# ON THE SIZE OF THE MESSAGE SPACE
# UNDER NON-CONVEXITIES

by

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## 1. Introduction

In the recent literature on resource allocation mechanisms, a great deal of attention has been paid to their informational properties. The knowledge about the economic environment is assumed to be dispersed among the agents and therefore, if consistent decisions are to be made, some communication is necessary. The size of the set of signals with which communication takes place has been taken as a measure of the degree of informational decentralization.

The term economic environment (denoted by $e$) is used to describe the basic data which the designer of an allocation mechanism considers as given (e.g., initial resource endowments, preferences, and technology). Let $E$ denote the class of environments over which the mechanism is supposed to operate. Not surprisingly, the minimal size of the message space which is sufficient to attain a Pareto-satisfactory performance depends crucially on the nature of $E$. For instance, if $E$ is the class of classical pure exchange environments (i.e., convex and without externalities or public goods) it has been shown by Hurwicz [6], Mount and Reiter [10], and Osana [11] that the competitive process is informationally efficient in the sense that the size of the message space of any other Pareto-satisfactory mechanism is not smaller than that of the competitive process. If we try to extend the class $E$ in order to cover certain kinds of non-convex environments, some informational losses are to be ex-

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pected. Thus, it is interesting to try to estimate the magnitude of these losses. In particular, we may ask whether it is possible to attain efficient allocations through decentralized processes which use finite dimensional vectors as messages. We show that the answer is in the negative.

The results presented in this paper are generalizations of previous results in [1] and [6] in the following respects: First, the class of environments for which the impossibility results hold is much nicer than that in [6] since we only consider production functions which go through the origin, and are strictly increasing, twice differentiable and either concave or convex. Second, we do not require the message space to be Euclidean or Hausdorff topological spaces. We only put an upper bound on its informational size. Third, we postulate regularity conditions which are weaker than those required in [6] and [11].

## 2. Resource Allocation Processes

Consider a simple economy with a single produced good, to be denoted by $y$, and a single primary good, $x$. There are two firms with production functions $y_i = \Psi_i(x_i)$. We choose the units of measurement in such a way that the total initial endowment of labor equals one. Hence, if we define $t = x_1$ and $x_2 = 1 - t$, the problem is to select the distribution of labor $t \in [0,1]$ among the two firms so as to maximize total output. If we define $g_1(t) = \Psi_1(t)$ and $g_2(t) = \Psi_2(1 - t)$ the problem is

$$(1) \qquad \begin{array}{l} \text{Max} \quad g_1(t) + g_2(t) \\ \text{subject to} \quad 0 \leq t \leq 1 \end{array}$$

Since the initial endowment is fixed and known by all agents, in order to specify the economic environment we just need a pair of functions $e = (g_1, g_2)$. We define the class of environments $E$ as the Cartesian product, $E = E^1 \times E^2$ where $E^1 = \{g \in \mathbb{C}^2_\infty\colon\ g(0) = 0,\ g' \geq 0,\ g'' \geq 0\}$, $E^2 = \{g \in \mathbb{C}^2_\infty\colon\ g(1) = 0,\ g' \leq 0,\ g'' \leq 0\}$ and $\mathbb{C}^2_\infty$ is the linear topological space of all twice continuously differentiable real valued functions defined on the closed unit interval with the supremum norm topology.

The optimality correspondence $\mathcal{O}\colon\ E \twoheadrightarrow [0,1]$, that assigns an optimal action to every environment, is given in this case by $\mathcal{O}(g_1, g_2) = \{\hat{t} \in I\colon\ g_1(\hat{t}) + g_2(\hat{t}) \geq g_1(t) + g_2(t) \text{ for all } t \in I\}$.

Following Hurwicz [5] the resource allocation process is decomposed in two phases. In the first phase there is an iterative process of information exchange. Let $M^i$ be the set of signals used by the i-th agent. The <u>message space</u> is defined as the Cartesian product $M = M^1 \times M^2$. A point of this set, $m = (m^1, m^2) \in M$, is called a message. Each agent is characterized by a <u>response function</u> $f^i\colon\ M \times E \to M^i$ indicating the message $m^i_{t+1}$ sent by the i-th agent at the t+1-th iteration as a function of the message $m_t$ received in the previous iteration and the economic environment $e$. The iterative process of information exchange is formalized by a system of temporally homogeneous first order difference equations

$$(2) \qquad m^i_{t+1} = f^i(m_t; e) \qquad i \in \{1,2\}.$$

The message $\bar{m}$ is said to be an equilibrium message if it is a stationary point of (2), that is, if $\bar{m}^i = f^i(\bar{m},e)$ for all $i$.

In that second phase, the equilibrium message is translated into an action to be taken according to an <u>outcome function</u> $h\colon \bar{M} \to A$, where $\bar{M} = \{m \in M\colon\ m = f(m,e)$ for some $e \in E\}$ is the set of stationary messages. A resource allocation process is then defined as the ordered triple $\pi = (M,f,h)$. We assume the initial dispersion of information so that every agent knows only his own characteristic $e^i$. If the information concerning other agent's characteristics has to come via formal messages, the i-th response function will be of the form $f^i(m,e^i)$. A process which uses such functions is said to be <u>privacy preserving</u>. A process $\pi = (M,f,h)$ is said to be decisive over $E$ if there is an equilibrium message for every $e \in E$. In other words, a process is decisive over $E$ if the <u>message correspondence</u>, $\mu(e) = \{m \in M\colon f^i(m,e^i) = m$ for all $i\}$, is non-empty valued for all $e \in E$. A process is said to be <u>non-wasteful</u> over $E$ if every outcome is optimal, that is, for every $e \in E$, $\tilde{m}^i = f^i(\tilde{m},e^i)$ for all $i$ implies $h(\tilde{m}) \in \mathcal{O}(e)$.

<u>Definition 1</u>. A subclass of environments $E^* \subseteq E$, where $E = E^1 \times E^2$, is said to possess the uniqueness property with respect to $\mathcal{O}$ if for any $(\bar{g}_1,\bar{g}_2)$ and $(\tilde{g}_1,\tilde{g}_2)$ in $E^*$, $\mathcal{O}(\bar{g}_1,\bar{g}_2) \cap \mathcal{O}(\bar{g}_1,\tilde{g}_2) \cap \mathcal{O}(\tilde{g}_1,\bar{g}_2) \cap \mathcal{O}(\tilde{g}_1,\tilde{g}_2) \neq \emptyset$ implies $\bar{g}_i = \tilde{g}_i$ for $i \in \{1,2\}$.

<u>Lemma 1</u> ([1],[6]). If the class of environments $E$ is a Cartesian product, $E = E^1 \times E^2$, the subclass of environments $E^*$ has the uniqueness

property with respect to $\mathcal{O}$ and $\pi = (M,f,h)$ is a privacy preserving process which is decisive and non-wasteful over $E$, then the lower inverse[1/] of the restriction of the message correspondence $\mu$ to $E^*$ is a (single-valued) function $\varphi$ that maps $M^* = \mu[E^*]$ <u>onto</u> $E^*$.

<u>Proposition 1</u>. The subclass of environments $E^* = \{(g_1,g_2) \in E:$ $g_1(t) + g_2(t) = K$ for all $t \in I$ and some $K \in R_+\}$ has the uniqueness property with respect to the optimality correspondence $\mathcal{O}$.

<u>Proof</u>. Let $(\bar{g}_1,\bar{g}_2)$ and $(\tilde{g}_1,\tilde{g}_2)$ be two environments in $E^*$ such that $(\bar{g}_1,\bar{g}_2) \neq (\tilde{g}_1,\tilde{g}_2)$. Let $\bar{K} = \bar{g}_1(t) + \bar{g}_2(t)$ and $\tilde{K} = \tilde{g}_1(t) + \tilde{g}_2(t)$. It suffices to show that for any $\hat{t}$, $\hat{t} \in \mathcal{O}(\bar{g}_1,\tilde{g}_2)$ implies $\hat{t} \notin \mathcal{O}(\tilde{g}_1,\bar{g}_2)$. Since the two environments are different, there exist some $\bar{t} \in I$ such that $\bar{g}_1(\bar{t}) = \tilde{g}_1(\bar{t})$. Assume without loss of generality that $\bar{g}_1(\bar{t}) > \tilde{g}_1(\bar{t})$. Then, by definition of $E^*$, $\bar{g}_1(\bar{t}) + \tilde{g}_2(\bar{t}) = g_1(\bar{t}) + \tilde{K} - \tilde{g}_1(\bar{t}) > \tilde{K}$. Hence, total optimal output for the "crossed" environment $(\bar{g}_1,\tilde{g}_2)$ is greater than $\tilde{K}$. Hence, if $\hat{t} \in \mathcal{O}(\bar{g}_1,\tilde{g}_2)$ it follows that $\tilde{K} < \bar{g}_1(\hat{t}) + \tilde{g}_2(\hat{t}) =$ $\bar{K} - \bar{g}_2(\hat{t}) + \tilde{K} - \tilde{g}_1(\hat{t})$ so that $\tilde{g}_1(\hat{t}) + \bar{g}_2(\hat{t}) < \bar{K}$ and $\hat{t}$ cannot be optimum for the other crossed environment, $\hat{t} \notin \mathcal{O}(\tilde{g}_1,\bar{g}_2)$, because $\tilde{g}_1(0) + \bar{g}_2(0) = \tilde{g}_1(0) + \bar{K} - \bar{g}_1(0) = \bar{K} > \tilde{g}_1(\hat{t}) + \bar{g}_2(\hat{t})$, that is, by using the zero-allocation at least $\bar{K}$ units of output can be guaranteed.

Q.E.D.

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[1/] The lower inverse of a correspondence $\mu\colon E \twoheadrightarrow M$ is the correspondence given by $\mu^{-1}(m) = \{e \in E\colon m \in \mu(e)\}$.

<u>Definition 2</u>. A process $\pi = (M,f,h)$ is informationally feasible iff

a) the message space is $\sigma$-compact[2/]

b) the function $\varpi$ is continuous.

<u>Remark 1</u>. The first condition of informational feasibility puts an upper bound on the information contained in the message space and tries to reflect the limitations of communication. Note that any finite set, a Euclidean set, a manifold or even $I^w$ (the Cartesian product of countably many copies of the closed unit interval) are admissible message spaces. On the other hand, an infinite dimensional Banach space is not an admissible message space.[3/] This limitation, however, would be meaningless without some additional restrictions on the complexity of the communication process. This is due to the fact that there are encoding procedures by which a single real number can be used to completely identify a continuous function: the set of all continuous functions has the power of the continuum (see [12], Theorem 1, p. 79). That means that by using a real number as a signal every agent can describe his environment. The second condition of informational feasibility rules out some of these encoding procedures.[4/] The hemicontinuity requirement is satisfied by the competitive process (see [11], Corollary 3, p. 71)

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2/ A topological space is $\sigma$-compact if it is the countable union of compact sets.

3/ Alternatively, the limitations on communication can be formalized by putting an upper-bound on the informational size of message spaces as defined in [10] or [13]. Using such a concept a message space would be admissible if it contains no more information than the denumerably dimensional cube $I^w$ (with the product topology). It is easily verified that any message space that has no more information than $I^w$, $M \leq_{MR} I^w$, is $\sigma$-compact. Hence, the notion of admissibility in Definition 1 is less restrictive than the one just proposed.

4/ In particular, we do not rule out message correspondences whose lower inverse is the infinite dimensional version of the Peano function that maps continuously $I$ onto $I^w$ (see [4], Theorem 4.4, p. 105).

and is weaker than the quasi-Lipschitzian condition imposed in [6] or the regularity conditions in [11].

Theorem. There is no informationally feasible process which is privacy preserving, decisive and non-wasteful over the class of environments $E$.

Proof. Suppose, by way of contradiction that there exists such a process, $\pi = (M,f,h)$. Since $E^*$ has the uniqueness property it follows by Lemma 1 that the lower inverse $\gamma$ of the correspondence $\mu: E^* \twoheadrightarrow M^*$, where $M^* = \mu[E^*]$, is a surjection of $M^*$ onto $E^*$ which, by condition b of informational feasibility, is continuous. Since $E^*$ is homeomorphic to $E^1$, it follows that there exists a continuous surjection $\beta$ mapping $M^*$ onto $E^1$.

Let $\hat{\mathbb{C}}^2_\infty$ be the topological subspace of all the functions $g$ in $\mathbb{C}^2_\infty$ such that $g(0) = 0$. The function $\alpha: E^1 \times E^1 \to \hat{\mathbb{C}}^2_\infty$ given by $\alpha(f_1,f_2) = f_1 - f_2$ is a continuous surjection.[5/] Then the function $\xi: M^* \times M^* \to \hat{C}^2_\infty$ given by $\xi(m_1,m_2) = \alpha[\beta(m_1,\beta(m_2)]$ is also a continuous surjection. Since, by condition a of the definition of informational feasibility, $M^* \times M^*$ is $\sigma$-compact, it follows that $\hat{C}^2_\infty$ can be expressed as the countable union of compact sets, $\hat{C}^2_\infty = \bigcup_{r=1}^{\infty} V_r$. We shall show that this is a contradiction by constructing a sequence of balls whose centers converge to a $C^2$ function which is not in any set $V_r$.

Observe first that $\hat{C}^2_\infty$ is not locally compact (see [7], Theorem 7.8,

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5/ To see this, given any $\varphi \in \mathbb{C}^2_\infty$ define $f_1(x) = \int_0^x \varphi'(t)\,dt + ax^2 + bx$ and $f_2(x) = f_1(x) - \varphi(x)$ where $2a > \|\varphi''\|_\infty$ $b > \|\varphi'\|_\infty$.

p. 62) and therefore each one of the sets $V_r$ is nowhere dense. Take $B(f_1,\varepsilon_1) \subseteq \overline{V}_1$, where the non-empty open set $\overline{V}_1$ is the complement of $V_1$. Now choose $f_2 \in \overline{V}_2$ such that $\|f_2 - f_1\|_\infty \leq \varepsilon_1$, $\|f_2' - f_1'\|_\infty \leq \varepsilon_1$, and $\|f_2'' - f_1''\|_\infty \leq \varepsilon_1$ 6/. Then, since $f_2 \in \overline{V}_2 \cap B(f_1,\varepsilon_1)$, which is open, we can choose $\varepsilon_2 < \frac{1}{2^2}$ such that $B_k = B(f_k,\varepsilon_k) \subseteq \overline{V}_k \cap B(f_{k-1},\varepsilon_{k-1})$. Proceeding in this way we get a nested sequence of balls such that $B_r \cap V_r = \emptyset$ for each $r$, and the centers are such that $\{f_k\}$, $\{f_k'\}$, and $\{f_k''\}$ are all Cauchy sequences. Hence, $\{f_k\}$ converges uniformly to a function $f \in \bigcap_{r=1}^{\infty} B_r$ and, therefore, $f \notin V_r$.

Q.E.D.

Note that these results can be applied to all the mechanisms in which the information exchange can be formalized by means of a system of temporally homogeneous first order difference equations and the stationarity of a message is the signal indicating that the optimum has been reached. And it is precisely the amount of information contained in the stationary message - that is, the amount of information needed to identify the optimum whenever it is generated in the course of the iterative procedure - what is measured in our Theorem. Although

6/ That we can choose $f_2$ in such a way can be seen as follows. The set $K(f,\delta) = \{g \in \hat{\mathbb{C}}_\infty^2 : \|f - g\|_\infty \leq \delta,\ \|f' - g'\|_\infty \leq \delta$ and $\|f'' - g''\|_\infty \leq \delta\}$ is closed. This follows from the fact that if a sequence of $C^2$ functions with bounded first and second derivatives converges uniformly to a $C^2$ function, then this function has also the same bounds on its derivatives. Now, suppose by way of contradiction, that it is not possible to choose such $f_2$. This implies that $K(f_1,\varepsilon_1)$ is a closed subset of the compact set $V_2$ and therefore $K(f_1,\varepsilon_1)$ is complete. This is a contradiction since it can be shown that $K(f_1,\varepsilon_1)$ is not complete.

many of the mechanisms in the literature can be put into this form, the present framework does not encompass all processes of interest. Foremost among the latter are processes with memory ([2], [8]) or stochastic adjustment processes ([9]).

As a consequence of the mathematical fact that a temporally homogeneous system of difference equations of the k-th order can be transformed into an equivalent system of the first order, the result presented here can be readily extended to processes with memory (i.e., the message in the last iteration depends on messages received in several previous iterations) as long as memory is finite.

The case of unbounded memory, which is common in mechanisms based on the idea that the center constructs increasingly accurate approximations of the production sets can be dealt with in two ways: a) constructing an equivalent system with an infinitely dimensional message space, or b) formalizing it by means of a system of difference equations which are not temporally homogeneous. This second approach is proposed in [3] to formalize the quantity-quantity process which converges to a global optimum in non-convex environments using finite dimensional messages at every iteration (this dimension increases without bounds with the number of iterations). Since the system is not temporally homogeneous the concept of stationary message is not well defined and the same can be said about the equilibrium message correspondence, so that our results are not applicable to this case.

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