GAMES WITH COMMUNICATION

by

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I. Introduction

The obvious and traditional method of modeling communication in a noncooperative game is to assume the existence of a message space. During the play of the game the agent selects not only a "real" move affecting the physical state of the game, but also a message which affects only the beliefs of the other agents concerning the private information of the agent making the selection.

Although this modeling technique is simple and straightforward, it introduces certain pathologies. The cardinality of the message space may restrict the possibilities for communication, and enlarging the message space does not eliminate the pathological equilibria, since many messages may have the same meaning.

Furthermore, the introduction of message spaces leads to the expression of strategies in terms of choices of messages, but game theory is concerned only with the strategic "meaning" of the messages. Put another way, the internal structure of the message space is irrelevant to game theory: the set of equilibria should not depend on whether the agents speak English or French. Occam's Razor suggests that strategies should be expressed in terms of choices from endogenously determined sets of meanings.

Starting with a formulation in terms of message spaces, we derive a method for expressing strategies in terms of choices from sets of decisions. A decision is technically a complicated object, but intuitively it simply summarizes all the local information relevant to the choice. A decision has as components the resulting node in the game tree, the posterior beliefs generated by the choice for all agents and all private information states, and the expected payoffs, again for all agents and all private information states. Collectively this trio constitutes the situation brought about by the decision. The other components of the decision are the real move chosen (which is actually redundant, being recoverable from the node), and a probability measure on the set of private information states for the agent choosing the move. This probability measure represents the normalized vector of the probabilities assigned by the strategy to the decision in the various private information states. Combined with the prior beliefs by Bayes rule, this probability measure generates the posterior beliefs associated with the decision.

Intuitively we imagine the agents keeping track of the local information, computing according to the known strategy vector. When an agent must choose a move - message pair, he finds himself at a situation as described above, and he must select a decision from the set of those available.

Although this new framework is superficially complex, it has several advantages. For one thing it provides a natural setting for the concept of sequential equilibrium defined recently by Kreps and Wilson [5]. Secondly, it is in this setting that games are solved by backward induction. More important and surprising, it is possible to state a natural restriction on the set of decisions associated, at a given situation, with a particular move. This restriction can be combined with the usual assumption of optimizing behavior to form a solution concept, called communication equilibrium. Before resuming the discussion of this concept we must describe its setting more precisely.

Our attention is restricted to games of asymmetric information. This means that the physical state of the game is known by all agents at all times, so there are no nontrivial information sets. The payoffs associated with terminal (game-ending) nodes are uncertain: each agent starts the game with a piece of private information, and the payoff for an agent at a terminal node depends on the private information of all agents.

We also restrict attention to public communication. All agents are aware at all times of all prior communication, so the selection of one message rather than another can reflect only the private information of the agent making the selection. The extent to which these restrictions can be relaxed is an open problem.

Suppose that the set of nodes is finite, and suppose that the universal message space, available at all times, is also finite. A behavior strategy (in the message space framework) is an assignment, to each node, each associated history of message selection, and each private information state of the agent choosing the move, of a probability distribution over the set of move - message pairs available at the node. Consider a particular strategy vector, and suppose that for all nodes, all histories of message selection, and all information states, positive probability is assigned to all move - message pairs. Given a set of prior beliefs for each agent and each of his information states, the strategy vector uniquely determines Bayesian posteriors for each node and each history of message selection. Furthermore, expected payoffs are also uniquely determined for all agents and all information states at all stages of play. It is intuitively clear that this strategy is expressible in a unique way in terms of probability distributions on sets of decisions.

Now consider a particular node and a particular move - message pair. Suppose the agent making the choice cannot avoid assigning a small but positive probability to this move - message pair, and suppose this involuntary probability depends on the private information of the agent. Intuitively this move - message pair may be thought of as a "revealing gesture" which can be actively imitated if that is advantageous. If the agent, regardless of his private information, never voluntarily assigns probability to the move - message pair, Bayesian posteriors will be determined by the priors and by the vector assigning an involuntary probability to each of the agent's possible information states. Multiplying this vector by a positive constant does not change the posteriors, so the information transmitted by the move - message pair can be represented by the vector of probabilities divided by the sum of the probabilities. A vector of probabilities normalized in this fashion will be referred to as a likelihood vector. Such a vector can be treated formally as a probability distribution over the set of private information states for the agent, and the final component of a decision is such a vector, but it is important to remember that likelihood vectors are *not* interpreted as probability distributions.

Given such a revealing gesture, there are two possibilities in equilibrium. The first is that the agent, regardless of his private information, never voluntarily assigns probability to this move - message pair. In this case there is an available decision that has the move and the associated likelihood vector as its final two components, and for each of the information states for the agent there is another decision that weakly dominates this decision. The other possibility is that there are some information states in which the agent assigns more than the associated involuntary probability to the move - message pair. Normalizing the vector of actual probabilities produces another likelihood vector, and this vector must assign less likelihood to those information states for which the move - message pair is strictly dominated.

Now imagine an increasing sequence of message spaces, and suppose that in the limit the set of revealing gestures "fills up" the cartesian product of the sets of moves and likelihood vectors. We are led to condition (C2) of Section IV. Consider a state of the game (i.e., a situation), a move feasible at that state, and a likelihood vector for the agent choosing the move. Either there is an available decision with the given move and likelihood vector as the final two components, or (C2) requires that there be a decision with the move and some other likelihood vector as final components. Furthermore (C2) requires that the second likelihood vector assign no more likelihood than the first to those information states for which the decision is suboptimal.

Condition (C2) is rather complicated, but it reflects a simple intuition. If one suspected that a revealing gesture was being voluntarily imitated, one would naturally suspect that the agent's private information made such behavior advantageous.

Combining (C2) with the usual assumption of maximizing behavior yields the communication equilibrium solution concept. If strategies were expressed in terms of choices from a set of move - message pairs, this concept would be difficult if not impossible to formulate. The apparent complexity of our proposed framework is justified by the ease with which it allows the communication equilibrium concept to be expressed.

The proof of existence is essentially the argument outlined above. This argument is a variant of the argument employed by Selten [8] to prove the existence of perfect equilibria, and in Section IV we discuss an intuitive interpretation of communication equilibrium similar to the "trembling hand" interpretation of perfect equilibrium.

In principle communication equilibrium is not a replacement for Nash equilibrium or any of its refinements. Rather, it is appropriate when communication is possible and Nash equilibrium is appropriate when communication is impossible. Communication equilibrium and sequential equilibrium (a variant of Nash equilibrium proposed by Kreps and Wilson [5]) can be combined in a straightforward fashion when communication is possible at some nodes of the game tree and not at others. In Section V, however, an example is given which suggests that it is unnatural to assume that communication is impossible when a continuum of moves is available.

Section VI illustrates the power of the solution concept by applying it to a bargaining example. This example is similar in structure to ones analyzed by Fredenberg and Tirole [1], and the payoff structure is similar to that analyzed by Harsanyi [3]. Whereas this example has a continuum of sequential equilibria, there is essentially a unique communication equilibrium. This communication equilibrium can be represented as a sequential equilibrium, so, rather paradoxically, we have an example of the communication equilibrium concept refining the set of equilibria in which there is no communication.

The organization of the remainder of the paper is as follows. Section II gives a formal description of the games and their message space augmentations. Section III discusses strategies and develops the notion of a configuration. Section IV defines the solution concept, proves existence for games with finite game trees, and gives a discussion of the intuitive underpinnings. Section V extends the existence theorem to include games with compact metric spaces of moves and nodes. Section VI analyzes the bargaining example.

II. Games

A game is an ordered octuple

$$G = \langle A, I, N, a, M, f, U, p^{*} \rangle \quad .$$

The objects of G are, respectively and mnemonically, the set of agents, the set of private information states, the set of nodes, the agent function, the set of moves, the consequence function, the utility function, and the vector of priors. We consider only the case of perfect but incomplete information.

The set of nodes is partitioned as

$$N = N^{C} \cup N^{T} \quad ,$$

the sets of communication nodes and terminal nodes. As this terminology suggests, we do not consider the possibility that communication is allowed at certain nodes and not at others. As will be seen, this could be allowed with the only real difficulty being the additional burden of notation. Similarly, we could include chance moves, and these moves could be correlated with agents' private information, so nature could be a source of new information during the play. Again, no subtleties are involved, so we choose to keep the notation simple.

We assume throughout that both A and I are finite, and we refer to

$$A \times I$$

as the set of informed agents. The assumption that all agents have the same set of private information states entails no loss of generality; I might be thought of as the disjoint union of the sets of private information states for the agents.

The set of <u>states of the world</u> is

$$\mathcal{J} = I^A$$

with generic element $\hat{i} = (i_a)_{a \in A}$.

The agent function

$$a : N^C \to A$$

determines the agent choosing the move at each communication node. For each $a \in A$ let $N^a = a^{-1}(a)$.

The utility function

$$U : A \times \mathcal{J} \times N^T \to \mathbb{R}$$

assigns a payoff to each agent in each state of the world at each terminal node.

The consequence function

$$f : N^C \times M \to N$$

describes the game tree: $f(n, m) \in N$ is the node resulting from m chosen at n . The following assumptions guarantee that f describes a tree.

(<u>T1</u>) There is a unique $n^* \in N$ such that

$$f(N^C \times M) = N \setminus \{n^*\} \quad ,$$

called the <u>origin</u>.

(<u>T2</u>) For all $n \in N \setminus \{n^*\}$ there is a unique pair

$$(\bar{n}(n), \bar{m}(n)) \in N^C \times M \quad \text{with} \quad f(\bar{n}(m), \bar{m}(n)) = n \quad .$$

(T3) If $N_0 = \{n^*\}$ and

$$N_h = f((N_{n-1} \setminus N^T) \times M), \quad h = 1, 2, \ldots,$$

then
$$N = \bigcup_{h=0}^{\infty} N_h \quad .$$

The description of the game is completed by describing the nature of p^*. Of course p^* will be a vector of probability measures, and it is therefore convenient to specify certain conventions regarding such measures. If C is a topological space then, following Myerson [6],

$$\Delta(C)$$

will denote the set of Borel probability measures on C. The probability of event $E \subseteq C$ under $p \in \Delta(C)$ is written $p[E]$, and if $x \in C$ then we write $p[x]$ rather than $p[\{x\}]$. (This use of square brakets is intended to aid the reader by automatically alerting him or her to the fact that a set is being measured.)

If C is a metric space then $\Delta(C)$ is assumed to be endowed with the Prokorov metric:

$$\begin{aligned} \delta(p_1, p_2) = \inf \{\varepsilon > 0 \mid\ & p_1[E] \leq p_2[B(E; \varepsilon)] + \varepsilon \\ & \text{and } p_2[E] \leq p_1[B(E: \varepsilon)] + \varepsilon \\ & \text{for all Borel events } E \subseteq C\} \quad . \end{aligned}$$

(Here

$$B(E; \varepsilon) = \{x \in C \mid \text{ there exists } y \in E \text{ with } \delta(x, y) < \varepsilon\} \quad ;$$

throughout all metrics will be denoted by δ, with the domain to be inferred from the context.) The Prokorov metric topologizes the weak topology on measures. As shown in Porasarathy [7], p. 45, a metric space C is compact if and only if $\Delta(C)$ is compact. If C is finite then the weak toplogy is the usual topology on the simplex.

For informed agent (a, i) let

$$P_{ai} = \{p \in \Delta(\mathcal{J}) \mid p[\{\hat{j} \mid j_a \neq i\}] = 0\} \quad ,$$

so P_{ai} is the set of probability distributions over $\mathcal{J}$ consistent with a knowing i. The prior

$$p^* = (p^*(a, i))_{(a,i)\in A\times I} \in P = \prod_{(a,i)\in A\times I} P_{ai}$$

specifies the beliefs of each informed agent at the beginning of the game. Elements of P other than p^* will be referred to as <u>posterior vectors</u>. Note that, although we assume that each informed agent knows the priors of all other informed agents, we do not assume consistency (see Harsanyi [2]). Consistency would require that all components of p^* be conditional distributions derived from some common prior $p^{**} \in \Delta(\mathcal{J})$.

The game G may be thought of as describing the physical possibilities of the environment. We now introduce a <u>message space</u> $\tilde{Q}$, an abstract set with no internal structure. Define $\tilde{M}$, $\tilde{N}_h$, $h = 0, 1, 2, \ldots,$ and $\tilde{N}$ by

$$\tilde{M} = \tilde{M}(\tilde{Q}) = M \times \tilde{Q}$$

$$\tilde{N}_0 = \tilde{N}_0(\tilde{Q}) = \{n^*\} ,$$

$$\tilde{N}_h = \tilde{N}_h(\tilde{Q}) = N_h \times \tilde{Q}^h , \quad h = 1, 2, \ldots, \qquad \text{and}$$

$$\tilde{N} = \tilde{N}(\tilde{Q}) = \bigcup_{h=0}^{\infty} \tilde{N}_h \quad .$$

Elements of $\tilde{N}$ are denoted by $\nu = (n(\nu), \tilde{q}_1(\nu), \ldots, \tilde{q}_h(\nu))$ and are called <u>pseudonodes</u>. We let $\tilde{N}^C = \{\nu \in \tilde{N} \mid n(\nu) \in N^C\}$ and $\tilde{N}^T = \tilde{N} \setminus \tilde{N}^C$. Elements of $\tilde{M}$ are denoted by $\mu = (m(\mu), \tilde{q}(\mu))$ and are called <u>move-message pairs</u>. The function

$$\tilde{f} = \tilde{f}(\tilde{Q}) : \tilde{N}^C \times \tilde{M} \to \tilde{N}$$

is defined by

$$\tilde{f}(\nu, \mu) = (f(n(\nu), m(\mu)), \tilde{q}_1(\nu), \ldots, \tilde{q}_h(\nu), \tilde{q}(\mu)) \quad .$$

Extending the agent function to $\tilde{N}$ by

$$a(\nu) = a(n(\nu))$$

and the utility function to $A \times \mathcal{J} \times \tilde{N}$ by

$$U(a, \hat{i}, \nu) = U(a, \hat{i}, n(\nu)) \quad ,$$

it is quite clear that

$$\tilde{G} = \tilde{G}(\tilde{Q}) = \langle A, I, \tilde{N}, a, \tilde{M}, \tilde{f}, U, p^*\rangle$$

is a game as we have defined the notion.

III. Strategies and Configurations

Throughout this section we assume that N, M, and $\tilde{Q}$ are finite, so that $\tilde{N}$ and $\tilde{M}$ are also finite. Let the set of behavior strategy vectors be

$$\tilde{\Sigma}(\tilde{G}) = (\Delta(\tilde{M})^{I})^{\tilde{N}} \quad .$$

If $\tilde{\sigma} \in \tilde{\Sigma}(\tilde{G})$ then for each $\nu \in \tilde{N}^{C}$, $\tilde{\sigma}(\nu)$ is a function from I to $\Delta(\tilde{M})$, and we shall write $\tilde{\sigma}(\nu, i)$ rather than $\tilde{\sigma}(\nu)(i)$. A behavior strategy vector is said to be totally mixed if it is an element of

$$\text{int } \tilde{\Sigma}(\tilde{G}) = ((\text{int } \Delta(\tilde{M}))^{I})^{\tilde{N}^{C}} \quad .$$

The immediate goal is to express totally mixed behavior strategy vectors in a way that makes no reference to the message space $\tilde{Q}$. Given

$$\tilde{\sigma} \in \text{ int } \tilde{\Sigma}(\tilde{G}) \quad ,$$

it is possible to define

$$p(\tilde{\sigma}) = (p(\tilde{\sigma};\, a, i))_{(a,i) \in A \times I} : \tilde{N} \to P$$

by the equations

$$p(\tilde{\sigma};\, a, i, n^{*}) = p^{*}(a, i) \qquad \text{and}$$

$$p(\tilde{\sigma};\, a, i, \tilde{f}(\nu, \mu))[\hat{j}] = \frac{p(\tilde{\sigma};\, a, i, \nu)[\hat{j}]\, \tilde{\sigma}(\nu, j_{a(\nu)})[\mu]}{\sum_{\hat{k} \in \mathcal{J}} p(\tilde{\sigma};\, a, i, \nu)[\hat{k}]\, \tilde{\sigma}(\nu, k_{a(\nu)})[\mu]} \quad . \tag{1}$$

This, of course, is simply Bayes' rule.

Letting

$$\ell(\tilde{\sigma};\, \nu, \mu) \in \Delta(I)$$

be defined by

$$\ell(\tilde{\sigma};\, \nu, \mu)[i] = \tilde{\sigma}(\nu, i)[\mu] \,/\, (\sum_{j \in I} \tilde{\sigma}(\nu, j)[\mu]) \quad ,$$

equation (1) can be rewritten in the following form:

$$p(\tilde{\sigma}; a, i, \tilde{f}(\nu, \mu))[\hat{j}] = \frac{p(\tilde{\sigma}; a, i, \nu)[\hat{j}]\,\ell(\tilde{\sigma}; \nu, \mu)[j_{a(\nu)}]}{\sum_{\hat{k} \in \mathcal{J}} p(\tilde{\sigma}; a, i, \nu)[\hat{k}]\,\ell(\tilde{\sigma}; \nu, \mu)[k_{a(\nu)}]} \quad .$$

Although $\ell(\tilde{\sigma}; \nu, \mu)$ is written as a probability measure, this is the result of a normalization, and it should really be viewed as a vector of relative likelihoods expressing the information transmitted by the choice of μ. Such elements of $\Delta(I)$ will be called <u>likelihood vectors</u>.

Let

$$V = \prod_{(a,i) \in A \times I} \left[\min_{(a,\hat{i},\nu) \in A \times \mathcal{J} \times \tilde{N}^T} U(a, \hat{i}, \nu), \max_{(a,\hat{i},\nu) \in A \times \mathcal{J} \times \tilde{N}^T} U(a, \hat{i}, \nu) \right]$$

be the set of <u>value vectors</u>. The payoffs expected by the various informed agents at the various pseudonodes, given $\tilde{\sigma}$, are specified by the function

$$v(\tilde{\sigma}) = (v(\tilde{\sigma}; a, i))_{(a,i) \in A \times I} : \tilde{N} \to V \quad .$$

This function is defined by the expected value calculations

$$v(\tilde{\sigma}; a, i, \nu) = \sum_{\hat{j} \in \mathcal{J}} p(\tilde{\sigma}; a, i, \nu)[\hat{j}]\, U(a, \hat{j}, \nu)$$

for $\nu \in \tilde{N}^T$ and

$$v(\tilde{\sigma}; a, i, \nu) = \sum_{\hat{j} \in \mathcal{J}} p(\tilde{\sigma}; a, i, \nu)[\hat{j}] \cdot \sum_{\mu \in \tilde{M}} \sigma(\nu, j_{a(\nu)})[\mu]\, v(\tilde{\sigma}; a, i, \tilde{f}(\nu, \mu))$$

for $\nu \in \tilde{N}^C$.

We now proceed to aggregate the information generated by the functions $p(\tilde{\sigma})$, $\ell(\tilde{\sigma})$, and $v(\tilde{\sigma})$. Let

$$S = N \times P \times V$$

be the set of situations, endowed with the product topology where N has the discrete topology and P and V have the topologies inherited from their natural embeddings in euclidean spaces. We assume, in fact, that S is endowed with a metric that generates the topology.

We write S^C for $\{s \in S \mid n(s) \in N^C\}$, and similarly $S^T = \{s \in S \mid n(s) \in N^T\}$, and we let $a(s)$ stand for $a(n(s))$.

Define

$$s(\tilde{\sigma}) : \tilde{N} \to S$$

by

$$s(\tilde{\sigma}; \nu) = (n(\nu), p(\tilde{\sigma}; \nu), v(\tilde{\sigma}; \nu)) \quad .$$

Let

$$D = S \times M \times \Delta(I)$$

be the set of decisions, again endowed with a metric generating the product topology.

N.B.: Both S and D are compact metric spaces.

If $d = (s(d), m(d), \ell(d))$ then we write $a(d)$, $n(d)$, $p(d)$, and $v(d)$ rather than $a(s(d))$, $n(s(d))$, $p(s(d))$, and $v(s(d))$ respectively.

Define

$$d(\tilde{\sigma}) : \tilde{N}^C \times \tilde{M} \to D$$

by

$$d(\tilde{\sigma}; \nu, \mu) = (s(\tilde{\sigma}; \tilde{f}(\nu, \mu)), m(\mu), \ell(\tilde{\sigma}; \nu, \mu)) \quad ,$$

and let the correspondence

$$\mathcal{D}(\tilde{\sigma}) : \tilde{N}^C \to D$$

be defined by

$$\mathcal{D}(\tilde{\sigma}; \nu) = \{d(\tilde{\sigma}; \nu, \mu) \mid \mu \in \tilde{M}\} \quad .$$

Now we can reinterpret $\tilde{\sigma}(\nu)$ in terms of a probability distribution over $\mathcal{D}(\nu)$ for each $i \in I$. Define

$$\sigma(\tilde{\sigma}) : \tilde{N}^C \to \Delta(D)^I$$

by

$$\sigma(\tilde{\sigma}; \nu, i)[d] = \tilde{\sigma}(\nu, i)[\{\mu \in \tilde{M} \mid d(\tilde{\sigma}; \nu, \mu) = d\}] \quad .$$

Intuitively we are taking the following point of view. When an agent chooses a move - message pair he does not care about the move - message pair per se, only about the consequences of this choice. Obviously the move will have direct consequences, but the message will matter only insofar as it affects subsequent behavior. The utility - relevant consequences of the choice are summarized by $v(\tilde{\sigma}; \nu)$, where ν is the pseudonode resulting from the choice, and the beliefs of all agents are summarized by $p(\tilde{\sigma}; \nu)$. The function $\ell(\tilde{\sigma})$ expresses the notion that the observation of the choice yields information only about the private information state of the agent making the choice, and this information must be derived from the strategy vector $\tilde{\sigma}$. The functions $s(\tilde{\sigma})$, $\mathcal{D}(\tilde{\sigma})$, and $\sigma(\tilde{\sigma})$ express the strategically relevant features of $\tilde{\sigma}$ without making direct reference to $\tilde{Q}$. We are led to the following definition. If Z is an abstract set (to be thought of, for the time being, as $\tilde{N}^C$), then a <u>configuration indexed by Z</u> is a triple $\langle s, \mathcal{D}, \sigma \rangle$ where

$$\begin{array}{ll} s : Z \to \mathbf{S}^C & \text{is a function,} \\ \mathcal{D} : Z \to D & \text{is a nonempty compact valued correspondence, and} \\ \sigma : Z \to \Delta(D)^I & \text{is a function.} \end{array}$$

These objects are required to have the following properties.

(Z1) There is a unique $z^* \in Z$ such that $n(s(z^*)) = n^*$; furthermore $p(s(z^*)) = p^*$, and

$$\{s(d) \mid d \in \mathcal{D}(z), \text{ some } z \in Z, \text{ and } n(d) \in N^c\} = s(Z \setminus \{z^*\}) \quad .$$

(Z2) For all $z \in Z$ and all $d \in \mathcal{D}(z)$,

(a) $n(d) = f(n(s(z)), m(d))$,

(b) if $d \in \bigcup_{i \in I} \text{supp } \sigma(z, i)$ then, for all $i \in I$,

$$\ell(d)[i] = \lim_{\varepsilon \to 0} \sigma(z, i)[B(d; \varepsilon)] / \sum_{j \in I} \sigma(z, j)[B(d; \varepsilon)] \quad ,$$

(c) for all $(a, i) \in A \times I$ and $\hat{j} \in \mathcal{J}$,

$$p(a, i, d)[\hat{j}] \left(\sum_{\hat{k} \in \mathcal{J}} p(a, i, s(z)) [\hat{k}] \ell(d)[k_{a(s(z))}] \right)$$
$$= p(a, i, s(z))[\hat{j}] \ell(d) [j_{a(s(z))}] \quad .$$

(Z3) For all $z \in Z$ and $(a, i) \in A \times I$,

$$v(a, i, s(z)) = \sum_{\hat{j} \in \mathcal{J}} p(a, i, s(z))[\hat{j}] \int_{\mathcal{D}(z)} v(a, i, d) \, d\sigma(z, j_{a(s(z))}) \quad ,$$

and if $d \in \mathcal{D}(z)$ with $n(d) \in N^T$, then

$$v(a, i, d) = \sum_{\hat{j} \in \mathcal{J}} p(a, i, d)[\hat{j}] U(a, \hat{j}, n(d))$$

(Z4) For all $z \in Z$ and all $m \in M$ there exists $d \in \mathcal{D}(z)$ with $m(d) \in m$.

(Z5) For all $z \in Z$, $\bigcup_{i \in I} \text{supp } \sigma(z, i) \subseteq \mathcal{D}(z)$.

Let $C(Z)$ denote the set of configurations indexed by Z; clearly

$$\langle s(\tilde{\sigma}) \mid_{\tilde{N}^c}, \mathcal{D}(\tilde{\sigma}), \sigma(\tilde{\sigma}) \rangle \in C(\tilde{N}^c) \quad .$$

Although expressing strategy vectors in terms of configurations appears to be complicated, it in fact allows for considerable flexibility. Furthermore, configurations allow the concepts of game theory to be expressed in terms of the local strategically relevant information. They are therefore in many ways more intuitive than the grand strategy vectors of $\tilde{\Sigma}(\tilde{G})$.

We now wish to topologize $\mathcal{C}(Z)$ for finite Z . This requires a topology on the set of correspondences from Z to D . If C is a metric space and $E_1, E_2 \subseteq C$ are compact, let

$$\delta(E_1, E_2) = \inf \{\varepsilon > 0 \mid E_1 \subseteq B(E_2; \varepsilon) \text{ and } E_2 \subseteq B(E_1; \varepsilon)\}$$

be the Hausdorff distance between E_1 and E_2 . This distance is easily shown to be a metric, and we let $\mathcal{K}(C)$ denote the set of compact subsets of C endowed with this metric. As is shown in Hildenbrand [4], p. 17, $\mathcal{K}(C)$ is compact if and only if C is compact.

Rather than viewing a configuration as a triple of functions on Z , it can be regarded as a function on Z with three components, i.e.,

$$\langle s, \mathcal{D}, \sigma \rangle \in (S \times \mathcal{K}(D) \times \Delta(D)^I)^Z \quad .$$

We therefore endow $\mathcal{C}(Z)$ with the subspace topology derived from the natural product topology on

$$(S \times \mathcal{K}(D) \times \Delta(D)^I)^Z \quad .$$

This latter set is compact when Z is finite, and it is easily (albeit tediously) checked that $\mathcal{C}(Z)$ is a closed subspace, so $\mathcal{C}(Z)$ is compact.

We now note that the index set Z (whether finite or infinite) has only a minor role. Its raison d'etre is that there may be two pseudonodes $\nu, \nu' \in \tilde{N}^c$ such that $s(\tilde{\sigma}; \nu) = s(\tilde{\sigma}; \nu')$ but either $\mathcal{D}(\tilde{\sigma}; \nu) \neq \mathcal{D}(\tilde{\sigma}; \nu')$ or $\sigma(\tilde{\sigma}; \nu) \neq \sigma(\tilde{\sigma}; \nu')$. From the point of view of informed agents, ν is strategically the same as ν', but $\tilde{\sigma}$ (counterintuitively) induces different behavior at ν and ν'.

We handle this problem as follows. If $\langle s, \mathcal{D}, \sigma\rangle$ is a configuration induced by Z, $Z' \subseteq Z$, and

$$\langle s|_{Z'}, \mathcal{D}|_{Z'}, \sigma|_{Z'}\rangle$$

is a configuration induced by Z', then we say that

$$\langle s|_{Z'}, \mathcal{D}|_{Z''}, \sigma|_{Z'}\rangle$$

is a subconfiguration of $\langle s, \mathcal{D}, \sigma\rangle$. A configuration is said to be minimal if it has no proper subconfiguration.

Lemma 1: $\langle s, \mathcal{D}, \sigma\rangle$ is minimal if and only if s is one - to - one.

Proof: If s is one - to - one then minimality follows from (Z1): all $s(z)$ except $s(z^*)$ are $s(d)$ for some $z' \in Z$ and $d \in \mathcal{D}(z')$.

Suppose that $\langle s, \mathcal{D}, \sigma\rangle$ is minimal, but s is not one - to - one. Let $Z_0 \subseteq Z$ be such that $s(Z_0) = s(Z)$ and $s|_{Z_0}$ is one - to - one. Let

$S_1 = \{s(z^*)\} \cup \{s(d) | d \in \mathcal{D}(z) \text{ for some } z \in Z_0\}$,

and let $Z_1 = s^{-1}(S_1) \cap Z_0$.

If S_{h-1} and Z_{h-1} have been determined, let

$S_h = \{s(z^*)\} \cup \{s(d) | d \in \mathcal{D}(z) \text{ for some } z \in Z_{h-1}\}$,

and let $Z_h = s^{-1}(S_h) \cap Z_0$. An induction argument shows that if $g < h$,

$$S_h \cap \{s \in S | n(s) \in N_g\} = S_g \cap \{s \in S | n(s) \in N_g\} .$$

For h sufficiently large $S_h = S_{h+1}$, and it follows that $Z_h = Z_{h+1}$, so

$$\langle s|_{Z_h}, \mathfrak{D}|_{Z_h} \ \sigma|_{Z_h}\rangle$$

is a proper subconfiguration of $\langle s, \mathfrak{D}, \sigma\rangle$. This contradiction of minimality completes the proof.

If $\langle s, \mathfrak{D}, \sigma\rangle$ is minimal, Z can be replaced by $\mathfrak{S} = s(Z)$ and s can be replaced by $id_{\mathfrak{S}}$. In this case we let the configuration be denoted by

$$\langle \mathfrak{S}, \mathfrak{D}, \sigma\rangle \quad .$$

IV. Communication Equilibrium

In this section we state the definition of the proposed solution concept and prove existence. Following the proof of existence there is a discussion of the intuitive motivation of the concept.

Definition: A communication equilibrium (CE) for G is a minimal configuration $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ such that (C1) and (C2) hold:

(C1) For all $s \in \mathcal{S}$ and $i \in I$,

$$\operatorname{supp} \sigma(s, i) \subseteq \operatorname*{argmax}_{d \in \mathcal{D}(s)} v(a(s), i, d) \quad .$$

(C2) For all $s \in \mathcal{S}$, $m \in M$, and $\ell \in \Delta(I)$ there exists $d \in \mathcal{D}(s)$ with $m(d) = m$, such that for all $i \in I$,

$$v(a(s), i, d) < v(a(s), i, s) \quad \text{implies} \quad \ell(d)[i] \leq \ell[i] \quad .$$

We will actually prove more than existence. We therefore state a stronger definition which is mathematically sound but more difficult to defend intuitively.

Definition: A strong communication equilibrium for G is a minimal configuration $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ such that (C1) and (C3) hold:

(C3) For all $s \in \mathcal{S}$, $m \in M$, and $\ell \in \Delta(I)$ there exists $d \in \mathcal{D}(s)$ with $m(d) = m$ such that for all $i, j \in I$

$$v(a(s), i, d) < v(a(s), i, s) \quad \text{implies}$$

$$\ell(d)[i] \leq \ell[i] \quad \text{and} \quad \ell(d)[i]\, \ell[j] \leq \ell(d)[j]\, \ell[i] \quad .$$

Before proving existence we give two very important consequences of (C2).

Proposition: Suppose $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a CE, $s \in \mathcal{S}$, $m \in M$, $\ell \in \Delta(I)$, and there does not exist $d \in \mathcal{D}(s)$ with $m(d) = m$ and $\ell(d) = \ell$. Then there does exist $d \in \mathcal{D}(s)$ with $m(d) = m$ and $i \in I$ such that

$$v(a(s), i, d) = v(a(s), i, s) \text{ and } \ell(d)[i] > \ell[i].$$

Proof: Suppose the contrary: for all $d \in \mathcal{D}(s)$ with $m(d) = m$ and all $i \in I$ either

$v(a(s), i, d) < v(a(s), i, s)$

($\leq$ follows from (C1)), or $\ell(d)[i] \leq \ell[i]$. (*)

By (C2), (*) holds for some $d \in \mathcal{D}(s)$ with $m(d) = m$ and all $i \in I$, implying that $\ell(d) = \ell$ and contradicting the hypotheses.

Corollary: Suppose $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a CE, $s \in \mathcal{S}$, $m \in M$, $\ell \in \Delta(I)$, and for all $d \in \mathcal{D}(s)$ with $m(d) = m$ and all $i \in I$,

$$v(a(s), i, d) < v(a(s), i, s) .$$

Then for all $\ell \in \Delta(I)$ there exists $d \in \mathcal{D}(s)$ with $m(d) = m$ and $\ell(d) = \ell$.

Proof: Immediate.

In words, if there is no private information state in which it makes any sense for you to choose m, people will be willing to believe whatever you want them to believe about your private information.

As a final remark before the proof of the existence theorem, we give a definition of equilibrium corresponding to the case of $\#\tilde{Q} = 1$, i.e., no communication. This definition is very closely related to the notion of sequential equilibrium introduced recently by Kreps and Wilson [ ].

We shall say that a triple $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a loose minimal configuration if it has all the properties of a minimal configuration except that $\mathcal{D}(s)$ need not be compact.

Definition: A weak sequential equilibrium (WSE) for G is a loose minimal configuration $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ satisfying (C1) and (C4):

(C4) For each $s \in \mathcal{S}$ and each $m \in M$ there is exactly one $d \in \mathcal{D}(s)$ with $m(d) = m$.

Note that this definition implicitly requires the maximums of (C1) to be well defined even when $\mathcal{D}(s)$ is not compact.

Definition: (Kreps and Wilson [5]). A sequential equilibrium (SE) for G is a WSE $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ that is the limit of a sequence of configurations $\{\langle s(\tilde{\sigma}_h), \mathcal{D}(\tilde{\sigma}_h), \sigma(\tilde{\sigma}_h)\rangle\}_{h=1}^{\infty}$ generated by a sequence $\{\sigma_h\}_{h=1}^{\infty}$ in int $\tilde{\Sigma}$ (G).

Remark: The last definition is slightly imprecise: $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a configuration indexed by $\mathcal{S}$ while each $\langle s(\tilde{\sigma}_h), \mathcal{D}(\tilde{\sigma}_h), \sigma(\tilde{\sigma}_h)\rangle$ is indexed by N. Of course (C4) guarantees a natural transformation between $\mathcal{S}$ and N. Kreps and Wilson use different notation of course, but they can be interpreted as indexing by N.

As suggested in the introduction, the communication equilibrium concept is appropriate when communication is possible; otherwise weak sequential equilibrium is the appropriate concept. When communication is possible at some nodes and not at others then one can formulate a hybrid concept by requiring that for each $s \in \mathcal{S}$, (C2) holds when communication is possible, and (C4) holds otherwise. The following proof could be expanded to prove existence for this hybrid concept.

We now prove existence. The following proof is to a large extent modeled on Selten's [8] proof of the existence of perfect equilibria. As will be seen, the intuitive motivation for communication equilibrium is therefore quite similar.

Theorem 1: If $A, I, N,$ and therefore M are finite, a strong communication equilibrium exists.

Proof: For integers $t > 0$ let

$$R^t = \{\varphi \in \Delta(I) \mid t \cdot \varphi[i] \in \{1, \ldots, t-1\} \text{ for all } i \in I\} \quad .$$

For each t let $\tilde{Q}^t$ be an arbitrary finite set with $\#(\tilde{Q}^t) \geq \#(R^T)$, and let

$$r^t : R^t \to \tilde{Q}^t$$

be one - to - one.

Define

$$\tilde{M}^t = \tilde{M}(\tilde{Q}^t) \ ,$$

$$\tilde{N}_h^t = \tilde{N}_h(\tilde{Q}^t) \ , \qquad h = 0, 1, 2, \ldots \ ,$$

$$\tilde{N}^t = \tilde{N}(\tilde{Q}^t) \ , \qquad \text{and}$$

$$\tilde{f}^t : \tilde{N}^{tC} \times \tilde{M} \to \tilde{N}^t$$

as in Section II. Clearly $\tilde{N}^t$ is finite for each t.

We now define a restricted strategy space. For $\varepsilon > 0$ we say that

$$\gamma : \tilde{M}^t \times I \to \mathbb{R}_{++}$$

is ε - admissible if

$$\gamma((m, r^t(\varphi)), i) = \varepsilon\varphi[i]$$

for $m \in M$, $\varphi \in R^t$, and $i \in I$, and

$$\gamma((m, \tilde{q}), i) \in (0, \varepsilon)$$

for $m \in M$, $\tilde{q} \in \tilde{Q}^t \backslash r^t(R^t)$, and $i \in I$. Given such a γ, set

$$\Delta_i^\gamma(\tilde{M}^t) = \{\tau \in \Delta(\tilde{M}^t) \mid \tau(\mu) \geq \gamma(\mu, i) \quad \text{for all} \quad \mu \in \tilde{M}^t\} \quad ,$$

and set

$$\tilde{\Sigma}(\tilde{G}^t, \gamma) = \{\tilde{\sigma} \in \tilde{\Sigma}(\tilde{G}^t) \mid \tilde{\sigma}(\nu, i) \in \Delta_i^\gamma(\tilde{M}^t) \quad \text{for all} \quad \nu \in \tilde{N}^{tC} \quad \text{and} \quad i \in I\} \quad .$$

The <u>γ - best reply correspondence</u> is

$$\beta_\gamma : \tilde{\Sigma}(\tilde{G}^t, \gamma) \to \tilde{\Sigma}(\tilde{G}^t, \gamma)$$

defined by

$$\beta_\gamma(\tilde{\sigma}) = \{\tilde{\sigma}' \in \tilde{\Sigma}(\tilde{G}^t, \gamma) \mid \quad \text{for all} \quad \nu \in \tilde{N}^{tC} \quad \text{and all} \quad i \in I$$

$$\tilde{\sigma}'(\nu, i) \in \underset{\tau \in \Delta_i^\gamma(\tilde{M}^t)}{\operatorname{argmax}} \int_{\tilde{M}^t} v(\tilde{\sigma};\, a(\nu), i, d(\tilde{\sigma}; \nu, \mu))\, d\tau\} \quad .$$

Another characterization of $\beta_\gamma(\tilde{\sigma})$ is the following.

(M) $\tilde{\sigma}' \in \beta_\gamma(\tilde{\sigma})$ if and only if, for all $\nu \in \tilde{N}^{tC}$, $i \in I$, and $\mu \in \tilde{M}^t$,

$$\tilde{\sigma}'(\nu, i)[\mu] > \gamma(\mu, i) \quad \text{implies} \quad d(\tilde{\sigma}; \nu, \mu) \in \underset{d \in \mathcal{D}(\tilde{\sigma}; \nu)}{\operatorname{argmax}} \nu(\tilde{\sigma}; a(\nu), i, d) \quad .$$

Following Selten [8], we say that a fixed point $\tilde{\sigma}^{t*}$ of β_γ is a <u>γ - perfect equilibrium</u>, and we say that the configuration $\langle s(\tilde{\sigma}^{t*}), \mathcal{D}(\tilde{\sigma}^{t*}), \sigma(\tilde{\sigma}^{t*})\rangle$ is a <u>γ - perfect message configuration</u>.

Provided $\varepsilon > 0$ is sufficiently small (so that $\tilde{\Sigma}(G^t, \gamma)$ is non-empty), the usual argument via the Kakutani fixed point theorem shows that γ - perfect equilibria exist, implying the existence of γ - perfect message configurations.

Suppose that $\{\varepsilon_h\}_{h=1}^{\infty}$ is a sequence in $\mathbb{R}_{++}$ with

$$\lim_{h\to\infty} \varepsilon_h = 0 \quad ,$$

for each h γ_h is ε_h - admissible, and for each h sufficiently large

$$\langle s(\tilde{\sigma}_h^{t*}), \mathscr{D}(\tilde{\sigma}_h^{t*}), \sigma(\tilde{\sigma}_h^{t*})\rangle$$

is a γ_h - perfect message configuration. If

$$\langle s^{t*}, \mathscr{D}^{t*}, \sigma^{t*}\rangle = \lim_{h\to\infty} \langle s(\tilde{\sigma}_h^{t*}), \mathscr{D}(\tilde{\sigma}_h^{t*}), \sigma(\tilde{\sigma}_h^{t*})\rangle$$

then we say that $\langle s^{t*}, \mathscr{D}^{t*}, \sigma^{t*}\rangle$ is a <u>perfect message configuration</u> for $\tilde{G}^t$. That such configurations exist follows directly from the existence of γ - perfect message configurations and the compactness of

$$(S \times \mathcal{H}(D) \times \Delta(D)^I)^{\tilde{N}^{tC}} \quad .$$

That such configurations satisfy (C1) follows directly from (M) .

Now consider $\nu \in \tilde{N}^{tC}$, $m \in M$, and $\varphi \in R^t$. Choosing H , an infinite subset of the integers, if necessary, we can let

$$\lim_{h\in H} d(\tilde{\sigma}_h^{t*}; \nu, (m, r^t(\varphi))) = d \in \mathscr{D}^{t*}(\nu) \quad .$$

Suppose that $i \in I$ is such that

$$v(a(\nu), i, d) < v(a(\nu), i, s^{t*}(\nu)) \quad .$$

Then for h sufficiently large

$$v(a(\nu), i, d(\tilde{\sigma}_h^{t*}; \nu, (m, r^t(\varphi)))) < v(a(\nu), i, s(\tilde{\sigma}_h^{t*}; \nu)) \quad ,$$

and by (M) and (C1) this implies that

$$\tilde{\sigma}_h^{t*}(\nu, i)\,[(m, r^t(\varphi))] = \gamma_h((m, r^t(\varphi)), i) = \varepsilon_h \cdot \varphi[i] \quad .$$

Since for all $j \in I$

$$\tilde{\sigma}_h^{t*}(\nu, j)\,[(m, r^t(\varphi))] \geq \gamma_h((m, r^t(\varphi)), j) = \varepsilon_h \cdot \varphi[j] \quad ,$$

it follows from the definition of $\ell(\tilde{\sigma}_h^{t*})$ that

$$\ell(\tilde{\sigma}_h^{t*}; \nu, (m, r^t(\varphi)))[i] \leq \varphi[i] \quad ,$$

and

$$\ell(\tilde{\sigma}_h^{t*}; \nu, (m, r^t(\varphi)))[i]\varphi[j] \leq \ell(\tilde{\sigma}_h^{t*}; \nu, (m, r^t(\varphi)))[j]\varphi[i] \quad .$$

In the limit we have

$$\ell(d)[i] \leq \varphi[i] \quad \text{and} \quad \ell(d)[i]\varphi[j] \leq \ell(d)[j]\varphi[i] \quad ,$$

i.e., that (C3) holds when ℓ is restricted to lie in R^t.

For each integer $t > 0$ let

$$\langle s^{t*}, \mathcal{D}^{t*}, \sigma^{t*} \rangle$$

be a perfect message equilibrium for $\tilde{G}^t$, and choose $\{\nu^t \in \tilde{N}^t\}_{t \in T}$ for some T, an infinite set of positive integers, such that

$$\lim_{t \in T} (s^{t*}(\nu^t), \mathcal{D}^{t*}(\nu^t), \sigma^{t*}(\nu^t)) = (s, \mathcal{D}(s), \sigma(s))$$

exists. (This is always possible since $S \times \mathcal{H}(D) \times \Delta(D)^I$ is compact.) For each $i \in I$ and each $t \in T$,

$$\operatorname{supp} \sigma^{t*}(s^{t*}(\nu^t), i) \subseteq \operatorname*{argmax}_{d \in \mathcal{D}^{t*}(\nu^t)} v(a(s^{t*}(\nu^t)), i, d) \quad ,$$

so for each $i \in I$

$$\operatorname{supp} \sigma(s, i) \subseteq \operatorname*{argmax}_{d \in \mathcal{D}(s)} v(a(s), i, d) \quad .$$

Now consider $m \in M$ and $\ell \in \Delta(I)$, and choose $\{\varphi^t \in R^t\}_{t \in T}$ such that

$$\lim_{t \in T} \varphi^t = \ell \quad .$$

For each t we can choose $d^t \in \mathcal{D}^{t*}(\nu^t)$ such that $m(d^t) = m$ and for all $i, j \in I$

$$v(a(s(\nu^t)), i, d^t) < v(a(s(\nu^t)), i, s(\nu^t))$$

implies

$$\ell(d^t)[i] \le \varphi^t[i] \tag{*}$$

and

$$\ell(d^t)[i]\varphi^t[j] \le \ell(d^t)[j]\varphi^t[i] \quad . \tag{**}$$

Choose $T_1 \subseteq T$ infinite such that

$$d = \lim_{t \in T_1} d^t \in \mathcal{D}(s)$$

exists. If $i, j \in I$ and

$$v(a(s), i, d) < v(a(s), i, s) \quad ,$$

then for $t \in T_1$ sufficiently large (*) and (**) hold. In the limit we find that

$$\ell(d)[i] \le \ell[i] \quad \text{and} \quad \ell(d)[i]\ell[j] \le \ell(d)[j]\ell[i] \quad .$$

Therefore a minimal configuration $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ will be a strong communication equilibrium if for all $s \in \mathcal{S}$ we can find T, an infinite set of positive integers, and $\{\nu^t \in \tilde{N}^t\}_{t \in T}$ such that

$$(s, \mathcal{S}(s), \sigma(s)) = \lim_{t \in T} (s^{t*}(\nu^t), \mathcal{S}^{t*}(\nu^t), \sigma^{t*}(\nu^t)) \quad .$$

For each $t > 0$ let

$$\mathcal{T}^t = \{(s^{t*}(\nu), \mathcal{S}^{t*}(\nu), \sigma^{t*}(\nu)) \mid \nu \in \tilde{N}^t\} \quad ,$$

and let

$$\mathcal{T} = \bigcap_{g=1}^{\infty} \mathrm{cl}\left(\bigcup_{t=g}^{\infty} \mathcal{T}^t\right) \quad .$$

Thus $\mathcal{T}$ is exactly the set of limit points of the form just given.

Since $\{n^*\} \times \mathcal{H}(D) \times \Delta(D)^I$ is compact, there must exist $(s^*, \mathcal{S}(s^*), \sigma(s^*)) \in \mathcal{T}$ with $n(s^*) = n^*$. Let

$$\mathbb{S}_0 = \{s^*\} \quad ,$$

and define $\mathcal{S}_0 : \mathbb{S}_0 \to \mathcal{H}(D)$ and $\sigma_0 : \mathbb{S} \to \Delta(D)^I$ by $\mathcal{S}_0(s^*) = \mathcal{S}(s^*)$ and $\sigma_0(s^*) = \sigma(s^*)$.

If $\mathbb{S}_{h-1}$ with associated $\mathcal{S}_{h-1}$ and σ_{h-1} has been chosen, let

$$\mathbb{S}_h = \{s(d) \mid d \in \mathcal{S}_{h-1}(s) \quad \text{for some} \quad s \in \mathbb{S}_{h-1} \quad \text{and} \quad n(d) \in N^C\} \quad .$$

For each $s \in \mathbb{S}_h$ there exists T an infinite set of integers and $\{\nu^t \in \tilde{N}^t\}_{t \in T}$ such that

$$\lim_{t \in T} s^{t*}(\nu^t) = s \quad .$$

Since $N \times \mathcal{H}(D) \times \Delta(D)^I$ is compact, the sequence

$$\{(s^{t*}(\nu^t), \mathcal{S}^{t*}(\nu^t), \sigma^{t*}(\nu^t))\}_{t \in T}$$

has a limit point. For each $s \in \mathbb{S}_h$ it is therefore possible to choose $\mathcal{S}(s) \in \mathcal{H}(D)$ and $\sigma(s) \in \Delta(D)^I$ such that

$$(s, \mathcal{S}(s), \sigma(s)) \in \mathcal{T} \quad .$$

It is therefore possible to choose

$$\mathcal{D}_h : \mathcal{S}_h \to \mathcal{H}(D)$$

and

$$\sigma_h : \mathcal{S}_h \to \Delta(D)^I$$

such that $(s, \mathcal{D}_h(s), \sigma_h(s)) \in \mathcal{T}$ for all $s \in \mathcal{S}_h$. Let

$$\mathcal{S} = \bigcup_{h=0}^{\infty} \mathcal{S}_h$$

and define $\mathcal{D} : \mathcal{S} \to \mathcal{H}(D)$ and $\sigma : \mathcal{S} \to \Delta(D)$ by

$$\mathcal{D}(s) = \mathcal{D}_h(s)$$

and

$$\sigma(s) = \sigma_h(s)$$

if $s \in \mathcal{S}_h$. Clearly $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a minimal configuration, so the proof is complete.

Definition: If a communication equilibrium can be constructed as in the proof, i.e., as a selection from the set of limit points in $S \times \mathcal{H}(D) \times \Delta(D)$ of a sequence of perfect message equilibria for a sequence $\{\tilde{G}^t\}_{t=1}^{\infty}$, then we say that it is a perfect communication equilibrium.

We are now in a position to discuss the methodological underpinnings of the communication equilibrium concept. The possible justifications for (C1) have been much discussed, and we have little to add: the question we address is whether (C2) is a "reasonable" restriction.

Ultimately the justification for a solution concept of game theory must be empirical. Two questions are possible. Does the solution concept describe observed behavior? Given observed behavior, does the solution concept have prescriptive value, i.e., does it lead to insights that allow agents informed of the solution concept to improve their performance? We have no data that suggest answers to either of these questions.

On the other hand the solution concept may be thought of as only a partial resolution of the gap between theory and reality, in which case empirical tests may be inaccurate measures of the concept's value. There may be no accepted statistical method for testing the concept. Finally there may simply be no available data. In these cases the solution concept must be evaluated on nonempirical grounds.

One approach is to examine the relationship between the solution concept and received methodology which, in the case of game theory, we take to be rational decision making. One can therefore ask whether the solution concept can be derived from a reasonable interpretation of rationality, as, for instance, Selten's [8] notion of subgame perfection is derived from the assumption that agents' expectations of behavior in all subgames are consistent with rationality. We do not see any such derivation of (C2), but this should not be surprising since (C2) restricts the strategic role of communication, and communication is a form of behavior that is inherently social and not attributable to single individuals.

If a solution concept is not derivable from rationality then one must at least insist that it be consistent with rationality. Theorem 1 can be read as saying that (C3) and therefore (C2) are consistent with (C1). The proof can be read as saying that (C3) and (C2) are consistent with a stronger "trembling hand" interpretation of (C1). Therefore this test poses no problem for our solution concept.

Having established that (C2) is consistent with the methodology of rationality, we can ask whether it has intuitive appeal. To some extent this notion is necessarily subjective, but what we seem to mean when we attribute intuitive appeal to some concept is that it reflects aspects of reality that would be difficult or impossible to model in a straightforward fashion. An example is the "trembling hand" argument in support of Selten's [8] concept of perfect equilibrium: recognizing that in reality people make mistakes with positive probability, the solution concepts of game theory should be consistent with this observation even if we do not want our models cluttered up with actual mistake probabilities.

A similar argument can be given for (C2), and to some extent (C3) as well. Recall that in the proof we assume the existence of a map r^t from

$$R^t = \{\varphi \in \Delta(I) \mid t \cdot \varphi[i] \in \{1, \ldots, t-1\} \quad \text{for all} \quad i \in I\}$$

to $\tilde{Q}^t$, the set of messages. The involuntary probability assigned to $r^t(\varphi)$ was assumed to be proportional to $\varphi[i]$ when i was the private information of the agent choosing the move. Given a pseudonode ν and a move m, if the agent choosing the move never chose $r^t(\varphi)$ in conjunction with m voluntarily, regardless of private information, then φ determined the likelihood vector associated with $(m, r^t(\varphi))$ chosen at ν. The counterpart to $r^t(\varphi)$ in experience is the "revealing gesture," or at least a certain class of such gestures. Condition (C2) can be seen as the consequence of assuming that for each $\ell \in \Delta(I)$ there is a corresponding revealing gesture. This may be thought of as a "stuttering tongue" interpretation of (C2).

This argument for (C2) gains force when it is turned around. An endorsement of (C2) does not necessarily entail a belief in a complete set of revealing gestures, but an endorsement of an equilibrium contradicting (C2) necessarily implies a belief in the absence of some revealing gesture. Put more mathematically, (C2) is robust with respect to perturbing the game by adding a revealing gesture. In the proof $\tilde{Q}^t$ was allowed to be a proper superset of $r^t(R^t)$, and the only reason for this was to establish this robustness property rigorously.

The additional strength of (C3) is more difficult to justify because it seems to depend more heavily on our implicit "cost of error avoidance function." Specifically, the apparatus of the proof assumes that the involuntary probability for i associated with $(m, r^t(\varphi))$ can be reduced to $\varepsilon \cdot \varphi[i]$ without cost, and that further reductions are infinitely costly. Condition (C2) can be seen as saying that agents will devote more resources to avoiding an error when the error is definitely suboptimal. It may be that (C3) could be justified by saying that the set of revealing gestures should approximate the cartesian product of $\Delta(I)$ and some set of "reasonable" cost of error avoidance functions, but in this case one would expect to derive a condition even stronger than (C3). Whether this is possible is an open problem.

V. Compact Metric Spaces of Moves

Theorem 1 can be extended easily to allow for games with continua of moves. After this result is proved an example is presented to suggest that the ease of this extension is not accidental.

Theorem 2: Suppose N and M are compact metric spaces, N^C and N^T are closed in N, $N_h = \phi$ for some h, and a, f, and U are continuous. Then a strong communication equilibrium exists.

Remark: The topological assumptions of Theorem 2 are in several ways redundant. Rather than stating a minimal set of assumptions, we simplify the presentation by assuming things that could be derived.

Proof: Let $\{m_1, m_2, \ldots\}$ be a countable dense subset of M, and for each integer t let $M^t = \{m_1, \ldots, m_t\}$. Let $N_0^t = \{n^*\}$, let

$$N_h^t = f((N_{h-1}^t \setminus N^T) \times M^t), \qquad h = 1, 2, \ldots,$$

and let $N^t = \bigcup_{h=0}^{\infty} N_h^t$. (Since $N_h = \phi$ for some h, a finite union would suffice.) Restricting a, f, and U appropriately,

$$G^t = \langle A, I, N^t, a, M^t, f, U, p^* \rangle$$

is a game satisfying the hypotheses of Theorem 1, so we let $\langle \mathcal{S}^t, \mathcal{D}^t, \sigma^t \rangle$ be a strong communication equilibrium.

As in the proof of Theorem 1, let

$$\mathcal{T} = \bigcap_{g=1}^{\infty} \mathrm{cl}\Big(\bigcup_{t=g}^{\infty} \{(s, \mathcal{D}^t(s), \sigma^t(s)) \mid s \in \mathcal{S}^t\}\Big) .$$

Note that S is compact, as are D, $\mathcal{H}(D)$, and $\Delta(D)^I$.

For each t there exists $s^{t*} \in \mathcal{S}^t$ with $n(s^{t*}) = n^*$, so we may choose $(s^*, \mathcal{D}(s^*), \sigma(s^*)) \in \mathcal{T}$ with $n(s^*) = n^*$. Let

$$\mathcal{S}_o = \{s^*\} \quad ,$$

and define $\mathcal{D}_0 : \mathcal{S}_0 \to \mathcal{K}(D)$ and $\sigma_0 : \mathcal{S}_0 \to \Delta(D)^I$ by $\mathcal{D}_0(s^*) = \mathcal{D}(s^*)$ and $\sigma_0(s^*) = \sigma(s^*)$.

If $\mathcal{S}_{h-1}$ with associated $\mathcal{D}_{h-1}$ and σ_{h-1} has been chosen, let

$$\mathcal{S}_h = \{s(d) \mid d \in \mathcal{D}_{h-1}(s) \text{ for some } s \in \mathcal{S}_{h-1} \text{ with } n(s) \in N^C\} \quad .$$

For each $s \in \mathcal{S}_h$ there exists an infinite set of integers T and a sequence

$$\{s^t \in \mathcal{S}^T\}_{t \in T}$$

such that

$$\lim_{t \in T} s^t = s \quad .$$

By compactness it is possible to choose a subsequence $T_1 \subseteq T$ such that

$$(s, \mathcal{D}(s), \sigma(s)) = \lim_{t \in T_1} (s^t, \mathcal{D}^t(s^t), \sigma^t(s^t)) \in \mathcal{T}$$

exists. We can therefore define $\mathcal{D}_h : \mathcal{S}_h \to \mathcal{K}(D)$ and $\sigma_h : \mathcal{S}_h \to \Delta(D)^T$ so that

$$(s, \mathcal{D}_h(s), \sigma_h(s)) \in \mathcal{T}$$

for all $s \in \mathcal{S}_h$. Let

$$\mathcal{S} = \bigcup_{h=0}^{\infty} \mathcal{S}_h$$

and define $\mathcal{D} : \mathcal{S} \to \mathcal{K}(D)$ and $\sigma : \mathcal{S} \to \Delta(D)^I$ by

$\mathcal{D}(s) = \mathcal{D}_h(s)$ and $\sigma(s) = \sigma_h(s)$

for $s \in \mathcal{S}_h$. Clearly $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a minimal configuration.

Clearly $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ satisfies (C1), and to show (C3) we choose arbitrary $s \in \mathcal{S}$ and $m \in M$, and $\ell \in \Delta(I)$. There is an infinite set of integers T and a sequence $\{s^t \in \mathcal{S}^t\}_{t \in T}$ such that

$$\lim_{t \in T} (s^t, \mathcal{D}^t(s^t), \sigma^t(s^t)) = (s, \mathcal{D}(s), \sigma(s)) \quad .$$

Since $\{m_1, m_2, \ldots\}$ is dense in M, we can choose $m^t \in M^t$ for $t \in T$ so that

$$\lim_{t \in T} m^t = m ,$$

and for each $t \in T$ we can choose $d^t \in \mathcal{D}^t(s^t)$ satisfying (C3) for m^t and ℓ. Choosing a subsequence $T_1 \subseteq T$ so that

$$d = \lim_{t \in T_1} d^t$$

exists, the definition of convergence in Hausdorff distance guarantees $d \in \mathcal{D}(s)$, and it is easily seen that (C3) is satisfied for m and ℓ by d. This completes the proof.

We now give an example to show that the same argument could not be used for the sequential equilibrium concept. The following game does not conform to the assumptions that all nodes have the same associated set of moves and all agents have the same set of private information states. As noted before these assumptions are convenient without being restrictive.

Set $A = \{a, b\}$, and let $I_a = \{i, j\}$. Agent b is uninformed so I_a will also be the set of states of the world. The rules of the game are that a first chooses $m_1 \in M_1 = [-1, 1]$, and then b, observing m_1, chooses $m_2 \in \{-1, 1\}$. Identifying N^T and $M^1 \times M^2$, let

$$U(a, k, m_1, m_2) = \begin{cases} -m_1^2 + 1, & k = i \text{ and } m_2 = -1 \quad \text{or} \\ & k = j \text{ and } m_2 = 1, \\ -m_1^2 & \text{otherwise,} \end{cases}$$

and let

$$U(b, k, m_1, m_2) = \begin{cases} 1, & k = i \text{ and } m_2 = -1 \quad \text{or} \\ & k = j \text{ and } m_2 = 1, \\ 0, & \text{otherwise} \quad . \end{cases}$$

Intuitively a would like to tell b the state of the world so that b can choose what is best for both a and b.

For odd integers t let

$$M^t = \{-1, -1 + \frac{2}{t}, \ldots, -1 + \frac{2t}{t}\} \quad .$$

For each odd t there exists a sequential equilibrium $\langle \mathcal{S}^t, \mathcal{D}^t, \sigma^t \rangle$ such that there exist $d_i, d_j \in \mathcal{D}^t(s^*)$ with

$$m(d_i) = -\frac{1}{t}, \; m(d_j) = \frac{1}{t}, \qquad \text{and}$$

$$\sigma(s^{t*}, i)[d_i] = 1 = \sigma(s^{t*}, j)[d_j] \quad .$$

Agent b, observing d_i, will choose $m_2 = -1$, and d_j leads to choosing $m_2 = 1$. The details involved in showing that such sequential equilibria exist are left to the reader.

If this sequence of sequential equilibria had a sequential equilibrium as a limit point (using the phrase "limit point" loosely to describe the process employed in proving Theorem 2), then $\sigma(s^*, i)$ and $\sigma(s^*, j)$ would both assign all probability to the unique $d \in \mathcal{D}(s^*)$ with $m(d) = 0$. Since both informed agents (a, i) and (a, j) must attain a payoff of one, this is impossible.

The sequential equilibrium concept (along with all usual solution concepts of noncooperative game theory) is derived from the assumption that communication is impossible. In this example that assumption is unnatural, since M_1 acts in effect like a message space with messages of variable cost. It should therefore not be surprising that the sequential equilibrium concept yields unnatural results.

VI. The Bargaining Example

The following example illustrates the power of the solution concept, and it is important in itself since it suggests the possibility of a deterministic theory of bargaining with asymmetric information.

We suppose that some indivisible object is being sold by a seller who is not informed of the reservation price of the buyer. The seller's reservation price is known by both agents to be zero, and the set of possible reservation prices for the buyer is $I = \{i, j\}$ where

$$0 < j < i \quad .$$

At the start of the game the buyer suggests a price, and if the seller accepts this price the transaction is carried out, ending the game. If the seller does not accept the first price he must suggest a second price, and if this price is accepted by the buyer a transaction occurs at this price with probability one-half. If neither price is accepted no transaction takes place.

The set of moves is

$$M = [0, i] \cup \{Y\} \quad ,$$

where Y represents acceptance. We define the following subsets of N:

$$N_1^C = f(\{n^*\} \times [0, i]) \quad ;$$

$$N_2^C = f(N_1^C \times [0, i]) \quad ;$$

$$N_2^Y = f(N_1^C \times \{Y\}) \quad ;$$

$$N_3^Y = f(N_2^C \times \{Y\}) \quad ;$$

$$N_3^N = f(N_2 \times [0, i]) \quad .$$

The rules of the game are specified by letting

$$N^a = N_1^C \quad ,$$

$$N^b = \{n^*\} \cup N_2^C, \quad \text{and}$$

$$N^T = \{f(n^*, Y)\} \cup N_2^Y \cup N_3^Y \cup N_3^N \quad .$$

This game tree is represented in Figure 2.

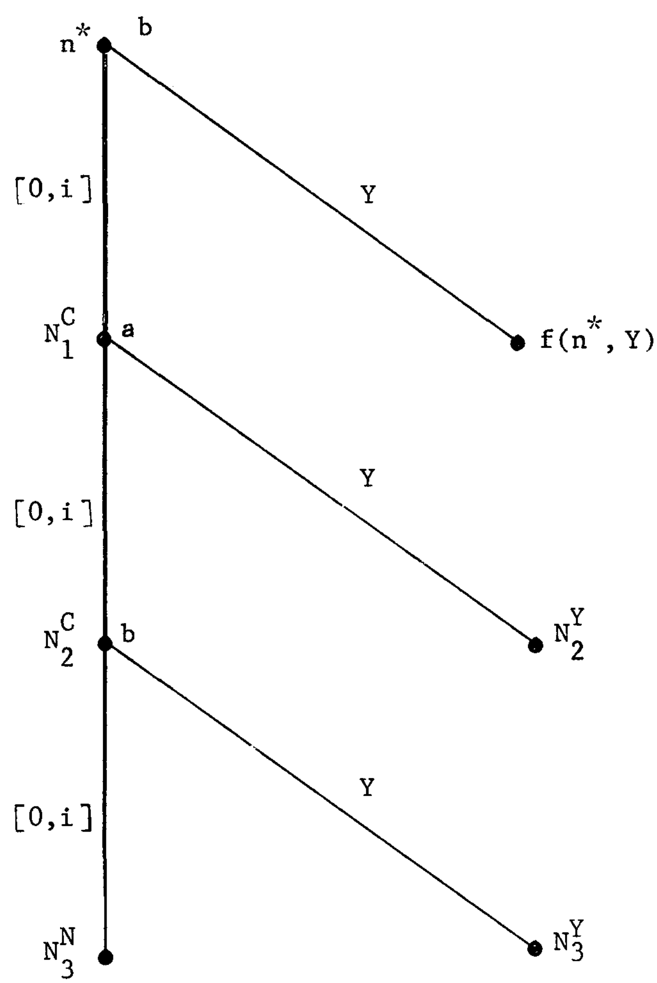

Figure 2

It is convenient to have notation for the sequence of offers preceding a node. Recall (from (T2)) that $(\bar{n}(n), \bar{m}(n))$ is the unique pair such that $f(\bar{n}(n), \bar{m}(n)) = n \in N \setminus \{n^*\}$. Let

$$m_1 : N \setminus \{n^*, f(n^*, Y)\} \to [0, i]$$

be defined by

$$m_1(n) = \begin{cases} \bar{m}(n) & , \quad n \in N_1^C , \\ \bar{m}(\bar{n}(n)) & , \quad n \in N_2 = N_2^C \cup N_2^Y , \\ \bar{m}(\bar{n}(\bar{n}(n))), & \quad n \in N_3 = N_3^N \cup n_3^Y . \end{cases}$$

Let

$$m_2 : N_2^C \cup N_3^N \cup N_3^Y \to [0, i]$$

be defined by

$$m_2(n) = \begin{cases} \bar{m}(n) & , \quad n \in N_2^C \\ \bar{m}(\bar{n}(n)) & , \quad n \in N_3 = N_3^N \cup N_3^Y . \end{cases}$$

For $s \in S$ and $d \in D$ we shall write $m_1(s)$, $m_2(s)$, $m_1(d)$, and $m_2(d)$ rather than $m_1(n(s))$, $m_2(n(s))$, $m_1(n(d))$, and $m_2(n(d))$.

Since only one agent has private information, the set of states of the world can be identified with I . The utility the seller associates with a terminal node depends only on the transaction price, not the state of the world, so we let

$$U(a, \cdot\,) : N^T \to \mathbb{R}$$

be given by

$$U(a,n) = \begin{cases} -1 & , \quad n = f(n^*, Y) , \\ m_1(n) & , \quad n \in N_2^Y , \\ m_2(n)/2 & , \quad n \in N_3^Y , \\ 0 & , \quad n \in N_3^N . \end{cases}$$

The utility function for the buyer is

$$U(b, \cdot, \cdot) : I \times N^T \to \mathbb{R}$$

given by

$$U(b, k, n) = \begin{cases} -1 & , \quad n = f(n^*, Y) \quad , \\ k - m_1(n) & , \quad n \in N_2^Y \quad , \\ (k - m_2(n))/2 & , \quad n \in N_3^Y \quad , \\ 0 & , \quad n \in N_3^N \quad . \end{cases}$$

There is, of course, nothing to accept at n^*, so the payoffs are specified in a way that makes the choice of Y at n^* dominated. This is an example of why no generality is lost by assuming that all of M is available at every communication node.

Since there are essentially three informed agents we can let the set of value vectors be

$$V = [-1, i] \times [-1, i] \times [\min\{-1, j - i\}, j]$$

with generic element $v = (v(a), v(b, i), v(b, j))$.

Since there is one uninformed agent and two states of the world, $p(a)[i]$ is a sufficient statistic for $p \in P$. It is convenient to let the symbol 'p' denote an element of $[0, 1]$, interpreted as the probability assigned by the seller to reservation price i. The prior p^* will also be interpreted in this fashion. Similarly, $\ell[i]$ is a sufficient statistic for a likelihood vector $\ell \in \Delta(I)$, so we let 'ℓ' denote an element of $[0, 1]$, interpreted as the relative likelihood of i. If $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a minimal configuration, $s \in \mathcal{S}$, and $d \in \mathcal{D}(s)$, then $n(s) \in N^a$ implies

$$p(d) = p(s) \quad ,$$

while $n(s) \in N^b$ implies

$$p(d) \cdot (p(s)\ell(d) + (1-p(s))(1-\ell(d))) = p(s)\ell(d) \quad .$$

We are now in a position to analyze the weak sequential equilibria and the communication equilibria of this game. The initial stages of the analysis apply equally to both solution concepts.

Lemma 2: If $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a loose minimal configuration satisfying (C1), $s \in \mathcal{S}$, and $n(s) \in N_2^C$, then

$$v(a, s) \begin{cases} = m_2(s)/2 & , \quad 0 \le m_2(s) < j , \\ \in [p(s)\, j/2,\, j/2] & , \quad m_2(s) = j \quad , \\ = p(s)\, m_2(s)/2 & , \quad j < m_2(s) < i , \\ \in [0,\, p(s)\, i/2] & , \quad m_2(s) = i \quad , \end{cases}$$

$$v(b, i, s) = (i - m_2(s))/2 , \qquad \text{and}$$

$$v(b, j, s) = \max\{0, (j - m_2(s))/2\} \quad .$$

Proof: The value vector at s depends only on the probability that the buyer accepts $m_2(s)$. The results in Lemma 2 simply express the restrictions derived from (C1).

As a further notational simplification we shall write $\sigma(s)[Y]$ rather than

$$\sigma(s)[\{d \in \mathcal{D}(s) \mid m(d) = Y\}]$$

for $s \in \mathcal{S}$ with $n(s) \in N_1^C$.

Lemma 3: If $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a loose minimal configuration satisfying (C1) $s \in \mathcal{S}$, and $n(s) \in N_1^C$, then

$$\sigma(s)[Y] \begin{cases} = 0 & , m_1(s) < \max\{j/2, p(s)\, i/2\} , \\ \in [0, 1] & , m_1(s) = \max\{j/2, p(s)\, i/2\} , \\ = 1 & , m_1(s) > \max\{j/2, p(s)\, i/2\} . \end{cases}$$

Proof: Again this is a straightforward consequence of (C1) combined with lemma 2.

We now proceed to give a class of weak sequential equilibria for this game. Although we do not attempt to show this, all these weak sequential equilibria are in fact sequential equilibria.

Claim: For each $\hat{m} \in [j/2, \min\{i/2, j\}]$ there is a weak sequential equilibrium $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ such that

$$\sigma(s^*, j)\,[\{d \in \mathcal{D}(s^*) \,|\, m(d) = \hat{m}\}] = 1 \quad .$$

Since our principal interest is the set of communication equilibria, we only sketch the construction. For each

$$m \in [0, i]$$

we let $d(m)$ be the unique element of $\mathcal{D}(s^*)$ with $m(d(m)) = m$. Assume that

$$p(d(m)) = 1 \quad \text{for} \quad m \in [0, i] \setminus \{\hat{m}\} \quad .$$

If $p^* i/2 \le \hat{m}$ then assume that $p(d(\hat{m})) = p^*$, while if $p^* i/2 > \hat{m}$ assume that $p(d(\hat{m})) = 2\hat{m}/i$. We now specify $\sigma(s^*, j)$ and $\sigma(s^*, i)$ by

$$\sigma(s^*, j)[d(\hat{m})] = 1 \quad ,$$

$$p(d(\hat{m}))(p^* \sigma(s^*, i)[d(\hat{m})] + (1 - p^*)) = p^* \sigma(s^*, i)[d(\hat{m})] \quad ,$$

and

$$\sigma(s^*, i)[d(i/2)] = 1 - \sigma(s^*, i)[d(\hat{m})] \quad .$$

In words informed agent (b, j) always chooses $d(\hat{m})$, and (b, i) assigns probability to $d(\hat{m})$ up to the point at which it becomes less attractive than having $i/2$ with certainty. This pair of strategies is optimal because the sequential equilibrium concept allows the seller, upon observing any deviation, to infer that the buyer has reservation price i.

We now analyze the communication equilibria for this example. Of course Theorem 2 guarantees existence, so our results are not vacuous. It will be seen that for this example communication equilibrium is a more restrictive solution concept than sequential equilibrium.

Lemma 4: If $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a communication equilibrium then $v(b, i, s^*) \geq i/2$.

Proof: For all $m > i/2$ there exists $d_m \in \mathcal{D}(s^*)$ with $m(d_m) = m$, by (Z4). By Lemma 3,

$\sigma(s(d_m))[Y] = 1$,

so $v(b, i, s^*) \geq v(b, i, d_m) = i - m$.

Lemma 5: If $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a communication equilibrium then

$$v(b, j, s^*)/v(b, i, s^*) \geq 1/(2 - j/i) .$$

Proof: By Lemma 3 informed agent (b, i) cannot do better than have his offer of $j/2$ accepted with probability 1, so

$v(b, i, s^*) \leq i - j/2$.

Therefore suppose that $v(b, j, s^*) < j/2$. For $m > j/2$ sufficiently small Lemma 3 implies that there does not exist $d \in \mathcal{D}(s^*)$ with $m(d) = m$ and $p(d) = 0$. For such m (C2) therefore implies the existence of $d \in \mathcal{D}(s^*)$ with $m(d) = m$ and $v(b, i, d) = v(b, i, s^*)$.

Since $v(b, i, s^*) \geq i/2$, $\sigma(s(d))[Y] > 0$, and Lemma 3 implies that $p(d) > j/i$. It follows from (C1) and Lemma 2 that

$$\sigma(s(d))[\{d' \in \mathcal{D}(s(d)) \mid m(d') < i\}] = 0 \quad .$$

Therefore

$$v(b, i, s^*) = v(b, i, d) = \sigma(s(d))[Y](i - m) \quad ,$$

$$v(b, j, s^*) \geq v(b, j, d) = \sigma(s(d))[Y](j - m) \quad ,$$

and

$$v(b, j, s^*)/v(b, i, s^*) \geq (j - m)/(i - m) \quad .$$

Since this holds for all $m > j/2$ sufficiently small, the proof is complete.

<u>Lemma 6</u>: If $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a communication equilibrium and $d \in \text{supp}\, \sigma(s^*, j)$ then $m(d) = j/2$ and

$$\hat{\sigma} = \sigma(s(d))[\{d' \in \mathcal{D}(s(d)) \mid m(d') < i\}] = 0 \quad .$$

<u>Proof</u>: If $m(d) < j/2$ then $\sigma(s(d))[Y] = 0$ and $v(b, j, d) = 0$, contradicting Lemmas 4 and 5. If $d' \in \text{supp}\, \sigma(s(d))$ and $m(d') < i$ then, by Lemma 3 and (C1), $m(d') = j$. Therefore

$$v(b, i, s^*) \geq v(b, i, d) = \sigma(s(d))[Y] \cdot (i - m(d)) + \hat{\sigma}(i - j)/2 \quad .$$

On the other hand

$$v(b, j, s^*) = v(b, j, d) = \sigma(s(d))[Y] \cdot (j - m(d)) \quad ,$$

so

$$v(b, j, s^*)/v(b, i, s^*) \le \frac{\sigma(s(d))[Y](j - m(d))}{\sigma(s(d))[Y](i - m(d)) + \hat{\sigma}(i - j)/2} .$$

Since $m(d) \ge j/2$, this is consistent with Lemma 5 only if $m(d) = j/2$ and $\hat{\sigma} = 0$.

<u>Lemma 7</u>: If $p^* > 0$, $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a communication equilibrium, and

$$d \in \operatorname{supp} \sigma(s^*, i) \setminus \operatorname{supp} \sigma(s^*, j) ,$$

then $m(d) = i/2$ and $\sigma(s(d))[Y] = 1$.

<u>Proof</u>: It follows from $p^* > 0$, (Z2b), and (Z2c) that $p(d) = 1$. If $m(d) < i/2$ then $v(b, i, d) = 0$, contradicting Lemma 4, while $m(d) > i/2$ or $\sigma(s(d))[Y] < 1$ imply $v(b, i, d) < i/2$, again contradicting Lemma 4.

We summarize and extend these results in Theorem 3 below.

<u>Theorem 3</u>: If $p^* \in (0, 1)$ and $\langle \mathcal{S}, \mathcal{D}, \sigma \rangle$ is a communication equilibrium then $d \in \operatorname{supp} \sigma(s^*, j)$ implies $m(d) = j/2$, $p(d) \le j/i$.

$$\sigma(s(d))[\{d' \in \mathcal{D}(s(d)) \mid m(d) < i\}] = 0 ,$$

and

$$\sigma(s(d))[Y] \begin{cases} = 1 & , \quad p^* < j/i , \\ \in [1/(2 - j/i),\ 1] & , \quad p^* = j/i , \\ = 1/(2 - j/i) & , \quad p^* > j/i . \end{cases}$$

Furthermore $d \in \operatorname{supp} \sigma(s^*, i) \setminus \operatorname{supp} \sigma(s^*, j)$ implies $m(d) = i/2$ and $\sigma(s(d))[Y] = 1$.

<u>Proof</u>: If $d \in \text{supp}\,\sigma(s^*, j)$ then Lemma 6 implies that $m(d) = j/2$ and that a never responds with a price less than i. Thus $p(d) \leq j/i$ follows from $v(b, j, d) > 0$ and Lemma 3.

If $d \in \text{supp}\,\sigma(s^*, j)$ and $\sigma(s(d))[Y] < 1/(2 - j/i)$, that a never responds with a price less than i implies that

$$v(b, i, d) < i/2 ,$$

so $d \notin \text{supp}\,\sigma(s^*, i)$ and $p(d) = 0$. But then Lemma 2 and (C1) imply

$$\sigma(s(d))[\{d' \in \mathcal{D}(s(d)) \mid m(d') = i\}] = 0 ,$$

i.e., $\sigma(s(d))[Y] = 1$, a contradiction. We conclude that $d \in \text{supp}\,\sigma(s^*, j)$ implies

$$\sigma(s(d))[Y] \geq 1/(2 - j/i) .$$

If $p^* < j/i$ then there must be some $d \in \text{supp}\,\sigma(s^*, j)$ with $p(d) \leq p^* < j/i$. We know that a never responds to this d with a price less than i, and responding with i leads to a payoff of $p(d)\,i/2 < j/2$, so $\sigma(s(d))[Y] = 1$. Since this is true for one $d \in \text{supp}\,\sigma(s^*, j)$, it must be true for all such d.

That $d \in \text{supp}\,\sigma(s^*, i) \setminus \text{supp}\,\sigma(s^*, j)$ implies $m(d) = i/2$ and $\sigma(s(d))[Y] = 1$ is the assertion of Lemma 7. If $p^* > j/i$ and

$$\text{supp}\,\sigma(s^*, i) \subseteq \text{supp}\,\sigma(s^*, j)$$

then there must exist $d \in \text{supp}\,\sigma(s^*, i)$ with $p(d) \geq p^* > j/i$. If $m(d) = j/2$, Lemma 3 would imply that $\sigma(s(d))[Y] = 0$. Therefore $p^* > j/i$ implies

$$\text{supp}\,\sigma(s^*, i) \setminus \text{supp}\,\sigma(s^*, j) \neq \emptyset \text{ and } v(b, i, s^*) = i/2 .$$

It follows immediately from (C1) that

$\sigma(s(d))[Y] \leq 1/(2 - j/i)$

for all $d \in \text{supp}\,\sigma(s^*, j)$. The proof is complete.

If $p^* \in (0, j/i) \cup (j/i, 1)$ then Theorem 3 shows that $v(b, i, s^*)$ and $v(b, j, s^*)$ are uniquely determined. Furthermore, if $p^* \in (j/i, 1)$ then supp $\sigma(s^*, j)$ contains a single uniquely determined element and supp $\sigma(s^*, i)$ contains two uniquely determined elements. (To see this simply check that all the components of $d \in \text{supp}\,\sigma(s^*, j)$ are uniquely determined.) If $p^* \in (0, j/i)$ then supp $\sigma(s^*, j)$ may contain a variety of elements, since $p(d)$ for $d \in \text{supp}\,\sigma(s^*, j)$ is not uniquely determined. Nonetheless, the variability of $p(d)$ has no effect on the seller's behavior. Any communication equilibrium can be replaced by another that is "essentially equivalent" in which supp $\sigma(s^*, j)$ is a singleton set. We know of no definition of uniqueness for the communication equilibrium concept that is both precise and meaningful. A meaningful definition would surely classify the communication equilibria for this game when $p^* \in (0, j/i) \cup (j/i, 1)$ as "unique."

We conclude the paper with a somewhat paradoxical observation. Consider a communication equilibrium in which supp $\sigma(s^*, j)$ is a singleton. <u>This communication equilibrium involves no communication</u>, in that for all

$$d \in \text{supp}\,\sigma(s^*, i) \cup \text{supp}\,\sigma(s^*, j) \quad ,$$

$m(d)$ uniquely determines $p(d)$. Of course this is only to be expected, since it is hard to imagine why either (b, i) or (b, j) might want to tell a that i is the reservation price. By allowing for communication and applying a new solution concept, we have been led in this case to a restriction on the set of sequential equilibria. This is surprising, but it is not unprecedented. Just as Selten [8] restricts the set of equilibria in which mistakes do not occur by allowing for the possibility of mistakes, in this example we are able to restrict the set of equilibria in which communication does not occur by allowing communication to be possible.

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