EQUITY IN EXCHANGE ECONOMIES

by

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We propose here a new concept of equity for economic environments and study the existence of equitable allocations.

A number of equity concepts have recently been proposed on the basis of which normative evaluations of allocations can be performed. Equity concepts for trades have also been studied.

If the initial allocation favors certain agents, this fact will in general be reflected in the allocations obtained through an equitable trade. Taking the Walrasian mechanism as an example, it is sometimes argued that its rules are fair (i.e., that it generates equitable trades) and that any inequity observed in Walrasian allocations should be attributed to the initial position. But how is one to judge the equity of the initial position?

The answer we propose here simply consists in permuting the initial bundles among the agents and in checking how this operation affects the final allocation obtained through an equitable mechanism. The central concept in our analysis is therefore that of an equitable mechanism, and an allocation is declared equitable relative to that mechanism if it is invariant under permutations of the initial bundles.

There are in fact several ways of formulating definitions in that spirit. Various alternatives are proposed and their logical relationships established. Finally the existence of allocations satisfying these definitions for several examples of equitable mechanisms is examined.

The paper is organized as follows: Section 1 is devoted to a description of the class of economies with which the analysis is concerned. Section 2 motivates the approach followed here. Section 3 spells out our first definitions of equity and applies these definitions to a number of well-known examples of mechanisms. Section 4 proposes several weakenings of the definitions. Section 5 develops yet another formulation based on somewhat different considerations, and examines the application of this formulation to the same list of examples as in Section 3. The definitions and the results are gathered and summarized in Section 6. The appendix develops certain peripheral issues.

1. Preliminaries

Let $\mathcal{E}$ be the class of (exchange) economies with ℓ commodities, n agents and aggregate endowment Ω. Agent i, indexed by the subscript i, is characterized for all i by his consumption set Z_i, taken to be R_+^ℓ, and a von Neumann - Morgernstern utility function $u_i : Z_i \to R$. Ω is a point in R_{++}^ℓ. A typical element e of $\mathcal{E}$ is therefore a list $((Z_i, u_i)_{i=1}^n, \Omega)$. An economy in an initial position[1] is a pair (e, ω), where e is an element of $\mathcal{E}$ and ω represents a distribution of the aggregate endowment Ω of e, i.e., ω belongs to $R_+^{\ell n}$ and satisfies $\Sigma \omega_i = \Omega$.[2] $\mathcal{E}'$ is the class of such pairs. Agent i's consumption is denoted z_i.

Most of the paper is concerned with $\mathcal{E}$ and various subclasses of $\mathcal{E}$. However, we will on occasions consider the class $\tilde{\mathcal{E}}$ of transferable utility (exchange) economies with ℓ commodities, n agents and aggregate endowment Ω. There, agent i is characterized for all i by his consumption set Z_i, taken to be $R \times R_+^{\ell-1}$, and a utility function $u_i : Z_i \to R$ separable additive and linear in the first commodity. $\tilde{\mathcal{E}}'$ is defined from $\tilde{\mathcal{E}}$ as $\mathcal{E}'$ was defined from $\mathcal{E}$. Agent i's consumption is denoted $z_i = (x_i, y_i)$, where x_i is his consumption of the first commodity (unrestricted in sign), and y_i is his consumption of the last $\ell - 1$ commodities. The assumptions made on u_i can be reformulated as follows: there exists a function $v_i : R_+^{\ell-1} \to R$ such that for all $z_i = (x_i, y_i)$ in Z_i, $u_i(z_i) = x_i + v_i(y_i)$.

The i^{th} agent's preference relation (the relation "represented" (Debreu [8]) by his utility function u_i) is denoted $\succsim_i$. 3/

The preference relation $\succsim_i$ is monotonic if for all z_i, z_i' in Z_i,

$$z_i > z_i' \Rightarrow z_i \succ_i z_i',$$

and strictly monotonic if for all z_i, z_i' in Z_i,

$$z_i \geq z_i' \Rightarrow z_i \succ_i z_i' \quad .$$

The preference relation $\succsim_i$ is convex if for all z_i, z_i' in Z_i and for all λ in $[0, 1]$,

$$z_i \succsim_i z_i' \Rightarrow \lambda z_i + (1 - \lambda) z_i' \succsim_i z_i' \quad ,$$

and strictly convex if for all z_i, z_i' in Z_i, and for all λ in $]0, 1[$,

$$z_i \neq z_i' \quad \text{and} \quad z_i \succsim_i z_i' \Rightarrow \lambda z_i + (1 - \lambda) z_i' \succ_i z_i' \quad .$$

The preference relation $\succsim_i$ satisfies local non-satiation if for all z_i in Z_i and for all neighborhoods B of z_i,

$$\exists z_i' \in B \cap Z_i \quad \text{s.t.} \quad z_i' \succ_i z_i \quad .$$

Given e in $\mathcal{E}$ or $\tilde{\mathcal{E}}$, a feasible allocation for e is a list $z = (z_1, \ldots, z_n)$ in $Z \equiv Z_1 \times \ldots \times Z_n$ such that $\Sigma z_i = \Omega$. The set of feasible allocations for e is denoted $Z(e)$. A feasible allocation for e is Pareto-efficient for e if there does not exist another allocation z' in $Z(e)$ such that for all i, $z_i \succsim_i z_i'$, with strict preference for at least one i. The set of Pareto-efficient allocations for e is denoted

$P(e)$. The sets of feasible allocations and Pareto-efficient allocations for (e, ω) where e is in $\mathcal{E}$ or $\tilde{\mathcal{E}}$ are similarly defined and also denoted $Z(e)$ and $P(e)$ respectively (by a slight abuse of notation).

Finally, we introduce the equity concept that plays the central role in the current literature on the topic.

Definition F: z in $Z(e)$ is an envy-free allocation for e if

$$\forall i, j, \quad z_i \succsim_i z_j \quad .$$

See Appendix 1 for bibliographical details. The set of envy-free allocations for e is denoted $F(e)$, and the set of envy-free and Pareto-efficient allocations for e is denoted $FP(e)$.

Π_n is the class of permutations of order n.

2. Equitable allocations and equitable trades

In this section, we observe that normative requirements of equity and fairness may be imposed on allocations (2-1), on trades (2-2), or on both, and we pose the question whether choices of equity criteria for allocations and trades can be made independently or whether certain consistency requirements should be satisfied in their selection. After a preliminary discussion of this problem (items (i) and (ii) of (2-3)), we point out that it may be conceptually necessary to differentiate between equity criteria for *initial* allocations and for *final* allocations, and to rephrase our original question accordingly. We then present several natural ways of establishing links between the three families of criteria (items (iii), (iv), (v) of (2-3), and introduce the approach followed in this paper, which consists in taking as a primary concept the criterion of equity on trades, and *deriving* from it equity criteria for initial allocations and for final allocations. Various such derivation procedures are defined and further motivated in Section 3.

2-1. A number of equity criteria for allocations have been recently defined and studied by Foley [10], Schmeidler and Yaari [25], Kolm [15], Pazner and Schmeidler [19], [20], Varian [30], [31], [32], Feldman and Kirman [9], Daniel [7], Pazner [18], and Goldman and Sussangkarn [13]. They are formally stated in Appendix 1.

2-2. Equity criteria can also be formulated for choice correspondences. A choice correspondence (CC) associates to each economy in an initial position a set of allocations feasible for that economy, among which a final choice will eventually be made:

Definition: A choice correspondence φ defined on $\mathcal{E}'$ (or $\tilde{\mathcal{E}}'$) is a mapping from $\mathcal{E}'$ (or $\tilde{\mathcal{E}}'$) into Z such that

$$\forall (e, \omega) \in \mathcal{E}' \text{ (or } \tilde{\mathcal{E}}'), \quad \varphi(e, \omega) \subset Z(e) \quad .$$

It is natural to demand of a CC that it satisfy certain fairness conditions. Much attention has been devoted to defining and analyzing such conditions in bargaining theory and in game theory. The symmetry axiom entering the definitions of the Nash solution and of the Shapley-value reflects this normative concern.

The weakest requirement in that spirit is perhaps the requirement of anonymity, a CC being said to be anonymous if its specification does not single anyone out for special consideration. Remembering that Π_n is the class of permutations of order n, we have:

Definition: A choice correspondence φ defined on $\mathcal{E}'$ (or $\tilde{\mathcal{E}}'$) is anonymous if for all $((Z_i, u_i)_{i=1}^n, \Omega, \omega)$ in $\mathcal{E}'$ (or $\tilde{\mathcal{E}}'$), and for all π in Π_n,

$$\varphi(\pi((Z_i, u_i)_{i=1}^n), \Omega, \pi(\omega)) = \pi \circ \varphi((Z_i, u_i)_{i=1}^n, \Omega, \omega) \ .$$

For an anonymous CC, the names of the agents do not matter. Most of the CC's encountered in economies have this property (Walrasian CC, core CC, Shapley-value CC ....). An example of a non-anonymous CC is the dictatorial (or lexicographically dictatorial) CC.

Note. In what follows, we will indifferently speak of the equity of a choice correspondence or of the trades that it generates.

2-3. We now ask whether equity criteria for allocations and choice correspondences can be chosen independently or whether their selection should satisfy certain consistency requirements. In each of the paragraphs (i) - (v) below, a particular approach to this question is discussed, each of them being symbolically represented by one line of Table 1.

(i) First, we discuss a difficulty that arises if equity criteria for allocations and choice correspondences are adopted independently.

Let α and τ be these criteria, and assume for example that the α-equitable allocations are the envy-free allocations and that the τ-equitable trades are the Nash-solution trades.[4] Since an equitable trade should presumably not "decrease" the equity of the distribution of the resources, it is somewhat disturbing that there exist economies admitting of equitable allocations from which an equitable trade leads to an allocation which is not equitable. The existence of such economies follows from a result due to Goldman and Sussangkarn [13].

(ii) In response to the above problem, the argument can be made that the pair (τ, α) it involves is not a "natural" pair and that whichever equity criterion is used for trades should be somehow derived from the equity criterion used for allocations. Such a derivation seems indeed possible from the concept of an envy-free allocation, leading to the concept of an envy-free trade, originally introduced by Schmeidler and Vind [24], and defined as follows:

<u>Definition</u>: t in $R^{\ell n}$ is an <u>envy-free trade for (e, ω)</u> if

(a) $z \equiv \omega + t \in Z(e)$

(b) $\forall i,j$, it is not the case that $\omega_i + t_j \succ_i \omega_i + t_i$.

Similarly, an equity criterion for trades can be associated to each of the equity criteria for allocations that have been proposed in the literature so far. (This is done in Appendix 4.)

Such derivation will however not resolve our problems, as revealed by an example, provided by Feldman and Kirman [9], of an economy where an envy-free trade from an envy-free allocation leads to an allocation with envy.

There may be other difficulties. Goldman and Sussangkarn [13] showed that there usually exist nonefficient and envy-free allocations that are Pareto-dominated by no envy-free allocation, precluding the existence of Pareto-improving trades (envy-free or not) that would lead to an envy-free and efficient allocation.

In our context, this difficulty may not be as serious as the previous one. The requirement on a trade that it be Pareto-improving is of interest mainly if individuals have a right to their initial endowments, as it increases the likelihood of their willing participation in the exchange. No such rights are here acknowledged. (On the other hand, this result suggests that it may be delicate to devise dynamic procedures leading to efficient and envy-free allocations.)

Perhaps more seriously, there exist economies with envy-free allocations from which no envy-free and efficient allocations can be reached through an envy-free trade (Thomson [29]).

How general are these phenomena? Do they extend to the other pairs of equity concepts as constructed in Appendix 4? An answer to these questions would go beyond the limits of this paper, and instead we will follow a different route:

First we remark that it may be illegitimate to use the same criteria to evaluate the equity of initial and final allocations. After all, initial allocations do not matter as such but only to the extent that they affect the final allocations. What makes it tempting to use the same criteria is of course the fact that in our context initial and final allocations happen to belong to the same mathematical space. That there is no necessity of doing so is perhaps brought out by considering the particular case of a production economy in which the initial position would be given by a distribution of factors of production, not susceptible of being directly consumed, and the final position by an allocation of the commodities made possible by the transformation of these factors of production through some production process. In such circumstances, it would clearly not be possible to use the same equity criteria for initial and final allocations.

Three equity criteria could then and maybe should be simultaneously used, α_i and α_f for initial and final allocations, and τ for choice correspondences, and the question we asked earlier about pairs (τ, α) should now be asked about triples $(\alpha_i, \tau, \alpha_f)$.

(iii) The independent choice of these criteria will clearly lead to conceptual problems since this permits in particular selecting $\alpha_i = \alpha_f$ which, as already observed in (i) , does create such problems.

(iv) An alternative approach consists in first selecting a pair (τ, α_f) and then declaring an initial allocation α_i - equitable if some

τ - equitable trade, (or perhaps all τ - equitable trades) would lead from it to an α_f - equitable allocation. Although such an approach is formally consistent, it may not be operationally very useful, since a characterization of the α_f - equitable final allocations is a necessary prior step to a characterization of the α_i - equitable initial allocations, information of little value if one's ultimate interest is in the equity of the final allocation.

(v) The opposite operation could be performed: after the selection of a pair (α_i, τ), one would declare a final allocation α_f - equitable if it were obtained from an α_i - equitable allocation via a τ - equitable trade. For instance, if the equal bundle allocation is accepted as an equitable initial allocation, and the Walrasian CC as generating equitable trades, then any Walrasian allocation from the equal bundle allocation could be considered as an equitable final allocation.

(vi) Finally, we come to the approach followed here. We will take the equity concept on trades as our central concept and formulate derived equity concepts for initial and final allocations. This approach is further developed and motivated in the next section.

	Choice Correspondence	Allocation	Described in Item
	τ	α	(i)
	$\tau(\alpha)$	α	(ii)
Initial Allocation	**Choice Correspondence**	**Final Allocation**	
α_i	τ	α_f	(iii)
$\alpha_i(\tau, \alpha_f)$	τ	α_f	(iv)
α_i	τ	$\alpha_f(\alpha_i, \tau)$	(v)
$\alpha_i(\tau)$	τ	$\alpha_f(\tau)$	(vi)

Table I

3. Acceptable allocations

3-1. B - acceptability

The central idea is the following: first, the choice is made of an equity criterion for trades, embodied in a choice correspondence φ. Then each agent is asked to evaluate the final allocations attainable through the operation of φ from the various permutations of the initial bundles. In definition B1 proposed below, a final allocation will be said to be an equitable final allocation if no permutation would be advantageous to some individual, in the sense that all the allocations to which φ would lead from there would be strictly preferred by him to the final allocation under consideration; at the same time, the initial allocation will be declared an equitable initial allocation.

Because φ may and will in general be multivalued, there are in fact other ways (alternative to B1) of formulating conditions in that spirit, and we will offer several definitions. Instead of attempting to find names for all of these, we will denote them by letter-number combinations followed by a symbol representing the performance correspondence under consideration; finally, we will use the term "acceptability" to avoid confusion with the terminologies of some other papers. For example, an allocation will be said to be "B1 - φ - acceptable" or "acceptable relative to the choice correspondence φ according to definition B1".

Note that in the approach followed here, equity (or acceptability) criteria are first formulated about ordered pairs of an initial and a final allocation. We will therefore start by defining acceptable configurations (i.e., pairs of this kind). In the rest of the paper, we will however concentrate exclusively on the formulation of acceptability criteria for final allocations; one should keep in mind though that a parallel derivation of acceptability criteria for configurations and for initial allocations could also be performed.

In the following definitions, φ is an anonymous choice correspondence, and $e = ((Z_i, u_i)_{i=1}^n, \Omega)$ is the generic element of $\mathcal{E}$ (or $\mathcal{E}'$).

Definition b1: A pair (ω, z) in $Z(e) \times Z(e)$ is a b1 - φ - acceptable configuration for e if

(a) $z \in \varphi(e, \omega)$

(b) $\forall \pi \in \Pi_n, \forall i, \exists z' \in \varphi(e, \pi(\omega))$ with $z \succsim_i z'$.

Note that if (ω, z) is a b1 - φ - acceptable configuration for e, so is $(\pi(\omega), z)$ for all π in Π_n.

From this definition can be derived two concepts of φ - acceptability for allocations.

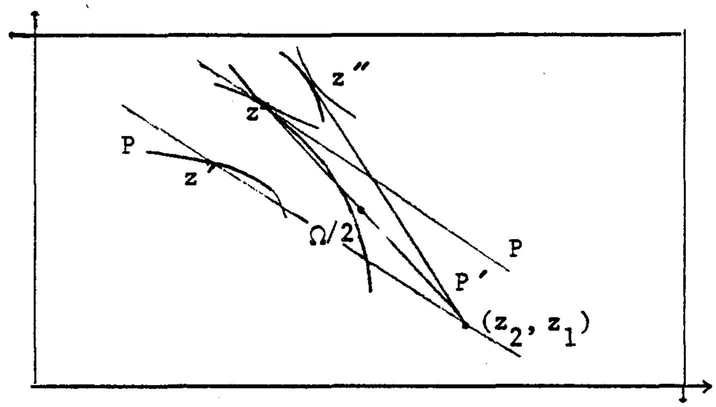
Definition B1: z in $Z(e)$ is a B1 - φ - acceptable allocation for e if there exists ω in $Z(e)$ such that the pair (ω, z) is a b1 - φ - acceptable configuration for e.

A stronger definition can be obtained from definition B1 by taking the permutations of z itself as initial endowments.

Definition B2: z in $Z(e)$ is a B2 - φ - acceptable allocation for e, if the pair (z, z) is a b1 - φ - acceptable configuration for e.

This last definition can be seen as an extension of the concept of opportunity fairness proposed by Varian [32] (see appendix 1 for a formal definition). A presentation of Varian's argument is therefore offered and illustrated in Figure 1.

Let e be a two-commodity, two-person economy with convex, monotonic and differentiable preferences; let $z = (z_1, z_2)$ be an interior allocation in $FP(e)$, and p be efficiency prices supporting z. Finally, assume that $p z_1 > p z_2$.



<u>Figure 1</u>

Varian observes that, although agent 2 would not want to swap bundles with agent 1 if he had to <u>consume</u> agent 1's bundle, he would favor the swap if <u>subsequently given the opportunity to trade at prices p</u>. Then, he could reach consumptions like z_2' that he prefers to z_2: agent 2 does not envy agent 1 but he envies his trading opportunities. Varian defines an <u>opportunity-fair</u> efficient allocation as an allocation where each agent prefers his bundle to any bundle in anyone else's budget set as determined via the efficiency prices associated with the allocation.

One should note however that if the competitive mechanism were actually operated from (z_2, z_1), p would in general not remain equilibrium prices, and that at the new competitive equilibrium (p, z''), agent 2 could in fact be worse off than at z, and not benefit from the swap. This possibility is illustrated in Figure 1.

If one postulates a commitment of society to efficiency along with an agreement on the competitive mechanism as providing equitable resolutions of inefficiencies, the comparison of z_2 with z_2' by agent 2 can therefore be considered irrelevant, while the comparison between z_2 and z_2'' reveals that he has been treated fairly.

One can go one step further since the rules of exchage have no reason to be the Walrasian rules: their specification is also part of the normative choices made by society. After society's adoption of some choice correspondence φ as providing equitable resolutions of inefficiencies, an efficient allocation z would be declared equitable on the basis of comparisons of z with the allocations to which φ would lead from (z_2, z_1).

Finally, although most of the examples of CC's studied later happen to select Pareto-efficient outcomes, this property is not necessary for the present argument, which leads us to definition B2, as stated above.

We now propose several variants of definitions B1 and B2.

First, we consider the stronger form taken by B1 when it is required that it be to the <u>same</u> allocation z' that the conclusion applies for all agents.

<u>Definition B1*</u>: z in $Z(e)$ is a <u>B1* φ-acceptable allocation for e</u> if there exists ω in $Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $\forall \pi \in \Pi_n, \exists z' \in \varphi(e, \pi(\omega))$ s.t. $\forall i, z \succsim_i z'$.

Next, we observe that if φ selects Pareto-efficient outcomes, the statement $[\forall i, z \succsim_i z']$ is equivalent to the statement $[\forall i, z \sim_i z']$, and we identify a property, satisfied by a large number of CC's, that will permit to considerably simplify the statements of our definitions:

<u>Property P</u>. φ satisfies Property P if

$$\forall z, z' \in Z(e),\ [z \in \varphi(e, \omega) \ \text{ and } \ \forall i, z \sim_i z'] \Rightarrow z' \in \varphi(e, \omega) \ .$$

If φ selects Pareto-efficient outcomes and satisfies Property P, then $B1^*$ takes the form:

<u>Definition B1*</u>: z in $Z(e)$ is a <u>$B1^*$-φ-acceptable allocation for e</u> if there exists ω in $Z(e)$ s.t.

$$\forall \pi \in \Pi_n,\ z \in \varphi(e, \pi(\omega))$$

where (a) has been eliminated since it is implied by the new form of (b) as written above when π is the identity. We also get:

<u>Definition B2*</u>: z in $Z(e)$ is a <u>$B2^*$-φ-acceptable allocation for e</u> if it is a $B1^*$-φ-acceptable allocation for e with $\omega = z$.

Next we propose to strengthen B1 by requiring that the conclusion applies to all (and not just to some) z' in $\varphi(e, \pi(z))$.

<u>Definition B3</u>: z in $Z(e)$ is a <u>B3-φ-acceptable allocation for e</u> if there exists ω in $Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $\forall \pi \in \Pi_n,\ \forall i,\ \forall z' \in \varphi(e, \pi(\omega)),\ z \succsim_i z'$.

If φ selects Pareto-optimal outcomes and satisfies Property P, this can be written as

<u>Definition B3*</u>: z in $Z(e)$ is a <u>$B3^*$-φ-acceptable allocation for e</u> if there exists ω in $Z(e)$ s.t.

$$\forall \pi \in \Pi_n,\ \varphi(e, \pi(\omega)) = c(e, z)$$

where

$$c(e, z) = \{z' \in Z(e) \mid \forall i,\ z' \sim_i z\} \ .$$

And finally, we derive

<u>Definition B4*</u>: z in $Z(e)$ is a <u>$B4^*$-φ-acceptable allocation for e</u> if z is a $B3^*$-φ-acceptable allocation for e with $\omega = z$.

In the next subsection, we examine the existence of φ-acceptable allocations according to definition $B2^*$, for a number of φ's. The existence of allocations according to the definitions $B1^*$, $B3^*$ and $B4^*$ will be discussed later. All of the results are presented in summary form in Section 6.

<u>Note</u>. The set of allocations satisfying acceptability definition D relative to φ for the economy e is denoted $D_\varphi(e)$.

3-2. <u>Applications</u>

In this subsection, we study several examples of CC's and investigate the existence of $B2^*$-acceptable allocations relative to them. These examples fall into two classes. The CC's of the first class (examples (a) - (c)) are ordinally defined while those of the second class (examples (e) - (h)) rely on the existence of von Neumann - Morgernstern utility functions.

First, we present some results of general interest:

<u>Proposition 1</u>: If φ and φ' are such that $\varphi'(e, \omega)$ is always a subset of $\varphi(e, \omega)$, then $B2^*_{\varphi'}(e)$ is a subset of $B2^*_\varphi(e)$.

<u>Proof</u>: We have:

$$z \in B2^*_{\varphi'}(e) \Rightarrow \forall \pi \in \Gamma_n,\ z \in \varphi'(e, \pi(z)) \Rightarrow \forall \pi \in \Gamma_n,\ z \in \varphi(e, \pi(z)) \Rightarrow z \in B2^*_\varphi(e),$$

where the first and third implications follow from the definitions of $B2^*$-φ'- and $B2^*$-φ-acceptability, and the second one from the hypothesis.

Q.E.D.

Proposition 2: Let I be the CC associating to each (e, ω) the set of allocations in $Z(e)$ that are individually-rational from ω. Then $B2_I^*(e)$ is the set of envy-free allocations for e, $F(e)$.

Proof: Let $z \in B2_I^*(e)$ be given. Let also i and j be arbitrary agents. By definition, for all $\pi \in \Pi_n$, z is individually-rational from $\pi(z)$, so in particular for π^o such that $\pi_i^o(z) = z_j$, we have $z_i \succsim_i \pi_i^o(z) = z_j$. As the argument applies for all i, j, it follows that $z \in F(e)$.

Conversely, let $z \in F(e)$ and $\pi \in \Pi_n$ be given. Since for all i, j, $z_i \succsim_i z_j$, in particular for all i, $z_i \succsim_i \pi_i(z)$, which implies that $z \in I(\pi(z))$. Since the argument applies for all $\pi \in \Pi_n$, it follows that $z \in B2_I^*(e)$.

Q.E.D.

Since all (except (d)) of the forthcoming examples of CC's select individually rational outcomes, it follows from Propositions 1 and 2 that their $B2^*$-acceptable allocations will be envy free.

(a) W, the Walrasian CC

Proposition 3: Let W be the Walrasian CC. Then $z \in Z(e)$ belongs to $B2_W^*(e)$ if and only if z is a Walrasian allocation from the equal bundle allocation $\bar{\omega} = (\Omega/n, \ldots, \Omega/n)$.

Proof: (i) Let $z \in Z(e) \cap W(e, \bar{\omega})$ be given, and p be associated equilibrium prices. Then, for all i, $p z_i \leq p\Omega/n$, which by addition yields $p \Sigma z_i \leq p\Omega$. Since $z \in Z(e)$, $\Sigma z_i = \Omega$, and therefore for all i, $p z_i = p\Omega/n$. In conjunction with the statement

$$\forall i, \forall z_i' \in \{z_i' \in Z_i \mid p z_i' \leq p\Omega/n\}, \quad z_i \succsim_i z_i' \quad ,$$

expressing that z_i is maximal in agent i's budget set, we obtain

$$\forall i\,,\, j\ \forall z_i' \in \{z_i' \in z_i \mid p\, z_i' \leqq p\, z_j\}, \quad z_i \succsim_i z_i' \quad .$$

This, along with $\Sigma z_i = \Omega$ implies that for all $\pi \in \Pi_n$, (p, z) is a Walrasian equilibrium for $(e, \pi(z))$ and consequently that $z \in B2^*_\varphi(e)$.

(ii) Conversely, let $z \in B2^*_\varphi(e)$ be given, and let $\pi^o \in \Pi_n$ be defined by: $\pi^o_i(z) = z_{i+1}$ if $i < n$, $\pi^o_n(z) = z_1$. Since $z \in B2^*_\varphi(e)$, z is a Walrasian allocation from $\pi^o(z)$, which implies the existence of some prices p^o such that

$$\forall i\,,\, \forall z_i' \in \{z_i' \in Z_i \mid p\, z_i' \leqq p^o\, \pi^o_i(z)\}\ , \quad z_i \succsim_i z_i' \quad .$$

Then $p^o z_1 \leqq p^o z_2$, $p^o z_2 \leqq p^o z_3$, ..., $p^o z_n \leqq p^o z_1$ which yields $p^o z_1 = p^o z_2 = \ldots = p^o z_n$. By addition, it follows that for all i, $p^o \pi^o_i(z) = p^o(\Sigma z_i)/n$, and since $\Sigma z_i = \Omega$, for all i, $p^o \pi^o_i(z) = p^o \Omega/n$ and therefore,

$$\forall i\,,\, \forall z_i' \in \{z_i' \in Z_i \mid p^o z_i' \leqq p^o \Omega/n\}\,,\ z_i \succsim_i z_i' \quad ,$$

which means that (p^o, z) is a Walrasian equilibrium for $(e, \bar{\omega})$.

Q.E.D.

(b) <u>IP, the CC selecting the set of individually-rational and Pareto-optimal allocations</u>

<u>Proposition 4</u>: Let IP be the CC associating to each (e, ω) the set of individually-rational from ω and Pareto-efficient allocations. Then $z \in Z(e)$ belongs to $B2^*_{IP}(e)$ if and only if z belongs to the set of envy-free and Pareto-efficient allocations for e , $FP(e)$.

<u>Proof</u>: It is a straightforward extension of the proof of Proposition 2.

Remark 1: The Walrasian CC is trivially individually-rational. If preferences are locally nonsatiated, Walrasian allocations are Pareto-efficient. By Proposition 1, we can therefore conclude that under such an assumption, $B2^*_W(e) \subset B2^*_{IP}(e)$.

(c) C, the core CC

In the two-person case, C coincides with IP , and the above result applies. Otherwise, we have

Proposition 5: Let C be the core CC , and assume that preferences are locally nonsatiated. Then $B2^*_C(e)$ is a superset of $W(e, \bar{\omega})$.

Proof: Under these assumptions, $C(e, \omega)$ is always a superset of $W(e, \omega)$. (standard argument). The result follows from Propositions 1 and 3.

Remark 2: An application of Proposition 1 also permits to conclude that $B2^*_C(e) \subset B2^*_{IP}(e)$.

(d) T , the CC selecting the set of allocations attainable via an envy-free trade Pareto-undominated by any other envy-free trade

(See Section 1 for the definition of an envy-free trade.)

Lemma 1: If for all i, u_i is continuous, $T(e, \omega)$ is nonempty.

Proof: First, we note that $t = 0$ is always an envy-free trade. The existence of maximal (i.e., Pareto-undominated) elements in the set of envy-free trades follows from continuity of preferences.

Q.E.D.

Lemma 2: If $z \in Z(e)$ is a competitive allocation for (e, ω) , then z is in $T(e, \omega)$.

Proof: If $z \in W(e, \omega)$, there exists p such that

$$\forall i, \forall z_i' \in \{z_i' \in Z_i \mid p z_i' \leqq p \omega\}, \quad z_i \succsim_i z_i' \quad .$$

Since $z \in Z(e)$, we can conclude as in Proposition 3 that for all i, $p z_i = p \omega_i$, and therefore for all i, $p t_i \equiv p(z_i - \omega_i) = 0$. This implies that for all i, j, $p(\omega_i + t_j) = p \omega_i$ and therefore the bundle $\omega_i + t_j$ is affordable by agent i. Because $z_i = \omega_i + t_i$ is maximal in agent i's budget set, it is not the case that agent i would strictly prefer agent j's trade t_j to his own trade t_i.

Q.E.D.

Proposition 6: Assume that for all i, u_i is continuous. Then $B2_T^*(e)$ is a superset of $W(e, \bar{\omega}) \cap Z(e)$.

Proof: It follows from Proposition 1 and Lemma 2.

Proposition 7: Assume that $n > 2$. Then any allocation in $B2_T^*(e)$ is envy-free.

Proof: Let $z \in B2_T^*(e)$ be given, i and j be arbitrary agents and π be the permutation $\in \Pi_n$ exchanging i and j, and only them. The net trade leading from $\pi(z)$ back to z has a zero component for every agent k different from i and j. Since $z \in B2_T^*(e)$, it is not the case that

$$z_j + z_k - z_k \succ_i z_j + z_i - z_j \quad ,$$

a statement expressing the fact that agent i, whose initial position is z_j, prefers agent k's net trade (agent k starts at z_k and ends up at z_k), to his own trade $z_i - z_j$, which leads him back to z_i. Since $z \in Z(e)$ and the operation can be performed for all i and j, this yields

$$\forall i, j, \quad z_i \succsim_i z_j$$

which is the definition of an envy-free allocation.

Q.E.D.

Remark 3: In the two-person case, an explicit characterization of $B2_T^*(e)$ can be given. Simple examples can be found for which $B2_T^*(e) \not\subset F(e)$.

(e) N, the Nash bargaining solution

First, we define this solution concept and show how it is traditionally applied to the problem of allocating economic resources. Then, we prove the nonemptiness of the set of $B2^*$-acceptable allocations relative to it in the case of two agents.

An n-person bargaining problem is a pair (S, d) where S is a compact and convex subset of R^n representing the utility levels measured in von Neumann - Morgernstern utility scales, attainable by the n agents through some joint action; and d is a point of S, strictly dominated by some other point of S. d, the disagreement point, is the outcome that would achieve if the agents failed to reach a compromise. A bargaining solution associates to every bargaining problem (S, d) a unique point of S interpreted as the compromise reached by the agents. Nash's bargaining solution [17] selects the point where the "Nash product" $\prod_i (u_i - d_i)$ is maximized in u over $\{u \in S \mid u \geqq d\}$.

Because of its appealing properties (it selects Pareto-efficient outcomes and is invariant under linear transformation of utilities; more importantly here, it satisfies a symmetry axiom), this concept is often suggested as a solution to the economic problem of dividing among n agents the gains they can achieve by trading. In such an application, S is the set of utility vectors attainable through arbitrary redistribution of all the resources available to the agents, assumed to be able to freely dispose of what they receive. We therefore introduce the notation:

$$Z^+(e) = \{z \in Z \mid \sum z_i \leqq \Omega\}$$

and write $S = u(Z^+(e))$. When agents are entitled to their initial endowments ω, it is natural to take d as the image under u of ω.

The following lemma tells us when the pair (S, d) so obtained is a well-defined bargaining problem. (See Figure 2.)

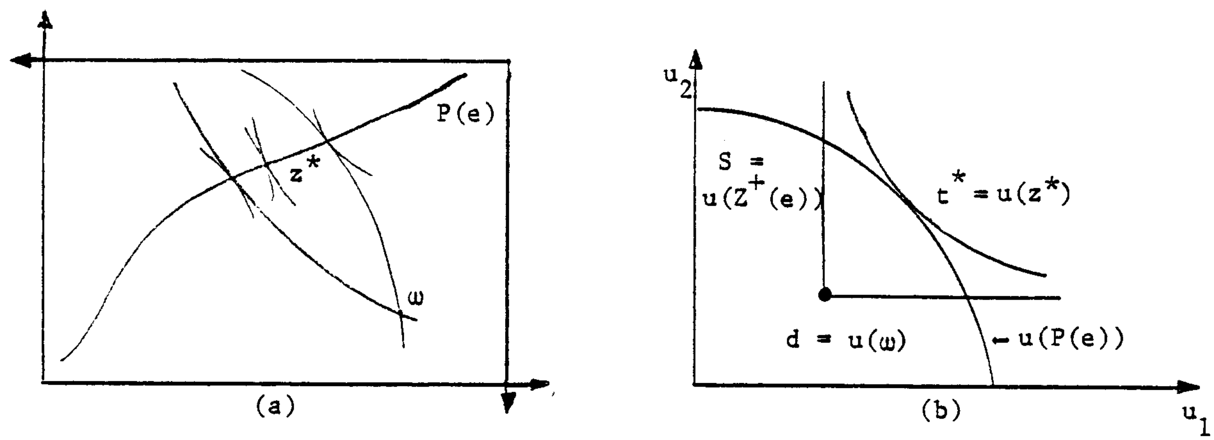


Figure 2

Lemma 3 : If ω is not Pareto-efficient for e , and for each i, u_i is (i) continuous, (ii) strictly monotonic, and (iii) concave, then $(u(Z^+(e)), u(\omega))$ is a well-defined bargaining problem.

Proof: $Z^+(e)$ is a compact set. Its image under the continuous function u (assumption (i)) is also compact. The convexity of $u(Z^+(e))$ under assumption (iii) is proved in Chipman and Moore [3] (lemma 5, p. 24).

The proof that if ω is not Pareto-efficient for e , $d = u(\omega)$ is strictly dominated by some other point of $u(Z^+(e))$ is standard.[5/]

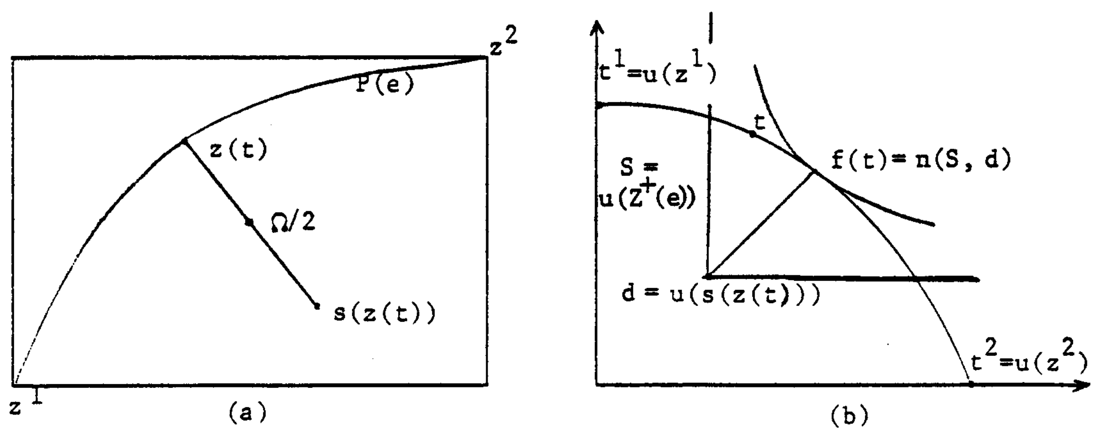
Remark 4 : As in the next lemmas, where continuity and monotonicity assumptions will be made, we normalize u_i by setting $u_i(0) = 0$, so that $u(Z^+(e))$ becomes a subset of R^n_+ .

Since we want to allow for the case when ω is Pareto-efficient, we will define an extended bargaining problem as a pair (S, d) where S and d are as defined above except for the requirement that d be strictly dominated by a point of S. We then trivially extend the Nash bargaining solution by selecting d as the solution outcome if d is Pareto-efficient. This extended Nash bargaining solution will be denoted $n(\cdot)$.

Finally, we define the <u>Nash CC</u>, that we denote N, by associating to each (e, ω) satisfying the assumptions stated in lemma 3 the allocations in $Z(e)$ whose image under u is the extended Nash solution outcome $n(S, d)$ for the extended bargaining problem $(S, d) = (u(Z^+(e)), u(\omega))$.

<u>Proposition 8</u>: Let $n = 2$. Assume that for $i = 1, 2$, u_i is (i) continuous, (ii) strictly monotonic, and (iii) concave. Then $B2^*_N(e)$ is nonempty.

<u>Proof</u>: (a) A heuristic argument is given here and illustrated in Figure 3. The complete proof relies on the Kakutani fixed point theorem and except for one step, follows the proof of Proposition 8′ of Section 5, which concerns a different concept of acceptability, in the n-person case. We refer the reader to that proof where the special step is also treated in detail.



<u>Figure 3</u>

(b) For simplicity, preferences are assumed to be strictly convex. The Pareto set is the curvilinear segment z^1, z^2, whose image in the utility space is the curvilinear segment t^1, t^2. Given each t in $u(P(e))$, we identify the unique allocation $z(t)$ in $P(e)$ such that $u(z) = t$, determine its symmetric $s(z(t)) = (z_2(t), z_1(t))$ with respect to the center of the Edgeworth box, the image under u of $s(z(t))$ being then taken as disagreement point.

(c) Then the extended Nash solution outcome for the extended bargaining problem $(u(Z^+(e)), u(s(z(t)))$ is determined. Let $f(t)$ denote this outcome.

(d) It is shown in Proposition 8′ that f is continuous. Also $f(t^1) = t^2$ and $f(t^2) = t^1$. By the intermediate value theorem, there exist t such that $f(t) = t$. The inverse image of t under u is a desired allocation.

Q.E.D.

(f) <u>R, the Raiffa-Kalai-Smorodinsky solution</u>

This two-person bargaining solution was proposed by Raiffa [21], and given an axiomatic characterization by Kalai and Smorodinsky [14], a similar argument appearing in Rosenthal [22]. In order to define it we introduce for each two-person bargaining problem (S, d), the point $v(S, d)$ of coordinates $v_i(S, d) = \max \{u_i \mid u \in S\ ;\ u \geqq d\}$ for $i = 1, 2$. The solution outcome is then given by the unique intersection of the segment connecting d and $u(S, d)$ with the boundary of S. The point so obtained is Pareto-efficient.

The generalization of this geometric operation to the n-person case is shown by Roth [23] to be incompatible with Pareto-efficiency. However, in the economic applications discussed here, where S and d are defined as in example (e), it happens to yield Pareto-efficient outcomes; moreover, this result holds even if the requirement of convexity on S is dropped. This is formalized by the following lemma, where $S = u(Z^+(e))$. Given some compact set S, and some point d in S, let $v(S, d)$ be the point of coordinates $v_i(S, d) = \max \{u_i | u \in S ; u \geqq d\}$ for all i.

<u>Lemma 4</u>: If ω is not Pareto-efficient for e, and for all i, u_i is (i) continuous and (ii) strictly monotonic, then the segment connecting d with the point $v(S, d)$ intersects the Pareto-efficient boundary of S at a unique point.

<u>Proof</u>: From the proof of lemma 3 we know that $S = u(Z^+(e))$ is compact and that $d = u(\omega)$ belongs to S and is strictly dominated by some other point t of S. This in turn implies that for all i, $v_i(S, d) \geq t_i > u_i(\omega)$. The conclusion follows from an argument similar to the one appearing in Chipman and Moore [3].

Q.E.D.

Again, we want to allow for the case when ω is Pareto-efficient. So we call a <u>generalized bargaining problem</u> any pair (S, d) as defined in example (e), except for the requirements that S be convex and that d be strictly dominated by a point of S. We then define the generalized Raiffa-Kalai-Smorodinsky solution $r(\cdot)$ by selecting d as the solution outcome if d is Pareto-efficient. Finally, we define the <u>Raiffa-Kalai-Smorodinsky CC</u>, that we denote R, by associating to each (e, ω) satisfying the assumptions stated in lemma 4, the allocations in $Z(e)$ whose image under u is the generalized Raiffa-Kalai Smorodinsky solution for the generalized bargaining problem $(u(Z^+(e)), u(\omega))$.

<u>Proposition 9</u>: Let $n = 2$. Assume that for $i = 1, 2$, u_i is (i) continuous and (ii) strictly monotonic. Then $B2_R^*(e)$ is nonempty.

Proof: We follow the steps of the heuristic argument given in Proposition 8:

(a) is identical to (a) of Proposition 8 with Proposition 9′ replacing Proposition 8′;

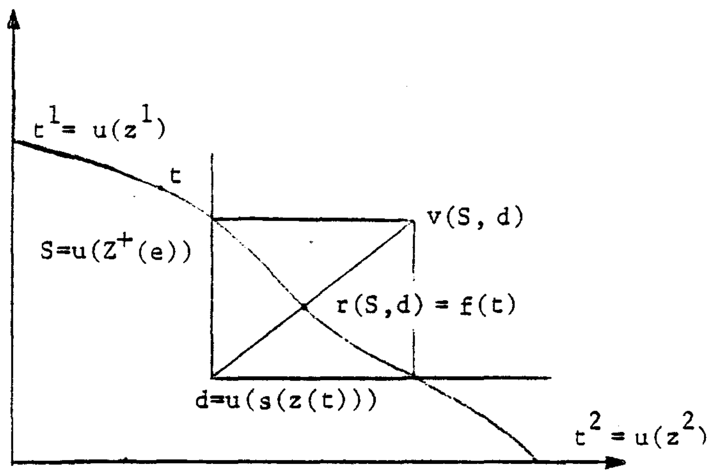


Figure 4

(b) is identical to (b) of Proposition 8 ;

(c) then the generalized RKS solution for the generalized bargaining problem $(u(Z^+(e)), u(s(z(t))))$ is determined. Let $f(t)$ denote this outcome;

(d) is identical to (d) of Proposition 8 .

(g) G, the equal gains CC

Next, we consider the solution associating with each generalized bargaining problem (S, d) as defined in example (f), the unique point $g(S, d)$ where the utility gains from d are equalized across agents, and there is no point of S with the same property dominating d . Formally, if $t = g(S, d)$, then for all i, j, $t_i - d_i = t_j - d_j$ and there is no t' in S with $t' > t$ and for all i, j, $t'_i - d_i = t'_j - d_j$. The next lemma gives conditions under which this operation yields Pareto-efficient outcomes.

Lemma 5: If for each i, u_i is (i) continuous and (ii) strictly monotonic, $g(u(Z^+(e)), u(\omega))$ is Pareto-efficient.

Proof: It is essentially the same as the proof of lemma 4.

Finally, we define the equal gain CC, that we denote G, by associating to each (e, ω) satisfying the assumptions stated in lemma 5, the allocations in $Z(e)$ whose image under u is the equal gain solution outcome for the generalized bargaining problem $(u(Z^+(e)), u(\omega))$.

Proposition 10: Let $n = 2$. Assume that for $i = 1, 2$, u_i is (i) continuous and (ii) strictly monotonic. Then $B2^*_G(e)$ is nonempty.

Proof: Again, we follow the steps of Proposition 8:

(a) is identical to (a) of Proposition 8, with Proposition 10′ replacing Proposition 8′;

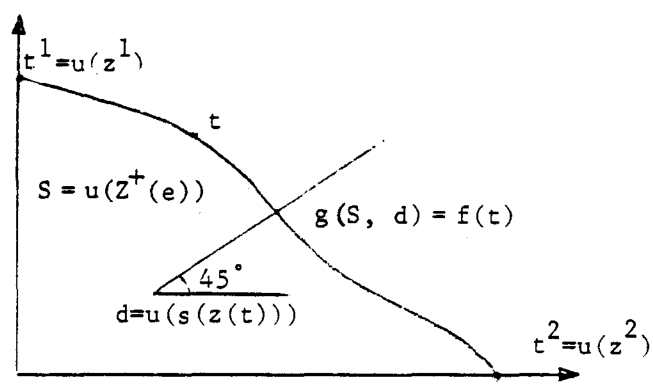


Figure 5

(b) is identical to (b) of Proposition 8;

(c) then the equal gain solution outcome for the generalized bargaining problem $(u(Z^+(e)), u(s(z(t))))$ is determined; let $f(t)$ denote this outcome;

(d) is identical to (d) of Proposition 8.

Q.E.D.

(h) The Shapley value

The Shapley value is a game-theoretic solution concept introduced by Shapley [26] and applied to the class $\tilde{\mathcal{E}}$ of transferable utility economies by Shapley and Shubik [27].

Let (e, ω) be some element of $\tilde{\mathcal{E}}'$. A coalition Γ is a subset of the n agents. The worth $w(e, \omega; \Gamma)$ of the coalition Γ is the maximum welfare achievable by the agents in that coalition through redistribution of the resources they control. Formally, let $Z(e, \omega; \Gamma) = \{z \in (R \times R_+^{\ell-1})^{|\Gamma|} \mid \sum_{i\in\Gamma} z_i = \sum_{i\in\Gamma} \omega_i\}$ where $|\Gamma|$ denotes the cardinality of Γ. Then

$$w(e, \omega; \Gamma) = \max\{\sum_{i\in\Gamma} (x_i + v_i(y_i)) \mid (x_i, y_i)_{i\in\Gamma} \in Z(e, \omega; \Gamma)\}.$$

The contribution of agent i to coalition Γ that contains him is the difference of the worths of Γ and $\Gamma\setminus i$. The value of agent i is a weighted average of his contributions to all the coalitions containing him.

$$w_i(e, \omega) = \sum_{\Gamma\ni i} k_\Gamma \quad (w(e, \omega; \Gamma) - w(e, \omega; \Gamma\setminus i)) \quad .$$

Finally, a value allocation for (e, ω) is an allocation such that the utility of each agent is equal to his value. The Shapley value CC, denoted V, associates to every (e, ω) in $\tilde{\mathcal{E}}'$, the set of value allocations for (e, ω).

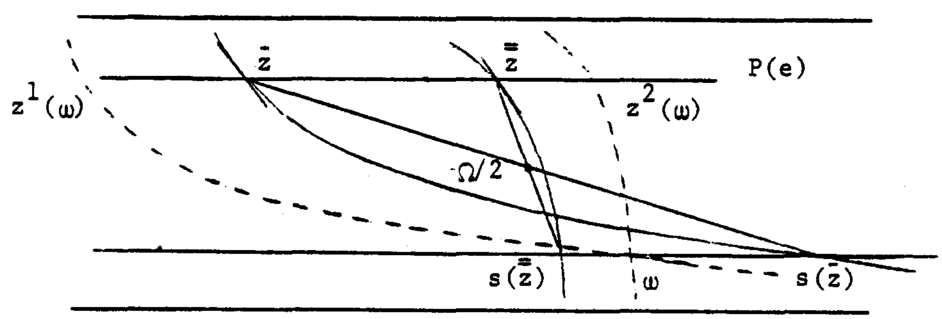
Lemma 6: Assume that for each i, u_i is (i) continuous and (ii) concave. Then value allocations exist.

Proof: Standard.

<u>Proposition 11</u>: Let $n = 2$. Assume that for $i = 1, 2$, u_i is (i) continuous and (ii) concave. Then $B2^*_V(e)$ is nonempty.

<u>Proof</u>: A heuristic argument is given here and illustrated in Figure 6 for the two-person case, with strictly convex preferences.

(a) First, we observe that in a transferable utility economy, the distribution of the first commodity is irrelevant to the determination of the efficient distribution of the $\ell - 1$ last commodities. $P(e)$ is therefore a horizontal line.



<u>Figure 6</u>

(b) Given some allocation ω in $Z(e)$, one determines the extremities $z^1(\omega)$ and $z^2(\omega)$ of the core for (e, ω). The length of the segment $[z^1(\omega), z^2(\omega)]$ gives the "gains from trade" in terms of the first commodity. Simple computations show that the unique value allocation for (e, ω) is the middle of that segment.

(c) Let $\bar{z}$ and $\bar{\bar{z}}$ in $P(e)$ be such that $\bar{z} \sim_1 s(\bar{z})$ and $\bar{\bar{z}} \sim_2 s(\bar{\bar{z}})$. $s(\bar{z})$ and $s(\bar{\bar{z}})$ belong to a same horizontal line to which will also belong $s(z)$ for any z in $P(e)$.

Using the facts that indifference curves are horizontal translates of one another, and that the value CC splits the gains from trade equally among the agents, one can deduce that the transformation associating to each z' in $[s(\bar{z}), s(\bar{\bar{z}})]$ the value allocation for (e, z') is linear, and that the images of $s(\bar{z})$ and $s(\bar{\bar{z}})$ are $\bar{\bar{z}}$ and $\bar{z}$. The composition of this transformation with the linear transformation s is consequently a linear transformation from $[\bar{z}, \bar{\bar{z}}]$ into itself, and $z^* = (\bar{z} + \bar{\bar{z}})/2$ is a fixed point.

Q.E.D.

<u>Remark 5</u>: Although it is true for some of the examples we examined (the Walrasian example is one of them), it need not be the case that an allocation in the image under φ of the economy e in the initial position $\bar{\omega}$ where $\bar{\omega} = (\Omega/n, \ldots, \Omega/n)$, is in $B2^*_\varphi(e)$ (cf. the Nash example).

<u>3.3 Nonexistence of $B2^*$ - φ - acceptable allocations if φ = N, R, G, V and $n > 2$</u>

Propositions 8, 9, 10 and 11 establish the existence of $B2^*$-acceptable allocations associated with the Nash, Raiffa-Kalai-Smorodinsky, equal gains and Shapley-value CC's, when the number of agents is equal to two. We now explain why existence is not guaranteed for $n > 2$.

We will heuristically illustrate the point with the Nash CC in the three-agent case; the argument would be the same for the other examples.

Π_3 contains six elements and for z in $Z(e)$ to be $B2^*$ - N - acceptable, it is necessary that its image in the utility space be a Nash solution for each of the six extended bargaining problems $(u(Z^+(e)), u(\pi(z)))$. It is well-known that the Nash solution t of a bargaining problem (S, d) is characterized by the property that S be supported at t by a hyperplane with a positive normal p satisfying

(i) $$\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\right) = k(t_1 - d_1, t_2 - d_2, t_3 - d_3) \quad \text{for} \quad k > 0 .$$

If the boundary of $u(P(e))$ is smooth, there is a unique supporting hyperplane at any point in its relative interior, and the points d satisfying (i) belong to a segment. It is however unlikely that all the images of $\pi(z)$ would ever be on that segment.

The source of the difficulty is that when the number of agents increases, the number of conditions that an allocation has to satisfy in order to be $B2^*$ - acceptable grows very rapidly, to be precise as the number of permutations of order n; these conditions are already too numerous when $n = 3$.

One should note however that how strict each of these conditions is depends on how "large" the image under φ of $(e, \pi(z))$ is. If $\varphi(e, \pi(z))$ is a large set, z will be more likely to belong to it and existence of $B2^*$ - φ - acceptable allocations will also be more likely. In that respect, it is therefore relevant to point out that under strict convexity of preferences, examples (e) through (h) generate unique φ optimal allocations. This may explain in part the negative results obtained earlier. However, this cannot be the whole story as clearly indicated by the positive result derived for the Walrasian CC since there are large subclasses of $\mathcal{E}$ for which Walrasian allocations are "few", and even unique.

The existence of $B4^*$ - φ - acceptable allocations can be proved on certain subclasses of $\mathcal{E}$ (W for any n, N, R, G, V for $n = 2$) and implies existence of $B3^*$ - φ - acceptable on the same subclasses. The nonexistence of $B3^*$ - φ - acceptable (IP, C, T) follows from the nonexistence of $C3^*$ - φ -

acceptable allocations (see next section) and implies the nonexistence of $B4^*$ - φ - acceptable allocations. These results are summarized in Section 6.

4. Weaker concepts of acceptability

4-1. Pairwise comparisons

In the previous section, we encountered several choice correspondences with no $B2^*$ - acceptable allocations, and we attributed this result to the large number of conditions that an allocation has to satisfy to be $B2^*$ - acceptable. In our first reformulation of the concept of acceptability, we specifically aim at reducing the number of these conditions by permitting only pairwise exchanges of initial bundles: given some initial allocation ω, each individual successively swaps his initial bundle with every other agent; the remaining agents keeping theirs. The choice correspondence is then operated, and the agent compares the resulting allocations to ω.

Given Π_n^2, the subset of Π_n of permutations exchanging at most two terms, we obtain the following counterparts of $B1^*$ and $B2^*$.

Definition $C1^*$: z in Z(e) is a $C1^*$ - φ - acceptable allocation for e if there exists ω in Z(e) s.t.

$$\forall \pi \in \Pi_n^2, \; z \in \varphi(e, \pi(\omega)) .$$

Definition $C2^*$: z in Z(e) is a $C2^*$ - φ - acceptable allocation for e if it is a $C1^*$ - φ - acceptable allocation for e with $\omega = z$.

$C3^*$ and $C4^*$ are obtained from $C1^*$ and $C2^*$ as $B3^*$ and $B4^*$ were obtained from $B1^*$ and $B2^*$.

Note that in the two-person case, $C1^*$ and $C2^*$ coincide with $B1^*$ and $B2^*$ respectively, since $\Pi_2 = \Pi_2^2$.

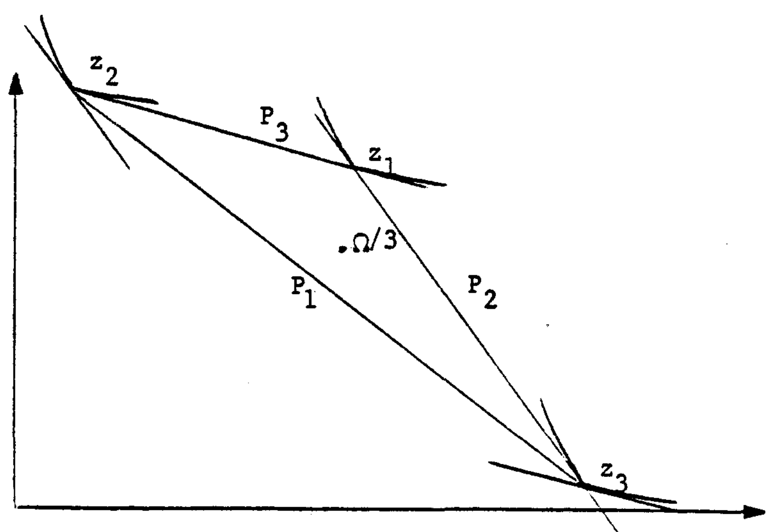
One could say that an allocation is acceptable according to these definitions if <u>no agent envies anybody else's initial position</u>.

We postpone the discussion of $C1^*$ to the next subsection, and concentrate here on $C2^*$.

-- Because $C2^*$ is weaker than $B2^*$, the existence of $B2^*$ - φ - acceptable allocations when φ is W, C, IP, T, proved in Section 3, implies the existence of $C2^*$ - φ - acceptable.

The next example shows that $C2^*$ is a genuine weakening of $B2^*$.

<u>Example</u>: We consider the Walrasian CC. From Proposition 3, we know that $B2^*_W(e) = W(e, \bar{\omega})$. From the statements of definition $B2^*$ and $C2^*$, we can conclude that $C2^*_W(e) \supset B2^*_W(e)$; we now establish that the inclusion may be strict.



<u>Figure 7</u>

This is a three-agent example. Π_3^2 contains three non-trivial elements. Aggregate endowment and preferences are chosen in such a way that $z = (z_1, z_2, z_3)$ is Pareto-efficient, and belongs to $W(e, \pi(z))$ for all π in Π_3^2. One can indeed check that z is a Walrasian allocation for (e, z), $(e, (z_1, z_3, z_2))$, $(e, (z_3, z_2, z_1))$, $(e, (z_2, z_1, z_3))$ associated with the equilibrium prices p_1 (or p_2 or p_3), p_1, p_2 and p_3, respectively. It follows that $z \in C2_W^*(e)$. However, $z \notin W(e, \bar{\omega})$.

-- If φ is either one of N, R, G, SV, the nonexistence results of the previous section persist whenever $n > 2$. Instead of increasing like $n!$, the number of conditions for acceptability increases like n^2 but this is still too much.

The next lemma shows that most economies will have no $C2^*$-acceptable allocations relative to the Nash CC. The same reasoning would apply as well to the Raiffa-Kalai-Smorodinsky, equal gains and Shapley-value CC's.

<u>Proposition 12</u>: If $\bar{\omega} = (\Omega/n, \ldots, \Omega/n)$ is not Pareto-efficient for e, and preferences are strictly convex, $C2_N^*(e)$ is empty.

<u>Proof</u>: It is illustrated in Figure 8 for three agents. Let $t = u(z) \in u(P(e))$ be given, and consider the element of Π_3^2 that exchanges z_1 and z_3.

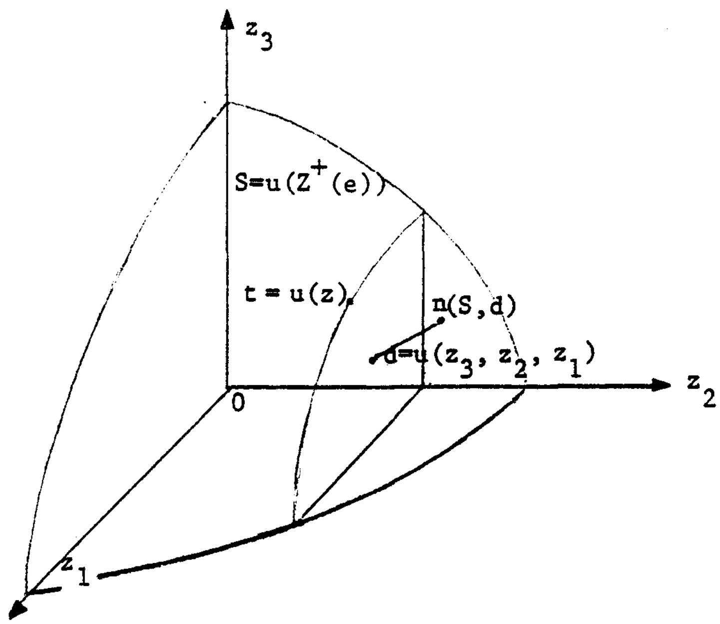


Figure 8

The image d of (z_3, z_2, z_1) under u is in a plane parallel to the plane determined by the first and third axes. If $d \notin u(P(e))$, the extended Nash solution outcome for $(u(Z^+(e)), d)$ strictly dominates d, and is therefore not equal to t. For t to be in $C2^*_N(e)$, it is therefore necessary that d be in $u(P(e))$. By strict convexity of preferences, this implies that $z_1 = z_3$. It could similarly be proved that $z_1 = z_2$ and $z_2 = z_3$, from which it follows that $z_1 = z_2 = z_3 = \Omega/3$. However it is excluded by hypothesis that $\bar{\omega}$ be in $P(e)$. This contradiction proves the lemma for $n = 3$. The proof of the general case is identical.

Q.E.D.

4-2. Relaxing the constraints on the initial endowment

We now turn to the odd-numbered definition $B1^*$ and $C1^*$ (already stated) which permit the free choice of ω.

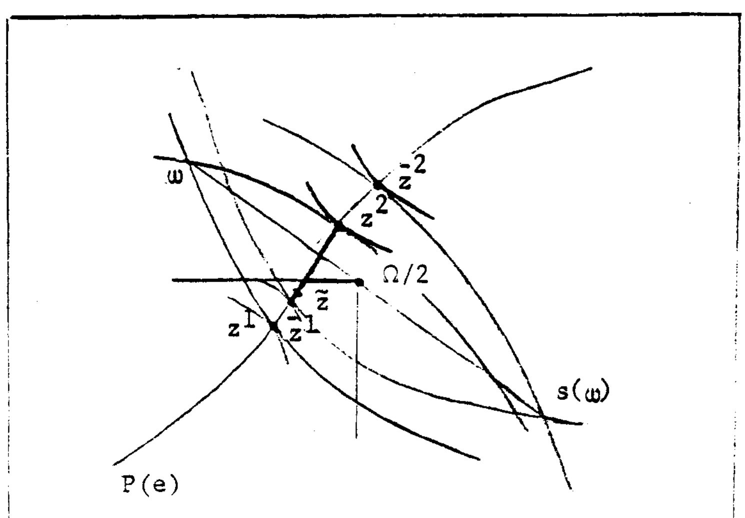
Since $B1^*$ and $C1^*$ are weaker than $B2^*$ and $C2^*$, any existence result for the latter definitions extends to the former. This is the case for W, C, IP, T.

However, the nonexistence results concerning $C2^*$ - φ - acceptable (and therefore $B2^*$ - φ - acceptable) allocations when φ is N, R, G or SV turn to existence results if the weaker concept of acceptability is adopted. To show this, it suffices to choose $\bar{\omega} = (\Omega/n, \ldots, \Omega/n)$ as initial allocation to prove that $B1^*$ - φ - acceptable (and therefore $C1^*$ - φ - acceptable) allocations exist.

On the negative side, it should be noted that the weaker definitions $B1^*$ and $C1^*$ lead to some unpalatable results for some CC's and some configurations of preferences, as illustrated by the following example.

<u>Example</u>

This is a two-person, two-commodity example with φ as the core CC, and where there exists a $B1^*$ - φ - acceptable allocation allocating strictly more of all the commodities to one of the two agents.



<u>Figure 9</u>

The core relative to ω, $C(e, \omega)$, is the curvilinear segment z^1, z^2. The core relative to $s(\omega)$, $C(e, s(\omega))$, is the curvilinear segment $\bar{z}^1, \bar{z}^2$. Their intersection contains an allocation $\tilde{z}$ with the property that $\tilde{z}_2 > \tilde{z}_1$.

This phenomenon, which is also exhibited by the Pazner-Schmeidler concept of egalitarian-equivalence (see Appendix 2) can be avoided by operating φ from $\bar{\omega}$ for any individually rational φ. On the other hand, note that if ω moves in a northwesternly direction, the set of $B1^*$-C-acceptable allocations widens and more and more allocations exhibit this unpleasant feature.

The nonexistence of $C3^*$-φ-acceptable allocations for IP, C, T (due to an essential lack of nonuniqueness of $\varphi(e)$), implies the nonexistence of $C4^*$-, $B3^*$- and $B4^*$-φ-acceptable allocations. The existence of $C3^*$-W-acceptable allocations follows from the existence of $C4^*$-W-acceptable allocations. The existence of $C3^*$- and (for $n = 2$) of $C4^*$-φ-acceptable allocations for N, R, G, V follows from the existence of $B3^*$- and (for $n = 2$) of $B4^*$-φ-acceptable allocations. The nonexistence of $C4^*$-φ-acceptable allocations for N, R, G, V and $n > 2$ follows from the non-existence of $C2^*$-φ-acceptable allocations.

5. Averaging Operation

5-1. Definitions

In this section, we propose another reformulation of the equity concepts presented earlier, in the spirit of an argument developed in Thomson [29]. It is argued there that an agent should not really care about the way the others have distributed among themselves the resources allocated to them, but only about what they consume on average; or that if an individual does care, his feelings in that respect should be disregarded.

An <u>A - envy - free allocation</u> (where A stands for average) is then defined as an allocation where no one would rather consume the average bundle consumed by everybody else. The informational efficiency of this definition (only n comparisons are necessary in order to verify that an allocation is A - envy-free versus the n(n - 1) comparisons required in order to check whether an allocation is envy free) is the analytic form taken by this requirement that agents not be "nosy."

Instead of performing on the initial allocations the permutations described in the preceding sections, we will here follow the spirit of the above argument and subject the initial allocation to this averaging operation, before applying φ. Formally, we define:

The <u>averaging transformation</u> $a : R^{\ell n} \to R^{\ell n}$ associates to every list z in $R^{\ell n}$ the list $a(z) = (a_1(z), \ldots, a_n(z)) = (\sum_{j\neq 1} z_j/(n-1), \ldots, \sum_{j\neq 1} z_j/(n-1))$. Note that if z is feasible for e, so is $a(z)$ since for all i, $a_i(z)$ belongs to Z_i, and

$$\Sigma\, a_i(z) = \Sigma[\sum_{j\neq i} z_j/(n-1)] = \Sigma\, z_i = \Omega .$$

<u>Definition A</u>: z in $Z(e)$ is <u>A - envy - free for e</u> if

$$\forall i ,\; z_i \succsim_i a_i(z) \quad .$$

The set of A - envy - free allocations for e is denoted $A(e)$. We can now introduce the counterparts of definitions $B1^*$ and $B2^*$.

<u>Definition $D1^*$</u>: z in $Z(e)$ is a <u>$D1^*$ - φ - acceptable allocation for e</u> if there exists ω in $Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $z \in \varphi(e, a(\omega))$.

Definition $D2^*$: z in $Z(e)$ is a $D2^*$ - φ - acceptable allocation for e if it is a $D1^*$ - φ - acceptable allocation for e with $\omega = z$.

$D3^*$ and $D4^*$ are obtained from $D1^*$ and $D2^*$ as $B3^*$ and $B4^*$ were obtained from $B1^*$ and $B2^*$.

5-2. Applications

In this subsection, we reexamine the examples presented in Subsection 3-2 and, for each of them propose conditions guaranteeing the existence of $D2^*$ - acceptable allocations. All of the examples (except sometimes (d)) select Pareto-optimal allocations, and have the property that if $z \in P(e)$, then $z \in \varphi(e, z)$. Therefore we will not explicitly check (a) in seven of our examples. (The proof for example (d) is straightforward.)

First, we have the analogue of Propositions 1 and 2.

Proposition 1′: If φ and φ' are such that $\varphi'(e, \omega)$ is always a subset of $\varphi(e, \omega)$, then $D2^*_{\varphi'}(e)$ is a subset of $D2^*_{\varphi}(e)$.

Proof: It is too similar to the proof of Proposition to be worth spelling out.

Proposition 2′: Let I be the CC associating to each (e, ω) the set of allocations in $Z(e)$ that are individually-rational from ω. Then $D2^*_I(e)$ is the set of A - envy - free allocations, $A(e)$.

Proof: We have

$$z \in A(e) \Leftrightarrow \begin{cases} \forall i, z_i \succsim_i z_i \\ \forall i, z_i \succsim_i a_i(z) \end{cases} \Leftrightarrow \begin{cases} z \in I(z) \\ z \in I(a(z)) \end{cases} \Leftrightarrow z \in D2^*_I(e) .$$

Q.E.D

We now consider the examples.

(a) <u>W, the Walrasian CC</u>

<u>Proposition 3'</u>: Let W be the Walrasian CC. Then $z \in Z(e)$ belongs to $D2^*_W(e)$ if and only if z is a Walrasian allocation from the equal bundle allocation $\bar{\omega} = (\Omega/n, \ldots, \Omega/n)$.

<u>Note</u>: In conjunction with Proposition 3, this implies $B2^*_W(e) = D2^*_W(e)$.

<u>Proof</u>: (i) It is closely patterned after the proof of Proposition 3. Let $z \in W(e, \bar{\omega})$ be given, and p be associated equilibrium prices. Then for all i, $p z_i \leqq p\,\Omega/n$, which by addition yields $p \Sigma z_i \leq p\,\Omega$. Since $z \in Z(e)$, $\Sigma z_i = \Omega$ and therefore for all i, $p z_i = p\,\Omega/n$ which in turn implies that for all i, $p\,a_i(z) = p\,\Omega/n$. In conjunction with the statement

$$\forall i\,,\ \forall z_i' \in \{z_i' \in Z_i \mid p z_i' \leqq p\,\Omega/n\}\,,\quad z_i \succsim_i z_i' \quad ,$$

expressing that z_i is maximal in agent i's budget set, we obtain

$$\forall i\,,\ \forall z_i' \in \{z_i' \in Z_i \mid p z_i' \leqq p\,a_i(z)\}\,,\ z_i \succsim_i z_i' \quad .$$

This, along with $\Sigma z_i = \Omega$ implies that (p, z) is a Walrasian equilibrium for $(e, a(z))$, and consequently $z \in D2^*_W(e)$.

(ii) Conversely, let $z \in D2^*_W(e)$ be given. Then $\Sigma z_i = \Omega$ and there exists prices p such that

$$\forall i\,,\ \forall z_i' \in \{z_i' \in Z_i \mid p z_i' \leqq p\,a_i(z)\}\,,\ z_i \succsim_i z_i' \quad .$$

This implies that for all i, $p z_i \leqq p\,a_i(z)$. Adding these inequalities yields $p \Sigma z_i \leqq p \Sigma a_i(z)$, and since $\Sigma z_i = \Sigma a_i(z)$ by definition of the averaging operation a, we conclude that for all i, $p z_i = p\,a_i(z)$. Fixing i, and adding $p z_i/(n-1)$ to both sides of this equality yields

$$p\, z_i + p \frac{z_i}{n-1} = p \frac{\sum_{j \neq i} z_j}{n-1} + p \frac{z_i}{n-1} \Rightarrow \frac{n}{n-1}\, p\, z_i = \frac{p\,\Omega}{n-1} \Rightarrow p\, z_i = p\,\Omega/n \ ,$$

and therefore $p\, a_i(z) = p\,\Omega/n$.

This derivation being true for all i , we finally obtain that:

$$\forall i,\ \forall z_i' \in \{z_i' \in Z_i \mid p\, z_i' \leq p\,\Omega/n\},\quad z_i \succsim_i z_i' \ ,$$

which in conjunction with $\sum z_i = \Omega$ implies that $z \in W(e, \bar{\omega})$.

Q.E.D.

(b) <u>IP, the CC selecting the set of individually-rational and Pareto-optimal allocations</u>

<u>Proposition 4'</u>: $D2^*_{IP}(e)$ is the set of A - envy-free and Pareto-efficient allocations for e .

<u>Proof</u>: Analogous to the proof of Proposition 4.

<u>Remark 1'</u>: Same as Remark 1 with $D2^*$ replacing $B2^*$.

(c) <u>C, the core CC</u>

<u>Proposition 5'</u>: Let C be the core CC , and assume that preferences are locally non-satiated. Then $D2^*_C(e)$ is a superset of $W(e, \bar{\omega})$.

<u>Proof</u>: Analogous to the proof of Proposition 5.

<u>Remark 2'</u>: Same as Remark 2 with $D2^*$ replacing $B2^*$.

(d) <u>T, the CC selecting the set of allocations obtained through an envy-free trade Pareto-undominated by any other envy-free trade</u>

<u>Proposition 6'</u>: Assume that for all i , u_i is continuous. Then $D2^*_T(e)$ is a superset of $W(e, \bar{\omega}) \cap Z(e)$.

<u>Proof</u>: It follows from Proposition 1' and lemma 2.

Remark 6: There is no equivalent of Proposition 7.

We now discuss the examples (e) - (h) for which we had previously proved the emptiness of the sets of $B2^*$ - acceptable allocations when $n > 2$. The definition proposed now turns all of these results from negative to positive.

(e) N, the Nash CC

Proposition 8′: Assume that for each i, u_i is (i) continuous, (ii) strictly monotonic, (iii) concave and (iv) u_i represents strictly convex preferences. Then $D2^*_N(e)$ is nonempty.

Proof: Here are the main steps of the proof: first, it is argued that $u(P(e))$, the image in the utility space of the set of Pareto-optimal allocations is homeomorphic to a simplex. Then a function from $u(P(e))$ into itself is defined as follows: to each t in $u(P(e))$ is associated the unique allocation z whose image under u is t; z is subjected to the averaging operation a, the image under u of $a(z)$ is determined and used as disagreement point in the computation of a Nash solution outcome. The function associating to t this Nash solution outcome is shown to be continuous and the Brouwer fixed point theorem is finally applied.

Turning to a formal proof, we have:

Under assumptions (i), (ii), (iii), the pair $(S, d) \equiv (u(Z^+(e)), u(\omega))$ is a well-defined extended bargaining problem for any ω in $Z(e)$, and admits of an extended Nash solution outcome (see the paragraph preceding Proposition 8).

Under assumptions (i) and (ii), there exists a homeomorphism between the $(n-1)$ - dimensional simplex S^{n-1} and $u(P(e))$ (Chipman and Moore [3]).[8] Let Ψ be such a homeomorphism.

Under the additional assumption (iii) the restriction u_P^{-1} of u^{-1} from $u(P(e))$ into $P(e)$ is a homeomorphism (Chipman and Moore [3]).

The function $a : Z(e) \to Z(e)$ is linear and therefore continuous.

The function $\bar{n} : S \to S$ defined by $\bar{n}(d) = n(S, d)$ is continuous. To show this, let $B : S \to S$ be defined by $B(d) = \{u' \in S | u' \geqq d\}$. Compactness of S and (ii) guarantee the continuity of B. The function $g : S \times S \to R$ defined by $g(u, d) = \prod(u_i - d_i)$ is continuous and for each d, $\bar{n}(d)$ is the maximizer of g in u when u varies in $B(d)$. All the conditions are satisfied to apply the maximum theorem (Berge [2], Debreu [8]) stating that $\bar{n}$ is an upper semi-continuous correspondence. In fact, $\bar{n}$ is a continuous <u>function</u>, by convexity of S, implied by (iii) (Chipman and Moore [3]).

Let now $\gamma : S^{n-1} \to S^{n-1}$ be defined by

$$\gamma = \Psi^{-1} \circ \bar{n} \circ u \circ a \circ u_P^{-1} \circ \Psi \quad .$$

γ is a continuous function and by the Brouwer theorem (Debreu [8]), there exists s^* in S^{n-1} such that $\gamma(s^*) = s^*$. Let then $z^* = u_P^{-1} \circ \Psi(s^*)$. We have $z^* = \bar{n} \circ u \circ a(z^*)$, or $z^* \in \varphi(e, a(z^*))$ which proves the proposition.

Q.E.D.

<u>Remark 7</u>: This remark completes the proof of Proposition 8, where assumption (iv) is not made, and $n = 2$ (if $n = 2$, $D2^*$ is equivalent to $B2^*$). Given t in $u(P(e))$, the set $L_1 = \{z \in Z(e) | u(z) = t\}$ is convex, and therefore connected, and its image under the continuous function $\Psi^{-1} \circ \bar{n} \circ u \circ a$ will also be connected. Since S^1 is a segment, $\gamma(\Psi^{-1}(t))$ is a convex subset of S^1. Upper semicontinuity of γ still holds as the composition of u_P^{-1}, which has this property, with Ψ on the right and $\Psi \circ n \circ u \circ a$ on the left, which are both continuous functions (Berge [2]). An application of the Kakutani fixed point theorem yields the desired result.

(f) R, the Raiffa-Kalai-Smorodinsky CC

Proposition 9': Suppose that for each i, u_i is (i) continuous, (ii) strictly monotonic, and u_i (iii) represents strictly convex preferences. Then $D2_R^*(e)$ is nonempty.

Proof: First, we observe that under assumptions (i) and (ii), $(S, d) \equiv (u(Z^+(e)), u(\omega))$ is a well-defined generalized bargaining problem for any ω in $Z(e)$, and admits of a generalized Raiffa-Kalai-Smorodinsky solution (see paragraph preceding Proposition 9).

The rest of the proof follows the logic of the preceding proof and the only step that differs is spelled out here.

The function $\bar{r} : S \to S$ defined by $\bar{r}(d) = r\ (S, d)$ is continuous. Given d in S, let $v(S, d)$ be defined as in Section 3-2, example f, (i.e., by $v_i(S, d) = \max\{u_i | u \in S,\ u \geqq d\}$). $v_i(S, d)$ is more conveniently seen as the maximizer of the projection function $p_i : S \times S \to R$ defined by $p_i(d, u) = u_i$ over the set $B(d) = \{u' \in B | u' \geqq d\}$ introduced in the previous proof. Since p_i is continuous, and that B is a continuous correspondence, an application of the maximum theorem (Berge [2], Debreu [8]) yields the needed continuity of v as a function of d. The segment connecting d to $v(S, d)$ varies continuously with d and so does its intersection with the boundary of S.

The proof concludes as for Proposition 8.

Q.E.D.

Remark 8: An argument identical to that presented in remark 7 following the proof of Proposition 8' permits us to conclude the proof of Proposition 9.

(g) G, the equal gains CC

Proposition 10′: Suppose that for each i, u_i is (i) continuous, (ii) strictly monotonic and u_i (iii) represents strictly convex preferences. Then $D2^*_G(e)$ is nonempty.

Proof: Again, the proof follows that of Proposition 8′. The only step that differs is the proof of the continuity of the function $\bar{g} : S \to S$ defined by $\bar{g}(d) = g(u(Z^+(e)), d)$, i.e., $\forall i, j$, $\bar{g}_i(d) - d_i = \bar{g}_j(d) - d_j$ and $\nexists\, u' \in S$ with $u' \geq \bar{g}(d)$. This property is however easy to verify.

The proof concludes as for Proposition 8′.

Remark 9: An argument identical to that presented in remark 7 following the proof of Proposition 8′ permits us to conclude the proof of Proposition 10.

Remark 10: In the case of transferable utilities, the equal gains CC coincides with the n-person Nash CC.

Remark 11: Let $W : S \times S \to S$ be a single-valued social welfare function continuous in both arguments. Reasoning similar to the one appearing in Propositions 8′ guarantees the nonemptiness of $D2^*_W(e)$.

(h) V, the Shapley-value CC

Proposition 11′: Let $\tilde{\mathcal{E}}$ be the class of economies e with transferable utility as defined in Section 1. Assume that for each i, u_i is (i) continuous, (ii) monotonic and (iii) concave. Then $D2^*_V(e)$ is nonempty.

The proof relies on the following theorem, where A is an (m, n) matrix, α an n-vector and β an m-vector.

Theorem 1: (Gale [12], p. 48). Exactly one of the following alternatives holds: Either the equation $A\alpha = 0$ has a semipositive solution, or the inequality $\beta A > 0$ has a solution.

Proof of the Proposition: It is organized in several steps. Given e as in the statement of Proposition 11′, we show that $P(e)$ contains a subset M_0 homeomorphic to R^n (step 1). We successively introduce the subset M_1 of M_0 of A-envy-free allocations (step 2), the image M_2 of M_1 under the averaging operation a, and the intersection M_3 of M_0 with the image under V of M_2. We define the selection V' of V associating to each allocation z in M_2, the unique allocation in M_0 such that $V'(z) \in \{V(z)\}$ (step 3). Finally we show that $V' \circ a : M_1 \to M_3$ is linear (step 4) and admits of a fixed point (step 5).

Reminder: For any economy in $\tilde{\mathcal{E}}$, for each i, there exists $v_i : R_+^{\ell-1} \to R$ such that $u_i(x_i, y_i) = x_i + v_i(y_i)$.

Step 1: Given the linear separable form of the utilities, z in $Z(e)$ is Pareto-efficient for e if and only if y maximizes $\Sigma v_i(y_i)$ over the set $\{y' \in R_+^{(\ell-1)n} | \Sigma y_i \leq \Sigma \omega_{iy} = \Omega_y\}$. Let y^* be a maximizer such that $\Sigma y_i^* = \Omega y$. The existence of y^* follows from (i) and (ii).

Since the distribution of the first commodity among the agents is irrelevant for optimality, the set

$$M_0 = \{z = (x_i, y_i)_{i=1}^n \in (R \times R_+^{\ell-1})^n | \Sigma x_i = \Omega_x , \quad y = y^*\}$$

is a subset of $P(e)$.

Step 2: We now introduce the subset of M_0 of A-envy-free allocations for e. To that end, for each i we let $\bar{x}_i$ be defined by:

$$\bar{x}_i + v_i(y_i^*) = \frac{\Omega_x - \bar{x}_i}{n-1} + v_i\left(\frac{\Omega_y - y_i^*}{n-1}\right) .$$

An allocation z in M_0 is A-envy-free if and only if for each i, $x_i \geq \bar{x}_i$.

Let p be some efficiency prices supporting an efficient allocation z' with $y = y^*$. For each i, let $\tilde{z}_i$ be defined by $p\tilde{z}_i = p\Omega/n$ and $\tilde{y}_i = y_i^*$. Clearly, $\tilde{z}$ is a Walrasian allocation for $(e, \bar{\omega})$. It is proved in Thomson [ ] that $\tilde{z}$ is A-envy-free. Therefore there are allocations in M_0 that are A-envy-free. It follows from this that $\Sigma \bar{x}_i \leq \Omega_x$, and consequently the set M_1 defined below is nonempty:

$$M_1 = \{z = (x_i, y_i)_{i=1}^n \in (R \times R_+^{\ell-1})^n \mid \forall i,\ x_i \geq \bar{x}_i;\ y = y^*\} \quad .$$

Step 3: Given each $z \in M_1$, let $V \circ a(z)$ be the set of value-allocations from $a(z)$. Let $z' \in V \circ a(z)$ be given and for each i, let x_i'' be defined by $x_i'' + v_i(y_i^*) = x_i' + v_i(y_i')$; the list $(x_i'', y_i^*)_{i=1}^n$ is a feasible allocation for e; it is also efficient; it is clearly in $V \circ a(z)$. It follows that it is in M_0. So let V' be the function associating to each allocation $a(z)$ the unique value-allocation for $(e, a(z))$ in M_0.

Step 4: We now prove that the mapping $V' \circ a : M_1 \to M_0$ is linear.

For all $z = (x, y)$ in M_1, $y = y^*$. Consequently, for all $z' = (x', y')$ in $a(M_1)$, $y' = a(y^*)$. Given z' in $a(M_1)$, let $w(e, z'; \Gamma)$ be the worth of the coalition Γ in (e, z'). It is given by

$$w(e, z'; \Gamma) = \max\{\sum_{i\in\Gamma} (x_i + v_i(y_i)) \text{ s.t. } \sum_{i\in\Gamma} x_i \leq \sum_{i\in\Gamma} x_i' \text{ and } \sum_{i\in\Gamma} y_i \leq \sum_{i\in\Gamma} y_i'\}$$

$$= \sum_{i\in\Gamma} x_i' + \max\{\sum_{i\in\Gamma} v_i(y_i) \text{ s.t. } \sum_{i\in\Gamma} y_i \leq \sum_{i\in\Gamma} y_i'\} \quad .$$

Given z' and z'' in $a(M_1)$, $y' = y'' = a(y^*)$ and the maximum appearing in this last expression is the same. This implies that

$$w(e, z''; \Gamma) - w(e, z'; \Gamma) = \sum_{i\in\Gamma} x_i'' - \sum_{i\in\Gamma} x_i' \quad .$$

Moreover, since

$$w_i(e, z') = \sum_{\Gamma \ni i} k_\Gamma [w(e, z'; \Gamma) - w(e, z'; \Gamma\backslash i)]$$

$$w_i(e, z'') = \sum_{\Gamma \ni i} k_\Gamma [w(e, z''; \Gamma) - w(e, z''; \Gamma)]$$

where $\sum_{S \ni i} k_\Gamma = 1$, we conclude that

$$w_i(e, z'') - w_i(e, z') = \sum_{\Gamma \ni i} k_\Gamma \left[\sum_{i \in \Gamma} x_i'' - \sum_{i \in \Gamma} x_i' - \sum_{i \in \Gamma\backslash i} x_i'' + \sum_{i \in \Gamma\backslash i} x_i' \right]$$

$$= \sum_{\Gamma \ni i} k_\Gamma [x_i'' - x_i'] = (x_i'' - x_i') \sum_{\Gamma \ni i} k_\Gamma = x_i'' - x_i' \quad .$$

This means that if the distribution of the first commodity among the agents varies, the value of each agent varies by exactly the same amount.

<u>Step 5</u>: Next, we identify the vertices of M_1.

Let $z^i \in M_1$ be defined by

$$\begin{cases} z_j^i = (\bar{x}_j, y_j^*) & \text{for all } j \neq i \quad \text{and} \\ z_i^i = (\Omega_x - \sum_{j \neq i} \bar{x}_j, y_i^*) & . \end{cases}$$

By construction, for all i, $\tilde{z}^i = V' \circ a(z^i) \in M_0$. Because the value is individually rational, for all i and for all $j \neq i$, $\tilde{z}_j^i \succsim_j a_j(z^i)$, which implies that $\tilde{x}_j^i \geq \bar{x}_j$. Since for any $z \in P(e)$, $\sum x_i = \Omega_x$, we have

$$\tilde{x}^i = \begin{pmatrix} \tilde{x}_1^i \\ \vdots \\ \tilde{x}_i^i \\ \vdots \\ \tilde{x}_n^i \end{pmatrix} = \begin{pmatrix} \bar{x}_1 + \varepsilon_1^i \\ \vdots \\ \Omega_x - \sum_{j \neq i} \bar{x}_j - \sum_{j \neq i} \varepsilon_j^i \\ \vdots \\ \bar{x}_n + \varepsilon_n^i \end{pmatrix} = \begin{pmatrix} x_1^i + \varepsilon_1^i \\ \vdots \\ x_i^i - \sum_{j \neq i} \varepsilon_j^i \\ \vdots \\ x_n^i + \varepsilon_n^i \end{pmatrix} \equiv x^i + \eta^i$$

where $\varepsilon_j^i \geq 0$ for all i and j.

Given α, an arbitrary point in the $(n-1)$ dimensional simplex S^{n-1}, let $z(\alpha) = \alpha_1 z^1 + \alpha_2 z^2 + \cdots + \alpha_n z^n$. It follows from the definition of M_1, that $z(\alpha) \in M_1$. We now claim that there exists $\alpha^* \in S^{n-1}$ such that $z(\alpha^*) = V' \circ a(z(\alpha^*))$. $z(\alpha^*)$ will be an element of $D2_V^*(e)$.

By the linearity of $V' \circ a$, $V' \circ a(z(\alpha)) = \alpha_1 V' \circ a(z^1) + \cdots + \alpha_n V' \circ a(z^n)$. It is easy to check that the y components of both vectors are equal for all $\alpha \in S^{n-1}$. Therefore, it remains to find a solution $\alpha^* \in S^{n-1}$ to:

$$\alpha_1 x^1 + \cdots + \alpha_n x^n = \alpha_1 \tilde{x}^1 + \cdots + \alpha_n \tilde{x}^n$$
$$= \alpha_1 (x^1 + \eta^1) + \cdots + \alpha_n (x^n + \eta^n) \quad ,$$

which can be equivalently written as

$$A \alpha = 0 ,$$

where $A = (\eta^1, \ldots, \eta^n)$. In view of Theorem 1, it is sufficient to prove that the inequality $\beta A > 0$, with $\beta \in R^n$, has no solution. Suppose, by way of contradiction, that it did, and let β_{j_o} be the biggest of its coordinates. Multiplying β by the j_o^{th} column of A, we obtain

$$(1) \quad \beta_1 \varepsilon_1^{j_o} + \cdots + \beta_{j_o} (- \sum_{j \neq j_o} \varepsilon_j^{j_o}) + \cdots + \beta_n \varepsilon_n^{j_o} > 0 \quad .$$

Since $\varepsilon_j^{j_o} \geqq 0$ for all $j \neq j_o$, we have $\beta_j \varepsilon_j^{j_o} \leqq \beta_{j_o} \varepsilon_j^{j_o}$ for all $j \neq j_o$, so that the left-hand side of inequality (1) is smaller than

$$\beta_{j_o} \varepsilon_1^{j_o} + \beta_{j_o} \varepsilon_2^{j_o} + \cdots + \beta_{j_o} (- \sum_{j \neq j_o} \varepsilon_j^{j_o}) + \cdots + = \beta_{j_o} 0 = 0 \quad ,$$

which yields a contradiction.

Q.E.D.

Remark 12: This existence proof actually provides a method of finding the $D2^* - V$ - acceptable allocations.

The existence or nonexistence of $D1^*$ - , $D3^*$ - , $D4^* - \varphi$ - acceptable allocations can be deduced from the logical relationship between these definitions as was done for the other sets of definitions. All the results are summarized in the next section.

6. Summary

All the definitions developed in the text are reproduced here, their logical interrelations indicated and the existence or nonexistence results summarized in Tableau 2.

Π_n : is the class of permutations of order n.

B1 : $z \in Z(e)$ is a B1 - φ - acceptable allocation for e if there exists $\omega \in Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $\forall \pi \in \Pi_n, \forall i, \exists z' \in \varphi(e, \pi(\omega)), z \gtrsim_i z'$

B2 : $z \in Z(e)$ is a B2 - φ - acceptable allocation for e if it is a B1 - φ - acceptable allocation for $\omega = z$

In the next definitions, the permutations are restricted to exchange only two terms. Π_n^2 is the subclass of Π_n with this property:

C1 : $z \in Z(e)$ is a C1 - φ - acceptable allocation for e if there exists $\omega \in Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $\forall \pi \in \Pi_n^2, \forall i, \exists z' \in \varphi(e, \pi(\omega)), z \gtrsim_i z'$

C2 : $z \in Z(e)$ is a C2 - φ - acceptable allocation for e if it is a C1 - φ - acceptable allocation for $\omega = z$

In the next definitions, z is subjected to the averaging operation:

<u>a</u> : $R^{\ell n} \to R^{\ell n}$ defined by $a(z) = (\sum_{j \neq 1} z_j/(n-1), \ldots, \sum_{j \neq n} z_j/(n-1))$

<u>D1</u> : $z \in Z(e)$ is a <u>D1 - φ - acceptable allocation for e</u> if there exists $\omega \in Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $\forall i$, $\exists z' \in \varphi(e, a(\omega))$, $z \succsim_i z'$

<u>D2</u> : $z \in Z(e)$ is a <u>D2 - φ - acceptable allocation for e</u> if it is a D1 - φ - acceptable allocation for $\omega = z$

Each of these definitions can in turn be strengthened by requiring that the conclusion be stated for all z' and not just for some z'. This yields

<u>B3</u> : $z \in Z(e)$ is a <u>B3 - φ - acceptable allocation for e</u> if there exists $\omega \in Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $\forall \pi \in \Pi_n$, $\forall i$, $\forall z' \in \varphi(e, \pi(\omega))$, $z \succsim_i z'$.

<u>B4</u> : $z \in Z(e)$ is a <u>B4 - φ - acceptable allocation for e</u> if it is a B3 - φ - acceptable allocation for $\omega = z$.

<u>C3</u>, <u>C4</u> and <u>D3</u>, <u>D4</u> are obtained from C1, C2 and D1, D2 respectively, as B3, B4 were obtained from B1, B2.

Each of the 12 definitions has the common structure schematically represented in Figure 10.

Given z in Z(e), z is said to satisfy a condition of φ - acceptability for e if

statement quantifying the initial allocation ω	,	statement describing and quantifying the operation(s) "op" performed on the initial allocation and yielding the allocation(s) op(ω)	, for all i,	statement quantifying the allocations z′ in φ(e,op(ω)) about which the final statement applies	,	Final statement $z \succsim_i z'$

Figure 10

The first and last boxed segments may take two forms and the middle segment three forms. These choices, which can be made independently, are summarized in the following three-way tableau.

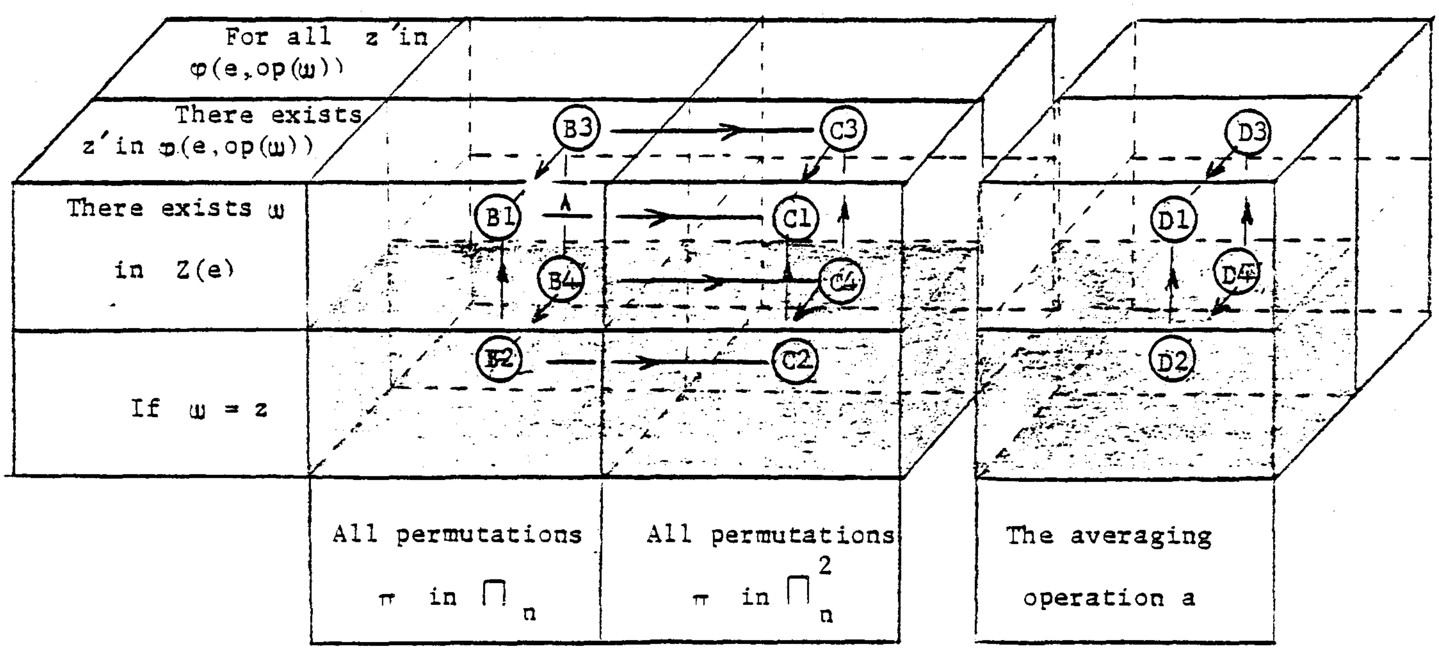


Figure 11

Each of the six definitions on the front of Figure 11 can be strengthened by exchanging the position of the last boxed statement and of the statement quantifying i. For instance B1 and B2 become

B1* : $z \in Z(e)$ is a B1*-φ-acceptable allocation for e if there exists $\omega \in Z(e)$ s.t.

(a) $z \in \varphi(e, \omega)$

(b) $\forall \pi \in \Pi_n$, $\exists z' \in \varphi(e, \pi(\omega))$ s.t. $\forall i$, $z \succsim_i z'$

B2* : $z \in Z(e)$ is a B2*-φ-acceptable allocation for e if it is a B1*-φ-acceptable allocation for e with $\omega = z$

C1*, C2* and D1*, D2* are obtained from C1, C2 and D1, D2 respectively, as B1*, B2* were obtained from B1, B2.

If φ selects Pareto-optimal outcomes and satisfies Property P (Section 3-1) B1* and B2* take the form:

B1* : $z \in Z(e)$ is a B1*-φ-acceptable allocation for e if there exists $\omega \in Z(e)$ s.t.

$$\forall \pi \in \Pi_n, \quad z \in \varphi(e, \pi(\omega)) \quad .$$

B2* : $z \in Z(e)$ is a B2*-φ-acceptable allocation for e if it is a B1*-φ-acceptable allocation for e with $\omega = z$.

Similarly,

B3* : $z \in Z(e)$ is a B3*-φ-acceptable allocation for e if there exists $\omega \in Z(e)$ s.t.

$$\forall \pi \in \Pi_n, \quad c(e, z) = \varphi(e, \pi(\omega))$$

B4* : $z \in Z(e)$ is a B4*-φ-acceptable allocation for e if it is a B3*-φ-acceptable allocation for e with $\omega = z$

C1*, C2*, C3*, C4* and D1*, D2*, D3*, D4* are then obtained from C1, C2, C3, C4 and D1, D2, D3, D4, the way B1*, B2*, B3*, B4* were obtained from B1, B2, B3, B4.

The existence of φ-acceptable allocations according to each of these 18 definitions was then examined when φ was one of eight possible examples. Fortunately, because of the logical connections between them no more than two or three had to be examined in depth for each φ.

The results are summarized in Tableau 2, which should be read as follows:

Each double row corresponds to a particular φ, indicated on the left along with a list of assumptions on (e, ω) guaranteeing the existence of φ-optimal outcomes.

Each double row contains 12 cells in groups of four corresponding to the subdivisions of Figure 11. An existence result is denoted by E followed by a parenthesis containing any assumption used in its proof and not already listed on the extreme left. A nonexistence result is denoted $\not{E}$.

The logical implications between the various definitions are schematically represented above the tableau. An existence result for one cell extends to any other cell that can be reached by following arrows. A nonexistence result for one cell extends to any other cell from which the first cell can be reached by following arrows.

The tableau contains the results for the starred definitions. The results for the unstarred definitions are obtained as follows:

(i) Six columns are identical for the starred and unstarred definitions (second, fourth and sixth columns), since the corresponding starred and unstarred definitions are then equivalent.

(ii) Since the starred definitions imply the unstarred definitions, any existence result for a starred definition yields an existence result for the corresponding unstarred definition.

(iii) The only remaining entries to consider are the non-existence results stated in the bottom half of the first and third columns. They concern φ's that, on very well-behaved environments, select unique outcomes. It is easily checked that the starred and unstarred definitions are equivalent for single-valued φ's. Therefore, the results in the tableau extend directly.

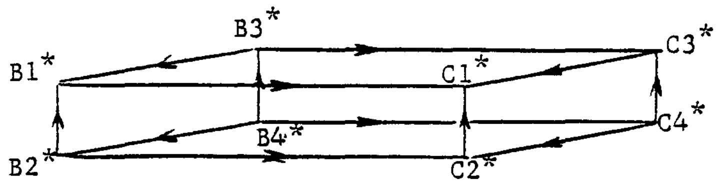
Key for Tableau 2

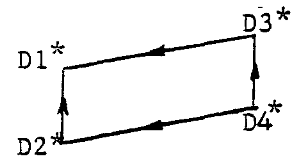
W	- Walrasian CC
IP	- CC selecting individually-rational and Pareto-optimal allocations
C	- Core CC
T	- CC selecting the allocations attainable through an envy-free trade undominated by any other envy-free trade
N	- Nash CC
R	- Raiffa-Kalai-Smorodinsky CC
G	- Equal-gains CC
V	- Shapley-value CC

* * * * * *

c	- continuity of preferences
m	- monotonicity of preferences
s · m	- strong monotonicity of preferences
ca	- concavity of utility functions
vx	- convexity of preferences
s · vx	- strict convexity of preferences
uni	- any assumption implying uniqueness of competitive equilibrium

SUMMARY OF THE RESULTS





	$\forall \pi \in \Gamma_n$		$\forall \pi \in \Gamma_n^2$	
W (c, m, vx)	B1* ↑ B2* E →	B3* ↑ B4* E(uni) →	C1* ↑ C2*	C3* ↑ C4*
IP (c)	↑ E(m, v x) →	↓	← ↑	$\not{E}$ ↓
C (c, vx)	↑ E →	↓	← ↑	$\not{E}$ ↓
T (c)	↑ E(m, v x) →	↓	← ↑	$\not{E}$ ↓

	a(ω)	
W	D1* ↑ D2* E	D3* ↑ D4* E(uni)
IP	↑ E(m, v x)	$\not{E}$ ↓
C	↑ E	$\not{E}$ ↓
T	↑ E(m, v x)	$\not{E}$ ↓

N (c,s·m,ca)	E → E(n = 2) ⇄	E(s · v x) → E(s · vx, n=2) →	$\not{E}$(n > 2) →	→
R (c, s · m)	E → E(n = 2) ⇄	E(s · vx) → E(s · vx, n=2) →	$\not{E}$(n > 2) →	→
G (c, s · m)	E → E(n = 2) ⇄	E(s · vx) → E(s · vx, n=2) →	$\not{E}$(n > 2) →	→
V (c, ca)	E → E(n = 2) ⇄	E(s · vx) → E(s · vx, n=2) →	$\not{E}$(n > 2) →	→

E	↑
E(n = 2) E(s·vx, n>2)	E(s·vx)
E	↑
E(n = 2) E(s·vx, n>2)	E(s·vx)
E	↑
E(n = 2) E(s·vx, n>2)	E(s·vx)
E	↑
E(n = 2) E(s·vx, n>2)	E(s · vx)

Tableau 2

Appendix 1

We regroup here the concepts of equity which can be found in the current literature on the topic. In these definitions, e is an arbitrary element of $\mathcal{E}$, and z an arbitrary feasible allocation for e.

Definition: z is envy-free for e if

$$\forall i, j \quad z_i \succsim_i z_j \quad .$$

This definition was first introduced by Foley [10]. It is extensively studied by Kolm [15] and Varian [30].

Definition: z is egalitarian equivalent for e if

$$\exists z_o \in R_+^{\ell} \text{ s.t. } \forall i \quad z_i \sim_i z_o \quad .$$

This concept, due to Pazner and Schmeidler [20], is generalized by Pazner [18] under the name of fair-equivalence, and studied by Crawford [5], [6].

Definition: z is per capita envy-free for e if

$$\forall i, \; z_i \succsim_i \Omega/n \quad .$$

This concept is introduced and discussed by Pazner [18].

Definition: z is c'-envy-free for e if

$$\forall \Gamma, \; \forall z' \in R_+^{\ell|\Gamma|}, \left[\sum_{i \in \Gamma} z_i' \leq \frac{|\Gamma|}{n-|\Gamma|} \sum_{i \in \Gamma} z_i ; \; z_i' \succsim_i z_i \; \forall i \in \Gamma \right] \Rightarrow z_i' \sim_i z_i \; \forall i \in \Gamma$$

where $|\Gamma|$ denotes the cardinality of the coalition Γ.

Varian introduces this concept [30] in a form requiring efficiency and uses the term c'-fairness. Two other concepts of coalitional equity are proposed by Varian [30] and Gabszewicz [11].

Definition: z is income-fair for e if

(a) z is Pareto-efficient, with efficiency prices p

(b) $\forall i, j \quad p z_i = p z_j$

This concept, suggested by Pazner and Schmeidler [20a] is discussed by Varian [30] [32] and Pazner [18].

Definition: z is opportunity-fair for e if

(a) z is Pareto-efficient, with efficiency prices p

(b) $\forall i, j \ \forall z'_j$, if $p z'_j \leqq p z_j$, then

$z_i \succsim_i z'_j$

Varian introduced and discussed this concept relating it to the concept of income-fairness (previous definition) [32].

Definition: z is A - envy-free for e if

$\forall i, z_i \succsim_i a_i(z)$

This concept, which is a weakening of Varian's concept of c'- envy-free is briefly discussed in Thomson [28], and further studied in Thomson [29].

Definition: The net trade t in $R^{\ell n}$ is an envy-free net trade for e if

$\forall i, j, \ \omega_i + t_j \in R^{\ell}_{+} \Rightarrow \omega_i + t_i \succsim_i \omega_i + t_j$.

Schmeidler and Vind [24] introduced this concept under the name of fair net trade as well as a related concept of a strongly fair net trade.

Appendix 2

The purpose of this appendix is to establish the following fact, a weaker version of which is used in Section 4-2.

Lemma: There exist economies admitting of Pareto-efficient and egalitarian-equivalent allocations attributing to one agent an arbitrarily high proportion of the aggregate endowment.

Proof: It is provided by an example of a two-commodity, two-agent economy defined as follow . Let Δ be the 45° line; agent 1 has right-angle indifference curves with vertices on Δ and agent 2 has symmetric Cobb-Douglas indifference curves. The aggregate endowment Ω is a point of Δ.

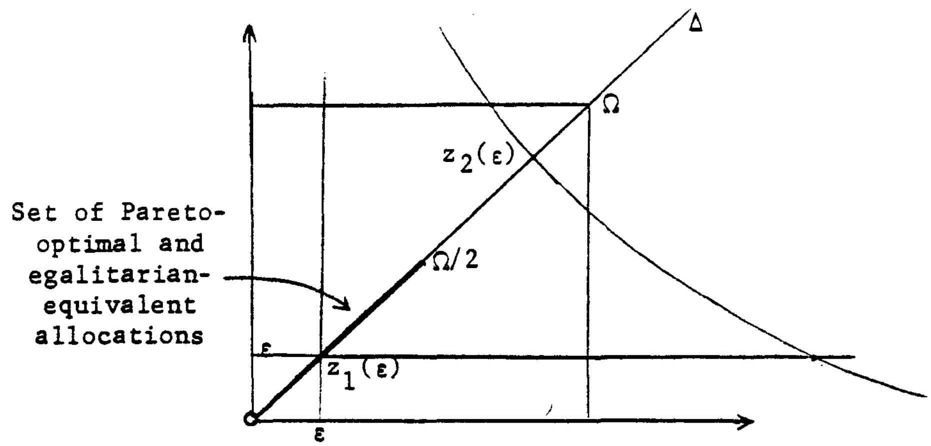


Figure 12

From monotonicity and symmetry of preferences, it follows that a feasible allocation $z = (z_1, z_2)$ is Pareto-optimal if and only if $z_1, z_2 \in \Delta$ and $z_1 + z_2 = \Omega$. Let $\varepsilon \in R_+$ be given, $z_1(\varepsilon) = (\varepsilon, \varepsilon)$, $z_2(\varepsilon) = \Omega - (\varepsilon, \varepsilon)$. For every $0 < \varepsilon \leq \Omega_1/2$, $z(\varepsilon)$ is egalitarian-equivalent since a rectangular hyperbola through $z_2(\varepsilon)$ with the axes as asymptotes will necessarily intersect a horizontal half line through $z_1(\varepsilon)$. The lemma is proved by choosing ε arbitrarily close to zero. This example reveals a certain form of inequality permitted by the concept.

Remark: This example also reveals that the set of Pareto-optimal and egalitarian-equivalent allocations is not closed. To see this, observe that $z(0)$ is not egalitarian-equivalent, although it is the limit of $z(\varepsilon)$ as $\varepsilon \to 0$, where each $z(\varepsilon)$ is egalitarian-equivalent.

Appendix 3

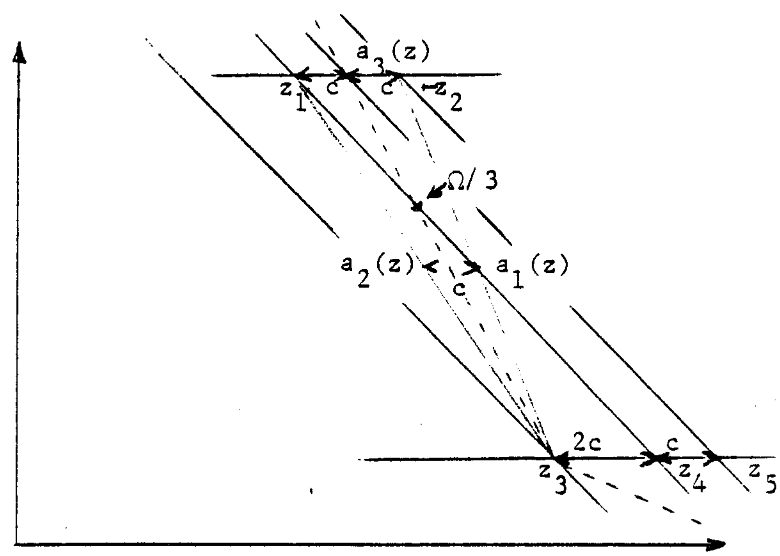
In the proof of Proposition 11, we saw that for two-person economies, the image under $V \circ a$ of $AP(e)$ is precisely $AP(e)$.. We prove here that this is not the case if $n > 2$, a fact which explains why the proof of

Proposition 11′ concerning the existence of $D2^{*}$ - V - acceptable allocations does not more closely resemble the equivalent proof for the Nash, RKS and Equal Gains CC's.

<u>Lemma</u>: There are economies e in $\tilde{\mathcal{E}}$ admitting of Pareto-efficient and A - envy-free allocations z such that no Shapley-value-allocation for (e, a(z)) is A - envy-free for e. Formally,

$$z \in AP(e) \not\Rightarrow V(e, a(z)) \cap AP(e) \neq \emptyset .$$

<u>Proof</u>: It is in the form of a two-commodity, three-agent example illustrated in Figure 13.



<u>Figure 13</u>

First, we recall that transferable utility economies have the property that the preference map of each agent can be obtained by translation of one sample indifference curve parallel to the money axis. In Figure 13, agents 1 and 2 have identical preferences represented by the heavy lines, and agent 3 has preferences represented by the broken line. Ω is the center of gravity of the triangle z_1, z_2, z_3. For each i, $a_i(z)$ is the middle of the side of that triangle facing z_i. z is Pareto-optimal.

Let us now compute the value allocations of $(e, a(z))$. In order to determine the "value" of agent i, one ranks the agents in all possible ways and for each of these orderings, one computes the difference of the "worths" of the coalitions composed of (a) all the agents preceding i plus i and (b) all the agents preceding i. The computations are summarized in the tableau, and can be easily understood by noting that gains from trade arise only when either agent 1 or agent 2 is matched with agent 3, and that the grand coalition can achieve nothing beyond that. c denotes the horizontal distance between $a_1(z)$ and $a_2(z)$. It is also the distance between z_1 and $a_3(z)$, z_2 and $a_3(z)$ (this follows from similarity of triangles), and between z_4 and z_5 (these points are introduced to facilitate the computations). The total gains from trade are $3c$.

		Net Contribution of Each Agent to the Coalition Preceding Him in the Ordering		
Ordering	Weights	Agent 1	Agent 2	Agent 3
1 2 3	1/6	0	0	3c
1 3 2	1/6	0	0	3c
2 1 3	1/6	0	0	3c
2 3 1	1/6	0	0	3c
3 1 2	1/6	3c	0	0
3 2 1	1/6	0	3c	0
	"Values"	c/2	c/2	2c

Tableau 3

Agent 2 starts at $a_2(z)$. His value is $c/2$. This means that he should move to the right by $c/2$. This is not enough to bring him up to a bundle which is A - envy-free for him, since no bundle below his indifference curve through $\Omega/3$ has this property.

Remark: The example could be slightly perturbed to guarantee uniqueness of optimal allocations.

Appendix 4

Other Definitions of Equity for Choice Correspondences

In Appendix 1, we presented the various equity criteria that have been proposed in the literature. Observe that the concept of a fair net trade defined and studied by Schmeidler and Vind [24], a concept which does not evaluate allocations but changes in allocations, is the counterpart for trades of the criterion of an envy-free allocation for allocations discussed earlier. We argue here that an equity criterion for trades can be associated in a natural way to every one of the equity criteria for allocations given in Appendix 1. Each of these definitions is now taken in turn.

Definition: t in $R^{\ell n}$ is an envy-free net trade for (e, ω) if

(a) $\Sigma t_i = 0$

(b) $\forall i, j$, it is not the case that $\omega_i + t_j >_i \omega_i + t_i$.

Definition: t in $R^{\ell n}$ is an egalitarian-equivalent net trade for (e, ω) if

(a) $\Sigma t_i = 0$

(b) $\exists t_o$ in R^{ℓ} s.t. $\forall i$, $\omega_i + t_i \sim_i \omega_i + t_o$.

<u>Definition</u>: t in $R^{\ell n}$ is a <u>per capita envy-free net trade for (e, ω)</u> if

(a) $\Sigma t_i = 0$

(b) $\forall i$, $\omega_i + t_i \gtrsim_i \omega_i$.

This simply means that t should be a Pareto-improving net trade.

<u>Definition</u>: t in $R^{\ell n}$ is a <u>c'-envy-free net trade for (e, ω)</u> if

(a) $\Sigma t_i = 0$

(b) $$\forall \Gamma,\ \forall t' \in R^{\ell|\Gamma|},\ \left[\sum_{i\in\Gamma} t'_i \leqq \frac{|\Gamma|}{n-|\Gamma|} \sum_{i\notin\Gamma} t_i\ ;\ \forall i \in \Gamma,\ \omega_i + t'_i \gtrsim_i \omega_i + t_i\right]$$

$$\Rightarrow \forall i \in \Gamma,\ \omega_i + t'_i \sim_i \omega_i + t_i \ .$$

<u>Definition</u>: t in $R^{\ell n}$ is an <u>income-fair net trade for (e, ω)</u> if

(a) $\omega + t$ is Pareto-efficient, with efficiency prices p

(b) $\forall i, j$, $p\,t_i = p\,t_j$.

<u>Definition</u>: t in $R^{\ell n}$ is an <u>opportunity-fair net trade for (e, ω)</u> if

(a) $\omega + t$ is Pareto-efficient, with efficiency prices p

(b) $\forall i, j, \forall t'_j$, if $p\,t'_j \leqq p\,t_j$, it is not the case that $\omega_i + t_i \gtrsim_i \omega_i + t'_j$.

<u>Definition</u>: t in $R^{\ell n}$ is an <u>A-envy-free net trade for (e, ω)</u> if

(a) $\Sigma t_i = 0$

(b) $\forall i$, it is not the case that $\omega_i + a_i(t) \succ_i \omega_i + t_i$.

Finally, all of the even-numbered definitions proposed in the paper give rise to corresponding concepts of φ-acceptability for net trades.

<u>Definition</u>: $\beta 2^*$: t in $R^{\ell n}$ is a <u>$\beta 2^*$-acceptable net trade for (e, ω)</u> if

(a) $\omega + t \in \varphi(e, \omega + t)$

(b) $\forall \pi \in \Pi_n$, $\omega + t \in \varphi(e, \omega + \pi(t))$.

Other definitions would be obtained in a similar manner.

Footnotes

1. It is in fact what Debreu [8] calls a private ownership exchange economy. We refrain from using this term here because of its implicit acceptance of each individual's claim to his component of the endowment vector.

2. A summation sign $\sum$ with no explicit bound represents a summation over all the agents. To exclude agent i from a summation, we use the symbol "$\sum_{j \neq i}$". The statements "$\forall i$" and "$\exists_i$" are to be read "for all i in $\{1, \ldots, n\}$" and "there exists i in $\{1, \ldots, n\}$.

3. Because preferences are assumed to be "selfish," the preference relation $\gtrsim_i$ defined on Z_i induces a preference relation $\gtrsim_i^o$ on $Z = \prod_i Z_i$ in the usual way:

$$z \gtrsim_i^o z' \Leftrightarrow z_i \gtrsim_i z_i' .$$

To simplify the notation we will use the same symbol $\gtrsim_i$ and write indifferently $z \gtrsim_i z'$ or $z_i \gtrsim_i z_i'$.

4. A precise definition of the Nash CC is given in Section 3-2, example (e).

5. By hypothesis, there exists $z \in Z(e)$ such that $u(z) \geq u(\omega)$. W.L.O.G., assume that $u_1(z_1) > u_1(\omega_1)$. Then $z_1 \geq 0$. W.L.O.G. assume that $z_{11} > 0$. By (i) applied to u_1, for ε small enough, $z_1' = z_1 - \varepsilon(1, 0, \ldots, 0)$ is such that $u_1(z_1') > u_1(\omega_1)$. For each $i \neq 1$, let $z_i' = z_i + \frac{\varepsilon}{n-1}(1, 0, \ldots, 0)$. By (ii) applied $n-1$ times, we have that for all $i \neq 1$, $u_i(z_i') > u_i(z_i)$; therefore $u(z') > u(\omega)$. Since $\sum z'_i = \sum z_i = \Omega$, the claim is proved.

Q.E.D.

6. A similar theorem is proved in Arrow and Hahn [1].

References

[1] Arrow, K. J. and F. H. Hahn. General Competitive Analysis, Holden-Day, San Francisco, 1970.

[2] Berge, C. Topological Spaces, McMillan, 1963.

[3] Chipman, J. S. and J. C. Moore. "The Compensation Principle in Welfare Economics," in Papers in Quantitative Economics, Vol. 2, Arvid M. Zarley, ed., University Press of Kansas, Lawrence, 1971, 1 - 77.

[4] Chipman, J. S. and J. C. Moore. "Social Utility and the Gains from Trade," Journal of International Economics, 2 (1972), 157 - 172.

[5] Crawford, V. P. "A Procedure for Generating Pareto-efficient Egalitarian Equivalent Allocations," Econometrica, 47 (1979), 49 - 60.

[6] Crawford, V. P. "A Self-Administered Solution of the Bargaining Problem," Review of Economic Studies, 47 (1980), 385 - 392.

[7] Daniel, T. E. "A Revised Concept of Distributional Equity," Journal of Economic Theory, 11 (1975), 94 - 109.

[8] Debreu, G. Theory of Value, 1959, Yale University Press.

[9] Feldman, A. and A. Kirman. "Fairness and Envy," American Economic Review, 64 (1974), 995 - 1005.

[10] Foley, D. K. "Resource Allocation and the Public Sector," Yale Economic Essays 7, 1967.

[11] Gabszewicz, J. J. "Coalition Fairness of Allocations in Pure Exchange Economies," Econometrica, 43 (1975), 661 - 668.

[12] Gale, D. The Theory of Linear Economic Models, McGraw-Hill, 1960.

[13] Goldman, S. and C. Sussangkarn. "On the Concept of Fairness," Journal of Economic Theory, 19 (1978), 210 - 216.

[14] Kalai, E. and M. Smorodinsky. "Other Solutions to Nash's Bargaining Problems," Econometrica, 43 (1979), 513 - 518.

[15] Kolm, S.-C. "Justice et Equité," Editions du Centre National de la Recherche Scientifique, Paris, 1972.

[16] Luce, D. and H. Raiffa. Games and Decisions, Introduction and Critical Survey, Wiley, New York, 1957.

[17] Nash, J. "The Bargaining Problem," Econometrica, 18 (1950), 155 - 162.

[18] Pazner, E. A. "Pitfalls in the Theory of Fairness," Journal of Economic Theory, 14 (1977), 458 - 466.

[19] Pazner, E. A. and D. Schmeidler. "A Difficulty in the Concept of Fairness," Review of Economic Studies, 41 (1974), 441 - 443.

[20] Pazner, E. A. and D. Schmeidler. "Egalitarian Equivalent Allocations: A New Concept of Economic Equity," Quarterly Journal of Economics, (1978), 671 - 687.

[20a] Pazner, E. A. and D. Schmeidler. "Decentralization and Income Distribution in Socialist Economies," Economic Inquiry, 16 (1978), 257 - 264.

[21] Raiffa, H. "Arbitration Schemes for Generalized Two-Person Games," in Contributions to the Theory of Games, II, H. W. Kuhn and A. W. Tucker, eds., Princeton University Press, 1953.

[22] Rosenthal, R. "An Arbitration Model for Normal-Form Games," Mathematics of Operations Research 1, 1976, 82 - 88.

[23] Roth, A. E. "An Impossibility Theorem for n-Person Games," International Journal of Game Theory, 8 (1980), 129 - 132.

[24] Schmeidler, D. and K. Vind. "Fair Net Trades," Econometrica, 50 (1972), 637 - 642.

[25] Schmeidler, D. and M. Yaari. "Fair Allocations," unpublished (1971).

[26] Shapley, L. "A Value for n-Person Games," in Contributions to the Theory of Games, H. W. Kuhn and A. W. Tucker, eds., Princeton University Press, 1953.

[27] Shapley, L. S. and M. Shubik. "Pure Competition, Coalitional Power and Fair Division," International Economic Review 10 (1969), 337 - 362.

[28] Thomson, W. "On Allocations Attainable Through Nash Equilibria, A Comment," in Aggregation and Revelation of Preferences, J.-J. Laffont, ed., North Holland (1979), 420 - 431.

[29] Thomson, W. "Anonymity and Equity," mimeograph, forthcoming.

[30] Varian, H. R. "Equity, Envy, and Efficiency," Journal of Economic Theory, 9 (1974), 63 - 91.

[31] Varian, H. R. "Distributive Justice, Welfare Economics and the Theory of Fairness," Philosophy and Public Affairs, 4 (1975), 223 - 247.

[32] Varian, H. R. "Two Problems in the Theory of Fairness," Journal of Public Economics, 5 (1976), 249 - 260.