between μ' and the map $m^{*\prime}: (U,U - \text{graph } \mu^*) \to (R^n,R^n - 0)$ defined by $m^{*\prime}(e,m) = m^*(e,m;1) - m$. Define the function $H: U \times [0,1] \to R^n$ by

$$H(e,\lambda) = \begin{cases} F(e,m) & \text{if } \lambda = 0 \\ \lambda^{-1}[m^*(e,m;\lambda) - m] & \text{if } \lambda > 0. \end{cases}$$

The continuity assumptions (2.3 above) together with a coordinate application of the Mean Value Theorem imply that H is a homotopy between F and $m^{*\prime}$, which completes the proof.

2.11 Remarks: In general the neighborhood U will not be Euclidean, and even if Euclidean, may have dimension much larger than n. This might seem to suggest that applications such as the comparison of determinants mentioned in Section 2.8 above are rare. However, the homotopy given by the Theorem is directly inherited by restrictions of μ' and F to subsets of U. Indeed, the applications of the Theorem in Theorems 2.14 and 3.6 below consist of such restrictions. The inheritance property is stated formally in Corollary 2.12 below, where the map h may be thought of as an inclusion map with $X \subset U$ and $A \subset U$ - graph μ^*.

2.12 Corollary: Let X be a topological space, let $A \subset X$ and let $h: (X,A) \to (U, U - \text{graph } \mu^*)$ be a map of pairs. Under the hypothesis of Theorem 2.10, the composition maps

$\mu' \circ h: (X,A) \to (R^n, R^n - 0)$ and

$F \circ h: (X,A) \to (R^n, R^n - 0)$ are homotopic.

2.13 Remarks: In the informational efficiency literature, a standard technique for establishing the minimal dimension of M involves the construction of a subspace $E^0 \subset E$ with the property that distinct environments in E^0 cannot have any equilibrium messages in common. If μ is "locally threaded", one can then conclude that $\dim M \geq \dim E^0$. Theorem 2.14 below states that in this case, if $\dim M = \dim E^0$ and the adjustment process is locally stable then $\dim C \geq \dim M$. In other words, if E contains a subspace E^0 with the same dimension as M and such that μ is, in a set-valued sense, 1-1 on E^0, then C must be large enough to communicate all directions of movement in M. The hypothesis that $\dim E^0 = \dim M$ is natural because if $\dim E^0 > \dim M$, μ could not be 1-1 in the above sense, and if $\dim E^0 < \dim M$, M may have redundant coordinates, so that the directions of movement required to reach equilibrium from any disequilibrium might lie in a proper linear subspace of R^n. We will see an example of this in Section 4 below.

2.14 Theorem: Suppose that E^0 is a topological subspace of E satisfying

i) E^0 is homeomorphic to an open subset of R^n, where $n = \dim M$; and

ii) for any e, $e' \in E^0$, $\mu(e) \cap \mu(e') = \emptyset$ if $e \neq e'$.

If the adjustment process is locally stable then $\dim C \geq n$.

Proof: Let $e^0 \in E^0$, $m^0 \in \mu(e^0)$, and let $U = U_1 \times U_2$ be a neighborhood of (e^0,m^0) in $E \times M$ given by Proposition 2.4. Let $U_1^0 = U_1 \cap E^0$ and let $\mu^{0'}: (U_1^0, U_1^0 - e^0) \to (R^n, R^n - 0)$ be defined by $\mu^{0'}(e) = \mu'(e,m^0)$, where μ' is defined in Theorem 2.10. By (ii) above, $\mu^*(\cdot,m^0)$ is 1-1 on U_1^0, so $\mu^{0'}$ is 1-1 and is thus well-defined as a map of pairs. Define $F^0: (U_1^0, U_1^0 - e^0) \to (R^n, R^n - 0)$ by $F^0(e) = F(e,m^0)$. Then $\mu^{0'} = \mu' \circ h$ and $F^0 = F \circ h$, where $h: (U_1^0, U_1^0 - e^0) \to (U, U - \text{graph } \mu^*)$ is defined by $h(e) = (e,m^0)$. Therefore Corollary 2.12 implies that $\mu^{0'}$ and F^0 are homotopic.

Let $C^0 = \{c \in C: \alpha(c,m^0) = 0\}$ and define the map

$\alpha^0: (C, C - c^0) \to (R^n, R^n - 0)$ by $\alpha^0(c) = \alpha(c, m^0)$, and define the map $f^0: (U_1^0, U_1^0 - e^0) \to (C, C - c^0)$ by $f^0(e) = (f^i(e^i, m^0))_i$. Then $\alpha^0 \cdot f^0 = F^0$.

Since $\mu^{0'}$ is 1-1 on E^0, and $\dim E^0 = \dim M = n$, the map $\mu^{0'}$ induces an isomorphism on homology groups $\mu_*^{0'}: H_n(U_1^0, U_1^0 - e^0) \to H_n(R^n, R^n - 0)$ (Dold (1972), Proposition 7.4, p. 79 and Proposition 3.2, p. 59). Since $\mu^{0'}$ is homotopic to F^0, F^0 induces the same isomorphism $F_*^0 = \mu_*^{0'}$. Since $\alpha^0 \cdot f^0 = F^0$, $\alpha_*^0 \cdot f_*^0 = F_*^0$, where $f_*^0: H_n(U_1^0, U_1^0 - e^0) \to H_n(C, C - c^0)$ and $\alpha_*^0: H_n(C, C - c^0) \to H_n(R^n, R^n - 0)$. Since $H_n(R^n, R^n - 0) \neq 0$ and F_*^0 is an isomorphism, $H_n(C, C - c^0) \neq 0$. Since α^0 is continuous, c^0 is closed so $\dim C \geq n$ (Dold (1972), Proposition 3.3(a), p. 260).

2.15 Remarks: If the hypothesis of the Theorem is strengthened with some smoothness and regularity assumptions, an elementary proof can be used. Suppose that μ is a C^1 function on E^0, let $e^0 \in E^0$, and suppose that $D\mu(e^0)$ is nonsingular. Let $m^0 = \mu(e^0)$, and suppose that F is C^1 on $E^0 \times M$ and that $D_mF(e^0, m^0)$ is nonsingular. Then $D_eF(e^0, m^0) = -D_mF(e^0, m^0)D\mu(e^0)$ so $D_eF(e^0, m^0)$ is nonsingular. However, if α and each f^i are C^1, then rank $D_eF(e^0, m^0) \leq$ rank $D_c\alpha(c^0, m^0) \leq \dim C$, where $c^0 = (f^i(e^{0i}, m^0))_i$, so $\dim C \geq n$.

This argument indicates that if our definition of local stability is replaced by the hypothesis of Proposition 2.7 above, it may be possible to replace applications of Theorem 2.10 with more elementary arguments. However, in constructing locally stable adjustment processes, our definition is much less burdensome to satisfy. An example of a simple locally stable adjustment process which violates the regularity required in the above argument is given in Section 4.3 below. Also, the more geometrical nature of our approach may be more fruitful in suggesting extensions to the study of global stability.

Theorem 2.14 can be applied to the adjustment process constructed in Section 2.2 above to find the joint maximum of several functions. The hypothesis of the Theorem is satisfied by the subspace E^0 defined in the fourth paragraph of Section 2.2. Since $\dim C = \dim M$ for that adjustment process, C has minimal dimension.

3. Competitive Adjustment Processes

This section is devoted to dynamic adjustment processes which stabilize the competitive equilibria of exchange environments. We will take the state message to be the competitive message as formulated by Mount and Reiter (1974) and consider adjustment processes which locally adjust the competitive message to its equilibrium value. We will define all notation formally below, but proceeding informally for the purpose of motivation, a competitive message is a price vector p together with a profile of trade vectors $(y^i)_{i=1}^N$ such that for each trader i, $py^i = 0$ and the feasibility condition $\Sigma_{i=1}^N y^i = 0$ is satisfied. Trader i's characteristic e^i consists of his preferences and endowment. If the space of environments is such that the aggregate excess demand function always satisfies the gross substitution property, which will be true, for example, if all preferences are Cobb-Douglas, then an adjustment process modelled after the familiar tatonnement process is locally (in fact globally) stable. For each trader i, let $\zeta^i(e^i,\cdot)$ denote the excess demand function determined by the characteristic e^i, and let f^i: $(e^i;p,y^1,\ldots,y^N) \to \zeta^i(e^i,p)$. That is, each trader's response is his excess demand at the current price. The adjustment function α is described by

$$\dot{p} = \Sigma_{i=1}^N \zeta^i(e^i,p)$$

$$\dot{y}^i = \zeta^i(e^i,p) - y^i \quad \text{if} \quad i < N$$

$$\dot{y}^N = -\Sigma_{i<N} \dot{y}^i .$$

Given the environment, the adjustment $\dot{p}$ depends only on p and is stable by the classic result of Arrow, Block, and Hurwicz (1959). The adjustment of the trade profile $(y^i)_{i=1}^N$, though not symmetric across traders, preserves feasibility even out of equilibrium.

Unfortunately it is well-known that in the absence of gross substitution or some other strong restriction on the aggregate excess demand function, the above process can fail to be even locally stable. For this reason Smale (1976a) and several subsequent authors (Saari and Simon (1978), Varian (1977) and Keenan (1981)) have studied "Newton methods" which use the derivatives of excess demand as well. Let $\zeta(e,\cdot)$ denote the aggregate excess demand function for the environment $e = (e^1,\ldots,e^N)$, and suppose that at least near an equilibrium, $D_p\zeta(e,p)$ is nonsingular. Then for each i, describe the response function by

$$f^i\colon (e^i;p,y^1,\ldots,y^N) \to (\zeta^i(e^i,p),\ D_p\zeta^i(e^i,p)).$$

That is, each trader reports his excess demand and its derivative at the current price. The adjustment in the competitive message is then

$$\dot{p} = -(\Sigma_i D_p\zeta^i(e^i,p))^{-1}\Sigma_i\zeta^i(e^i,p) = -(D_p\zeta(e,p))^{-1}\zeta(e,p)$$

$$\dot{y}^i = \zeta^i(e^i,p) - y^i,\quad i < N$$

$$\dot{y}^N = -\Sigma_{i<N}\dot{y}^i.$$

Given an environment, $\dot{p}$ again depends only on p. At an equilibrium, $\zeta(e,p) = 0$ and $D_p\dot{p} = -I$ so local stability is ensured. This process is well-defined only near "regular equilibria", that is, equilibria with

$D_p \zeta(e,p)$ nonsingular. However, in the absence of this regularity condition the equilibrium may not possess the "structural stability" implicit in our definition of local stability. Some such regularity condition is therefore needed to ensure the local stability of any adjustment process.

The communication of excess demand at the current price, or a dual message such as the demand price at the current trade, seems a natural requirement. However, the communication of the derivative of excess demand causes a drastic increase in the dimension of the control message, and has no apparent behavioral interpretation. Hence it would be desirable to find some locally stable adjustment process with a communication requirement closer to that of the tatonnement process. Unfortunately, Theorem 3.6 below states that any adjustment process which makes all regular equilibria locally stable must use control messages at least as large as those of the Newton method.

We will first introduce, in Section 3.1 below, a special parametric class of exchange environments. Since we are only interested in local stability, we will use a class of preferences representable by quadratic utility functions. These preferences possess all of the classical properties locally, although of course satiation will occur within the nonnegative orthant. Presumably these preferences can be extended from the neighborhood of interest in such a way as to preserve the classical properties globally, but we will not construct such extensions here. Our analysis is complicated by the well-known properties of the Slutsky matrix, which imply that not every matrix of the appropriate dimension can be the derivative of an individual excess demand function. For this reason we will parameterize the endowments and quadratic utility functions in such a way that the parameters become the value of the excess demand function, the Slutsky matrix, and the income derivative, all at a given price. Hence this class of preferences and endowments generates in a transparent way the full range of classical demand behavior at a given price. We hope that this class of environments will also be useful in studying other issues in informational decentralization theory.

3.1 Definitions: Let $K \geq 2$, $N \geq 2$, and let M be an open subset of $R^{K-1}_{++} \times R^{(K-1)(N-1)}$, with generic element $m = (p, y^1, \ldots, y^{N-1})$, where p is interpreted as a price vector with the price of commodity K normalized to unity, and y^i_j is trader i's net trade of commodity j, with y^i_K being determined by the budget equation $py^i + y^i_K = 0$. Trader N's net trade is determined by the feasibility constraint $\Sigma^{N-1}_{i=1} y^i + y^N = 0$. Hence M is $N(K-1)$-dimensional but each $(p, y^1, \ldots, y^{N-1}) \in M$ determines a vector $(\bar{p}, \bar{y}^1, \ldots, \bar{y}^{N-1}, \bar{y}^N) \in R^K_{++} \times R^{NK}$ by $\bar{p} = (p_1, \ldots, p_{K-1}, 1)$, $\bar{y}^i = (y^i_1, \ldots, y^i_{K-1}, -\Sigma^{K-1}_{j=1} p_j y^i_j)$ for each $i \leq N-1$, and $\bar{y}^N = -\Sigma^{K-1}_{i=1} \bar{y}^i$.

Choose $m^0 = (p^0, y^{01}, \ldots, y^{0N-1}) \in M$ to satisfy the following two properties. For each i

i) $y^{0i}_j \neq 0$ for all $1 \leq j \leq K-1$; and

ii) the $(K-1) \times (N-1)$ matrix $(y^{0j})_{j \neq i}$ has full rank.

For each $1 \leq i \leq N$, let $\omega^{0i} \in R^K_{++}$ with $\omega^{0i} + \bar{y}^{0i} \in R^K_{++}$. Fix i, and, suppressing the superscript i, let $x^0 = \omega^0 + \bar{y}^{0i}$. We will construct a parametric family of characteristics for trader i. Let $E^i_{m^0} = \{(\omega, a, B)$: $\omega \in R^K_{++}$, $\bar{p}^0\omega = \bar{p}^0\omega^0$; $a \in R^K$, $\bar{p}^0 a = 1$; and B is a symmetric negative definite $(K-1) \times (K-1)$ matrix$\}$, where ω denotes an endowment, and a and B determine a utility function as follows. Let A denote the $K \times K$ matrix

$$A = \left(\begin{array}{c|c} I_{K-1} & a \\ \hline -p^0 & \end{array}\right)$$

where I_{K-1} is the $(K-1) \times (K-1)$ identity matrix. Thus the matrix A^{-1} transforms the standard coordinates on R^K to the basis consisting of the vectors $(0,\ldots,0,1_j,0,\ldots,0,-p_j^0)_{j=1}^{K-1}$, which span the null space of $\bar{p}^0$, together with a. Let $\overline{B}$ denote the $K \times K$ matrix

$$\overline{B} = \left(\begin{array}{c|c} B^{-1} & 0 \\ \hline 0 & \end{array}\right),$$

and define the utility function $u\colon R_+^K \to R$ by $u(x) = \bar{p}^0 x + (\tfrac{1}{2})(x - x^0)(A^{-1})'\overline{B}A^{-1}(x - x^0)$, where $(A^{-1})'$ is the transpose of A^{-1}. With x^0 fixed, a triple $e^i = (\omega,a,B) \in E^i_{m^o}$ determines an excess demand function $\zeta^i(e^i,\cdot)\colon R_{++}^{K-1} \to R^{K-1}$ for commodities $1 \leq j \leq K-1$. For p near p^0, a tedious but conventional computation establishes that for each $1 \leq j \leq K$,

(1) $\zeta_j^i(\omega,a,B;p) =$

$$x_j^0 - \omega_j + (\bar{p}a)^{-1}[\Sigma_{k=1}^{K-1} b_{jk}(p_k - p_k^0) - a_j\bar{p}(x^0 - \omega) - a_K(\bar{p}a)^{-1}(p-p^0)B(p-p^0)],$$

where b_{jk} is the element in row j and column k of B.

Define the function $f^{*i}\colon E^i_{m^0} \to R^{K-1} \times R^{(K-1)^2}$ by $f^{*i}(e^i) = (\zeta^i(e^i,p^0), D_p\zeta^i(e^i,p^0))$, where D_p denotes the derivative with respect to p. We need to establish the rank of Df^{*i}. It follows directly from (1) that

(2) $f^{*i}(\omega,a,B) =$

$$= ((x_1^0-\omega_1,\ldots,x_{K-1}^0-\omega_{K-1});\ (b_{jk} - a_j(x_k^0-\omega_k))_{\substack{1\leq j\leq K-1 \\ 1\leq k\leq K-1}}).$$

Thus, at p^0, B is the Slutsky matrix and a is the tangent to the income expansion path. From (2) we have that $D_\omega\zeta^i(\omega,a,B;p^0)$ has full rank and $D_{(a,B)}\zeta^i(\omega,a,B;p^0) = 0$. Hence the rank of Df^{*i} is $K-1$ plus the rank of $D_{(a,B)}(D_p\zeta^i(\omega,a,B;p^0))$. To compute the latter, note that the map $(a,B) \mapsto D_p\zeta^i(\omega,a,B;p^0)$ is linear by (2). We will show that, at ω^{0i}, this map has rank $K(K-1)/2 + (K-2)$, which is one less than full rank. Given (a,B), for each $1\leq j,k\leq K-1$, let d_{jk} denote the entry in row j and column k of $D_p\zeta^i(\omega,a,B;p^0)$; and let $y_k = x_k^0 - \omega_k$. Then $d_{jk} = b_{jk} - a_jy_k$ for each j,k. Suppose that $y_k \neq 0$ for all k, which is true at ω^{0i} by property (i) above. We will show that the $K(K-1)/2 + (K-2)$ entries $\{d_{jk}: k\leq j+1\}$ determine the entire matrix. Since B is symmetric, for each $j\leq K-2$, $(d_{j(j+1)} - d_{(j+1)j})/y_jy_{j+1} = (a_{j+1}/y_{j+1}) - (a_j/y_j)$. Hence by adding and subtracting such terms we can obtain $(a_j/y_j) - (a_k/y_k)$ for any j,k. Also, if $k>j+1$, $d_{jk} = d_{kj} - [(a_j/y_j) - (a_k/y_k)]y_jy_k$, so d_{jk} is determined by the above collection of entries. It is straightforward to show that the linear map from (a,B) to this collection of $K(K-1)/2 + (K-2)$ entries has full rank, which proves that for any (a,B),

(3) $Df^{*i}(\omega^{0i},a,B)$ has rank $(K-1) + (K-2) + K(K-1)/2$.

We will need one more fact about the derivatives of excess demand at p^0. Using property (ii) of m^0, it is straightforward to show that if $N\geq K$, then for each i, $D_{(a^j)_{j\neq i}}(\Sigma_{j\neq i}D_p\zeta^j(\omega^{0j},a^j,B^j))$ is surjective for

$(a^j,B^j)_{j\neq i}$. Hence, for each i and all $(B^j)_{j\neq i}$,

(4) if $N \geq K$ the function $(a^j)_{j\neq i} \mapsto \Sigma_{j\neq i} D_p \zeta^j(\omega^{0j},a^j,B^j)$ is an open map.

<u>3.2 Competitive Adjustment Processes</u>: Given $e \in E_m^0$ and $m = (p,y) \in M$, m is a <u>competitive equilibrium</u> for e if $\zeta^i(e^i,p) = y^i$ for each $i \leq N-1$, and $\zeta^N(e^N,p) = -\Sigma_{i=1}^{N-1} y^i$. If in addition, the matrix $\Sigma_{i=1}^N D_p \zeta^i(e^i,p)$ has full rank, then m is a <u>regular equilibrium</u> for e. Define the correspondence $\mu_r: E_m^0 \twoheadrightarrow M$ by $\mu_r(e) = \{m \in M$: m is a regular equilibrum for $e\}$. Of course $\mu_r(e) = \emptyset$ for some environments e.

Given the space of environments E_m^0 and the space of state messages M, a <u>competitive adjustment process</u> is an adjustment process as defined in Section 2.1 above, with the additional properties that for each $(e,m) \in E_m^0 \times M$,

i) $F(e,m) = 0$ if and only if m is a competitive equilibrium for e; and

ii) for each i, f^i is C^1 and rank $D_{e^i} f^i(\cdot,m^0)$ is constant on a neighborhood of the set $\{e^i \in E_m^{i0}: \zeta^i(e^i,p^0) = y^{0i}\}$, where $y^{0N} = -\Sigma_{i=1}^{N-1} y^{0i}$.

A competitive adjustment process stabilizes regular equilibria if, in addition, for each $e \in E_{m^0}$ and each $m \in \mu_r(e)$, there is a neighborhood M' of m, and for each i there is a neighborhood E'^i of e^i in $E^i_{m^0}$ and an open subset $C'^i \subset C^i$ with $f^i(E'^i \times M') \subset C'^i$, satisfying

iii) α restricted to $C' \times M'$ is Lipschitzian, where $C' = \Pi_{i=1}^N C'^i$; and

iv) the adjustment process restricted to $E' \times M'$, where $E' = \Pi_{i=1}^N E'^i$, is locally stable. (That is, Definition 2.4 above is satisfied with $E \times M$ replaced by $E' \times M'$.)

3.3 Remarks: The constant rank condition in 3.2 (ii) is a regularity condition which enables the information communicated by trader i to be measured by the rank of $D_{e^i} f^i(e^i, m^0)$. Since the latter is a lower bound on the dimension of C^i, we will prove Theorem 3.5 below by showing that rank $D_{e^i} f^i(e^i, m^0) \geq$ rank $D_{e^i} f^{*i}(e^i, m^0)$. As a regularity condition, the constant rank condition is fairly strong, implying that any competitive adjustment process, locally stable or not, has $\dim C^i \geq K - 1$ for all i (see the first paragraph of the proof of Theorem 3.5). However, we can obtain this inequality for locally stable adjustment processes without the rank condition by applying Theorem 2.14 above, and in any case we are primarily interested in the additional required dimensions. I expect that the rank condition can be dropped, but I have not attempted to do this.

The neighborhoods C'^i are introduced so that the adjustment function

α is not required to be well-behaved on all of C. For example, the price adjustment $\dot{p}$ in the Newton method is not well-defined if the matrix $\Sigma_{i=1}^{N} D_p \zeta^i(e^i,p)$ is not invertible. However, the Newton method will satisfy the definition of a competitive adjustment process which stabilizes regular equilibria if we set

$$\dot{p} = \Sigma_{i=1}^{N} \zeta^i(e^i,p)$$

in this case.

It should be emphasized at this point that the definition of a competitive adjustment process is quite general, so the communication of excess demand information is not imposed by definition. In particular, since the state message includes the profile of trades, there may be dual versions of the Newton method in which traders communicate first and second derivatives of their utility functions at the current trades.

<u>3.4 Definitions</u>: Let $e^0 \in E_{m^0}$ such that m^0 is a regular equilibrium for e^0, and let E' and M' be neighborhoods of e^0 and m^0 respectively, such that for each $e \in E'$, $\mu_r(e) \cap E'$ is a singleton and the selection from μ_r on E' to M' is C^1. We will denote this selection $\mu_r: E' \to M'$.

Let V^N be a neighborhood of zero in R^{K-1}, let $(\omega^0,a,B) = e^{0N}$, and let E'^N be a neighborhood of e^{0N} in E^N with $\{e^{01},\ldots,e^{0N-1}\} \times E'^N \subset E'$. Define a function $\varepsilon: V^N \to E^N$ by

$$\varepsilon^N(Z) = (\omega^0 - \bar{z},a,B),$$

where $\bar{z} = (z_1,\ldots,z_{K-1},-p^0 z) \in R^K$. For each $1 \leq i \leq N-1$, let V^i be a neighborhood of zero in R^{K-1} and define $\mu_r'\colon V \to R^{N(K-1)}$, where $V = \Pi_i V^i$, by

$$\mu_r'(z^1,\ldots,z^N) = \mu_r(e^{01},\ldots,e^{0N-1};\varepsilon^N(z^N)) - (p^0,y^{01} - z^1,\ldots,y^{0N-1}-z^{N-1}).$$

<u>3.5 Lemma</u>: The function μ_r' is a map of pairs $\mu_r'\colon(V,V - 0) \to (R^{N(K-1)},R^{N(K-1)} - 0)$ and $\det D\mu_r'(0) = (-1)^{(N+1)(K-1)}\det(\Sigma_{i=1}^N D_p\zeta^i(e^{0i},p^0))^{-1}$.

<u>Proof</u>: If $\mu_r'(z^1,\ldots,z^N) = 0$, then by the definition of μ_r', p^0 is a competitive equilibrium price for the environment $(e^{01},\ldots,e^{0N-1},\varepsilon(z^N))$. Since $\zeta^i(e^{0i},p^0) = y^{0i}$ for each $i < N$, we must have $z^i = 0$ for each $i < N$. Since $\zeta^N(\varepsilon(z^N),p^0) = z^N$, and $\Sigma_{i=1}^{N-1}\zeta^i(e^{0i},p^0) + \zeta^N(\varepsilon(z^N),p^0) = 0$, we must have $z^N = 0$. Hence if $(p^0,y^{01} - z^1,\ldots,y^{0N-1} - z^N)$ is a competitive equilibrium for $(e^{01},\ldots,e^{0N-1},\varepsilon(z^N))$ then $(z^1,\ldots,z^N) = 0$. (Note for future reference that this implication does not rely on the fact that m^0 is a regular equilibrium for e^0). This proves the first assertion.

Direct computation shows that $D\mu_r'(0)$ is represented by the matrix

$$\left(\begin{array}{c|c} 0 & Z^0 \\ \hline & Z^1Z^0 \\ I & \vdots \\ & Z^{N-1}Z^0 \end{array}\right)$$

where I is the $(N - 1)(K - 1) \times (N - 1)(K - 1)$ identity matrix, $Z^0 = -(\Sigma_{i=1}^N D_p\zeta^i(e^{0i},p^0))^{-1}$, and for each $i < N$, $Z^i = D_p\zeta^i(e^{0i},p^0)$. Hence $\det D\mu_r'(0) = (-1)^{N(K-1)}\det Z^0$, which completes the proof.

<u>3.6 Theorem</u>: Let $N \geq K$, and let $(\alpha,(f^i,C^i)_{i=1}^N)$ be a competitive adjustment process which stabilizes regular equilibria. Then for each i,

$$\dim C^i \geq (K-1) + (K-2) + K(K-1)/2 = \text{rank } Df^{*i}(e^{0i}).$$

<u>Proof</u>: Let $e^0 \in E$ such that m^0 is a competitive equilibrium for e^0 and $\det(\Sigma_{i=1}^N D_p\zeta^i(e^{0i},p^0)) = 0$. We will show that

$$\text{rank } Df^{*1}(e^{01}) = \text{rank } D_{e^1}f^1(e^{01},m^0),$$

where f^{*i} is defined in Section 3.1 above. This will prove the theorem for $i = 1$. Let $c^{01} = f^1(e^{01},m^0)$ and let $h \in R^{2(K-1)+K(K-1)/2}$ with $Df^{*1}(e^{01})h \neq 0$, and suppose by way of contradiction that $D_{e^1}f^1(e^{01},m^0)h = 0$. Since $D_{e^1}f^1(\cdot,m^0)$ has constant rank, the implicit function theorem implies the existence of a C^1 function $\pi\colon [-1,1] \to E^1_{m^0}$ with $\pi(0) = e^{01}$, $D\pi(0) = h$, and $f^1(\pi(\cdot),m^0) \equiv c^{01}$. Since m^0 is a competitive equilibrium for e^0 and the adjustment process is competitive, $\alpha(c^{01},f^2(e^{02},m^0),\ldots,f^N(e^{0N},m^0),m^0) = 0$, so $\zeta^1(\pi(\cdot),p^0) \equiv y^{01}$. Hence $(D_{e^1}\zeta^1(e^{01},p^0))h = 0$ so $[D_{e^1}(D_p\zeta^1(e^{01},p^0))]h \neq 0$. For each $i > 1$, let $(\omega^{0i},a^i,B^i) = e^{0i}$. We noted in 3.1(4) above that the map $(a'^i)_{i>1} \to \Sigma_{i>1}D_p\zeta^i(\omega^{0i},a'^i,B^i;p)$ is an open map, so we can choose $(a^2,\ldots,a^N)$ so that

$$D_{e^1}(\det[\Sigma_{i=1}^N D_p\zeta^i(e^{0i},p^0)])h \neq 0.$$

In particular, we can assume that

$$(*)\quad \begin{aligned} &\det[D_p\zeta^1(\pi(1),p^0) + \Sigma_{i>1}D_p\zeta^i(e^{0i},p^0)] > 0 \quad \text{and} \\ &\det[D_p\zeta^1(\pi(-1),p^0) + \Sigma_{i>1}D_p\zeta^1(e^{0i},p^0)] < 0. \end{aligned}$$

Since m^0 is a regular equilibrium for the environment $(\pi(1),e^{02},\ldots,e^{0N})$ and the adjustment process stabilizes regular equilibria, there is a neighborhood M' of m^0, a neighborhood C'^1 of C^{01} with $f^1(\{\pi(1)\} \times M') \subset C'^1$, and for each $i > 1$, a neighborhood C'^i of $f^i(e^{0i},m^0)$ and a neighborhood E^i of e^{0i} with $f^i(E^i \times M') \subset C'^i$, such that α is continuous on $C' \times M'$ where $C' = \Pi_i C'^i$. Since $f^1(\pi(\cdot),m^0) \equiv c^{01}$, we can choose M' small enough so that $f^1(\pi([-1,1]) \times M') \subset C'^1$. Hence the function $(\lambda,e^N,m) \to F(\pi(\lambda),e^{02},\ldots,e^{0N-1},e^N;m)$ is continuous on $[-1,1] \times E^N \times M'$.

For each i, let V^i be a neighborhood of zero in R^{K-1}, let $V = \Pi_i V^i$, and define the function $H\colon V \times [-1,1] \to R^{N(K-1)}$ by $H(z^1,\ldots,z^N;\lambda) = F(\pi(\lambda),e^{02},\ldots,e^{0N-1},\varepsilon(z^N);p^0,y^{01}-z^1,\ldots,y^{0N-1}-z^{N-1})$, where $\varepsilon\colon V^N \to E^N$ is the function defined in Section 3.4 above. By the first paragraph of the proof of Lemma 3.5 above, $H(z^1,\ldots,z^N;\lambda) = 0$ only if $(z^1,\ldots,z^N) = 0$ so H is a map of pairs,

$$H\colon (V \times [-1,1],(V - 0) \times [-1,1]) \to (R^{N(K-1)},R^{N(K-1)} - 0).$$

Therefore $H(\cdot,1)$ and $H(\cdot,-1)$ are homotopic maps of pairs. Since the adjustment process stabilizes regular equilibria, Corollary 2.11 implies that $H(\cdot,1)$ is homotopic to the map

$$\begin{aligned} \mu'_{r1}\colon (z^1,\ldots,z^N) \to\ &\mu_r(\pi(1),e^{02},\ldots,e^{0N-1};\varepsilon^N(z^N)) \\ &- (p^0,y^{01} - z^1,\ldots,y^{0N-1} - z^{N-1}) \end{aligned}$$

and $H(\cdot,-1)$ is homotopic to the map.

$$\mu'_{r(-1)}: (z^1,\dots,z^N) \to \mu_r(\pi(-1),e^{02},\dots,e^{ON-1};\varepsilon^N(z^N))$$

$$-(p^0,y^{01}-z^1,\dots,y^{ON-1}-z^{N-1}),$$

so μ'_{r1} and $\mu'_{r(-1)}$ are homotopic. However, (*) and Lemma 3.5 above imply that $\det D\mu'_{r1}(0)$ has the sign $(-1)^{(N+1)(K-1)}$ and $\det D\mu'_{r(-1)}(0)$ has the sign $(-1)^{(N+1)(K-1)+1}$, so μ'_{r1} and $\mu'_{r(-1)}$ cannot be homotopic. This contradiction completes the proof for $i = 1$.

The same argument clearly applies to any $i < N$. For $i = N$, let $(\omega^{ON},a^N,B^N) = e^{ON}$ as above. If we define $\pi: [-1,1] \to E^N_{m^0}$ exactly as in the case $i = 1$, the fact that $\zeta^N(\pi(\cdot),p^0) \equiv -\Sigma_{i=1}^{N-1}y^{0i}$ implies that π takes the form $\pi(\lambda) = (\omega^{ON},a^N(\lambda),B^N(\lambda))$. Hence we can define the function $\varepsilon'^N: [-1,1] \times V^N \to E'^N$ by $\varepsilon'^N(\lambda,z) = (\omega^{ON}-\bar{z},a^N(\lambda),B^N(\lambda))$. Then defining

$$H'(z^1,\dots,z^N;\lambda) = F(e^{01},\dots,e^{ON-1},\varepsilon'^N(z^N,\lambda);p^0,y^{01}-z^1,\dots,y^{ON-1}-z^{N-1}),$$

and proceeding as in the case $i = 1$ completes the proof.

<u>3.7 Remarks</u>: Although disappointing, the theorem should not be surprising in view of a related result by Saari and Simon (1978). These authors studied the informational requirements of stability for nondecentralized price adjustment procedures. In their models, a "price mechanism" is a function M which, given an aggregate excess demand function ζ, determines a price adjustment according to the differential equation

$$\dot{p} = M(z,D),$$

where $z = \zeta(p)$ and $D = D_p\zeta(p)$. To quantify the information required by M, they introduce the notion of an "ignorable coordinate". A coordinate d_{ij} of $D_p\zeta(p)$ is ignorable if for all D and all z in a neighborhood of zero, $\frac{\partial}{\partial d_{ij}} M(z,D) = 0$, so that M is independent of d_{ij} near equilibria. Also, M is assumed to be C^1 and for some excess demand function ζ and some p with $\zeta(p) = 0$, the derivative

$$D_p M(\zeta(p), D_p\zeta(p))$$

is assumed to be nonsingular. A price mechanism M is "locally"

effective" if all regular

equilibria are locally stable. They prove that a locally effective M must be sensitive to the sign of the determinant of $D_p\zeta(p)$ at a regular equilibrium. Since this can generically be changed by a sufficiently large change in any coordinate of $D_p\zeta(p)$, a locally effective price mechanism admits no ignorable coordinates.

Our approach differs from that of Saari and Simon in several important respects. First, we conclude the sensitivity of $F(e,m^0)$ to the sign of $\det D_p\zeta(e,p^0)$ directly from the fact of local stability, via the homotopy equivalence established in Theorem 2.10, without using a nonsingularity assumption on $D_m F(e,m^0)$. Second, since we do not place any restrictions on the qualitative nature of the information communicated by traders, we do not discuss ignorable coordinates. Rather, we observe that since a trader cannot know how or whether a change in a coordinate of his own derivative $D_p\zeta^i(e^i,p^0)$ will affect the sign of $\det D_p\zeta(e,p^0)$, he must report all coordinates. Nonetheless, it should be clear that the work of Saari and Simon was an important source of inspiration for this section of the present paper.

The fact that Theorem 3.6 applies even to the two commodity case emphasizes the strength of the requirement that all regular equilibria be locally stable. At the opposite extreme, for example, one might require that every regular economy have at least one locally stable equilibrium. Unfortunately, the analysis of this limited stability requirement is beyond the scope of the present paper. In particular, the local utility functions constructed above would have to be explicitly extended to a more conventional domain, such as R^K_{++}, so that all potentially stable equilibria could be analysed. In a nondecentralized context, the information required to ensure the stability of at least one equilibrium has been studied by Saari and Simon (1978), and more recently by the present author (Jordan (1982)) using the homotopy approach embodied in Theorem 2.10 above. It seems likely that the results of these two papers could eventually be applied to decentralized adjustment processes, but this will have to await further research.

4. Pareto Adjustment Processes

This section is a brief discussion of adjustment processes which stabilize Pareto optimal allocations for exchange environments. Since we will not need the parameterization used in the last section, we will make this section formally independent on Section 3 by using a more conventional class of smooth exchange environments.

In Section 4.1 below, a Pareto adjustment process is defined as an adjustment process whose stationary messages are Pareto optimal trade profiles. Theorem 4.2 states that every Pareto adjustment process, locally stable or not, has $\dim C^i \geq K - 1$ for each i, so that $\dim C \geq N(K - 1)$. Hence equilibrium considerations alone give the dimension of the competitive message space as a lower bound on the dimension of the control message space for Pareto adjustment processes. The proof is very similar to the standard proof of the Hurwicz-Mount-Reiter informational efficiency theorem.

In Section 4.3 we define a locally stable Pareto adjustment process which achieves this lower bound. Each trader reports his demand price (normalized utility gradient) at the current trade, and a Pareto improving trade adjustment is made. Together with Theorem 3.6 above, this shows that no competitive adjustment process which stabilizes regular equilibria can be informationally efficient among the class of locally stable adjustment processes with Pareto optimal equilibria.

<u>4.1 Definitions</u>: For each $1 \leq i \leq N$, trader i's consumption set is R^K_{++}. Trader i is characterized by a C^2 utility function $u^i\colon R^K_{++} \to R$ satisfying, for each $x \in R^K_{++}$,

i) $Du^i(x) \in R^K_{++}$;

ii) $h'D^2u^i(x)h < 0$ for every nonzero $h \in R^K$ with $Du^i(x)h = 0$; and

iii) the closure in R^K of the set $\{x' \in R^K_{++}: u^i(x') \geq u^i(x)\}$ is contained in R^K_{++};

and an endowment $\omega^i \in R^K_{++}$. Let E^i denote the set of possible characteristics (ω^i, u^i), with the Euclidean topology on endowments and the topology of C^2 uniform convergence on compact subsets of R^K_{++} for utility functions. Let $E = \Pi_i E^i$.

Let $M = \{y = (y^1, \ldots, y^N) \in R^{KN}: \Sigma_{i=1}^N y^i = 0\}$, and define the Pareto correspondence $\pi: E \twoheadrightarrow M$ by $\pi(e) = \{y \in M: \omega^i + y^i \in R^K_{++}$ for each i, and the allocation $(\omega^i + y^i)_{i=1}^N$ is Pareto optimal for $e\}$. Given the space of environments E and the state message space M, a Pareto adjustment process is an adjustment process as defined in Section 2.1 above and possessing the regularity properties of Section 2.3, with the additional property that for each (e,y),

P) $F(e,y) = 0$ if and only if $y \in \pi(e)$.

4.2 Theorem: If $(\alpha, (f^i, c^i)_{i=1}^N)$ is a Pareto adjustment process then for each i, $\dim C^i \geq K - 1$.

Proof: For each i, let $\omega^{0i} \in R^K$ and let $E^{0i} = \{(\omega^i, u^i) \in E^i$: $\omega^i = \omega^{0i}$ and there is some $\beta^i \in R^{K-1}_{++}$ such that $u^i(x) = \Sigma_{j=1}^{K-1} \beta_j^i \ln x_j + \ln x_K$, where $\ln$ denotes the natural logarithm$\}$. Then each characteristic in E^{0i} is determined by the parameter vector β^i, and the Euclidean topology on parameters agrees with the topology on E^{0i} as a subspace of E^{i}. Hence we can identify E^{0i} with the parameter space R^{K-1}_{++}. Let y denote the zero trade, $y = (0,\ldots,0) \in M$, and for any i, let $\beta^1, \beta'^1 \in E^{0i}$ with $f^1(\beta^1,y) = f^1(\beta'^1,y)$. Let $(\beta^2,\ldots,\beta^N) \in \Pi_{i>1}E^{0i}$ such that $y \in \pi(\beta^1,\ldots,\beta^N)$. Then by 4.1 (P), $\alpha(f^1(\beta^1,y),\ldots,f^N(\beta^N,y),y) = 0$ so $\alpha(f^1(\beta'^1,y),f^2(\beta^2,y),\ldots,f^N(\beta^N,y),y) = 0$. Again by (P), this implies that $y \in \pi(\beta'^1,\beta^2,\ldots,\beta^N)$, so $\beta'^1 = \beta^1$. Hence f^1 is 1-1, so $\dim C^1 \geq K - 1$. The same argument clearly applies to each $i > 1$, which completes the proof.

4.3 <u>A Locally Stable Pareto Adjustment Process</u>: For each i, let $C^i = R^{K-1}_{++}$ and define the function $f^i\colon E^i \times M \to C^i$ by $f^i(\omega^i,u^i;y) = (\frac{\partial u^i}{\partial x_K}(\omega^i + x))^{-1}Du^i(\omega^i + y^i)$. Although f^i is well-defined only when $\omega^i + y^i \in R^K_{++}$, this suffices to define f^i near any Pareto optimum. Let $C = \Pi^N_{i=1}C^i$ and define $\alpha\colon C \times M \to R^{K(N+1)}$ by $\alpha(c^1,\ldots,c^N;y) = \dot{y}^*$ where $\dot{y}^*$ maximizes $\Sigma^N_{i=1}c^i\dot{y}^i$ subject to

a) $\Sigma^N_{i=1}\dot{y}^i = 0$;

b) $c^i\dot{y}^i \geq 0$ for all i; and

c) $\|\dot{y}\| \leq \sum_{1\leq i,j\leq N} \|c^i - c^j\|$,

where $\|\cdot\|$ denotes the Euclidean norm.

4.4 <u>Remarks</u>: This adjustment process maximizes the normalized sum of individual utility improvements subject to the feasibility constraint (a), the constraint that no individual suffer a utility loss (b), and the constraint (c) which causes $\dot{y}$ to converge to zero as the normalized utility gradients converge to equality. Because these constraints impose boundaries on the choice of $\dot{y}$, α is not C^1 although it is easily seen to be Lipschitzian. The proof that the adjustment process is locally stable is straightforward so we give a fairly condensed outline below. The definition of the adjustment process and the proof of local stability would be much more complicated if the regularity conditions in the hypothesis of Proposition 2.6 above were included in the definition of local stability. This illustrates the convenience of the weaker definition.

<u>4.5 Proposition</u>: The adjustment process $((f^i,c^i)_{i=1}^N,\alpha)$ defined in Section 4.3 above is a locally stable Pareto adjustment process.

<u>Proof</u>: It is clear that $((f^i,c^i)_i,\alpha)$ is a Pareto adjustment process. To verify local stability, let $e^0 \in E$ and $y^0 \in \pi(e^0)$. It is straightforward to show that for any (e,y) sufficiently near (e^0,y^0) so that for each i, $\omega^i + y^i \in R^K_{++}$, where $(\omega^i,u^i) = e^i$, we will have $\lim_{t\to\infty} m^*(e,y;t) \in \pi(e)$, using the notation of Section 2.4 above. To verify the structural stability condition 2.4(a), note that for any $\varepsilon > 0$ a sufficiently small perturbation of (e^0,y^0) will cause the "lens" of Pareto superior trades to have diameter less than ε. This proves local stability and completes the proof of the theorem.

<u>4.6 Remarks</u>: If $N > K$, then

$$\dim C = N(K - 1) < (N - 1)K = \dim M,$$

so this adjustment process violates the conclusion of Theorem 2.14 above. Although E contains subspaces E^0 of arbitrarily large dimension, $\pi(e)$ is too large to permit 2.14 (i) and 2.14 (ii) to be satisfied similtaneously.

There are clearly many more locally stable Pareto adjustment processes which use a control message space of the same dimension. For example, one could choose $\dot{y}^*$, subject to the same constraints, to maximize $\Sigma_i \lambda^i c^i \dot{y}^i$ where $(\lambda^i)_i$ is any N-tuple of fixed nonnegative weights with $\lambda^i > 0$ for some i. Also, one could append a price vector $p \in R^{K-1}_{++} \times \{1\}$ to the state message, add the adjustment rule

$$\dot{p} = [(1/N)\Sigma_i c^i] - p,$$

and add the constraint

d) $p\dot{y}^i = 0$ for all i.

In this way one obtains an example of an "exchange process with price adjustment" as studied by Smale (1976b). Miller (1978) gives a general study of the stability of Pareto improving processes.

5. Conclusion

The model of dynamic adjustment processes developed here seems sufficiently general to permit applications to a wide range of allocation problems. The point of view we have taken in the applications presented here is that any performance correspondence or equilibrium concept implicitly determines a minimal amount of communication needed by any adjustment process which makes the given equilibria locally stable. In Sections 3 and 4 above, we applied the techniques developed in Section 2 to calculate this communication requirement for regular competitive exchange equilibria and Pareto optima respectively. We hope that similar calculations can be made in future research to help evaluate the informational efficiency of the many new equilibrium concepts which have appeared in recent years.

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