ALLOCATION AND PREDICTION IN STOCHASTIC

ENVIRONMENTS: AN ELEMENTARY THEORY

by

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1. Introduction

Economic predictions differ from predictions of other phenomena in that an economic prediction, through its influence on the decisions of firms and households, can affect the predicted events. This problem was originally thought to be a major impediment to the development of predictive economic models.[1] However, this difficulty was essentially resolved by Grunberg and Modigliani [10]. They showed that in the absence of exogenous randomness, the existence of a correct prediction could be established by a fixed-point theorem. Indeed, through the "rational expectations" hypothesis, the causal influence of predictions has been widely exploited in the specification of econometric models.

The rational expectations hypothesis is that the expectations formed by economic agents are the statistically correct expectations conditioned on all of the information available to them. Because of the causal influence of predictions, rational expectations entail an equilibrium condition as well as a specification of expectations, so the term "expectations equilibrium" will be used synonymously below.

An additional equilibrium condition is also implicit in the rational expectations concept. In most models of allocation under uncertainty, the uncertainty is induced by exogenous randomnesss in the fundamental variables of the economy, such as preferences and technology. However,

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[1] See, for example, Morgenstern [21], reviewed in [20].

these variables are seldom directly observable. Instead, agents (and econometricians) must attempt to infer these variables from data, such as price and quantity data, generated by the allocation process. In general, each agent has his own sources of exogenous information which he attempts to augment with the information revealed by endogenous variables. As Lucas [19] has emphasized, the relation between exogenous and endogenous variables depends on agents' expectations. Hence the rational expectations hypothesis also entails an equilibrium in the information revealed by endogenous variables.

Unfortunately, this equilibrium condition may be impossible to satisfy. This was first noticed by Radner [24] who observed that if traders condition their expectations on prices, demand functions may be discontinuous. For example, suppose there are two states, a and b, which would influence a trader's demand if he could infer them from prices. In event a he observes the price p_a and in event b he observes p_b. If $p_a \neq p_b$ he can make the desired inference, but if $p_a = p_b$ he cannot. In particular, his demand in event a as a function of p_a is discontinuous at the point $p_a = p_b$. Explicit examples of the nonexistence of equilibrium are given in [9] and [18]. The informational discontinuity persists even if traders' state-dependent preferences satisfy all of the classical regularity conditions, and if there are a continuum of events distributed according to a smooth probability density function, although Allen, in [1] and [2], has generalized a result of Radner [26] to show that if the dimension of the state space is sufficiently small then an equilibrium will exist generically for smooth environments.

The fact that strong regularity conditions do not eliminate the informational discontinuity suggests that the theory might be clarified by abandoning such conditions and the attendant mathematical structure entirely. This paper attempts to verify this conjecture by presenting a mathematically elementary analysis and exposition of some of the major issues in expectations equilibrium theory. In addition to explaining the conceptual foundations of known results, our model is intended to provide researchers with a simple context for developing the intuition for future results.

Having been promised a simple exposition, the reader may be surprised to find no Edgeworth boxes, no demand or supply curves, and no coefficients of risk aversion. Most results in expectations equilibrium theory do not rely on competitive behavior as such, but turn on the use of information. Most economic data are generated not for the purpose of scientific observation, but to guide the allocation of resources. This statement is as true of socialist or mixed allocation mechanisms as of the market mechansim. The existence and other properties of expectations equilibria depend more on the information used and revealed by an allocation mechanism than on the allocation itself. Accordingly, all our results will be stated for general allocation mechanisms, although most were motivated by analogous results obtained by the author and others for the competitive mechanism.

Section 2 contains some preliminary definitions and results from abstract allocation theory. Each agent is presumed to have only a finite number of feasible allocations. Hence the number of possible preference profiles is finite. This restriction is the principal source of simplification. All possible preference profiles are included in the set of environments, so the set of environments is analogous to the unrestricted domain of discrete preference profiles which provides the setting of much of social choice theory.

Sections 3-5 are devoted to the three central problems of rational expectations theory: When does an equilibrium exist? How can an equilibrium be implemented? How can agents learn enough about their environment to form rational expectations? An allocation mechanism which generally admits equilibria is said to be admissible. Proposition 3.4 states that admissibility requires an allocation mechanism to be informationally decentralized. In terms of modelling, this requirement is essentially a prohibition against omitting individual decision variables or aggregating endogenous variables across nonidentical economic agents. Proposition 3.6 states that decentralized mechanisms which have an additional property associated with individual maximizing behavior are admissible. Section 4 provides a dynamic adjustment process which achieves expectations equilibria. This process requires agents to have even more knowledge of their environment than is required by the equilibrium. Section 5 describes a method of inference which enables agents to learn the required knowledge from repetitions of the adjustment process. Unlike sections 3 and 4, the analysis of section 5 relies heavily on the finiteness of the domain of preference profiles, and does not appear to have a direct extension to the competitive mechansim on general stochastic exchange environments.

Section 6 is devoted to multiperiod environments in which agents' expectations are conditioned on past endogenous variables. Proposition 6.5 states that admissibility in this context essentially requires an allocation mechansim to be trivial in that each agent determines his own allocation independently. Section 7 studies multiperiod environments which evolve as a stationary Markov process. In this case equilibria will generally exist with expectations conditioned on past data, but Proposition 7.6 states that the stochastic process of equilibria will not generally be stationary unless the allocation mechansim is dictatorial. If expectations are conditioned on current data, the equilibrium process is a stationary Markov process if the allocation mechansim satisfies the hypothesis of Proposition 3.6. The conclusion of sections 6 and 7 is that in multiperiod environments, the existence and other desireable properties of expectations equilibria can only be ensured for reasonable allocation mechansims if expectations are conditioned on current and past data.

Section 8 discusses the existence of equilibrium public predictions, a problem first studied by Grunberg and Modigliani [10] in a deterministic context. Suppose that agents do not form their own expectations based on private and public information, but rely instead on public predictions based on public information. Proposition 8.4 states that in this case, the existence of equilibrium cannot be ensured for nontrivial allocation mechansims. The informational discontinuity creates a marked difference between the stochastic and deterministic public prediction models.

Some open problems and directions for future research are discussed in the conclusion to this paper.

2. Allocation Mechanisms

Most "short-run equilibrium" models of economic behavior have the property that the mechanism which chooses allocations is stationary. For example, in the single market model presented by Muth [23], equilibrium in each period is determined by the intersection of supply and demand curves. The fact that these curves are time and state-dependent and are obtained through dynamic stochastic optimization rather than static optimization does not influence the way they are used to determine current equilibria. Thus a short-run competitive equilibrium is an ordinary competitive equilibrium of a short-run environment. We now define allocation mechanisms on short-run environments.

<u>2.1 Definitions</u>: There are N agents, indexed by the superscript i , $1 \leq i \leq N$, with $N \geq 2$. For each i, let A^i be a finite set of allocations for the i^{th} agent, with $\#A^i \geq 2$, and let $A \subset \Pi_i A^i$ be the set of social allocations, with $\#A \geq 2$. For each i, let U^i denote the set of all utility functions $u^i\colon A^i \to R$, and let $E = \Pi_i U^i$, with generic element $u = (u^1,\ldots,u^N)$. Then E is the set of short-run environments and A is the set of short-run allocations.

A <u>message process</u> is a pair (μ, M), where M is a set of abstract messages, with generic element m, and μ is a function on E to M. Message processes will be assumed to be <u>ordinal</u> in the sense that if u, $u' \varepsilon E$ such that for each i and each a^i, $a'^i \varepsilon A^i$, $u^i(a^i) \geq u^i(a'^i)$ if and only if $u'^i(a^i) \geq u'^i(a'^i)$, then $\mu(u) = \mu(u')$. An <u>allocation mechanism</u> is a triple (μ, M, g), where (μ, M) is a message process and g: $M \to A$. The function g is called an <u>outcome function</u>.

<u>2.2 Remarks</u>: In the usual context of social choice theory, A^i is the same abstract set of alternatives for each i, and A is the diagonal of $\Pi_i A^i$. In the usual context of allocation theory, A^i is a set of net trades for the i^{th} agent, and A is the set of N-tuples $(a^i)_{i=1}^N$ with $\Sigma a^i = 0$. In the latter case, an agent's utility of trades is derived from his initial endowment and his utility of final consumption. Since A^i is finite for each i, E cannot be interpreted as a set of classical exchange environments, but we will continue to refer to the competitive mechanism for exchange environments by way of analogy and motivation.

A message m determines an allocation $g(m)$, but may also contain other information such as prices, quotas, parameters of incentive schemes; etc. In general competitive equilibrium theory, for example, the message is the equilibrium price and N-tuple of trades, $m = (p, a)$, and g is the projection $(p, a) \to a$.

We now consider some properties of allocation mechanisms that arise in expectations equilibrium theory as well as general allocation theory. For motivation, the reader might prefer to first read section 3 up to Proposition 3.4.

2.3 Definitions: For each $u, u' \in E$, define $u'\theta^i u =$ $(u^1,\ldots,u^{i-1}, u'^i, u^{i+1},\ldots,u^N) \in E$. A message process (μ, M) is decentralized if

θ) for each $u, u' \in E$, $\mu(u) = \mu(u')$ if and only if, for each i, $\mu(u) = \mu(u'\theta^i u)$.

2.4 Lemma: Suppose that a message process (μ, M) is decentralized and define the correspondence μ^i: $U^i \twoheadrightarrow M$ for each i by

$$\mu^i(u^i) = \{m: \text{ for some } (u^1,\ldots,u^{i-1}, u^{i+1},\ldots,u^N) \in \Pi_{j \neq i} U^j \ ,$$
$$m = \mu(u^1,\ldots,u^{i-1}, u^i, u^{i+1},\ldots,u^N)\}.$$

Then for each $u \in E$, the intersection $\cap_{i=1}^N \mu^i(u^i)$ is the singleton $\{\mu(u)\}$.

Proof: Direct.

2.5 Remarks: If (μ, M) is decentralized, to check that $m = \mu(u)$ it suffices to check that $m \in \mu^i(u^i)$ for each i independently. Thus μ^i associates with u^i a set of messages that can be interpreted as equilibria for the i^{th} agent with the utility function

u^i. Mount and Reiter [22, pp. 170-171] showed that if (μ, M) is decentralized then $\mu(u)$ can be represented as the equilibrium of an iterative message adjustment process where each agent's message response correspondence depends on the previous message and his own characteristic alone. This property was originally introduced by Hurwicz under the term "externality" [13, Definition 4, p. 32]. The term "privacy respecting" or "privacy preserving" is now more frequently used. The "crossing condition" ($\otimes$) was introduced by Mount and Reiter [22], and Lemma 2.4 is a consequence of their Lemma 5.

In general, decentralization can be viewed as a requirement that the message space contain enough variables. Given an arbitrary message process (μ, M), one can always construct a set D and a function $\mu_1: E \to D$ such that the augmented message process $(\mu_1 \times \mu, D \times M)$ satisfies the privacy condition. For a drastic example, let $D = E$ and let μ_1 be the identity. As applied to models of economic time series, decentralization can thus be interpreted as a condition on the data set, rather than a condition on the economic institutions which constitute the allocation process.

Before discussing examples we define another property which will be useful in ensuring the existence of an expectations equilibrium.

<u>2.6 Definition</u>: A decentralized message process (μ, M) is <u>convex</u> if for each i and each $m \in M$, $\{u^i\colon m \in \mu^i(u^i)\}$ is convex, where μ^i is defined in Lemma 2.4.

2.7 Remarks: Decentralization and convexity do not formally entail any behavioral principles but they are naturally associated with maximizing behavior. For example, suppose that M can be written as the product of individual message sets, $M = \Pi_i M^i$, and that μ is the Nash equilibrium function of the noncooperative game with the outcome function g. That is, for each i and each u^i let $\mu^i(u^i) = \{m \in M\colon\ m^i \text{ maximizes } u^i(g(m^1,\ldots,m^{i-1},\ \cdot, m^{i+1},\ldots,m^N))\}$, and for each $u \in E$, let $\mu(u) = \bigcap_i \mu^i(u^i)$. The message process (μ, M) is clearly decentralized, and is also easily seen to be convex. A second example is the competitive message process for exchange environments. For each i and each u^i let $\mu^i(u^i) = \{(p, a)\colon a^i$ maximizes $u^i(y)$ s.t. $py \leq 0\}$ and for each u let $\mu(u) = \bigcap_i \mu^i(u^i)$. Again (μ, M) is decentralized and convex. These two examples emphasize that the convexity of a message process does not involve the convexity of preferences or feasible sets.

These examples also serve to indicate the restrictiveness of our requirement that μ be singlevalued. Noncooperative and competitive equilibria are seldom unique. In studying the competitive mechanism on exchange environments this problem can be finessed by taking single-valued selections from the equilibrium correspondence [see 15], and the results we will obtain below have analogues involving selections. However, the assumption of uniqueness makes the results sharper and the exposition easier, and also avoids a conceptual ambiguity which is discussed in section 3.7 below.

The subset of E consisting of "strict preference environments" will play an important role in sections 3 and 6 below, so we now give a formal definition.

2.8 Definitions: For each i, let $U^{\circ i} = \{u^i \in U^i: u^i(a^i) \neq u^i(a'^i)$ for all $a^i, a'^i \in A^i$ with $a^i \neq a'^i\}$, and let $E^\circ = \Pi_i U^{\circ i}$. A message process is decentralized on E° if it satisfies ($\otimes$) with E replaced by E°. If a message process (μ, M) is decentralized on E°, then the conclusion of Lemma 2.4 remains true with E replaced by E° and U^i replaced by $U^{\circ i}$ for each i.

2.9 Remarks: The following property is associated with the expectations equilibrium concept studied in section 6. For motivation, the reader may prefer to first read that section up to Proposition 6.5.

2.10 Definitions: Let (μ, M, g) be an allocation process. A subset I of the set $\{1,\ldots,N\}$ of agents is minimal if

i) there is a function $\tilde{\mu}: \Pi_{i\varepsilon I} U^i \to M$ such that for each $u \in E$, $\mu(u) = \tilde{\mu}[(u^i)_{i\varepsilon I}]$; and

ii) I contains no proper subset for which (i) is satisfied.

It follows that there is a unique minimal set I. The allocation process is unilateral on E° if for each $i \in I$ there is a function $\alpha^i: U^{\circ i} \to A^i$ such that for each $u \in E^\circ$, the i^{th} coordinate of $g[\mu(u)]$ is equal to $\alpha^i(u^i)$.

2.11 Remarks: Agents who are not members of the minimal set I can be ignored, since their characteristics do not influence the messages generated in any environment. If (μ, M, g) is unilateral on E°, the allocation for each agent in I is determined by his own characteristic independently, for short run environments u with $u^i \in U^{\circ i}$

for each $i \in I$. An allocation process is trivially unilateral if I consists of a single agent or if g is a constant function.

The following definition is used in Section 7 below.

<u>2.12 Definition</u>: A message process (μ, M) is <u>dictatorial</u> if there is some i and some function $\mu^i \colon U^i \to M$ with $\mu(u) = \mu^i(u^i)$ for all $u \in E$.

3. Stationary Stochastic Environments

A stationary stochastic environment is essentially a random variable taking values in E. The utility functions are not realized until after the allocation is determined, so agents participate in the allocation mechanism according to expected utility functions. The term stationary is used because agents have a one period time horizon, so the environment can be interpreted as being generated independently and identically over time. Initially each agent i observes a signal y^i which represents his private exogenous information about the future state. Then the message process occurs, culminating in a message m and an allocation $g(m)$. Since different agents have different private information, each agent attempts to infer the private information of others from the message m. An expectations equilibrium message is a message which is consistent with the agents' expected utility conditioned on private information and the message itself. We now define this structure formally.

3.1 Definitions: Let X denote a set of abstract future states containing at least two elements. For each i, let Y^i denote an abstract set containing at least two elements. Let $Y = \Pi_i Y^i$, with generic element $y = (y^i)_i$. A stationary stochastic environment is specified in part by a probability distribution π on $Y \times X$, which will always be assumed to have finite support. Let π_1 denote the marginal distribution on Y determined by π and let π_2 denote the marginal distribution on X. The phrase "for each $y \varepsilon Y$" will be understood to mean "for each $y \varepsilon Y$ with $\pi_1(y) \neq 0$. The i^{th} agent's initial observation of y^i determines the conditional distribution $\pi(\cdot|y^i)$ on X. That is, for each $y \varepsilon Y$ and each $x \varepsilon X$,

$$\pi(x|y^i) = \sum_{\{(y', x):\ y'^i = y^i\}} \pi(y', x) \Bigg/ \sum_{\{\ \}} \pi(y', x)$$

There are two special cases of interest. Agent i is said to be <u>informed</u> if for each $y \varepsilon Y$, $\pi(\cdot|y^i)$ is a degenerate distribution, that is, $\pi(x|y^i) \ \varepsilon\ \{0, 1\}$ for all $x \varepsilon X$. Agent i is said to be <u>uninformed</u> if $\pi(\cdot|y^i)$ is equal to the marginal distribution on X for each $y \varepsilon Y$. For each i, let V^i denote the set of all state dependent utility functions v^i: $A^i \times X \to R$. Let $V = \Pi_i V^i$, generic element $v = (v^i)_i$. A <u>stationary stochastic environment</u> is a pair (π, v). We will usually drop the term stationary. Let S denote the set of stochastic environments, and note that the only restriction we have placed on S is the requirement that π have finite support.

<u>3.2 Expectations Equilibrium</u>: Given a message process (μ, M), a <u>reduced form</u> is a function f: $Y \to M$. A reduced form f associates with each initial state of information y a message $f(y)$. Given a stochastic environment (π, V), define the i^{th} agent's expected utility function $w^i(f)$: $Y \to U^i$ by

$$w^i(f)(y)(a^i) = \sum_{\{(y',x):\, y'^i = y^i \quad \text{and} \quad f(y') = f(y)\}} v^i(a^i, x)\pi(y', x) \Bigg/ \sum_{\{\quad\}} \pi(y', x)$$

for each $y \in Y$ and each $a^i \in A^i$. Let $w(f) = (w^i(f))_i$. An <u>expectations</u> <u>equilibrium</u> is a reduced form f satisfying

$$(*) \qquad f(y) = \mu[w(f)(y)] \quad \text{for each } y \in Y.$$

A message process (μ, M) is <u>admissible</u> if it admits an expectations equilibrium for every stochastic environment.

<u>3.3 Remarks</u>: Condition (*) states that an agent's expectations are determined by the objective probability distribution π, the initial information y^i, and the reduced form f. Hence expectations are determined by those variables which would naturally appear in an empirical model of the environment and allocation merchanism, and are not influenced by psychological factors peculiar to each agent. This is perhaps the principal empirical advantage of the rational expectations hypothesis.

<u>3.4 Proposition</u>: Every admissible message process is decentralized on E°.

<u>Proof</u>: Let $u_a, u_b \in E^\circ$, and suppose that $\mu(u_a) = \mu(u_b) = m$ for some $m \in M$. Let $x_a, x_b \in X$ and for each i, let $v^i \in V^i$ be defined by

$$v^i(a^i, \ x) = \begin{cases} u_a^i(a^i) & \text{if} \quad x = x_a; \\ u_b^i(a^i) & \text{otherwise.} \end{cases}$$

We need to show that for any i, say $i = 1$, $\mu(u_b \otimes^1 u_a) = m$. For each $i > 1$, let y_a^i, $y_b^i \in Y^i$, and let $y_o^1 \in Y^1$, and let $y_a = (y_o^1, \ y_a^2, \ldots, y_a^N)$, $y_b = (y_o^1, \ y_b^2, \ldots, y_b^N)$. Let $\pi(y_a, \ x_a) = \lambda$ and $\pi(y_b, \ x_b) = 1 - \lambda$ for some $0 < \lambda < 1$. That is, agent 1 is uninformed and each agent $i > 1$ is informed. Since $(\mu, \ M)$ is admissible, there is an expectations equilibrium f for $(\pi, \ v)$. For each $i > 1$, $w^i(f)(y_a) = u_a^i$ and $w^i(f)(y_b) = u_b^i$. If $f(y_a) \neq f(y_b)$ then $w^1(f)(y_a) = u_a^1$ and $w^1(f)(y_b) = u_b^1$. But $\mu(u_a) = \mu(u_b)$ by supposition, so we must have $f(y_a) = f(y_b)$. In this case $w^1(f)(y_a) = w^1(f)(y_b) = u_\lambda^1$, where $u_\lambda^1 = \lambda u_a^1 + (1 - \lambda) u_b^1$. Therefore $\mu(u_\lambda^1, \ u_a^2, \ldots, u_a^N) = \mu(u_\lambda^1, \ u_b^2, \ldots, u_b^N)$. However, for λ sufficiently near 0, u_λ^1 and u_b^1 represent the same ordering on A^1. Since μ is ordinal, this implies $\mu(u_b \otimes^1 u_a) = \mu(u_b) = m$, which completes the proof.

3.5 Remarks: As mentioned in 2.5 above, decentralization is essentially a requirement that M contain enough variables. Since outcome functions play no direct role in the equilibrium condition (*) or in Proposition 3.4, the Proposition has the following immediate and important generalization. Suppose that instead of observing the entire message m, agents observe a statistic $h(m)$. For example, if m is the competitive message, $h(m)$ could be the price alone, or the price and the volume of trade. Since $(h \cdot \mu, M')$ is a message process, where M' is the range space of h, the Proposition implies that the "admissibility" of h requires that $(h \cdot \mu, M')$ be decentralized on E°. This requirement will commonly fail to be met if the statistic $h(m)$ omits individual decision variables or aggregates decision variables across agents with different characteristics.

Theorem 5.11 of [14] establishes that if (μ, M) is the competitive message process and h is continuous, then h must be either 1-1 or constant. Theorem 3.5 of [15] drops the continuity hypothesis from this result, and also addresses the case in which different agents may have different data functions, establishing that the i^{th} agent's data function must either be constant or must be informative enough to identify p and a^i.

Decentralization on E is neither necessary nor sufficient for admissibility. However, the following proposition states that decentralized message processes with the convexity property (Definition 2.6) are admissible.

<u>3.6 Proposition</u>: Every convex message process is admissible.

<u>Proof</u>: Let (μ, M) be a convex message process and (π, v) a stochastic environment. For each i, let w^{*i}: $Y \to U^i$ be defined by $w^{*i}(y)(a^i) = \sum_{x \in X} v^i(a^i, x)\pi(y, x) / \sum_{x \in X} \pi(y, x)$. That is w^{*i} is the i^{th} agent's expected utility conditioned on all of the initial information in the environment. Let $w^* = (w^{*i})_i$, and for each $y \in Y$, define $f(y) = \mu[w^*(y)]$. We will show that $f(y) = \mu[w(f)(y)]$

for each $y \varepsilon Y$. Let $m = f(y)$ for some y with $\pi_1(y) > 0$. By the definition of f, $m \varepsilon \mu^i(w^{*i}(y'))$ for each y' with $f(y') = m$, where μ^i is defined in Lemma 2.4. For each i, $w^i(f)(y)$ is a convex combination of the utility functions $\{w^{*i}(y'): y'^i = y^i$ and $f(y') = m\}$. The convexity of (μ, M) then implies that $m \varepsilon \mu^i[w^i(f)(y)]$ for each i, so $m = \mu[w(f)(y)]$, which completes the proof.

3.7 Remarks: Proposition 3.6 can be loosely interpreted as stating that message processes which involve decentralized maximizing behavior are admissible.

The reduced form constructed in the proof was obtained by conditioning each agent's expectations on the entire private information N-tuple y. Thus we have also proved that convex message processes admit expectations equilibria which give the same messages as if every agent had complete private information.

If the message correspondence μ is not single-valued, 3.2(*) would have to be written

$$f(y) \varepsilon \mu[w(f)(y)] \quad \text{for each } y \varepsilon Y.$$

But then the question of how $f(y)$ rather than some other message comes to be selected from $\mu[w(f)(y)]$ becomes important. We have avoided this question by assuming μ to be single-valued.

4. Dynamic Equilibrium

We have assumed that a message process associates a unique message $\mu(u)$ with each $u \varepsilon E$, but this does not ensure the uniqueness of expectations equilibria. For example, consider a two-state, two-agent environment in which agent 1 is informed but $v^1(\cdot, x_a) = v^1(\cdot, x_b)$, and agent 2 is uninformed but $v^2(\cdot, x_a) \neq v^2(\cdot, x_b)$. Suppose that $\mu(v^1(\cdot, x_a), v^2(\cdot, x_a)) \neq \mu(v^1(\cdot, x_b), v^2(\cdot, x_b))$. Let $y_a = (y_o^1, y_a^2)$ and $y_b = (y_o^1, y_b^2)$, and let $\pi_a = \pi(y_a, x_a)$ and $\pi_b = \pi(y_b, x_b)$ with $\pi_a + \pi_b = 1$. Then there will be two expectations equilibria, one with the reduced form

$$y_a, y_b \rightarrow \mu(v^1(\cdot, x_a), \pi_a v^2(\cdot, x_a) + \pi_b v^2(\cdot, x_b)) =$$

$$= \mu(v^1(\cdot, x_b), \pi_a v^2(\cdot, x_a) + \pi_b v^2(\cdot, x_b)),$$

which reveals no information; and one with the reduced form

$$y_a \rightarrow \mu(v^1(\cdot, x_a), v^2(\cdot, x_a)),$$

$$y_b \rightarrow \mu(v^1(\cdot, x_b), v^2(\cdot, x_b))$$

which reveals the information possessed by agent 1. Intuitively, only the first equilibrium is reasonable, since the information generated in the second equilibrium is initially possessed exclusively by an agent whose behavior would never reveal this information.

This discussion clearly indicates the need for some formalization of the process by which the agents' information influences the equilibrium message. One model of such a process has been described by Reiter [27] in a more general setting. We will now adapt Reiter's process to the present context.

The process can be described informally as follows. Initially, each agent conditions his expectations on his exogenous information, which yields an equilibrium message $f_1(y)$ for each state of information y. Before the allocation $g(f_1(y))$ is made, however, each agent adds the information revealed by $f_1(y)$ to his initial information. The resulting change in expected utility functions leads to a new message $f_2(y)$, and so on. The process terminates when the message $f_t(y)$ reveals no new information to any agent. Since agents retain the information revealed at each stage, the number of iterations will never exceed the number of states of information in the support of π_1.

4.1 Definitions: Given a stochastic environment (π, v) and a message process (μ, M), let $\{f_t\}_{t \geq 1}$ be a sequence of reduced forms, $f_t\colon Y \to M$. For each i and each $t \geq 1$, define $w_t^i(\{f_s\}_{s<t})\colon Y \to U^i$ by

i) $$w_1^i(y)(a^i) = \sum_{\{(y^i, x):\ y'^i = y^i\}} v^i(a^i, x)\pi(y', x) \Bigg/ \sum_{\{\ \}} \pi(y', x) \text{ ; and}$$

ii) $$w_t^i(\{f_s\}_{s<t})(y)(a^i) = \sum_{\{(y', x):\ y'^i = y^i \text{ and } f_s(y')=f_s(y) \text{ for all } s<t\}} v^i(a^i, x)\pi(y', x) \Bigg/ \sum_{\{\ \}} \pi(y', x), \text{ for each } t>1,$$

for each $y \in Y$ and each $a^i \in A^i$. That is, $w_t^i(\{f_s\}_{s<t})(y)$ is the i^{th} agent's expected utility conditioned on his private information

and the previous messages $f_s(y)$, $s < t$. Let $w_t = (w_t^i)_i$ and define a sequence of reduced forms iteratively by

iii) $f_t(y) = \mu[w_t(\{f_s\}_{s<t})(y)]$ for each $y \in Y$.

Let $T = \min\{t: f_{t+1} = f_t\}$. The reduced form f_T is a dynamic equilibrium for (π, v).

4.2 Remarks: It is immediate that for each environment and message process there exists a unique dynamic equilibrium, so this process resolves both the existence and uniqueness issues without any restrictions on message processes. Unfortunately, the i^{th} agent's dynamic equilibrium expectations cannot generally be derived from y^i and f_T alone. Thus dynamic equilibria do not have one of the principal desireable properties of expectations equilibria. However, the following proposition shows that if the message process (μ, M) is convex (2.6), dynamic equilibria are also expectations equilibria, so this property is recovered.

4.3 Proposition: Suppose that (μ, M) is convex message process. For each stochastic environment (π, v), if f_T is a dynamic equilibrium then f_T is an expectations equilibrium.

Proof: Let (μ, M) be a convex message process and let f_T be a dynamic equilibrium for a stochastic environment (π, v). For any $y \in Y$ with $\pi_1(y) > 0$, let $m = f_T(y)$. We need to show that $m = \mu[w(f_T)(y)]$. By the definition of dynamic equilibrium,

$m \in \mu^i[w^i_{T+1}(\{f_t\}_{t \leq T})(y')]$ for each $y' \in Y$ with $y'^i = y$ and $f_t(y') = f_t(y)$ for all $t \leq T$, for each i. Since $w^i(f_T)(y)$ is a convex combination of the utilities $w^i_{T+1}(\{f_t\}_{t \leq T})(y')$ for such y', $m \in \mu^i[w^i(f_T)(y)]$ for all i, so $m = \mu[w(f_T)(y)]$.

4.4 Remarks: For convex message processes we have described a mechanism which selects a unique element of the set of expectations equilibria. How can a dynamic equilibrium be distinguished from other expectations equilibria? One natural conjecture is that the information revealed by a dynamic equilibrium reduced form is in some sense minimal. However, the following example shows that another equilibrium may reveal strictly less information. Suppose there are two agents and three equiprobable states, and that agent 1 is informed and agent 2 is uninformed. Let $y_a = (y^1_o, y^2_a)$, $y_b = (y^1_o, y^2_b)$, and $y_c = (y^1_o, y^2_c)$, and let $\pi(y_a, x_a) = \pi(y_b, x_b) = \pi(y_c, x_c) = 1/3$. Suppose that $\mu(v^1(\cdot, x_a), u^2)$, $\mu(v^1(\cdot, x_b), u^2)$, and $\mu(v^1(\cdot, x_c), u^2)$ are all distinct when $u^2 = (1/3)v^2(\cdot, x_a) + (1/3)v^2(\cdot, x_b) + (1/3)v^2(\cdot, x_c)$; and let $m_a = \mu(v^1(\cdot, x_a), v^2(\cdot, x_b))$, $m_b = \mu(v^1(\cdot, x_b), v^2(\cdot, x_b))$, and $m_c = \mu(v^1(\cdot, x_c), v^2(\cdot, x_c))$. Then $T = 2$ and $f_T(y_a) = m_a$, $f_T(y_b) = m_b$, and $f_T(y_c) = m_c$. If m_a, m_b, and m_c are distinct messages, f_T reveals complete information to agent 2. Strictly speaking, agent 2 learns the state of information y by observing $f_1(y)$, but in this case, the dynamic equilibrium f_T itself reveals complete information. However it may also be the case that

$\mu(v^1(\cdot, x_a), u^2) = \mu(v^1(\cdot, x_b), u^2) \neq m_c$ when $u^2 = (1/2)v^2(\cdot, x_a) + (1/2)v^2(\cdot, x_b)$. Denoting the former message by m_{ab}, we have a second expectations equilibrium defined by $f(y_a) = f(y_b) = m_{ab}$ and $f(y_c) = m_c$, and f reveals less to agent 2 than f_T.

The characterization of dynamic equilibria probably should await a more axiomatic investigation of the adjustment mechanism itself. There are undoubtedly many other iterative processes which might result in different equilibria. For example, if the message $\mu(u)$ for each $u \in E$ is itself the outcome of a dynamic adjustment process, as suggested in 2.5 above, agents could adjust their information and message response simultaneously. The existence of a dynamic equilibrium might then depend on the stability of the message response process.

The dynamic equilibrium concept has been applied in [15] to stochastic exchange environments with the competitive message process, and extended to the case of arbitrarily many states. In this context, the dynamic theory provides a way of recovering the notion of price-conditional expectations equilibrium. Suppose that at each stage, agents adjust their expectations to the information generated by the price alone, rather than the entire competitive message. A dynamic equilibrium of this process is an expectations equilibrium in which each agent conditions his expectations on the dynamic equilibrium price and his own trade. Thus the information generated by pre-equilibrium prices which is not revealed by the dynamic equilibrium price is, to the extent that it is necessary, revealed by the agent's dynamic equilibrium trade.

Independently of Reiter and the present author, Kobayashi [17] has studied a similar process in a financial asset market model. Kobayashi assumes that there is a single risky asset and a riskless asset and that each trader i initially observes a single random variable y^i. The joint distribution of $(y^i)_i$ and the future value of the risky asset is assumed to be Normal, and traders are assumed to have utility functions of future wealth with constant absolute risk aversion. Kobayashi proves that a rational expectations equilibrium will be achieved in at most $N + 1$ iterations, where N is the number of traders. The adjustment process is similar to the one described in [15] except that pre-equilibrium trades are actually made, although traders are assumed not to anticipate capital gains or losses during the trading process. It is shown in [15] that in general stochastic exchange environments, the implementation of pre-equilibrium trades when traders have rational expectations about capital gains and losses can cause the process to break down in the sense that the intermediate or "temporary" equilibria may fail to exist.

5. The Estimation of Expectations

An expectations equilibrium requires an agent to know enough about his environment and the equilibrium reduced form to associate the correct preferences with each observed pair (y^i, m). Until recently, the question of how agents could acquire this knowledge was widely regarded as impossibly difficult. The source of this difficulty is the fact that the problem of learning rational expectations differs from a conventional problem of statistical estimation in that the estimates influence the relation being estimated. In the present context, the true relation between messages and future states is influenced by the expectations about future states which agents associate with messages. However, several recent papers have suggested that, at least in certain special cases, the correct expectations can be learned. I will only refer to a few of these papers; a more complete survey is given in [6].

All of the papers on this estimation problem that I am aware of treat the problem of learning equilibrium expectations directly. In particular, in [4], [6], and [25], agents attempt to learn the relation between prices and future states by choosing an estimate of this relationship, which determines the expectations they associate with each price, which in turn determines an actual relation between states and prices. In [25] and [6 , section 2.4], the estimates are held fixed until an arbitrarily large number of observations permits the actual relation to be learned exactly. This becomes the new estimate, which yields a new actual relation and the process repeats. In [4], and [6 , section 2.5], estimates are revised after each observation. In each case, the resulting sequence of estimated and actual relations is shown to converge to a rational expectations equilibrium in some circumstances and not in others.

This approach to the estimation problem is extremely, and possibly unnecessarily, restrictive. In the present context, suppose that agents begin with an estimated reduced form which is not an expectations equilibrium. Then in a given state of information y, any proposed message m may result in a conditional expected utility N-tuple u with $\mu(u) \neq m$. That is, the "actual" reduced form determined by the estimate does not exist. For this reason it is necessary to restrict estimates so that the actual relation being estimated exists, and to restrict environments and estimation procedures so that the restrictions on estimates are preserved by the estimation process. Hence this approach prevents any general possibility or impossibility theorem on the estimation of rational expectations from being stated much less proved.

We describe below a very different and conceptually much more general approach to the estimation of expectations. Our approach guarantees the convergence to a dynamic equilibrium for every message process and every stochastic environment, and thus convergence to an expectations equilibrium for convex message processes. However, the reader should be warned that, unlike the other results in this paper, analogues of this section have not yet been developed for classical environments, so the general effectiveness of the estimation procedure introduced here has not yet been demonstrated.

The estimation problem will be studied here in the framework of the dynamic process described in the previous section. This process has the advantage of avoiding the simultaneous determination of messages and expectations which creates the difficulties mentioned above, and the disadvantage of requiring agents to learn the correct conditional expectations for many reduced forms. We will show that the advantage predominates.

Given a stochastic environment (π, v), suppose that the pair (y, x) is drawn independently and identically over time from the distribution π. Henceforth the subscript t will denote sampling time. In sample period t, the i^{th} agent initially observes y^i_t and uses an estimate of the conditional distribution of x_t given y^i_t to form an expected utility u^i. The first stage message $m_{1t} = \mu(u)$ is then observed by all agents, and each agent i uses an estimate of the conditional distribution of x_t given (y^i_t, m_{1t}) to form a new expected utility function which determines a second stage message m_{2t}, and so on. After the final iteration is reached, all agents observe x_t and re-estimate their conditional distributions. Then (y_{t+1}, x_{t+1}) is drawn and the process repeats. We will postpone until section 5.9 below the question of which reduced form in period t is the final iteration when agents have incorrect expectations.

Initially the i^{th} agent knows only his state-dependent utility function v^i and his set of private observations Y^i. Thus agents do not know π or the characteristics of other agents. It is interesting to note that an agent i never observes any other agent's private information y^j_t, even after x_t is observed, so that the reduced forms themselves cannot be estimated. Only the conditional distributions on X given y^i and the observed messages can be learned. In order to avoid repeated separate references to null sets, we will suppose that $\pi_1(y) > 0$ for all y.

5.1 Stochastic Environments: We assume that for each i, Y^i is a finite set, and we consider only those stochastic environments (π, v) with $\pi_1(y) > 0$ for all $y \in Y$.

5.2 Remarks: The initial stage of the dynamic process requires an agent i to estimate the conditional distribution of x_t given y_t^i. The true conditional distribution is determined by π, so this estimation problem is straightforward. However, we will develop a particular solution to this problem which leads to a solution of the latter stage estimation problems as well. For the moment we will concentrate on the first stage problem.

5.3 Estimation Procedures: Let $D(X)$ denote the set of probability distributions on X. Given a probability distribution π, for each i there is a function $\delta^i(\pi,\cdot):Y^i \to D(X)$ defined by setting $\delta^i(\pi,y^i)$ equal to the conditional distribution on X given y^i. For each i, let $D(X)^{Y^i}$ denote the set of all functions on Y^i to $D(X)$. An estimation procedure is a sequence of functions $e_t^i:\Pi_{s=1}^t(Y^i \times X) \to D(X)^{Y^i}$ for each $t \geq 1$. With each π is associated the distribution on

$\Pi_{t=1}^{\infty}(Y\times X)$ such that for each $s,t \geq 1$, (y_s,x_s) and (y_t,x_t) are distributed independently and identically according to π. A natural estimation procedure is given by the sample frequency. For each $t \geq 1$, each i, and each $y^i \in Y^i$, $x \in X$, let

$$(5.3.1) \quad e_t^i(\{y_s^i,x_s\}_{s=1}^t)(y^i)(x) = \#\{1 \leq s \leq t:(y_s^i,x_s) = (y^i,x)\}/\#\{1 \leq s \leq t:y_s^i = y^i\},$$

if the denominator is positive. If the denominator is zero, $e_t^i(\{y_s^i,x_s\}_{s=1}^t)(y^i)(\cdot)$ is arbitrary.

The law of the iterated logarithm [e.g., 7, p. 64] provides bounds on the rate of convergence of this estimation procedure. Given π, for each (y^i,x)

$$(5.3.2) \quad \limsup_{t\to\infty} \frac{|\#\{1 \leq s \leq t:(y_s^i,x_s) = (y^i,x)\} - t\pi(y^i,x)|}{\sigma\sqrt{t\log\log t}} = 1 \quad \text{a.s.},$$

where $\sigma = \pi(y^i,x)[1 - \pi(y^i,x)]$. It follows that for each (y^i,x) there exists a constant $K > 0$ such that

$$(5.3.3) \quad \limsup_{t\to\infty} \frac{|e_t^i(\{y_s^i,x_s\}_{s=1}^t)(y^i)(x) - \delta^i(\pi,y^i)(x)|}{K[(\log\log t)/t]^{1/2}} < 1 \quad \text{a.s.}$$

5.4 Remarks: The i^{th} agent needs to estimate $\delta^i(\pi,\cdot)$ only in order to determine his correct conditional expected utility. Moreover, since allocation processes are assumed to depend only on preferences, only the conditional preferences need to be estimated. For this reason, the conditional frequency (5.3.1) may not be an appropriate estimation procedure.

By way of illustration, suppose that A^i and X are two element sets $\{a_1^i, a_2^i\}$ and $\{x_1, x_2\}$ respectively and v^i is such that $v^i(a_1^i, x_1) > v^i(a_2^i, x_1)$, $v^i(a_1^i, x_2) < v^i(a_2^i, x_2)$, and $(\sqrt{2}/2)v^i(a_1^i, x_1) + (1 - \sqrt{2}/2)v^i(a_1^i, x_2) = (\sqrt{2}/2)v^i(a_2^i, x_1) + (1 - \sqrt{2}/2)v^i(a_2^i, x_2)$. Thus, for a given y^i, if the conditional probability of x_1 is, respectively, greater than, equal to, or less than $\sqrt{2}/2$, then a_1^i is respectively better than, indifferent to, or worse than a_2^i. Given y^i, let $\hat{\delta}_t$ denote the conditional frequency of x_1 after t realizations. If $\delta^i(\pi, y^i) = \sqrt{2}/2$, $\hat{\delta}_t$ will converge to $\sqrt{2}/2$, but since $\hat{\delta}_t$ is always a rational number, the preferences determined by $\hat{\delta}_t$ will never be the correct preferences. However, a simple modification of $\hat{\delta}_t$ removes this difficulty.

For each t, let $\varepsilon_t = [(\text{loglog}t/t)]^{1/3}$ and define

$$\delta_t^* = \begin{cases} \sqrt{2}/2 \text{ if } |\sqrt{2}/2 - \hat{\delta}_t| \leq \varepsilon_t, \text{ and} \\ \hat{\delta}_t \quad \text{otherwise.} \end{cases}$$

The law of the iterated logarithm implies that for almost every infinite sample there is some T such that for t > T, $|\delta^i(\pi, y^i) - \hat{\delta}_t| < \varepsilon_t$. If $\delta^i(\pi, y^i) \neq \sqrt{2}/2$, let $T' = \min\{t: |\delta^i(\pi, y^i) - \sqrt{2/2}| > 2\varepsilon_t\}$, and if $\delta^i(\pi, y^i) = \sqrt{2}/2$, let T' = 0. For each π and almost every sample, the preferences determined by δ_t^* will be the correct preferences for all

$t > \max\{T,T'\}$. Hence, for large t, δ_t^* is a much better estimator than $\hat{\delta}_t$.

This estimation procedure is generalized in section 5.5, and Proposition 5.6 gives the general result.

5.5 Definitions: For each i, let $P(A^i)$ denote the set of preferences on A^i, with generic element p. The strict preference relation determined by p will be denoted $\succ p$. For each i, each v^i, and each $d \in D(X)$, let $\phi^i(d,v^i) \in P(A^i)$ denote the preference relation determined by the expected utility function $a^i \mapsto \sum_{x \in X} v^i(a^i,x)d(x)$. Let $n(d,v^i)$ denote the number of elements in the maximal subset of A^i which is completely ordered by $\succ p$, where $p = \phi^i(d,v^i)$.

We now define an estimation procedure for each agent. For each t, let $\varepsilon_t = [(\log\log t)/t]^{1/3}$ and for each $d,d' \in D(X)$, let $\|d,d'\| = \max\{|d(x) - d'(x)|: x \in X\}$. For each i, each v^i, each $t \geq 1$, and each $d \in P(A^i)$, let $h_t^i(d,v^i) = d' \in D(X)$ for some d' with $\|d',d\| \leq \varepsilon_t$ and for any d'' with $\|d,d''\| \leq \varepsilon_t$, $n(d'',v^i) \geq n(d',v^i)$. That is, $h_t^i(d,v^i)$ minimizes $n(d',v^i)$ subject to $\|d,d'\| \leq \varepsilon_t$. For each i, given v^i, let e_t^i be defined by (5.3.1) above, and define a new estimation procedure by

$$e_t^{*i}(\{y_s^i,x_s\}_{s=1}^t)(y^i)(\cdot) = h_t^i[e_t^i(\{y_s^i,x_s\}_{s=1}^t)(y^i)(\cdot),v^i]. \tag{5.5.1}$$

5.6 Proposition: Let (π,v) be a stochastic environment. For almost every $(y_t,x_t)_{t=1}^\infty \in \Pi_{t=1}^\infty(Y\times X)$ there is an integer $T > 0$ such that for each i and each $y^i \in Y^i$, $\phi^i(e_t^{*i}(\{y_s^i,x_s\}_{s=1}^t)(y^i)(\cdot),v^i) = \phi^i(\bar{\delta}^i(\pi,y^i),v^i)$

for every $t > T$.

Proof: Given i and $y^i \in Y^i$, let $d^* = \delta^i(\pi, y^i)$ and let $p^* = \phi^i(d^*, v^i)$. The definition of ϕ^i implies the existence of some $\varepsilon^\circ > 0$ such that for any $d' \in D(X)$ with $\|d^*, d'\| < \varepsilon^\circ$, and any $a^i, a'^i \in Y^i$ with $a^i \succ_{p^*} a'^i$, we have $a^i \succ_{p'} a'^i$, where $p' = \phi^i(d', v^i)$. It follows that d^* minimizes $n(d', v^i)$ subject to $\|d^*, d'\| < \varepsilon^\circ$, and that for any d' with $\|d^*, d'\| < \varepsilon^\circ$ and $n(d', v^i) = n(d^*, v^i)$, $\phi^i(d', v^i) = \phi^i(d^*, v^i)$. Equation (5.3.3) implies that for a.e. $\{y_t^i, x_t\}_{t=1}^\infty$ there is some $T(i, y^i) > 0$ such that for each $t > T(i, y^i)$

$$\begin{aligned} |e_t^i(\{y_s^i, x_s\}_{s=1}^t)(y^i)(\cdot) - d^*| &< 2K[(\mathrm{loglog}t)/t]^{1/2} \\ &< [(\mathrm{loglog}t)/t]^{1/3} = \varepsilon_t < \varepsilon^\circ \end{aligned}$$

and thus, by (5.5.1), $\phi^i(e_t^{*i}(\{y_s^i, x_s\}_{s=1}^t)(y^i)(\cdot), v^i) = \phi^i(d^*, v^i)$. Setting $T = \max\{T(i, y^i) : 1 \leq i \leq N,\ y^i \in Y^i\}$ completes the proof.

5.7 Remarks: Given a message process (μ, M) and agents' estimated conditional expectations, for each sample t there is a first stage reduced form $f_{1(t-1)} : (y_t^1, \ldots, y_t^N) \mapsto m_{1t}$.[1/] The second problem facing the i^{th} agent is the estimation of the conditional distribution of x given y_t^i and m_{1t}. It is this estimate which will determine the second stage reduced form $f_{2(t-1)}$. This estimation problem is complicated by the fact that if agents' estimated conditional

[1] The reduced form which determines m_{1t} is denoted $f_{1(t-1)}$ rather than f_{1t} since the conditional distributions which determine this reduced form are estimated using the previous samples.

distributions of x given y^i change as more data is collected, the reduced forms $f_{1(t-1)}$ will differ over time, so the <u>true</u> conditional distributions of x given y_t^i and m_{1t} will differ over time. However, Proposition 5.6 states that for large t, the estimated conditional preferences are the same, so the conditional distributions given y_t^i and m_{1t} will be unchanged for large t. Hence this estimation problem is analogous to the initial problem, and can be solved in the same way. Note that this would not be the case if agents estimated $\delta^i(\pi,\cdot)$ using the conditional frequencies.

These remarks are formalized below.

<u>5.8 Reduced Forms</u>: Given a message process (μ,M) and an environment (π,v), define the reduced form $f_{1t}:(Y\times X)^t \times Y \to M$ for each $t \geq 1$ by

$$f_{1t}(\{y_s,x_s\}_{s=1}^t,y) = \mu([\phi^i(e_t^{*i}(\{y_s^i,x_s\}_{s=1}^t)(y^i)(\cdot),v^i)]_{i=1}^N).$$

Note that μ is being applied to preference profiles rather than utility N-tuples, but this abuse of notation is justified by the assumption in definition 2.1 above that μ depends only on preference profiles.

Proposition 5.6 implies that for almost every $(y_t,x_t)_{t=1}^\infty \in \Pi_{t=1}^\infty(Y\times X)$ there exists some T such that for all $t > T$

$$(5.8.1)\quad f_{1t}((y_s,x_s)_{s=1}^t,y) = \mu([\phi^i(\delta^i(\pi,y^i),v^i)]_{i=1}^N)$$

for each $y \in Y$. Let π_3 denote the probability distribution on $Y \times M \times X$ induced by the distribution π on $Y \times X$ and the function

$(y,x) \mapsto (y,\mu([\phi^i(\delta^i(\pi,y^i),v^i)]_{i=1}^N),x)$. For each i, let $\delta_1^i(\pi_3,\cdot)$: $Y^i \times M \to D(X)$ be defined by setting $\delta_1^i(\pi_3,y^i,m)$ equal to the conditional distribution on X given y^i and m.

For each $t \geq 1$ and each $(y_s,x_s)_{s=1}^t \in \Pi_{s=1}^t(Y\times X)$ let $\pi_{3t}((y_s,x_s)_{s=1}^t)$ denote the distribution on $Y \times M \times X$ induced by π and the function $(y,x) \to (y,f_{1t}((y_s,x_s)_{s=1}^t, \ ),x)$. For each i let $\delta_{1t}^i(\pi_{3t}((y_s,x_s)_{s=1}^t),\cdot):Y^i \times M \to D(X)$ be defined in the same way as δ_1^i with π_{3t} in place of π_3. It follows for almost every $(y_t,x_t)_{t=1}^\infty$ there is some T such that $\pi_{3t}((y_s,x_s)_{s=1}^t) = \pi_3$ for all $t > T$, and thus $\delta_{1t}^i(\pi_{3t}((y_s,x_s)_{s=1}^t),\cdot) = \delta_1^i(\pi_3,\cdot)$ and $\phi^i[\delta_{1t}^i(\pi_{3t}((y_s,x_s)_{s=1}^t), y^i,m),v^i] = \phi^i(\delta_1^i(\pi_3,y^i,m),v^i)$ for all $t > T$ and all (y^i,m). Thus the estimation of $\phi^i(\delta_{1t}^i(\pi_{3t}((y_s,x_s)_{s=1}^t),\cdot),v^i)$ for large t reduces to the estimation of $\phi^i(\delta_1^i(\pi_3,\cdot),v^i)$.

For each i, let $D(X)^{Y^i\times M}$ denote the set of functions on $Y^i \times M$ to D(X) and for each $t \geq 1$, let $e_{1t}^i:\Pi_{s=1}^t(Y^i\times M\times X) \to D(X)^{Y^i\times M}$ be defined by

$$(5.8.2) \quad e_{1t}^i((y_s^i,m_s,x_s)_{s=1}^t)(y^i,m)(x) =$$
$$\#\{1 \leq s \leq t:(y_s^i,m_s,x_s) = (y^i,m,x)\}/\#\{1 \leq s \leq t:(y_s^i,m_s) = (y^i,m)\}$$

if the denominator is positive. If the denominator is zero, $e_{1t}^i((y_s^i,m_s,x_s)_{s=1}^t)(y^i,m)(\cdot)$ is arbitrary. Given the distribution on $\Pi_{s=1}^\infty(Y\times X)$ associated with π, a distribution on $\Pi_{s=1}^\infty(Y\times M\times X)$ is induced by the functions $(y_s,x_s)_{s=1}^t \mapsto (y_t,f_{1(t-1)}((y_s,x_s)_{s=1}^{t-1},y_t),x_t)$ for each $t \geq 2$, and setting m_1 equal to some constant m_1^o with probability one.

By (5.8.1) and the law of the iterated logarithm, for each i and each (y^i,m,x) there is some constant $K > 0$ such that for almost every $(y^i_s,m_s,x_s)^{\infty}_{s=1}$,

$$(5.8.3)\quad \limsup_{t\to\infty} \frac{\left|e^i_{1t}((y^i_s,m_s,x_s)^t_{s=1})(y^i,m)(x) - \delta^i_1(\pi_3,y^i,m)(x)\right|}{K[(\log\log t)/t]^{1/2}} < 1$$

Thus (5.8.3) is exactly analogous to (5.3.3) and the estimation of $\phi^i(\delta^i_1(\pi_3,\cdot),v^i)$ is analogous to the estimation of $\phi^i(\delta^i(\pi,\cdot),v^i)$.

5.9 Remarks: The estimated conditional distribution of x given y^i_t and m_{1t} for each i determines the second stage reduced form $f_{2(t-1)}:(y^1_t,\ldots,y^N_t) \mapsto m_{2t}$, and so on. The estimation of the conditional distribution of x given $(y^i_t;m_{1t},\ldots,m_{jt})$ can be accomplished for each j by extending the above procedure in the obvious way. However, since the estimated conditionals differ across periods, the terminal iteration J_t may also differ. Moreover, there is no assurance that J_t will be finite in all periods. The finiteness of J was ensured in section 4 by the fact that expectations were adjusted by conditioning on successively finer partitions of a finite set. In the present context, for large t, each agent i will have the correct conditional preferences given $(y^i,m_1,\ldots,m_j)$ for each j, so preferences will adjust between iterations by, in effect, conditioning on successively finer partitions of the finite set Y. Hence J_t will be finite for large t. However, in general we have not imposed enough consistency on the estimated conditional distribution given $(y_t^i, m_{1t}, \ldots, m_{jt})$ and the estimated conditional distribution given $(y_t^i, m_{1t}, \ldots, m_{(j+1)t})$ to ensure that the iterative process will not continue indefinitely. This could be avoided by requiring that if an iteration j is greather than the final iteration in any previous period, agents do not adjust their expectations, so that this iteration is the final one. Since the conditional frequency distribution is arbitrary if an iteration has never been experienced, this is consistent with the estimation procedure we have been using. Under this restriction, $J_t \leq t$ for each t.

The estimation procedure described here works for agent i because it is also used by other agents. For example, if some agent were to use conditional frequencies, the reduced forms f_{1t} may differ across time even for large t, so the estimated conditional distributions given y_t^i and m_{1t} need not converge to be true conditionals. Given an allocation mechanism, one agent may prefer an estimation procedure which causes the procedure described here to fail to converge for other agents. All we have shown here is that there exists a simple procedure which enables agents to acquire the knowledge required by the dynamic process described in section 4. The comparison of different estimation procedures has not been attempted.

6. Nonstationary Stochastic Environments

This section is devoted to stochastic environments with a time horizon of more than one period. In period t , the allocation mechanism determines an allocation a_t based on agents' expected utility of t^{th} period allocations. However, at this level of abstraction, an agent's t^{th} period expected utility function is difficult to define. In general, a_t^i influences the i^{th} agent's utility of future allocations, and, through the allocation mechanism, influences the allocations he receives in the future. The definition of an agent's t^{th} period expected utility is affected by the extent to which the agent recognizes this dependence. In models of competitive equilibrium over time, agents believe that their current trades have no influence on future prices but that they can adjust future trades at a given price. This dichotomy cannot be expressed for general allocation processes.

There is one particularly simple class of environments for which this complication does not arise. Suppose that agents have a two period time horizon and state dependent utility functions of the form $v_1^i(a_1^i, v_2^i(a_2^i,x))$, with $v_2^i \in V^i$, and $v_1^i(a_1^i, r)$ a function of the form $\alpha^i(a_1^i)r + \beta^i(a_1^i)$, with $\alpha^i(a_1^i)$, $\beta^i(a_1^i) \in R$ and $\alpha^i(a_1^i) > 0$. Then an agent's second period expected preferences will be represented by the expectation of $v_2^i(\cdot, x)$ regardless of the first period allocation.

6.1 Definitions: Let S_2 denote the set of stochastic environments (π, v) with $v^i\colon (A^i)^2 \times X \to R$ for each i such that for some $v_2^i \in V^i$ and some function $v_1^i\colon A^i \times R \to R$ with $v_1^i(a_1^i, \cdot)$ a strictly increasing affine function for each $a^i \in A^i$, $v^i(a_1^i, a_2^i, x) = v_1^i(a_1^i, v_2^i(a_2^i, x))$ for each (a_1^i, a_2^i, x).

6.2 Remarks: The purpose of introducing nonstationary environments is to define a concept of expectations equilibrium with expectations conditioned on past messages. The equilibrium concept defined in section 3 involves the simultaneous determination of agents' expectations and the data on which they are conditioned. Current expectations influence and are influenced by current messages. In contrast, econometric applications of the rational expectations hypothesis commonly exclude current endogenous variables from the information sets which determine the expectations which directly influence current endogenous variables (e.g. Shiller [29]). Hellwig and Rothschild [12], in the context of a securities market model, have used this restriction, along with the assumption of constant absolute risk aversion and a normality assumption on the distribution of the state, to derive a rational expectations equilibrium with current expectations of future returns conditioned on past prices.

Suppose that in period one, agents condition their expectations on their exogenous information, leading to a first period reduced form f_1. In period 2, agents condition their expectations on exogenous information and the information revealed by the first period message. This process differs from the dynamic process described in section 4 in that the time periods represent real time rather than adjustment time, so the state dependent utility functions are not identical in different periods. In particular, an agent's first period expected utility is influenced by his second period allocation, which is viewed from the first period as a random variable. For notational simplicity, we will suppose that agents do not receive additional exogenous information at the beginning of period 2.

6.3 Definitions: Given an allocation mechanism (μ, M, g), and a stochastic environment $(\pi, v) \in S_2$, and a second period reduced form f_2: $Y \to M$, define the first period expected utility function $w_1^i(f_2)$: $Y \to U^i$ by

$$w_1^i(f_2)(y)(a^i) = \sum_{\{(y', x):\ y'^i = y^i\}} v_1^i(a^i, v_2^i(a_2^i(y'), x))\pi(y', x) \Big/ \sum_{\{\ \}} \pi(y', x)$$

for each y and each a^i, where $a_2^i(y')$ is the i^{th} coordinate of $g(f_2(y'))$. Given a first period reduced form f_1: $Y \to M$ we can define the second period expected utility function $w_2^i(f_1)$: $Y \to U^i$ by

$$w_2^i(f_1)(y)(a^i) = \sum_{\{(y', x):\ y'^i = y^i \text{ and } f_2(y') = f_1(y)\}} v_2^i(a^i, x)\pi(y', x) \Big/ \sum_{\{\ \}} \pi(y', x),$$

for each y and each a^i. Let $w_1(f_2) = (w_1^i(f_2))_i$ and $w_2(f_1) = (w_2^i(f_1))_i$.

An <u>equilibrium with past data</u> is a pair of reduced forms (f_1, f_2) satisfying

1) $f_1(y) = [w_1(f_2)(y)]$ for each y; and

2) $f_2(y) = [w_2(f_1)(y)]$ for each y.

An allocation mechansim is <u>admissible for past data</u> if it admits an equilibrium with past data for every stochastic environment in S_2.

<u>6.4 Remarks</u>: Since the outcome function g influences first period expected utility functions, admissibility for past data depends on the outcome function as well as the message process. Proposition 6.5 shows that admissibility for past data essentially requires that each agent determine his own allocation independently.

<u>6.5 Proposition</u>: An allocation mechanism is admissible for past data only if it is unilateral on E^o.

<u>Proof</u>: Let (μ, M, g) be an allocation mechanism which is admissible for past data. Let I be minimal for (μ, M, g) and suppose by way of contradiction that (μ, M, g) is not unilateral on $E°$. Then there is some $u_c \in E°$ and some $i, j \in I$ with $i \neq j$, say $i = 1$, and $j = 2$, and some $\bar{u}^1 \in U^1$ such that if $a_c = g[\mu(u_c)]$ and

$\bar{a}_c = g[\mu(\bar{u}^1, u_c^2, \ldots, u_c^N)]$ then $a_c^2 \neq \bar{a}_c^2$. Let $u_d^1 = 2\bar{u}^1 - u_c^1$, and let $u_d^2 \in U^2$ with $u_d^2(a^2) \neq u_c^2(\bar{a}_c^2)$ for all $a^2 \in A^2$. Let $u_d^i = u_c^i$ for each $i > 2$. Since I is minimal, there is some $u_1 \in E$ with $u_1^2 \in U^{\circ 2}$, and some $\bar{u}^2 \in U^2$ such that $\mu(u_1) \neq \mu(u_1^1, \bar{u}^2, u_1^3, \ldots, u_1^N)$.

We can now construct a stochastic environment S_2 which has no equilibrium with past data. Let $x_a, x_b \in X$, and for each $i \neq 2$, suppose that for each $(a_1^i, a_2^i, x) \in (A^i)^2 \times X$, $v^i(a_1^i, a_2^i, x) = u_1^i(a_1^i) + v_2^i(a_2^i, x)$ where $v_2^i(\cdot, x_c) = u_c^i$, and $v_2^i(\cdot, x_d) = u_d^i$. Suppose that for each $(a_1^2, a_2^2, x) \in (A^2)^2 \times X$, $v^2(a_1^2, a_2^2, x) = v_1^2[a_1^2, v_2^2(a_2^2, x)]$ where $v_2^2(\cdot, x_c) = u_c^2$, $v_2^2(\cdot, x_d) = u_d^2$, $v_1^2(\cdot, u_c^2(a_c^2)) = u_1^2$, $v_1^2(\cdot, u_c^2(\bar{a}_c^2)) = \bar{u}^2$, and $v_1^2(\cdot, u_d^2(a^2)) = u_1^2$ for all $a^2 \in A^2$. It is clear that for each $i \neq 2$, v^i satisfies 6.1, and it is shown in an appendix to this paper that v^2 can be chosen to satisfy 6.1 also. Let $y_o^1 \in Y^1$ and for each $i > 1$, let $y_c^i, y_d^i \in Y^i$, and $y_c = (y_o^1, y_c^2, \ldots, y_c^N)$, $y_d = (y_o^1, y_d^2, \ldots, y_d^N)$. Define π by $\pi(y_c, x_c) = \pi(y_d, x_d) = 1/2$. Suppose that (π, v) has an equilibrium with past data (f_1, f_2). First suppose that $f_1(y_c) \neq f_1(y_d)$. Then for each i, $w_2^i(f_1)(y_c) \sim u_c^i$ and

$w_2^i(f_1)(y_d) \sim u_d^i$, where "$u^i \sim u'^i$" means that the two functions represent the same preference relation on A^i. Therefore $g(f_2(y_c)) = a_c$. Hence for each i, $w_1^i(f_2)(y_c) \sim w_1^i(f_2)(y_d) \sim u_1^i$, so $f_1(y_c) = f_1(y_d) = \mu(u_1)$, which contradicts our supposition that $f_1(y_c) \neq f_1(y_d)$. If $f_1(y_c) = f_1(y_d)$, then $w_2^1(f_1)(y_c) \sim w_2^1(f_1)(y_d) \sim \bar{u}^1$, and for each $i > 1$, $w_2^i(f_1)(y_c) \sim u_c^i$ and $w_2^i(f_1)(y_d) \sim u_d^i$. Hence $g(f_2(y_c)) = \bar{a}_c$. Therefore $w_1^i(f_2)(y_c) \sim w_1^i(f_2)(y_d) \sim u_1^i$ for each $i \neq 2$, but $w_1^2(f_2)(y_c) \sim \bar{u}^2$ and $w_1^2(f_2)(y_d) \sim u_1^2$, so $f_1(y_c) = \mu(u_1^1, \bar{u}^2, u_1^3, \ldots, u_1^N) \neq \mu(u_1) = f_1(y_d)$. Thus (π, v) has no equilibrium with past data, which completes the proof.

6.6 <u>Remarks</u>: Hence admissibility for past data essentially prohibits any economic interaction between agents, at least in strict preference environments.

Unlike Proposition 3.6, the above necessary condition involves the outcome function g. In fact, if g is a constant function, an allocation process is easily seen to be admissible for past data irrespective of the message process (μ, M). As this example indicates, (μ, M, g) can be unilateral on E° without (μ, M) being decentralized on E°. However, if (μ, M, g) is unilateral on E° and the set of all agents is minimal, let $\mu' = g \cdot \mu$, and let g' be the identity function on A. Then (μ', A) is decentralized on E^O and $g' \cdot \mu' = g \cdot \mu$.

Since the property of being unilateral on E° depends on g, Proposition 6.5 cannot be directly applied to the case in which agents observe some statistic computed from past messages. However, [5] presents a stochastic exchange model in which the i^{th} agent observes a statistic $h^i(m)$, where m is the past competitive message. Under the assumption that h^i is continuous, it is shown that admissibility requires h^i to be constant. In the model presented in [5], utility functions are assumed to be additively separable over time, but one of the commodities exchanged in each period is durable. This causes decisions to be sufficiently intertemporally dependent to create counterexamples similar to the one constructed in the proof of 6.5.

Finally, suppose that the equilibrium concept studied in this section is modified so that in each period, agents augment their initial information with the information revealed by the current and past period reduced forms. With this definition of expectations equilibrium, Propositions 3.4 and 3.6 are easily extended to nonstationary environments. Thus the restrictiveness of Proposition 6.5 is due not to the longer time horizon but to the exclusion of current messages from agents' endogenous information.

Appendix

The construction of v_1^2 in the proof of **6.5** can be refined to **ensure that 6.1 is satisfied.** Since $u_c \in E^o$, $u_c^2(a_c^2) \neq u_c^2(\bar{a}_c^2)$. Suppose that $u_c^2(a_c^2) < u_c^2(\bar{a}_c^2)$. Let $r_c = u_c^2(a_c^2)$, let $\bar{r}_c = u_c^2(\bar{a}_c^2)$, and, adding a negative constant to u_d^2 if necessary, suppose that $u_d^2(a^2) < u_c^2(a_c^2)$ for all $a_c^2 \varepsilon A^2$. We can also assume that $\bar{u}^2(a^2) > 0$ and $u_1^2(a^2) < 0$ for all $a^2 \varepsilon A^2$. If $\bar{u}^2$ is not constant, define v_1^2: $A^2 \times R \to R$ by

$$v_1^2(a^2,r) = \left(\frac{r-r_c}{\bar{r}_c-r_c}\right)\left(\frac{H_1K_1}{H_2K_2}\right) \bar{u}^2(a^2) + \left(\frac{\bar{r}_c-r}{\bar{r}_c-r_c}\right) u_1^2(a^2) ,$$

where

$$H_1 = \min \{|\bar{r}_c-r|: \ r \ \varepsilon \ [u_c^2(A^2) \cap (-\infty,\bar{r}_c)] \cup u_d^2(A^2)\} ,$$

$$H_2 = \max \{|r-r \ |: \ r \ \varepsilon \ [u_c^2(A^2) \cap (-\infty,\bar{r}_c)] \cup u_d^2(A^2) \ \} ,$$

$$K_1 = \min \{|u_1^2(a^2) - u_1^2(a'^2)|: \ a^2 \neq a'^2 \ \varepsilon \ A^2\} , \text{ and}$$

$$K_2 = \max \{|\bar{u}^2(a^2)-\bar{u}^2(a'^2)|: \ a^2, \ a'^2 \ \varepsilon \ A^2\} .$$

Then H_1 and H_2 are clearly positive, K_1 is positive since $u_1^2 \ \varepsilon \ U^{\circ 2}$ and K_2 is positive if $\bar{u}^2$ is not constant. If $\bar{u}^2$ is constant, define v_1^2: $A^2 \times R \to R$ by

$$v_1^2(a^2,r) = (\bar{r}_c-r)u_1^2(a^2) .$$

In either case, the resulting utility function v^2 **satisfies 6.1 as well as the other properties asserted in the proof of 6.5.** The case in which $u_c^2(a_c^2) > u_c^2(\bar{a}_c^2)$ can be treated symmetrically.

7. Markov Environments

The previous section shows that if expectations are conditioned on past data, the existence of equilibrium cannot be ensured without restrictions on the intertemporal dependence of utilities. A natural question is whether or not restrictions placed on the evolution of the environment over time will be inherited by the expectations equilibria. For example, in sections 3-5 we supposed that the pair (y, x) was independently and identically distributed over time. Since expectations equilibrium messages m are generated as a function of y, the triple (y, m, x) is also independently and identically distributed. Of course the independence assumption makes past data uninformative, so we cannot study expectations conditioned on past data in this context. Alternatively, suppose that the state evolves as a stationary Markov chain and that each agent's utility of an allocation made in period t depends on x_{t+1}, but is independent of past and future allocations. If agent's expectations are conditioned on private information and past messages, will the resulting state-message stochastic process be Markovian? If in period t, agents know x_t, then past messages provide no additional information about x_{t+1}, so the question needs to be more carefully stated. Suppose instead that the state x_t is a vector $(x_t^i)_{i=1}^N$, and that the i^{th} agent's utility of an allocation made in period t depends only on x_{t+1}^i. Suppose that the i^{th} agent's private information in period t consists of the observations $\{x_s^i\}_{s \leq t}$, and that each agent's expected utility is also conditioned on past messages. Since previous messages will generally not reveal the previous states completely, an agent's expectations in period t will be conditioned on a partial observation of past states $\{x_s\}_{s \leq t}$. As Michael Rothschild has noted in [28], partially observed Markov chains need not be Markovian. In particular the i^{th} agent's expectations about x_t^i may depend on the entire past sequence $\{x_s^i, m_{s-1}\}_{s \leq t}$. Hence the message m_t may depend on the entire past sequence $\{x_s, m_{s-1}\}_{s \leq t}$, so we cannot expect the Markov property of $\{x_t\}$ to be inherited by $\{x_t, m_t\}$. Additional impediments to the derivation of Markovian expectations equilibria for Markov environments have been observed by L. Blume [3], D. Easley [8], and M. Hellwig [11].

Although a partially observed stationary Markov chain need not be Markovian, it is stationary. For this reason one might hope that even though the dependence of current messages on previous states and messages cannot be limited to the immediate past, the joint distribution of states and messages might be stationary. Proposition 7.6 states that this property can be generally obtained only for dictatorial message processes. We now define the class of Markov environments which we have been discussing informally.

<u>7.1 Definitions</u>: For each i, let X^i be an abstract set containing at least two elements, and let $X = \Pi_i X^i$. For each i, let V^i denote the set of state-dependent utility functions v^i: $A^i \times X^i \to R$. For each i and each integer t, let $X_t^i = X^i$ and $X_t = X$. Let $X^\infty = \Pi_{t=0}^\infty X_t$, and for each i and each t define the projections z_t^i: $X^\infty \to X_t^i$ by $z_t^i(x^\infty) = x_t^i$ and z_t: $X^\infty \to X_t$ by $z_t(x^\infty) = x_t = (x_t^i)_i$. Let P denote the set of probability measures on X with finite support. A <u>Markov environment</u> is a triple (π, p, v), where

$v \in \Pi_i V^i$, $\pi \in P$, and $p\colon X \to P$ is a stationary transition probability such that $\{z_t\}_{t=0}^{\infty}$ is an indecomposable Markov chain with π as its unique stationary initial distribution.

Given a message process (μ, M), for each integer t let $M_t = M$, and let $M^{\infty} = \Pi_{t=0}^{\infty} M_t$. For each integer t define the projection $\tilde{m}_t\colon X^{\infty} \times M^{\infty} \to M_t$ by $\tilde{m}_t(x^{\infty}, m^{\infty}) = m_t$. The projections z_t^i and z_t for each i, t will sometimes be written as projections on $X^{\infty} \times M^{\infty}$ to X_t^i and X_t respectively.

<u>7.2 Reduced Forms</u>: Given a message process (μ, M), a <u>reduced form</u> is a sequence of functions $f_t\colon X^{\infty} \times M^{\infty} \to M_t$ such that for each integer t and each (x^{∞}, m^{∞}), $(x'^{\infty}, m'^{\infty}) \in X^{\infty} \times M^{\infty}$ with $x_s = x'_s$ for each $s \leq t$ and $m_s = m'_s$ for each $s < t$ then $f_t(x^{\infty}, m^{\infty}) = f_t(x'^{\infty}, m'^{\infty})$. A reduced form $(f_t)_{t=0}^{\infty}$ is denoted f^{∞}.

Given a stochastic environment (π, p, v) and a reduced form f^{∞}, a probability distribution π^* on $X^{\infty} \times M^{\infty}$ is said to be <u>consistent with</u> π, p, and f^{∞} if the marginal distribution on X^{∞} agrees with π and p, and for each t, the conditional probabilities satisfy

$$\pi^*(\tilde{m}_t = m_t \,|\, (z_s)_{s \leq t}, (\tilde{m}_s)_{s < t})(x^{\infty}, m^{\infty}) =$$

$$= \begin{cases} 1 & \text{if } m_t = f_t(x^{\infty}, m^{\infty}) \\ 0 & \text{otherwise.} \end{cases}$$

7.3 Expected Utility: Given a Markov environment (π, p, v), a reduced form f^∞ and a distribution π^* which is consistent with π, p, and f^∞, define the expected utility function $w_t^i(\pi^*, v)\colon X^\infty \times M^\infty \to U^i$ by

$$w_t^i(\pi^*, v)(x^\infty, m^\infty)(a^i) = E\{v^i(a^i, z_{t+1}^i(\cdot)) | (z_s^i(\cdot))_{s \leq t},$$

$$(\tilde{m}_s(\cdot))_{s \leq t-1}\}(x^\infty, m^\infty)$$

for each i, t. For each t let $w_t(\pi^*, v) = (w_t^i(\pi^*, v))_i$.

7.4 Expectations Equilibrium: An expectations equilibrium for a Markov environment (π, p, v) and a message process (μ, M) is a reduced form f^∞ and a probability distribution π^* consistent with π, p and f^∞ such that for each t

$$(*) \qquad f_t(x^\infty, m^\infty) = \mu[w_t(\pi^*, v)(x^\infty, m^\infty)] \quad \text{a.e.}$$

The equilibrium is said to be stationary if π^* is stationary.

7.5 Remarks: Since current utilities are independent of past and future allocations, the definition of equilibrium does not involve allocations directly. The general existence of expectations equilibrium for any message process and any Markov environment is immediate, since current expectations are independent of current and future messages. However, unless the message process is dictatorial (Definition 2.12), the underlying information structure will change over time, changing the joint distribution of states and messages. If the message process is dictatorial, the only agents who learn from the messages have no influence on π^*.

7.6 Proposition: A message process admits a stationary equilibrium for every Markov environment if and only if it is dictatorial.

Proof: Sufficiency is straightforward so we will only prove necessity. Let (μ, M) be any nondictatorial message process. We will construct a Markov environment which has no stationary equilibrium. For notational convenience, we will temporarily assume that $N = 2$. Let $x_a^1, x_b^1 \in X^1$ and $x_a^2, x_b^2 \in X^2$ and define the transition probability p by the following matrix, where, for example, the number in row ab and column ba is $p(x_b^1, x_a^2 | x_a^1, x_b^2)$.

	aa	ab	ba	bb
aa	1/4	1/4	1/2	0
ab	1/2	0	0	1/2
ba	1/2	0	1/4	1/4
bb	0	0	7/8	1/8

This transion matrix yields the unique stationary initial distribution:

$\pi(x_a^1, x_a^2) = 1/3, \quad \pi(x_a^1, x_b^2) = 1/12, \quad \pi(x_b^1, x_a^2) = 5/12, \quad \pi(x_b^1, x_b^2) = 1/6.$

We also record the following conditional probabilities determined by π and p:

1) prob $(z_1^1 = x_a^1 \mid z_0^1 = x_a^1) = 1/2$

2) prob $(z_1^1 = x_a^1 \mid z_0^1 = x_b^1) = 5/14$

3) prob $(z_1^2 = x_a^2 \mid z_0^2 = x_a^1) = 3/4$

4) prob $(z_1^2 = x_a^2 \mid z_0^2 = x_b^1) = 3/4$

5) prob $(z_2^1 = x_a^1 \mid z_1^1 = x_a^1,\ z_0^1 = x_a^1) = 1/2$

6) prob $(z_2^2 = x_a^2 \mid z_1^2 = x_b^2,\ z_0 = (x_a^1, x_a^2)) = 1/2.$

Since (μ, M) is nondictatorial, there are utility functions u^1, u^{o1}, u^2, and u^{o2} with $\mu(u^1, u^2) \neq \mu(u^{o1}, u^2)$ and $\mu(u^1, u^2) \neq \mu(u^1, u^{o2})$. Let v^1 satisfy $v^1(\cdot, x_a^1) = (9/2)u^1 - (7/2)u^{o1}$ and $v^1(\cdot, x_b^1) = (7/2)u^{o1} - (5/2)u^1$, and let v^2 satisfy $v^2(\cdot, x_a^2) = 2u^2 - u^{o2}$ and $v^2(\cdot, x_b^2) = 3u^{o2} - 2u^2$. We will now show that the Markov environment (π, p, v) has no stationary equilibrium.

Let (f^∞, π^*) be an expectations equilibrium for (π, p, v). Since the pair of expected utility functions $w_o^i(\pi^*, v)$ depends only on (π, p, v), the conditional probabilities given in (1-4) above yield

$$w_0(\pi^*, v)(x^\infty, m^\infty) = \begin{cases} (u^1, u^2) & \text{if } z_0(x^\infty) = (x_a^1, x_a^2) \\ (u^1, u^2) & \text{if } z_0(x^\infty) = (x_a^1, x_b^2) \\ (u^{o1}, u^2) & \text{if } z_0(x^\infty) = (x_b^1, x_a^2) \\ (u^{o1}, u^2) & \text{if } z_0(x^\infty) = (x_b^1, x_b^2)\ . \end{cases}$$

Therefore 7.4(*) implies that

$$\text{prob}\ (\{(x^\infty, m^\infty)\colon\ \tilde{m}_0(x^\infty, m^\infty) = \mu(u^1, u^2)\} | z_0(x^\infty) = (x_a^1, x_b^2)) = 1.$$

Since $\mu(u^1, u^2) \neq \mu(u^{o1}, u^2)$, $\tilde{m}_0$ reveals z_0^1 to agent 2, but does not reveal z_0^2 to agent 1. Hence it follows from the conditional probabilities given in (5-6) that if $z_1(x^\infty) = (x_a^1, x_b^2)$ and $z_0(x^\infty) = (x_a^1, x_a^2)$ then $w_1(\pi^*, v)(x^\infty, m^\infty) = (u^1, u^{o2})$, so in this case, $\tilde{m}_1(x^\infty, m^\infty) = \mu(u^1, u^{o2})$. Since $\mu(u^1, u^{o2}) \neq \mu(u^1, u^2)$, $\text{prob}(\{(x^\infty, m^\infty)\colon\ \tilde{m}_1(x^\infty, m^\infty) = \mu(u^1, u^2)\} | z_1(x^\infty) = (x_a^1, x_b^2)) < 1$ so the distribution of $(z_1, \tilde{m}_1)$ determined by π^* differs from the distribution of $(z_0, \tilde{m}_0)$. Hence π^* is not stationary. This completes the proof for the case $N = 2$.

In general, since μ is nondictatorial, there exists an agent i^* and utility functions u^i and u^{oi} for each i such that $\mu(u^1,\ldots,u^{i^*-1}, u^{i^*}, u^{i^*+1},\ldots, u^N) \neq \mu(u^1,\ldots,u^{i^*-1}, u^{oi^*}, u^{i^*+1},\ldots,u^N)$ and $\mu(u^1,\ldots,u^{i^*-1}, u^{i^*}, u^{i^*+1},\ldots,u^N) \neq \mu(u^{o1},\ldots,u^{oi^*-1}, u^{i^*}, u^{oi^*+1},\ldots,u^{oN})$

Renumbering agents if necessary, let $i^* = 1$. The two agent environment (π, p, v) constructed above can be extended to an N agent environment by treating agents $i > 2$ essentially as copies of agent 2. More precisely, for each $i > 2$, let $X^i = \{x_a^i, x_b^i\}$. Extend p by imposing that for each $i > 2$,

$$p(\{x\colon x^2 = x_a^2, x^i = x_b^i\} | x) = p(\{x\colon x^2 = x_b^2, x^i = x_a^i\} | x) = 0$$

for all x. For each $i > 2$, define $v^i(\cdot, x_a^i) = 2u^i - u^{oi}$ and

$v^i(\cdot, x_b^i) = 3u^{oi} - 2u^i$. With these definitions, the proof for the two agent case extends directly.

7.7 Remarks: As in section 6, the difficulty is caused by conditioning expectations on past messages alone. Suppose instead that expectations are conditioned on current as well as past messages. If the message process is convex, the proof of Proposition 3.6 can be imitated to obtain an equilibrium reduced form in period t by conditioning each agents expectations on the entire N-tuple of previous states x_t. Then m_t is determined as a function of x_t alone, the same function for each t, so $\{x_t, m_t\}$ is a stationary Markov process.

8. Public Predictions

Critics of the rational expectations hypothesis commonly argue that forecasting is a specialized activity, requiring sophisticated econometric techniques, so the hypothesis that economic agents have statistically correct expectations conditioned on all of the information available to them is unreasonable. This section is devoted to an alternative equilibrium concept which is intended to meet this objection. Suppose that the probability distribution of the future state is predicted by an independent forecasting agency rather than by individual agents. This prediction is then used by agents to determine their expected utility.

The forecasting agency observes only publicly available information, so the prediction will be conditioned only on the observed message. However, this implies that the i^{th} agent's expectations are not directly influenced by his private information y^i. Hence there is a trivial equilibrium in which the predicted probability distribution is the unconditional distribution on X, agents have the same expected utility for all realizations of y, and the equilibrium message is the same for all y. To make the model nontrivial we will suppose that the i^{th} agent's private information variable y^i is also an argument of his state dependent utility function v^i. Then if $y^i \neq y'^i$, the expected utility $Ev^i(a^i, y^i, \cdot)$ may differ from $Ev^i(a^i, y'^i, \cdot)$ even if the predicted probability distribution on X is the same in both cases.

It seems reasonable to suppose that an agent's source of information is a variable which directly influences his behavior even if he ignores its predictive content. For example, a farmer's decision to purchase new equipment will be influenced by the size of his recent harvests, but in assessing the costs of delaying the purchase, his estimate of the future price of farm equipment may be based on a published forecast which is too highly aggregated to be sensitive to regional harvest size. We now introduce this modification formally.

<u>8.1 Stochastic Environments:</u> For each i, let V^{oi} denote the set of utility functions v^{oi}: $A^i \times Y^i \times X \to R$, and let $V^o = \Pi_i V^{oi}$. For the purpose of this section a stochastic environment is a pair (π, v^o) with $v^o \in V^o$ and π a probability distribution on $Y \times X$ with finite support.

<u>8.2 Public Prediction Equilibrium:</u> Given a message process (μ, M) and a stochastic environment (π, v^o) we can define the i^{th} agent's expected utility function. For each i and each reduced form f: $Y \to M$, define $w^{oi}(f)$: $Y \to U^i$ by

$$w^{oi}(f)(y)(a^i) = \sum_{\{(y', x):\ f(y') = f(y)\}} v^{oi}(a^i, y^i, x)\pi(y', x) \Big/ \sum_{\{\ \}} \pi(y', x)$$

for each $y \in Y$ and each $a^i \in A^i$. Let $w^o(f) = (w^{oi}(f))_i$. A <u>public prediction equilibrium</u> is a reduced form f which satisfies

(*) $\quad f(y) = \mu[w^o(f)(y)]$ for each $y \in Y$.

<u>8.3 Remarks:</u> The definition of expected utility embodies the assumption that an agent's private information variable influences his utility

directly, but expectations are conditioned only on the publicly observable message. The following Proposition states that public prediction equilibria generally exist only in the trivial case in which μ is a constant function.

8.4 Proposition: A message process (μ, M) admits a public prediction equilibrium for every stochastic environment if and only if μ is constant.

Proof: Sufficiency is immediate. To prove necessity, suppose that (μ, M) admits a public prediction equilibrium for every stochastic environment and let $m, m' \varepsilon \mu(E)$. We will prove that $m = m'$. Let $u, u' \varepsilon E$ with $\mu(u) = m$ and $\mu(u') = m'$. Let $x_a, x_b \varepsilon X$ and for each i, let $y_a^i, y_b^i \varepsilon Y^i$ with $y_a^i \neq y_b^i$, and let $y_a = (y_a^i)_i$ and $y_b = (y_b^i)_i$. Define π by $\pi(y_a, x_a) = \pi(y_b, x_b) = 1/2$. Let the state dependent utility N-tuple v^o satisfy $v^o(\cdot, y_a, x_a) = v^o(\cdot, y_a, x_b) = v^o(\cdot, y_b, x_b) = u$ and $v^o(\cdot, y_b, x_a) = 2u' - u$. By supposition, (π, v^o) has a public prediction equilibrium f. If $f(y_a) \neq f(y_b)$, then $w^o(f)(y_a) = v^o(\cdot, y_a, x_a) = u$ and $w^o(f)(y_b) = v^o(\cdot, y_b, x_b) = u$, so $f(y_a) = f(y_b) = m$. This contradiction implies that $f(y_a) = f(y_b)$. Then $w^o(f)(y_a) = (1/2)v^o(\cdot, y_a, x_a) + (1/2)v^o(\cdot, y_a, x_b) = u$, and $w^o(f)(y_b) = (1/2)v^o(\cdot, y_b, x_a) + (1/2)v^o(\cdot, y_b, x_b) = u'$. Hence $f(y_a) = m = m' = f(y_b)$ which completes the proof.

8.5 Remarks: An analogous result is obtained in [16] for stochastic exchange environments with the competitive allocation mechansim. Each agent's private information variable is his endowment, and expectations are conditioned on a publicly available statistic. However, the statistic is permitted to depend on the endowments as well as on the allocations and prices. Theorem of [16] states that a public prediction equilibrium will generally exist only if the statistic can essentially be written as a function of the exogenous endowments alone. The conclusion of [16] and Proposition 8.4 seems to be that if agents rely on public predictions, the general existence of equilibrium requires the predictions to be independent of the message process.

9. Conclusion

The theory we have outlined raises several natural questions. We have assumed that each agent conditions his expectations on the entire message m, which represents all of the endogenous variables of the allocation mechanism. In some mechanisms, such as the competitive mechansim, the message takes the form $m = (m_o, m^1, \ldots, m^N)$ and the i^{th} agent's allocation is determined by the pair (m_o, m^i). It would be interesting to develop a specialized theory for mechanisms of this type subject to the restriction that the i^{th} agent's expectations are conditioned on private information and the variables (m_o, m^i) alone. Results of this nature were obtained in [15] for the competitive mechanism. The need for further study of information adjustment processes like the one described in section 4 was emphasized in that section. The proof in section 5 that rational expectations can be learned raises two questions. First, can an analogous result be proved for classical environments? Second, what is the class of estimation procedures which ensure convergence to an expectations equilibrium? There is also the question of whether the results of sections 4 and 5 extend to multi-period environments such as those studied in sections 6 and 7. It is apparent that the results obtained here constitute merely an introductory outline of the rich relationship between rational expectations and general allocation theory.

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