

THE ANALYSIS OF MACROECONOMIC POLICIES
IN PERFECT FORESIGHT EQUILIBRIUM

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1. INTRODUCTION*

Recent developments in macroeconomic theory have stressed what one might call the intrinsic dynamics of the system; that is the dynamics arising from the creation of assets required to finance certain current activities (see e.g. Blinder and Solow (1973), Tobin and Buiter (1976), Turnovsky (1977)). The emphasis of this literature is on analyzing the short-run, and more particularly the long-run effects of various government policies, as well as insofar as this is feasible, on considering the transitional dynamic adjustments in response to such policies. This literature, like most of macroeconomic theory which precedes it, suffers from two major deficiencies.

The first is in its simplistic treatment of the corporate sector. The traditional textbook macro model typically assumes that all private investment is financed through borrowing. More recently, authors such as Tobin (1969) and Mussa (1976) have introduced very crude stock markets into their models, doing so by identifying one unit of equity with one unit of physical capital. Yet this typically is not how real world stock markets operate. Physical capital on the one hand, and the paper claims to these physical assets on the other, are distinct entities, the relative prices of which are continually changing, thereby invalidating the one-to-one relationship assumed by these authors and others. Recent attempts to differentiate explicitly between physical capital and the corresponding paper claims in a macroeconomic context are contained in Turnovsky (1977, 1978), although this analysis is based on the extreme case of all-equity financing.

The second major shortcoming is that these analyses are all based on arbitrary behavioral relationships. Despite the increasing awareness of the need to ground macroeconomic theory on firm microeconomic foundations, in practice the underlying behavioral relationships remain arbitrarily specified. The deficiencies of following this procedure become particularly severe when one wishes to augment the model to include a corporate sector embodying rational behavior. Indeed as will become clear

below, the effects of alternative government policies depend critically upon the capital structure employed by firms, which in turn is a function of the tax structure and the differential tax treatments of different securities. To incorporate this aspect adequately requires an explicit model of optimizing behavior.

Moreover, the policy implications derived from models based on ad hoc behavioral relationships may be misleading. This point has been made recently by Fair (1978) in his critique of rational expectations models which find fully anticipated government policies to be ineffective in generating real output effects. Fair argues that one of the main reasons that this is so is that the behavioral relationships in these models -- particularly the labor supply function -- are arbitrarily specified and that when these are grounded in an optimizing framework, the proposition is no longer true.

Accordingly, the purpose of this paper is to introduce a more complete corporate sector into a contemporary dynamic macro model, embedding it in an explicit optimizing framework. The approach we shall adopt is an extension of the general equilibrium framework developed by Brock (1974), 1975). This includes the following key features: (a) All demand and supply functions of households and firms are derived from maximizing behavior; (b) expectations are determined simultaneously with perfect foresight continually holding; (c) all markets are continually cleared. Under these conditions all expectations will be "self-fulfilling" and for this reason an equilibrium characterized by (a), (b) and (c) has been termed a "perfect foresight equilibrium"; see Brock (1974).

The earlier Brock analysis essentially abstracted from the government and corporate sectors and it is these aspects (particularly the latter) which we wish to emphasize in our present discussion. More specifically we shall assume that the government can finance its deficit either by issuing bonds or by the creation of money. Likewise, firms may finance their investment plans either by issuing debt, which we take to be perfect substitutes for government bonds, by issuing equities, or out of retained earnings. We shall introduce corporate and personal taxes on various forms of

income. In doing so, we shall specify the tax rates so as to approximate what might be considered to be "real world" tax structures; see e.g. Feldstein, Green, and Sheshinski (1979), Auerbach (1979).

The question of the impact of various corporate and personal taxes on real and financial decisions of the firm continues to be widely discussed in the literature; see e.g. Stiglitz (1973, 1974), Feldstein, Green, and Sheshinski (1979), Auerbach (1979) for recent contributions. These taxes impinge on the firm primarily through the cost of capital and hence it is important to make sure that this is defined appropriately. It would not be entirely unfair to say that traditionally the literature has been less than fully lucid on the appropriate definition of the cost of capital in the presence of taxes.

The approach taken in this paper is to derive the appropriate cost of capital facing firms and hence determining their decisions, from the optimizing decisions of households. The basic idea is that, beginning with the budget constraint for firms, a differential equation determining the market value of firms may be derived. This equation is then integrated to determine a cost of capital for firms, expressed in terms of market yields. Using the optimality conditions for consumers, these market yields can be related to consumers' rate of return on consumption and the various tax parameters. The expression for the cost of capital thereby obtained, embodies the underlying optimizing behavior of consumers. The resulting objective function for firms separates into two parts. First, financial decisions are made to minimize the cost of capital; secondly having determined the minimized cost of capital, real productive decisions may then be made. The implications of our approach turn out to be fully consistent with the existing treatments of taxes in the corporate finance literature; see e.g. Modigliani and Miller (1958), Miller (1977). Except in special circumstances when the debt to equity ratio is indeterminate, the optimal financial structure will consist of all bond financing or all equity financing depending upon relevant tax rates. Thus while the conclusions

on this aspect of our work are not particularly novel, the derivation of the cost of capital from underlying consumer behavior, within an intertemporal general equilibrium framework, does appear to us to be new and in any event seems to us to be particularly instructive.

We should emphasize that our objective in this paper is primarily the development of an integrated framework, rather than with any detailed manipulations of such a model. To do this, we shall assume that perfect certainty holds throughout. While this is obviously a restriction, it seems to us that the construction of a consistent and more complete macro model under certainty is a necessary prerequisite for proceeding further to an analysis in a world characterized by uncertainty.

The remainder of the paper proceeds as follows. Section 2 develops in some detail the structure of the model. In Section 3 we characterize the notion of a perfect foresight equilibrium, which we then apply to our model. Section 4 derives the optimality conditions for consumers, which are then used in the derivation of the optimality conditions for firms in the following section. Section 6 discusses the equilibrium structure of the system and gives a few comments on the dynamics. Section 7 and 8 deal with the steady state. The first of these two sections deals with its more general characteristics, while the second considers the steady state in further detail for a specific government financial policy. Section 9 develops the dynamics in the case of fixed employment and a simplified tax structure. Finally, the main body of the paper concludes with some general remarks.

2. THE STRUCTURE

The model contains three basic sectors -- households, firms, and the government -- all of which are interrelated through their respective budget constraints. We shall consider these sectors in turn, in all cases expressing their behavioral constraints directly in *real* terms.

A. Household Sector

We assume that households can be aggregated into a single consolidated unit. The objective of this composite unit is to choose its consumption demand, its supply of labor, the rates at which it wishes to add to its real holdings of money balances, government bonds, corporate bonds, and equities, to maximize the intertemporal utility functions^{1/}

$$(1a) \quad \int_0^{\infty} e^{-\beta t} U(c, g, \ell^s, m^d) dt \quad U_c > 0, U_g > 0, U_{\ell} > 0$$

subject to

$$(1b) \quad \dot{c} + \dot{b}_g^d + \dot{b}_p^d + \dot{m}^d + q\dot{E}^d = w\ell^s + s(b_g^d + b_p^d) - p^*(b_g^d + b_p^d + m^d) + iqE^d - T_h$$

$$(1c) \quad M(0) = M_0, B_g(0) = B_{g0}, B_p(0) = B_{p0}, E(0) = E_0$$

$$(1d) \quad \lim_{t \rightarrow \infty} e^{-\int_0^t \theta(s) ds} b_g \geq 0, \lim_{t \rightarrow \infty} e^{-\int_0^t \theta(s) ds} b_p \geq 0; \lim_{t \rightarrow \infty} e^{-\int_0^t \theta(s) ds} qE \geq 0$$

where c = real private consumption plans by households,
 g = real government expenditure, taken to be fixed exogenously,
 ℓ^s = real supply of labor by households,
 $m^d = M^d/P$ = demand for real money balances,
 M = nominal stock of money,
 P = nominal price of output,
 b_g^d = demand for real government bonds,
 B_g = nominal stock of government bonds,
 b_p^d = demand for real corporate bonds,

B_p = nominal stock of corporate bonds,
 E = number of shares outstanding,
 q = relative price of equities in terms of current output,
 $i = D/qE$ = dividend yield, taken to be parametrically given to
the household sector,
 D = real dividends,
 qE^d = real stock demand for equities,
 w = real wage rate,
 s = nominal interest rate on both government and private bonds,
 p^* = instantaneous anticipated rate of inflation,
 T_h = personal income tax in real terms, specified more fully below,
 β = social rate of discount,
 $\theta(s)$ = instantaneous rate of return on consumption at time s , to be
determined endogenously below

The utility function is assumed to be concave in its four arguments c , g , ℓ , and m . The introduction of c and g as separate arguments reflects the assumption that consumers view private and public goods as imperfect substitutes. At appropriate places below it is convenient to focus on the case where they are perfect substitutes, in which case they enter additively as $c + g$. This corresponds to the assumption of "ultrarationality" in the sense discussed by Buiter (1977). Labor affects utility of leisure. The introduction of real money balances into the utility function is a convenient device for capturing the reasons for holding money in a certainty world. As Brock (1974) has shown, one can "justify" the inclusion of money in the utility function by means of transactions costs arguments.^{2/} We shall assume that for given values of c , g , ℓ , the marginal utility of money balances satisfies

$$\text{sgn } U_m = \text{sgn } (m^* - m)$$

so that m^* denotes the corresponding satiation level of real money balances, such as in Friedman (1969). For the real stock of money less than this level, the marginal utility of money is positive; for real stocks of money in excess of m^* , the holding costs outweigh the benefits and the net marginal utility of money becomes negative. While most of our discussion will focus on the general utility function U , it is expositionally convenient at appropriate places below to assume that it is additively separable in m , enabling us to separate out the real part of the system from the monetary part.

The household sector's budget constraint is given by (1b), which we have expressed in *real* flow terms. At each point of time the households are assumed to acquire income from a variety of sources. They supply labor to firms, at a real wage rate w ; they earn interest income on their holdings of real private and government bonds; they suffer instantaneous capital gains or losses on their holdings of financial wealth denominated in nominal terms (money and bonds); they receive dividend payments at a rate i on their holding of equities. This rate is taken as parametrically given to households, but is one of the decision variables of the corporate sector. This income can be used in a variety of ways. They may use it to purchase real consumption goods, to add to their real holdings of money, government bonds, corporate bonds, and equities (the relative price of which is q), and to pay taxes to the government.

It is important to note that the decisions derived from this optimization procedure are *planned* demands (or supply in the case of labor). We have recorded this fact explicitly by the inclusion of the superscripts d (s). Finally, the restraints (1d) must be added if borrowing is allowed, in order to prevent the present value of debt from becoming unbounded as $t \rightarrow \infty$.

B. Corporate Sector

As noted in the introduction, the corporate sector is "driven" by households in the sense that the optimizing decisions of the households determine the appropriate cost of capital facing the firms, which in turn governs their real and financial decisions. Thus before the firm's optimization problem can be explicitly formulated

and solved, it is first necessary to solve the optimization problem for the household sector. At this stage we shall simply record the financial and production constraints facing the firm and note the general form of the objective function which we shall derive.

The constraints facing the firm are summarized as follows

$$(2a) \quad y^s = f(k^d, \ell^d)$$

$$(2b) \quad \pi = y^s - w\ell^d$$

$$(2c) \quad \pi = sb_p + D + RE + T_f$$

$$(2d) \quad \dot{k}^d = RE + q\dot{E}^s + \dot{b}_p^s + p^*b_p^s$$

$$(2e) \quad k(o) = k_o, E(o) = E_o, B_p(o) = B_{po}$$

where ℓ^d = real demand for labor by firms,

k^d = real demand for physical capital by firms,

y^s = real output

π = real gross profit

b_p^s = supply of corporate bonds by firms, in real terms,

E^s = quantity of equities, issued by firms,

RE = retained earnings, in real terms,

T_f = corporate profit taxes in real terms, specified more fully below,

and all other symbols are as defined above.

Equation (2a) describes the production function, which is assumed to have the usual neoclassical properties of positive but diminishing marginal productivities and constant returns to scale. Equation (2b) is the conventional definition of gross profits in real terms as being revenue less payments to labor. Equation (2c) describes the allocation of gross profits. After paying corporate income taxes, this may be used to pay interest to bond holders, to pay dividends to stock holders, or retained within the firm. Equation (2d) expresses the firm's financial constraint.

Any additions to capital stock must be financed out of retained earnings, by issuing additional equities, or by issuing additional bonds. The final term p^*b_p is essentially the revenue on private bonds accruing to the firm by virtue of the fact that these bonds are presumed to be denominated in nominal terms. It is precisely analogous to the inflation tax generated on financial wealth issued by the government and which also appears in the household sector's budget constraint. Finally, equations (2e) are initial conditions on the real stock of capital, the number of equities outstanding, and the nominal stock of corporate bonds.

We define the market value of the firm's securities outstanding at time t by

$$(3) \quad V(t) = b_p(t) + q(t)E(t)$$

(where we suppress superscripts) and shall assume that the firm's objective is to maximize the initial real market value of its securities, $V(0) = b_p(0) + q(0)E_0$. Given the constraints in (2a) - (2e) and the optimality conditions for households, we shall show in Section 5 below, that this objective function leads to the following optimization problem for firms. Their problem is to choose production decisions k^d , l^d , and financial decisions b_p^s , E^s , i to maximize an objective function of the form ^{3/}

$$(4) \quad V(0) = \int_0^{\infty} e^{-\int_0^t \theta^*(\tau) d\tau} \gamma(t) dt$$

where $\gamma(t)$ = real net cash flow,

$\theta^*(t)$ = instantaneous cost of capital at time t .

The precise forms of the functions $\gamma(t)$, $\theta^*(t)$, will be developed in Section 5 below. At this point we wish to emphasize that $\gamma(t)$ is a function of *real* production decision variables, k , l ; the *financial* decision variables are all embodied in the cost of capital $\theta^*(\tau)$. As a result of this, the two sets of decisions can be obtained in a convenient, sequential manner.

C. The Government

The government is assumed to provide real goods and services g , which it finances out of real tax receipts, or by issuing some form of government debt. Its budget constraint is described in real terms by ^{4/}

$$(5) \quad \dot{m}^s + \dot{b}_g^s = g + sb_g^s - T_h - T_f - (m^s + b_g^s)p^*$$

where the superscript s denotes the planned supply by the government. This equation defines the real deficit net of the inflation tax by the right hand side of (3) and asserts that this is financed either by increasing the real money supply or by increasing the real stock of government bonds. The choice between these two alternatives, or any other monetary policy for that matter, represents a policy decision which needs to be specified in order to close the model.

Finally, we must specify the tax functions T_h, T_f . These are hypothesized as follows

$$(6a) \quad T_h = \tau_y [w\ell^s + s(b_g^d + b_p^d) + iqE^d] + \tau_c (\dot{q} + qp^*)E$$

$$(6b) \quad T_f = \tau_p [y^s - w\ell^d - sb_p^s] \quad 0 \leq \tau_y \leq 1, \quad 0 \leq \tau_c \leq 1, \quad 0 \leq \tau_p \leq 1$$

where for simplicity all tax structures are assumed to be linear. According to (6a), "ordinary" personal income -- income from wages, interest, and dividends -- are taxed at the flat rate τ_y . Nominal capital gains on equities are assumed to be taxed at the rate τ_c , which may, or may not, equal τ_y , and indeed in many economies $\tau_c = 0$. Notice that (6a) implies that capital gains are realized at each point in time; i.e. the portfolio is continuously "rolled over". Alternatively, one may view (6a) as representing taxes on unrealized capital gains. Turning to corporate income taxes, gross profit is assumed to be taxed at the proportional rate τ_p , with the interest payments to bond holders being fully deductible. In all cases full loss offset provisions are assumed.

We have specified these tax functions as reasonable approximations to real world tax structures. As we shall demonstrate below, this in general implies non-neutrality of the various tax rates. In order to restore neutrality it would be necessary to introduce appropriate tax deductions for the capital losses arising from inflation on the holdings of money and bonds as well as appropriate offset provisions for firms. Since such taxes do not generally characterize real world tax structures, we do not incorporate them, although it would be straightforward to do so.

3. PERFECT FORESIGHT EQUILIBRIUM

The *perfect foresight equilibrium* (PFE) we shall consider is defined as follows. First consider the household sector's maximization problem defined by (1a), (1b) subject to (1c) with T_h defined by (6a). Carrying out this maximization yields a set of demand functions for consumption and various securities together with a labor supply function, in terms of p^* , w , s etc. and other parameters which consumers take as given. Likewise, the corporate sector's optimization problem defined by (2a) - (2e) and (4), with T_f defined by (6b) yields a set of demand functions for capital and labor, and a set of supply functions for various securities together with output, which are also functions of w , s , etc. which firms too treat as parametrically given. Thirdly, the government policy decisions constrained by (5) generate supplies of money and government bonds and a demand for goods.

The perfect foresight equilibrium is defined as a situation in which the planned demands for output, labor, and the various securities in the economy all equal their corresponding real supplies and in addition all anticipated variables are correctly forecast. In this case $m^d = m^s$ etc. and where no confusion can arise we shall simply drop the superscript. Thus the quantity m say will denote the real money supply in a perfect foresight equilibrium. Henceforth we shall focus our attention on these equilibrium quantities. Nothing will be done on the question of existence of equilibrium in this paper. In fact equilibrium may not exist under all tax structures - especially when borrowing, lending, and new security creation is allowed.

4. DETERMINATION OF OPTIMALITY CONDITIONS FOR HOUSEHOLDS

As indicated above, the household sector's optimization problem is to choose c, m, b_g, b_p, E to maximize its utility function (1a), subject to the budget constraint (1b), with T_h defined by (6a), and subject to the initial conditions (1c). Substituting for T_h enables us to define the Hamiltonian function ^{5/}

$$(7) \quad \mathcal{H} \equiv e^{-\beta t} U(c, g, \ell, m) \\ -v\{c + (\dot{b}_g + \dot{b}_p) + \dot{m} + q\dot{E} - (1 - \tau_y)[w\ell + s(b_g + b_p) + iqE] \\ + [b_g + b_p + m]p + \tau_c(\dot{q} + qp)E\}$$

Since we are dealing with a perfect foresight equilibrium we have set $p^* = p$, the actual rate of inflation. Also, for notational convenience, all superscripts have been dropped.

We shall assume that $[c(t), m^d(t), b_g^d(t), b_p^d(t), E^d(t)] \geq 0$ for all t . ^{6/} The Hamiltonian \mathcal{H} is observed to be linear in the financial decision variables b_p, b_g, E . In view of this, depending upon the precise tax structure assumed, some of these securities may or may not appear in strictly positive quantities in the equilibrium demands of the household sector. To allow for the possibility that some may turn out to be zero in equilibrium it is necessary to solve the optimization problem by using Euler inequalities rather than in terms of the more familiar Euler equations. These are simply the analogues to the Kuhn-Tucker conditions in conventional non-linear programming. ^{7/} Performing the optimization yields the following conditions

$$(7a) \quad e^{-\beta t} U_c \leq v$$

$$(7b) \quad c[e^{-\beta t} U_c - v] = 0$$

$$(8a) \quad e^{-\beta t} U_\ell \leq -vw(1 - \tau_y)$$

$$(8b) \quad \ell[e^{-\beta t} U_\ell + vw(1 - \tau_y)] = 0$$

$$(9a) \quad e^{-\beta t} U_m - \nu p \leq -\dot{\nu}$$

$$(9b) \quad m[e^{-\beta t} U_m - \nu p + \dot{\nu}] = 0$$

$$(10a) \quad \nu[s(1 - \tau_y) - p] \leq -\dot{\nu}$$

$$(10b) \quad b_p \{ \nu[s(1 - \tau_y) - p] + \dot{\nu} \} = 0$$

$$(10c) \quad b_g \{ \nu[s(1 - \tau_y) - p] + \dot{\nu} \} = 0$$

$$(11a) \quad \nu[i(1 - \tau_y) + \frac{\dot{q}}{q}(1 - \tau_c) - \tau_c p] \leq -\dot{\nu}$$

$$(11b) \quad E \{ \nu[i(1 - \tau_y) + \frac{\dot{q}}{q}(1 - \tau_c) - \tau_c p + \dot{\nu}] = 0$$

where $\nu > 0$ is the Lagrange multiplier associated with the household sector budget constraint.

Inequalities (7a), (8a), (9a), (10a), (11a) are the Euler inequalities with respect to c , l , m , b_g and b_p (which are identical) and E respectively. If any of these inequalities are met *strictly*, then the corresponding decision variable is set equal to zero. Conversely if any of the decision variables are strictly positive in equilibrium, the corresponding constraint is satisfied with equality. These duality type relationships are reflected in the equations (7b), (8b), (9b), (10b), (10c), (11b).

Throughout the analysis we shall assume in equilibrium $c > 0$, $l > 0$, $m > 0$, so that (7a), (8a), (9a) all hold with equality. The strict positivity of these quantities seems reasonable on economic grounds and can be ensured by imposing appropriate Inada conditions on the utility function of consumers. Introducing the above equality conditions the optimality conditions (7) - (11) can be simplified and interpreted more readily. The Lagrange multiplier is simply the discounted marginal utility of consumption, so that

$$(12) \quad \frac{-\dot{\nu}}{\nu} = \beta - \frac{\dot{U}_c}{U_c} \equiv \theta \quad \text{say}$$

represents the rate of return on consumption. The optimality conditions for consumers thus may be written

$$(13a) \quad \frac{U_\ell}{U_c} = -w(1 - \tau_y)$$

$$(13b) \quad \frac{U_m}{U_c} = p + \theta$$

$$(13c) \quad s(1 - \tau_y) - p \leq \theta$$

$$(13c') \quad b_i \{s(1 - \tau_y) - p - \theta\} = 0 \quad i = p, g$$

$$(13d) \quad i(1 - \tau_y) + \frac{\dot{q}}{q} (1 - \tau_c) - \tau_c p \leq \theta$$

$$(13d') \quad E[i(1 - \tau_y) + \frac{\dot{q}}{q} (1 - \tau_c) - \tau_c p - \theta] = 0$$

These optimality conditions will now be used to derive explicitly the objective function for the firms.

5. DETERMINATION OF OPTIMALITY CONDITIONS FOR FIRMS

To derive the objective function for firms, we begin by eliminating RE from the firm's financial constraints (2c), (2d) to yield

$$\pi + q\dot{E} + \dot{b}_p + pb_p = sb_p + D + T_f + \dot{k}$$

where, because we are dealing with PFE, the superscripts have been dropped. Adding $\dot{q}E$ to both sides of this equation and noting the definition of \dot{V} given in (3) (and hence \dot{V}), we obtain

$$(14) \quad \dot{V} + \pi = (s - p)b_p + D + T_f + \dot{k} + qE$$

we now define the firm's real net cash flow $\gamma(t)$ to be

$$(15) \quad \begin{aligned} \gamma(t) &\equiv (1 - \tau_p)\pi - \dot{k} \\ &= (1 - \tau_p)[f(k, \ell) - w\ell] - \dot{k} \end{aligned}$$

That is, $\gamma(t)$ equals the gross profit after tax, less the cost of the additional capital purchased. Now using the definition of T_f together with (15), and recalling the definitional relation $D/qE = i$, equation (14) becomes

$$(16) \quad \dot{V} + \gamma = [s(1 - \tau_p) - p]b_p + iqE + \dot{q}E$$

We now define the firm's debt-to-equity ratio

$$(17) \quad \lambda = b_p/qE$$

enabling us to write (16) in the form

$$(18) \quad \dot{V} + \gamma = \left[[s(1 - \tau_p) - p] \frac{\lambda}{1 + \lambda} + \left(i + \frac{\dot{q}}{q} \right) \frac{1}{1 + \lambda} \right] V$$

Next, letting

$$\theta^*(\lambda, i, s, \dot{q}/q) \equiv [s(1 - \tau_p) - p] \frac{\lambda}{1 + \lambda} + \left(i + \frac{\dot{q}}{q} \right) \frac{1}{1 + \lambda}$$

(18) can be written more conveniently as

$$(19) \quad \dot{V}(t) + \gamma(t) = \theta^*(\lambda, i, s, \dot{q}/q) V(t)$$

where θ^* is in general a function of t , but is independent of V . Equation (19) can now be readily integrated to yield a general solution

$$V(s) = e^{\int_0^s \theta^*(t) dt} \left[A - \int_0^s \gamma(t) e^{-\int_0^t \theta^*(\tau) d\tau} dt \right]$$

where A is an arbitrary constant. Suppose for the moment that $\theta^* > 0$ (an assumption which will be justified below and certainly holds in steady state), then in order for $V(s)$ to remain finite as $s \rightarrow \infty$, we require^{8/}

$$A = \int_0^\infty \gamma(t) e^{-\int_0^t \theta^*(\tau) d\tau} dt$$

and hence the value of the firm at any arbitrary time s is

$$V(s) = e^{\int_0^s \theta^*(t) dt} \left[\int_s^\infty \gamma(t) e^{-\int_0^t \theta^*(\tau) d\tau} dt \right]$$

The *initial* value of the firm, which we assume the firm seeks to maximize, is therefore

$$(20) \quad V(o) = \int_0^{\infty} e^{-\int_0^t \theta^*(\tau) d\tau} \{ (1 - \tau_p) [f(k, \ell) - w\ell] - \dot{k} \} dt$$

where we have substituted for $\gamma(t)$. Thus we have derived the objective function (4) and shown that $V(o)$ is an appropriately discounted integral of the future net real cash flows.

The fact that $\gamma(t)$ depends only upon the real production variables (k, ℓ) , while the discount rate depends upon the financial variables $(\lambda, i, s, \dot{q}/q)$ means that the firm's optimization can be conducted sequentially. First it chooses its financial decisions to minimize θ^* ; having determined the optimal θ^* , it then chooses the optimal production decisions.

The critical factor in the firm's objective function is θ^* . Using the definitions of λ and V it may be written as

$$(21) \quad \theta^* = [s(1 - \tau_p) - p] \frac{b_p}{V} + \left(i + \frac{\dot{q}}{q} \right) \frac{qE}{V}$$

In other words, θ^* is simply a weighted average of the real costs of debt capital and equity capital to the firm, and hence surely will be positive. The real cost of debt capital is the after-corporate income tax nominal interest rate, less the rate of inflation; the real cost of equity capital to firms is the dividend payout ratio plus the real rate of capital gains on equity. Hence (21) is an expression which turns out to be familiar from basic corporate finance theory.

However, the expression for θ^* given in (21) is inappropriate from the viewpoint of determining the firm's financial decisions. The reason is that these decisions are themselves constrained by the preferences of households. These preferences, which are embodied in the optimality conditions (13c'), (13d'), impose constraints on the components of the financial rates of return. To obtain an appropriate expression for θ^* , we invoke the optimality conditions (13c'), (13d') for consumers to eliminate s and \dot{q}/q

$$(13c'') \quad sb_p = \frac{(\theta + p)b_p}{1 - \tau_y}$$

$$(13d'') \quad \dot{q}E = \frac{(\theta + \tau_c p)qE}{1 - \tau_c} - \frac{(1 - \tau_y)iqE}{1 - \tau_c}$$

Substituting these expressions into (21), we may express θ^* as

$$(21') \quad \theta^*(\lambda, i) = \theta + \frac{(\tau_y - \tau_p)(\theta + p)}{1 - \tau_y} \frac{\lambda}{1 + \lambda} + \frac{(\theta + p)\tau_c + i(\tau_y - \tau_c)}{1 - \tau_c} \frac{1}{1 + \lambda}$$

It becomes evident from this expression how the cost of capital θ^* provides the means whereby consumers "drive" the firms. As written in (21') it is seen that the relevant cost of capital is equal to the consumer's rate of return on consumption θ , adjusted by the various corporate and personal income tax rates, the adjustments themselves being weighted by the share of bonds and equities in the firm's financial structure. But (21') may also be written in the form

$$(21'') \quad \theta^* = \left[\theta + \frac{(\tau_y - \tau_p)(\theta + p)}{1 - \tau_y} \right] \frac{\lambda}{1 + \lambda} + \left[\frac{(\theta + p)\tau_c + i(\tau_y - \tau_c)}{1 - \tau_c} \right] \frac{1}{1 + \lambda}$$

The expression (21') (or (21'')), being expressed in terms of the firm's financial decision variables, together with other variables parametric to the firm is now suitable for determining the firm's optimal financial policies. This is done by calculating the partial derivatives with respect to λ, i

$$(22a) \quad \text{sgn } \frac{\partial \theta^*}{\partial i} = \text{sgn } \left(\frac{\tau_y - \tau_c}{1 - \tau_c} \right)$$

$$(22b) \quad \text{sgn } \frac{\partial \theta^*}{\partial \lambda} = \text{sgn } \left\{ \frac{(\tau_y - \tau_p)(\theta + p)}{1 - \tau_y} - \frac{(\theta + p)\tau_c + i(\tau_y - \tau_c)}{1 - \tau_c} \right\}$$

The optimal dividend policy and the optimal capital structure will therefore involve corner solutions. ^{9/} Given that the capital gains in the model really reflect accruals, whereas tax rates in reality apply to realized capital gains, it seems reasonable to assume $\tau_c < \tau_y$; see Auerbach (1979). Thus if the firm is to minimize its cost of capital, it should minimize the dividend payout ratio i . In the absence of any constraints this would involve the repurchase of shares, as long as D were

positive. In fact such behavior is prohibited, (at least in the U.S.) by Section 302 of the Internal Revenue Code. To model the legal constraints include in this code fully would be rather complicated and we do not attempt to do so here. Rather, we shall simply argue that the firm minimizes its dividend payments by setting $i = \bar{i}$, where \bar{i} is the legal minimum payout rate, which we take to be exogenous.^{10/} Thus setting $i = \bar{i}$, the optimal financial mix (λ) is determined as follows

$$(23a) \quad \frac{(\tau_y - \tau_p)(\theta + p)}{1 - \tau_y} < \frac{(\theta + p)\tau_c + \bar{i}(\tau_y - \tau_c)}{1 - \tau_c} \quad \text{set } \lambda = \infty$$

i.e. all bond financing ($E = 0$)

$$(23b) \quad \frac{(\tau_y - \tau_p)(\theta + p)}{1 - \tau_y} > \frac{(\theta + p)\tau_c + \bar{i}(\tau_y - \tau_c)}{1 - \tau_c} \quad \text{set } \lambda = 0$$

i.e. all equity financing (i.e., $b_p = 0$)

Defining the average tax rate on income from stock τ_s , to be a weighted average of the tax rates on income from dividends and income from capital gains,

$$\tau_s = \frac{\bar{i}\tau_y + (\dot{q}/q + p)\tau_c}{(\bar{i} + \dot{q}/q + p)}$$

and using the optimality condition (13d'), the criterion for determining the optimal financial mix can be rewritten as follows ^{11/}

$$(24a) \quad (1 - \tau_y) > (1 - \tau_p)(1 - \tau_s) \quad \text{set } \lambda = \infty$$

$$(24b) \quad (1 - \tau_y) < (1 - \tau_p)(1 - \tau_s) \quad \text{set } \lambda = 0$$

Thus written in this way, we see that our criterion is identical to that of Miller (1977).^{12/} In effect, (24a) asserts that if the net after tax income on bonds exceeds the net after tax income from equity, where the latter are taxed twice, first as corporate profits, secondly as personal income to stockholders, no investor will wish to hold equities and the firm must engage in all bond financing. The opposite applies if the inequality is reversed as in (24b).

If $(1 - \tau_y) = (1 - \tau_p)(1 - \tau_s)$, then the optimal debt to equity ratio is indeterminate. One special case of this arises when all tax rates are zero, when the conditions (23) simply reduce to a statement of the familiar Modigliani and Miller (1958) no-tax case. ^{13/} Thus while our conclusions for the firm's financial policy turn out to be familiar, we do feel that the derivation we have given for the cost of capital, in terms of the underlying optimality conditions for consumers has merit in making explicit the role played by consumers in determining the optimality conditions for value maximizing firms.

Thus using (24), the firm's minimum cost of capital may be expressed as

$$(25) \quad \theta_{\min}^* = \theta + \min \left[\frac{(\tau_y - \tau_p)(\theta + p)}{1 - \tau_y}, \frac{(\theta + p)\tau_c + \bar{i}(\tau_y - \tau_c)}{1 - \tau_c} \right]$$

with all bond financing this reduces to

$$(25') \quad \theta_{\min}^* = \frac{\theta(1 - \tau_p) + (\tau_y - \tau_p)p}{1 - \tau_y} = s(1 - \tau_p) - p$$

while with all equity financing it becomes

$$(25'') \quad \theta_{\min}^* = \frac{\theta + \tau_c p + \bar{i}(\tau_y - \tau_c)}{1 - \tau_c} = \bar{i} + \dot{q}/q$$

In deriving the second equalities in these two equations, use has been made of (13c''), (13d''). These expressions are similar to those given by Auerbach (1979), although there is one difference. He comments that with all equity financing, the cost of capital is independent of \bar{i} , and the rate at which dividends are taxed. But it is seen from (25'') that this really not so; the reason is that these are absorbed in his discount rate for income from equity. Once one explicitly relates such a parameter ($\bar{i} + \dot{q}/q$ in our notation) to the rate of discount for consumers, these parameters do indeed enter explicitly into the cost of capital.

We are now finally in a position to state and solve the real part of the firm's optimization problem. Having chosen i, λ , to minimize the cost of capital and determined θ_{\min}^* , it must next choose k, ℓ , to maximize

$$(26) \quad \int_0^{\infty} e^{-\int_0^t \theta^* \min} [(1 - \tau_p) [f(k, \ell) - w\ell] - \dot{k}] dt$$

subject to the initial condition $k(0) = k_0$. The optimality conditions to this latter problem are simply

$$(27a) \quad (1 - \tau_p) f_k(k, \ell) = \theta_{\min}^*$$

$$(27b) \quad f_{\ell}(k, \ell) = w$$

That is, the after tax marginal physical product of capital should be equated to the minimized cost of capital while the marginal physical product of labor should be equated to the real wage.

Moreover, substituting (27a), (27b) back into (20), we are able to use the transversality condition at ∞ for the above optimization problem to establish the familiar balance sheet constraint

$$(28) \quad V = b_p + qE = k$$

This condition requires that the capitalized value of the state variables be zero in the limit, which in effect rules out the possibility of the values of the claims becoming divorced from the underlying sources of earnings. The demonstration of this is relegated to a footnote.^{14/} Thus we may conclude by noting that if (24a) holds $b_p = k$; if (24b) applies $qE = k$; while in the knife edge case where (24) holds with equality, b_p and qE are indeterminate. This completes the formal optimization of the firm.

6. EQUILIBRIUM STRUCTURE AND DYNAMICS OF SYSTEM

The optimality conditions for the households and firms, together with the government budget constraint can now be combined to describe the perfect foresight equilibrium in the economy and to determine its dynamic evolution.

Combining the optimality conditions in (13), (27), we may write

$$(29a) \quad U_c(c, g, \ell, m) = \alpha$$

$$(29b) \quad \frac{U_\ell(c, g, \ell, m)}{U_c(c, g, \ell, m)} = -f_\ell(k, \ell)(1 - \tau_y)$$

$$(29c) \quad \frac{U_m(c, g, \ell, m)}{U_c(c, g, \ell, m)} = \theta + p$$

$$(29d) \quad (1 - \tau_p)f_k(k, \ell) = \theta_{\min}^*$$

$$(29e) \quad \theta_{\min}^* = \theta + \min\left[\frac{(\tau_y - \tau_p)(\theta + p)}{1 - \tau_y}, \frac{(\theta + p)\tau_c + \bar{i}(\tau_y - \tau_c)}{1 - \tau_c}\right]$$

$$(30a) \quad \dot{\alpha} = \alpha(\beta - \theta)$$

$$(30b) \quad \dot{k} = f(k, \ell) - g - c$$

$$(30c) \quad \dot{m} + \dot{b}_g = g + \theta b_g - \tau_y f - mp + [\theta - (1 - \tau_y)f_k]k$$

Equation (29a) simply defines a short hand notation for the marginal utility of consumption. Equations (29b) equates the marginal rate of substitution between consumption and leisure to the after-tax real wage, while (29c) requires that the marginal utility derived from holding a dollar in cash balances must equal the marginal utility that would be derived from spending the dollar on consumption. Equations (29d), (29e), restate the marginal productivity condition for capital, and the minimum cost of capital respectively. These five equations may be used to solve for the short-run solutions for the five variables $\ell, c, p, \theta, \theta^*$, in terms of the dynamically evolving variables α, k, m, b_g . The dynamics governing these latter variables are expressed in equations (30a) - (30c). The first of these is a restatement of (12), in terms of the new notation, while (30b) describes the rate of capital accumulation required to maintain product market equilibrium. The final equation (30c) is the government budget constraint. The derivation of this form of the equation involves several steps, which are omitted. Essentially it is obtained by first substituting the tax functions (6a), (6b) into (5); then using the optimality conditions for consumers (13c'), (13d'), the optimality conditions for firms (27a), (27b), together

with the linear homogeneity of the production function and the balance sheet constraint (28) to simplify the resulting expression for the real deficit. The end product of this process is (30c).^{15/}

As part of the specification of the dynamics of the system, something must be said about government financial policy. In this respect, the options open to the government depend critically upon the inequalities (23) determining firms' financial policy. Since government bonds are assumed to be perfect substitutes for corporate bonds, the government can issue bonds if and only if firms are able to issue bonds (i.e. have them willingly purchased by households). Thus if (23a) holds (including the case of equality), so that firms engage in bond financing, the government can issue bonds. In this case, if government expenditure and tax policy are taken to be exogenous policy parameters, we are still able to specify an independent financial policy for the government and ensure that the government budget constraint is met. On the other hand, suppose that (23b) applies, implying that firms engage in all equity financing. In this case, government bonds will be driven out of the system as well, for the simple reason that being dominated from the viewpoint of consumers by equities, nobody will be willing to hold them. The government therefore loses a degree of freedom insofar as its policy options are concerned. If government expenditure and tax rates are exogenous, then the money supply must adjust endogenously to meet the constraint. Alternatively, if an independent monetary policy, such as a monetary growth rule is adopted, then some other variable such as government expenditure must accommodate to meet the budget.

To complete the description of the system we must consider the initial conditions, $k(0)$, $E(0)$, $m(0)$, $b_p(0)$, $b_g(0)$. The first two of these are exogenously determined, being given by $k(0) = k_0$, $E(0) = E_0$. In the case of money and bonds, the initial nominal stocks are assumed to be given, with the initial real stocks being endogenously determined by an initial jump in the price level. The size of this initial jump, and therefore the initial values $m(0)$, $b_p(0)$, $b_g(0)$ are obtained from the transversality

conditions at infinity for households and firms. Unfortunately, given the dimensionality of the system a full dynamic analysis of the system (29), (30), turns out to be rather complex and not very enlightening. Rather than pursue it further with the model specified at the present level of generality, we shall proceed directly to a discussion of the steady state. Then in Section 9 below, we shall analyze a simplified version of the dynamics, which arises when labor is supplied exogenously. This simplification suffices to give a good indication of the nature of the transitional dynamics for the general case.

But before concluding the present discussion, we should also explain how the equilibrium stocks of bonds and equities and their respective rates of return are determined. In the case of all bond financing, the balance sheet constraint (28) implies $b_p = k$. Note that with the nominal stocks of bonds given, a jump may occur at time zero, through the price level, so that $k_o = b_p(o) = B_{po}/P(o)$. The nominal rate of interest can then be determined by inserting the known values into the consumers' optimality condition (13c). Similarly, with all equity financing, the price of equities can be obtained by substituting known values into (13d) and integrating. Again an initial jump may be required in q , in order to satisfy the balance sheet constraint (28), which is now $qE = k$. With both the value of equities and their price determined, the quantity of shares outstanding can be immediately inferred.^{16/}

7. STEADY STATE

The steady state of the system is attained when ^{17/}

$$\dot{\alpha} = \dot{k} = \dot{m} = \dot{b}_g = 0$$

implying that $\theta = \beta$, and $f(k, \ell) = c + g$. Accordingly, the long-run equilibrium of the system can be reduced to the following four equations

$$(31a) \quad \frac{U_{\ell} [f(k, \ell) - g, g, \ell, m]}{U_c [f(k, \ell) - g, g, \ell, m]} = -f_{\ell}(k, \ell)(1 - \tau_y)$$

$$(31b) \quad \frac{U_m [f(k, \ell) - g, g, \ell, m]}{U_c [f(k, \ell) - g, g, \ell, m]} = \beta + p$$

$$(31c) \quad (1 - \tau_p)f_k(k, \ell) = \beta + \min \left[\frac{(\tau_y - \tau_p)(\beta + p)}{1 - \tau_y}, \frac{(\beta + p)\tau_c + \bar{i}(\tau_y - \tau_c)}{1 - \tau_c} \right]$$

$$(31d) \quad g + \beta b_g - \tau_y f(k, \ell) - mp + [\beta - (1 - \tau_y)f_k]k = 0$$

The first three equations involve the four variables k , ℓ , m , and p . Thus if for given exogenous values of g , and the tax rates it is possible to specify an independent government financial policy in terms of the real stock of money or the inflation (this will be possible only as long as government bonds are not driven out of the system), then these three equations, together with the policy specification will determine the four variables k , ℓ , m , p . Inserting these stationary values into the steady-state government budget constraint, determines the required real stock of government bonds to maintain the budget in balance. On the other hand, if government bonds along with private bonds are driven out of the system, then for given g and tax rates, then four equations determine the equilibrium values of k , ℓ , m , p . There is no scope for an independent monetary policy.

It is of interest to note that in general, the system summarized by (31) is interdependent; real production decisions and financial decisions are jointly determined. This is in part a consequence of the fact that only the nominal component of the real interest rate is taxed and the tax deductibility provisions assumed. It is also in part a consequence of the interdependence between real money balances m on the one hand, and capital and labor on the other, in the consumers' utility function. Under certain conditions, however, the system dichotomizes into two recursive subsystems; the first determining the real decisions k , ℓ ; the second determining the financial variables m , p , conditional on these initially chosen real variables.

Finally, the equilibrium nominal interest rate in the case of bond financing, can be obtained by substituting the solutions from (31) into the appropriate arbitrage condition for consumers (13c'), (13d').

8. CHARACTERIZATIONS OF ALTERNATIVE STEADY STATES

In order to discuss the steady state of the system in further detail, it is necessary to introduce some government financial policy. We shall restrict most of our attention to the case where the monetary authorities maintain a constant rate of the nominal money supply, namely

$$(32) \quad \frac{\dot{M}}{M} = \mu$$

The real money supply $m = M/P$ therefore evolves in accordance with

$$\dot{m} = m(\mu - p)$$

so that in steady state we have

$$p = \mu$$

It is evident from previous sections that the steady state will be dependent upon the capital structure employed by the firms. This is determined by the inequality conditions (23) and the corresponding minimized cost of capital. Thus for the government monetary policy specified by (31), the following steady states may be characterized.

A. All Bond Financing by Firms

In this case inequality (23a) holds and the steady state reduces to the following

$$(33a) \quad \frac{U_{\ell} [f(k, \ell) - g, g, \ell, m]}{U_c [f(k, \ell) - g, g, \ell, m]} = - f_{\ell}(k, \ell)(1 - \tau_y)$$

$$(33b) \quad \frac{U_m [f(k, \ell) - g, g, \ell, m]}{U_c [f(k, \ell) - g, g, \ell, m]} = \beta + \mu$$

$$(33c) \quad (1 - \tau_p) f_k(k, \ell) = \frac{\beta(1 - \tau_p) + (\tau_y - \tau_p)\mu}{1 - \tau_y}$$

$$(33d) \quad g + b_g - \tau_y f(k, \ell) - \mu m + [\beta - (1 - \tau_y) f_k] k = 0$$

Thus the steady state in the case where firms engage in all bond financing can be obtained in the following recursive manner. First, given the parameters β , μ , τ_p , τ_y , (33c) yields the marginal physical product of capital. With the linear homogeneity of the production function, this establishes the capital-labor ratio, which in turn determines the real wage $f_\ell(k, \ell)$. Having determined k/ℓ , the two marginal rate of substitution conditions (33a), (33b), together, determine the employment of labor ℓ , and the real stock of money balances m . With k/ℓ and now ℓ fixed, the real stock of capital k is known, while the level of output y immediately follows from the production function. The government budget then determines the real stock of government bonds necessary to balance the budget.

Being a perfect foresight equilibrium, equations (33a) - (33d) have important implications for the debate concerning the effectiveness of fully anticipated government policies under rational expectations. It is seen from these equations that the real productive decisions, k , ℓ , are in general dependent upon the rate of growth of the nominal money supply, as well as upon both the corporate and personal income tax rates. Also, to the extent that public and private goods are viewed as imperfect substitutes by households, so that c and g enter as separate arguments in the utility function, an expansion in real government expenditure will have real effects as far as output and employment is concerned. This confirms the argument advanced recently by Fair (1978) that in a fully rational expectations model generated from underlying optimizing behavior, anticipated government policies are indeed able to have real effects. On the other hand, the range of effective government policies may be restricted. For example, if public and private goods are perfect substitutes, so that c and g enter additively in U , g disappears from the expressions U_c etc. It can then be easily seen that an increase in

government expenditure will cease to have any effect on the real part of the system.^{18/} It will simply displace an equal volume of private activity, resulting in complete "crowding out". This result is in agreement with the analogous property obtained by Buiter (1977), although his analysis was not conducted within an optimizing framework. Also, real activity will be neutral with respect to the monetary growth rate μ if: (i) the corporate and personal income tax rates are equal, and (ii) the utility function is separable in real money balances, so that the marginal rate of substitution U_ℓ/U_c is independent of m .^{19/} In this case the only government policy parameter able to influence real activity is the personal income tax rate, which does so by affecting the consumption-leisure choice.

To preserve simplicity and because of space limitations, we shall restrict our analysis of the comparative static properties of (33) to the effects of tax rates and the monetary growth rate on the capital-labor ratio. These effects have been discussed extensively in the literature over the years and are among the more interesting. They operate through the cost of capital, which therefore provides the critical channel through which alternative tax structures impinge on the system. Also when the results are compared to those we shall derive in Section 8.B below, they serve to highlight very clearly how comparative static effects depend upon the equilibrium financial structure employed by firms and emphasizes again the need to derive equilibria from underlying optimizing procedures.

From equation (33c) we may derive

$$(34a) \quad \text{sgn} \frac{\partial(k/\ell)}{\partial\mu} = \text{sgn} (\tau_p - \tau_y)$$

$$(34b) \quad \frac{\partial(k/\ell)}{\partial\tau_y} < 0$$

$$(34c) \quad \frac{\partial(k/\ell)}{\partial\tau_p} > 0$$

To understand these results it is useful to recall the expression for the nominal rate of interest (13c''), which in steady state is $s = (\beta + \mu)/(1 - \tau_y)$. An

increase in the rate of nominal monetary growth raises the nominal before tax interest rate by $1/(1 - \tau_y)$. The effect on the after-tax real rate of interest to firms, which with all bond financing is their effective cost of capital, is thus equal to $\{(1 - \tau_p)/(1 - \tau_y) - 1\}$, so that the effect on the capital-labor ratio depends upon $(\tau_p - \tau_y)$. In order for all bond financing to be optimal, (23a) imposes a lower bound on this quantity. Thus for example, if the rate of taxation on capital gains and the required minimum rate of dividend payments are both zero, then $\tau_p > \tau_y$ and k/ℓ will rise. But the reverse cannot be ruled out. An increase in the personal income tax rate raises the nominal interest rate s and hence the after-tax interest rate to firms, thereby inducing them to lower their capital-labor ratio. On the other hand, an increase in the corporate tax rate τ_p has no effect on the nominal interest rate. It therefore leads to a reduction in the after-tax real interest rate for firms, inducing them to increase their capital-labor ratio. Given the effects summarized in (34), the implications for the other endogenous variables can be easily obtained by taking appropriate differentials of (33).

B. All Equity Financing by Firms

As already commented in Section 6 above, the case where (23b) holds, so that firms engage in all equity financing, government bonds are also driven out of the system. Hence the steady state becomes

$$(35a) \quad \frac{U_\ell [f(k, \ell) - g, g, \ell, m]}{U_c [f(k, \ell) - g, g, \ell, m]} = -f_\ell(k, \ell)(1 - \tau_y)$$

$$(35b) \quad \frac{U_m [f(k, \ell) - g, g, \ell, m]}{U_c [f(k, \ell) - g, g, \ell, m]} = \beta + \mu$$

$$(35c) \quad (1 - \tau_p)f_k(k, \ell) = \frac{\beta + \tau_c \mu + \bar{i}(\tau_y - \tau_c)}{1 - \tau_c}$$

$$(35d) \quad g + \beta k - \mu m - \tau_y f(k, \ell) - (1 - \tau_y)f_k(k, \ell)k = 0$$

In this case, given the specification of money supply by (32) and the tax rates, the level of government expenditure g must adjust endogenously to satisfy the government budget constraint. Alternatively if g is specified exogenously, along with the tax rates, then the money supply is constrained by the government budget constraint and cannot be described by an independent rule such as (32). In this latter case, it is apparent that the comments made in Section 8.A regarding the equilibrium real effects of an exogenous change in government expenditure no longer apply. Even if c and g enter additively in the utility function, the fact that with $b_g = 0$ any change in g must be met by compensating changes in k , l , and m in order to maintain the government budget in balance, ensures that in general it will generate real effects in the economy. Hence, the "crowding out" effects of an exogenous increase in government expenditure in fact depend critically upon the capital structure employed by firms and emphasizes once again the desirability of casting these issues within an explicit optimizing framework.

Returning to (35) we see that equilibrium is now attained as follows. The capital labor ratio is first determined by equating the marginal physical product of capital to the appropriate cost given in (35c). Then given k/l , the remaining variables g , l , m are determined jointly from (35a), (35b) and (35d), enabling the level of k to be determined. The exogenous government policy parameters μ , τ_y , τ_p , τ_c , all have real effects, again confirming the proposition of Fair. In this case neutrality with respect to the monetary growth rate will pertain if and only if $\tau_c = 0$ and the utility function is separable in m .

The effects of changes in the rate of monetary growth and the various tax rates on the capital-labor ratio are obtained from (35c) and have the following general characteristics

$$(36a) \quad \frac{\partial(k/l)}{\partial\mu} < 0$$

$$(36b) \quad \frac{\partial(k/l)}{\partial\tau_y} < 0$$

$$(36c) \quad \frac{\partial(k/\ell)}{\partial\tau_p} < 0$$

$$(36d) \quad \text{sgn}\left(\frac{\partial k/\ell}{\partial\tau_c}\right) = \text{sgn}[\bar{i}(1 - \tau_y) - (\mu + \beta)]$$

It will be observed that the effects of a change in τ_p and also probably that of an increase in μ are opposite to those obtained under bond financing. The reason for these results can be understood by considering the expressions for the equilibrium rate of capital gains, which from (13d'') is

$$\dot{q}/q = \frac{\beta + \tau_c \mu - \bar{i}(1 - \tau_y)}{1 - \tau_c}$$

Thus an increase in either the rate of monetary growth or the rate of personal income tax τ_y will raise the equilibrium rate of capital gain on equities and hence the equilibrium rate of equity costs $(\bar{i} + \dot{q}/q)$, inducing firms to reduce their capital-labor ratio. On the other hand, an increase in the corporate profit tax rate τ_p leaves equity costs unchanged. The after-tax marginal physical product of capital must remain fixed, so that as τ_p increases, k/ℓ must fall. Finally, the effect of an increase in the capital gains tax on the rate of capital gains is ambiguous, depending upon the expression in (36d).

C. Both Bonds and Equities in Firms' Financial Structure

The third possible steady state equilibrium is one in which both bonds and equities appear in the firms' capital structure. This will arise only in the "knife edge" case where (23) is satisfied with equality, in which case λ will be indeterminate. The only point we wish to make is that government bonds will again appear in equilibrium, restoring the ability to specify an independent monetary policy rule. The steady state is thus essentially the same as in Section 8.A.

9. DYNAMICS IN SIMPLIFIED CASE

Finally, we turn to a brief analysis of the dynamics of the system where labor is assumed to remain fixed at an exogenously set level \bar{l} say. To keep things simple we shall assume that $\tau_y = \tau_p (= \tau)$, in which case firms find it optimal to engage in all bond financing. The monetary authorities maintain a fixed rate of growth of the nominal money supply, as specified by (32). Also, in order to simplify the dynamics as much as possible, we assume that the government maintains the real stock of government bonds fixed, financing its deficit with an endogenous lump sum tax.

With labor supplied exogenously, the marginal rate of substitution condition determining the labor supply is no longer applicable. Thus omitting \bar{l} from the relevant functions, the dynamics of the system (29), (30), corresponding to the present set of assumptions becomes

$$(37a) \quad U_c(c, g, m) = \alpha$$

$$(37b) \quad \frac{U_m(c, g, m)}{U_c(c, g, m)} = \theta + p$$

$$(37c) \quad (1 - \tau_p)f_k(k) = \theta$$

$$(38a) \quad \dot{\alpha} = \alpha(\beta - \theta)$$

$$(38b) \quad \dot{k} = f(k) - c - g$$

$$(38c) \quad \dot{m} = (\mu - p)m$$

$$(38d) \quad g + \theta \bar{b}_g = \tau_y f + \mu m + u$$

The first three equations determine c, θ, p , in terms of α, k, m the dynamics of which are then determined by equations (38a) - (38c). Given the tax rates, monetary growth rate, and g , the final equation determines the endogenous lump sum tax u required to meet the government deficit. Since this equation is a residual, it can henceforth be ignored.

The solutions for c , p may be written in the form

$$(39a) \quad c = c^0(\alpha, m, g)$$

$$(39b) \quad p = \frac{1}{\alpha} U_m [c^0(\alpha, m, g), g, m] - (1 - \tau_p) f_k(k)$$

while θ is given explicitly by (37c). From (39a) we have $c_\alpha^0 = 1/U_{cc} < 0$. In addition, introducing the mild restriction that consumption and money balances are complementary in utility, i.e. $U_{cm} > 0$, we also have $c_m^0 = -U_{cm}/U_{cc} > 0$. Now substituting (39a), (39b), (37c) into (38), the dynamics of the system becomes

$$(40a) \quad \dot{\alpha} = \alpha(\beta - (1 - \tau_p) f_k(k))$$

$$(40b) \quad \dot{k} = f(k) - c^0(\alpha, m, g) - g$$

$$(40c) \quad \dot{m} = m[\mu + (1 - \tau_p) f_k(k) - \frac{1}{\alpha} U_m [c^0(\alpha, m, g), g, m]].$$

Linearizing the system about its steady state equilibrium, we find that its local stability depends upon the eigenvalues of the matrix

$$(41) \quad \begin{pmatrix} 0 & -\alpha(1 - \tau_p) f_{kk} & 0 \\ -c_\alpha^0 & f_k & -c_m^0 \\ -mI_\alpha & m(1 - \tau_p) f_{kk} & -mI_m \end{pmatrix}$$

where all derivatives are evaluated at steady state and

$$I_\alpha = (U_{mc} U_c - U_m U_{cc}) / U_c^2 U_{cc} < 0$$

$$I_m = (U_{mm} U_{cc} - U_{mc}^2) / U_{cc} < 0$$

Denoting the roots of (41) by λ_i , and using properties of cubic equations, the sign restrictions we have imposed suffice to ensure that

$$\lambda_1 + \lambda_2 + \lambda_3 > 0$$

$$\lambda_1 \lambda_2 \lambda_3 < 0$$

Thus we may deduce that there are two unstable roots (possibly complex) and just one stable root. By invoking the transversality conditions we may argue that in response to any disturbance, the system will jump so as to always be on the stable locus associated with the stable eigenvalue. Thus for example, the capital stock will evolve in accordance with the stable first order adjustment process

$$(42) \quad \dot{k} = \Delta_1(k - \bar{k})$$

where \bar{k} denotes the steady state value and Δ is a function of the single stable root $\lambda_1 < 0$; and likewise for the other variables in the system. It is apparent that λ_1 will be a function of the various government policy parameters. In particular, the rate of monetary growth μ will affect the speed of adjustment of the capital stock, as well as its steady state level. To calculate the response of the adjustment speed λ_1 explicitly involves solving the cubic characteristic equation to (41) and is rather tedious to do.

The same kind of analysis can be carried out for the case of equity financing. One can also use the above method to consider the case where labor supply is endogenous. In all cases the strategy is the same, namely to invoke the transversality conditions to eliminate unstable adjustment paths. One would expect that this will lead to stable first order adjustment paths similar to (42), although the extent to which this is in fact the case remains an open question.

10. CONCLUDING COMMENTS

In this paper we have extended some of the recent macrodynamic models in two directions. First, we have specified a more complete corporate sector than such models usually contain; secondly, the relationships describing the private sector are derived from explicit optimizing procedures by households and firms.^{20/}

While much of our attention has been on the development of the model, we have discussed its steady-state structure in detail for one particular form of monetary policy. Probably the most important general conclusion to emerge from this analysis is the need to ground such models in an optimizing framework. It is shown how this will lead to three possible equilibrium capital structures, the choice of which depends upon relevant tax rates, and that these equilibria have very different implications for the effectiveness of various monetary and fiscal instruments. These differences are highlighted most clearly when one considers the effects of an increase in the monetary growth rate on the capital-labor ratio. If all bond financing is optimal, the capital-labor ratio will most probably (but not necessarily) rise; if all equity financing is optimal, it will definitely fall. The difference is even more clear-cut with respect to an increase in the corporate profit tax rate. The equilibrium capital-labor ratio will rise with all bond financing and fall with all equity financing. The effects of government expenditure are also sensitive to the choice of capital structure. For example, we have shown that for a steady state in which bonds exist, a change in real government expenditure will have no effect on real activity if and only if private and public goods are perfect substitutes in the utility function of consumers. This proposition is *not* true if the steady state equilibrium consists entirely of equities; in this case an increase in government expenditure will always generate real effects. Taken together, these results confirm Fair's view that in a complete rational expectations model -- one in which the underlying behavioral relations are obtained from optimizing behavior and fiscal policy is precisely specified -- anticipated government policies are able to generate real output effects.

But the model is capable of dealing with many other kinds of issues. For instances, more attention could be devoted to considering the comparative static properties of alternative monetary and fiscal policies under alternative tax structures. Furthermore, it is apparent from our analysis that currently popular policy discussions of pegging the monetary growth rate at say 4% are seriously

incomplete. The predictions of such policies are indeterminate in our model until fiscal policy is specified. Even then, the 4% rule may not be consistent with *given* tax rates and government expenditure. This is another reason why a richer model such as ours is essential for understanding the *full* implications of policies of this type.

Perhaps more interestingly, one has a framework capable of evaluating the welfare effects of alternative government policies. To take one example, one can look at questions concerning tradeoffs among alternative tax rates. Specifically, the choice of monetary expansion can be viewed as representing the choice of an inflation tax rate. If one can also choose the various income tax rates, this raises the question of the optimal mix between these two forms of taxation. This issue turns out to be extremely complicated to analyze and we have considered it elsewhere using a much simpler version of the present model, which abstracts from the corporate sector and the issues being stressed here.^{21/} But the point is that the model we have developed is capable of providing a framework within which these kinds of important issues can be discussed.

The model can be extended in different directions. Alternative assets such as rented capital, can be introduced. But undoubtedly the most important extension relates to uncertainty. Problems of corporate financing really become interesting only in a world of uncertainty. Recent developments in stochastic calculus indicate that it should not be too difficult a task to extend our framework to include various forms of stochastic disturbances.

FOOTNOTES

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1. We shall follow the convention of denoting partial derivatives by corresponding lower case letters.
 2. We do not mean to suggest that putting money into the utility function is good monetary theory. What we are saying is that until fully satisfactory micro foundations of the demand for money are developed, the procedure we are following is a useful analytical tool.
 3. There is a technical point here which should be noted. In general, the interests of bondholders may conflict with those of stockholders and hence maximizing the market value of all claims against the firm is not the same as maximizing the market value of equity. Later on we shall see that the value maximizing firm will specialize in either all debt or all equity financing, depending upon the tax treatment. It will want to jump to the optimal level of debt or equity immediately at date zero. Hence it may not be possible to satisfy the initial conditions on $b_p(0)$, E_0 in any equilibrium. There are two natural ways around this. One is to impose exogenous bounds on the rate of change of debt or equity. Then the equilibrium would involve eliminating the non-optimal security at the most rapid rate. This introduces a lot of messy mathematical detail. The second is to allow jumps in the appropriate variables directly. More will be said about this problem in footnote 17 below.

4. This formulation of the government budget constraint parallels that of our treatment of the two private sectors' budget constraints by assuming that government decisions are based on the anticipated rate of inflation. This is somewhat in contrast to much of the treatment of macroeconomics which typically replaces p^* in (5) with p , the actual rate of inflation. Since we are concerned with a perfect foresight equilibrium (defined below) where $p^* = p$, these two formulations are for our purposes identical.
5. The problem specified in (1a), (1b) is a standard calculus of variations problem. This can be seen by substituting c into the objective function by using the budget constraint. Define

$$F[b_g^d, b_p^d, m^d, E^d, \dot{b}_g^d, \dot{b}_p^d, \dot{m}^d, \dot{E}^d, \ell^S]$$

$$\equiv U[w\ell^S + s(b_g^d + b_p^d) - p^*(b_g^d + b_p^d + m^d) + iqE^d - T_h - \dot{b}_g^d - \dot{b}_p^d - \dot{m}^d - q\dot{E}^d + \dot{g}, \ell^S]$$

$$+ V(m^d) \equiv F(x, \dot{x}, \ell^S); \bar{F}(x, \dot{x}) = \text{Max}_{\ell^S \geq 0} F(x, \dot{x}, \ell^S)$$

$$x \equiv (b_g^d, b_p^d, m^d, E^d), \quad \dot{x} = (\dot{b}_g^d, \dot{b}_p^d, \dot{m}^d, \dot{E}^d).$$

With this notation (1a) becomes the standard calculus of variations problem

$$\text{Max}_{x(\cdot)} \int_0^{\infty} e^{-\beta t} \bar{F}(x(t), \dot{x}(t)) dt$$

where the maximum is taken over the set of all absolutely continuous

functions $x(\cdot)$ such that $x_0 \equiv \left(\frac{B_{g0}}{P(0)}, \frac{B_{p0}}{P(0)}, \frac{M_0}{P(0)}, E_0 \right)$.

The introduction of the "Lagrangian" \mathcal{H} allows us to write the necessary conditions for optimality in the symmetric form (7) - (11) below.

6. The assumption that financial stock variables are non-negative rules out short selling.
7. The use of Euler inequalities is discussed in further detail in the Appendix below.

8. Note that we are assuming that $\lim_{S \rightarrow \infty} \int_0^S \theta^* dt = \infty$.
9. Note that in the absence of taxes θ^* is independent of i . Dividend policy is therefore irrelevant, confirming the well known Modigliani and Miller Proposition in this case.
10. There are two technical points that need to be made concerning \bar{i} . First, if \bar{i} is set too large, then it will be beyond the firm's ability to pay \bar{i} ; i.e. \bar{i} must be set compatible with the firm's budget constraint. But take care to note that the "real side" i.e. $\gamma(t)$, depends upon the cost of capital, which in turn depends upon \bar{i} .

Secondly, if the constraint $i \geq \bar{i}$ is replaced by an equally (or perhaps more) realistic constraint, for example

$$iqE \geq \alpha\gamma \quad 0 < \alpha < 1$$

i.e. the IRS requires at least a fraction of after-tax net cash flow to be paid out in dividends, then the derivation of θ^* will be slightly different. One must replace the term iqE in (16) by $\alpha\gamma$ and then proceed to (18). The important thing to recognize is that γ will be replaced by $(1 - \alpha)\gamma$ in (20), with θ^* also being subject to a corresponding adjustment. Using (13d") it turns out that θ^* is a function of γ , so that the separation between real production and financial decisions obtained for the constraint $i \geq \bar{i}$ in the text breaks down. The reason for the difference arises from the fact that in the treatment in the text, the quantity \bar{i} is given *exogenously*, whereas the alternative constraint given in this footnote, being a function of γ , is *endogenous*. For space limitations, we restrict ourselves to the simpler constraint given in the text.

11. The derivation of (24) from (23) is rather messy. The details are not given but are available from the authors. The key step involves substituting for \dot{q}/q from (13d') into the definition of τ_s .
12. This is true for the case of a single agent. See Miller (1977) and also Miller and Scholes (1977) for the multi-agent case.
13. Our results are also consistent with Miller's (1977) extension of this proposition, again in the case of a single agent. He shows that the value of any firm in equilibrium will be independent of the amount of debt in its capital structure, provided that the marginal rate of tax payable by stockholders on income from shares is sufficiently far below the marginal rate of tax on personal income. This result follows immediately from the equality $(1 - \tau_y) = (1 - \tau_p)(1 - \tau_s)$.

14. The proof of the assertion in the text is as follows. For notational ease denote θ_{\min}^* by θ^* . From (20) we may write

$$V(t) = \lim_{T \rightarrow \infty} e^{\int_0^t \theta^*(s) ds} \int_t^T e^{-\int_0^s \theta^*(\tau) d\tau} [(1 - \tau_p)[f - wl] - \dot{k}] ds$$

Using the linear homogeneity of the production function and the optimality conditions (27a), (27b), this expression may be simplified to yield

$$V(t) = \lim_{T \rightarrow \infty} e^{\int_0^t \theta^*(s) ds} \int_t^T e^{-\int_0^s \theta^*(\tau) d\tau} [\theta^* k - \dot{k}] ds$$

Now integrating by parts and cancelling, we obtain

$$V(t) = k(t) - e^{\int_0^t \theta^*(s) ds} \lim_{T \rightarrow \infty} e^{-\int_0^T \theta^*(\tau) d\tau} k(T)$$

But the limit of the second term is zero, by the necessity of the transversality condition at infinity for infinite horizon concave programming problems of this type; see Weitzman (1973) for the discrete time case and Benveniste-Scheinkman (1978) for the continuous time version. Hence we deduce $V(t) = k(t)$.

15. The details of the derivation of this equation are available from the authors. The interpretation is as follows. The first two terms on the right hand side

- denote real government expenditures and the real interest payments owing on the outstanding government debt, the net of tax real interest rate on which is equal to the rate of return on consumption θ . The third term is the tax collected on personal income. Since income generated in the economy is ultimately accrued by households, this is taxed at the personal rate τ_y . The fourth term is the inflation tax revenue generated on real money balances. The final term is a complex one. Since $(1 - \tau_p)f_k = \theta_{\min}^*$, it reflects the differential tax rates between firms and households. Also, to the extent that the firms employ equity financing, it reflects the fact that income from shares is taxed twice, first as profit, when it is taxed at the corporate profits rate τ_p , then as personal income when it is included in y and taxed at the rate τ_y . To the extent that these sources of revenue and finance do not balance, the resulting real deficit must be financed either by issuing real bonds or by increasing real money balances.
16. We should also note since θ and p appear in the cost of capital expression (29e), switching in the financial mode may occur during the transition to steady state. To incorporate this will require us to permit jumps in bonds and equities, in order to satisfy the inequality conditions (23).
17. The fact that steady state requires $\dot{k} = \dot{m} = \dot{\alpha} = 0$ is readily apparent from (29), (30a) - (30c). The requirement that $\dot{b}_g = 0$ is less immediate, since b_g is determined residually by (30d). However, the need for bond accumulation to cease in steady state can be established by integrating the government budget constraint and imposing the transversality condition at infinity for consumers, $\lim_{t \rightarrow \infty} U_c(t)b_g(t)e^{-\beta t} = 0$. The argument may be sketched as follows.

First integrate (12) to yield

$$U_c(t) = U_c(0)e^{-\int_0^t \theta(\tau) d\tau} e^{\beta t}$$

Next, integrate the government budget constraint (30c), to obtain

$$b_g(t) = e^{-\int_0^t \theta(s) ds} \left[B + \int_0^t x(s) e^{-\int_0^s \theta(\tau) d\tau} ds \right]$$

where $x(s) \equiv g - \tau_y f - mp + [\theta - (1 - \tau_y)f_k]k - \dot{m}$ and is independent of b_g , while B is an arbitrary constant. Inserting these solutions for $U_c(t)$, $b_g(t)$ into the transversality condition and taking the limit we require (with $U_c(o)$ finite)

$$B = - \int_0^{\infty} x(s) e^{-\int_0^s \theta(\tau) d\tau} ds$$

so that the implied time path for real government bonds is

$$b_g(t) = -e^{-\int_0^t \theta(s) ds} \frac{\int_0^{\infty} x(s) e^{-\int_0^s \theta(\tau) d\tau} ds}{t}$$

which converges to (31d) in steady state.

The implied endogenous initial value $b_g(o)$ is not necessarily equal to $B_{g0}/P(o)$, even after allowing for the initial jump in the price level. Thus in order to ensure an equilibrium we require the monetary authorities to undertake an initial open market exchange of money for bonds in order to ensure that the solution for $b_g(t)$ is consistent with the consumers' optimality conditions.

18. The assumption that c and g are perceived as being perfect substitutes may be held as part of the assumption of rationality.
19. This form of neutrality in a full employment model such as this is often referred to as being "super-neutrality".
20. One might argue that an unsatisfactory feature of our analysis is that it is carried out in a framework of infinite-lived households rather than the more realistic case of overlapping finite-lived households. This argument is more apparent than real. First, the method of determining the cost of capital and the correct problem for a value maximizing capital accumulating firm to solve from the consumers' side of the economy will proceed in much the same way for overlapping generations as in our model; see Scheinkman (1978) for such an analysis in the no-tax case.

Secondly, it is not all that clear that the specification our consumers' objective function is as "unrealistic" as it looks at first sight. It depends

upon the treatment of bequests. If individuals give utility (in the manner of Barro (1974)) to their descendents, then the perfect foresight overlapping generations equilibrium problem can be shown to be equivalent to ours. Hence we believe it is fruitful to analyze both the infinite horizon and the overlapping generations cases. We have chosen the former because it seems easier and the transversality conditions at infinity are available.

21. See Turnovsky and Brock (forthcoming).

APPENDIX

1. Euler Inequalities

We now discuss in further detail the use of Euler inequalities in the derivation of the optimality conditions in Section 3. These are required in a situation where a vector $x(t)$ say, is required to be non-negative for each $t \geq 0$. Conditions corresponding to (6a) - (10b) in the text are derived by writing down the Kuhn-Tucker condition for the maximization problem in one variable:

$$\text{Max}_{\epsilon \geq 0} \int_0^T e^{-\beta t} \bar{F}(\bar{x} + \epsilon \eta, \dot{\bar{x}} + \epsilon \dot{\eta}, t) dt \equiv \text{Max}_{\epsilon \geq 0} J_{\eta}(\epsilon) \quad (\text{A.1})$$

for all absolutely continuous variations $\eta(\cdot)$ such that $\eta(0) = 0$, $\eta(T) = 0$, $\eta(t) \geq 0$ for t such that $\bar{x}(t) = 0$, $\eta(t)$ arbitrary for t such that $\bar{x}(t) > 0$. Here $\bar{x}(t)$ denotes an optimal path. The problem specified in (A.1) is considered for each T since any $\bar{x}(\cdot)$ that is optimal over $[0, \infty)$ must be optimal over $[0, T]$ subject to $x(0) = x_0$, $x(T) = \bar{x}(T)$.

Now the first order necessary condition for (A.1) is

$$\begin{aligned} 0 \geq J'_{\eta}(0) &= \int_0^T e^{-\beta t} [\bar{F}_x \eta + \bar{F}_{\dot{x}} \dot{\eta}] dt = \int_0^T \left\{ e^{-\beta t} \bar{F}_x - \frac{d}{dt} e^{-\beta t} \bar{F}_{\dot{x}} \right\} dt + \bar{F}_x \eta \Big|_0^T \\ &= \int_0^T \left\{ e^{-\beta t} \bar{F}_x - \frac{d}{dt} [e^{-\beta t} \bar{F}_{\dot{x}}] \right\} \eta dt. \end{aligned} \quad (\text{A.2})$$

Since (A.2) holds for absolutely continuous $\eta(\cdot)$ such that $\eta(t) \geq 0$, for $\bar{x}(t) = 0$, we deduce from (A.2)

$$e^{-\beta t} \bar{F}_x(t) - \frac{d}{dt} \left[e^{-\beta t} \bar{F}_{\dot{x}}(t) \right] \leq 0 \quad (= 0 \text{ for } t \text{ such that } \bar{x}(t) > 0) \quad (\text{A.3})$$

where $\bar{F}_x(t)$, $\bar{F}_{\dot{x}}(t)$ denote $\partial \bar{F} / \partial x$, $\partial \bar{F} / \partial \dot{x}$ evaluated at $(\bar{x}(t), \dot{\bar{x}}(t))$.

Note that the "proof" we wrote down in the text was purely *formal*, since for example, we *assumed* $\frac{d}{dt} [e^{-\beta t} \bar{F}_{\dot{x}}(t)]$ *exists* for each t . Hence the "necessary" conditions (6a) - (10b) have not really been established in a truly rigorous way. However, (6a) - (10b) may be used to generate *candidates* for optima. This is so because of the following

Theorem: Let $\bar{x}(t)$ satisfy (A.3), $\bar{F}(x, \dot{x})$ be concave in (x, \dot{x}) ,
 $\bar{F}_x + \beta \bar{F}_{\dot{x}} - \dot{\bar{F}}_{\dot{x}} \leq 0$ for all t and $e^{-\beta t} \bar{F}_x(t) \cdot \bar{x}(t) \rightarrow 0$ as
 $t \rightarrow \infty$, then $x(\cdot)$ is optimal for (A.1) with $T = \infty$.

Proof: Let $x(\cdot)$ be any other path starting from x_0 . Then

$$\begin{aligned} & \int_0^T e^{-\beta t} [F(x, \dot{x}) - F(\bar{x}, \dot{\bar{x}})] dt \leq \int_0^T e^{-\beta t} [\bar{F}_x(x - \bar{x}) + \bar{F}_{\dot{x}}(\dot{x} - \dot{\bar{x}})] dt \\ & = \int_0^T \left\{ e^{-\beta t} \bar{F}_x - \frac{d}{dt} e^{-\beta t} \bar{F}_{\dot{x}} \right\} (x - \bar{x}) dt + e^{-\beta t} \bar{F}_{\dot{x}} (x - \bar{x}) \Big|_0^T \\ & \equiv \int_{T^+} E(t) \xi(t) dt + \int_{T^0} E(t) \xi(t) dt + e^{-\beta T} \bar{F}_{\dot{x}} [x(T) - \bar{x}(T)] \end{aligned} \quad (A.4)$$

where

$$\xi = x(\cdot) - \bar{x}(\cdot), \quad E(t) = e^{-\beta t} \bar{F}_x(t) - \frac{d}{dt} [e^{-\beta t} \bar{F}_{\dot{x}}(t)]$$

$$T^+ = \{0 \leq t \leq T \mid \bar{x}(t) > 0\}, \quad T^0 = \{0 \leq t \leq T \mid \bar{x}(t) = 0\}.$$

Hence the RHS of (A.4) is given by

$$\begin{aligned} \text{RHS(A.4)} & = \int_{T^0} E(t) [x(t) - \bar{x}(t)] dt + e^{-\beta t} \bar{F}_{\dot{x}}(t) [x(t) - \bar{x}(t)] \\ & = \int_{T^0} E(t) x(t) dt + e^{-\beta t} \bar{F}_{\dot{x}}(t) [x(t) - \bar{x}(t)] \\ & \leq -e^{-\beta t} \bar{F}_{\dot{x}}(t) \bar{x}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (A.5)$$

This sequence of equalities and inequalities follows from

$$E(t) = 0, t \in T^+; \quad E(t) \leq 0, t \in T^0, \quad \bar{x}(t) = 0, t \in T^0, \quad \bar{F}_{\dot{x}}(t) \leq 0$$

respectively. This ends the proof.

Under the hypothesis maintained in this paper, conditions (6a) - (10b), together with $\text{TVC}_\infty: 0 = \lim_{t \rightarrow \infty} e^{-\beta t} \bar{F}_{\dot{x}} x$ are *sufficient* first order conditions for an optimum. The task of finding sufficient conditions on $F(\cdot)$ so that these are *necessary* for an optimum is beyond the scope of this paper. This is so

because this involves the study of *bounded state* problems, which are difficult. For further discussion see Hestenes (1966, p. 52) where a Fundamental Lemma of the Calculus of Variations is developed for problems where $x(t)$ is required to be non-negative for all t . Perhaps it may be used to derive (A.3) when $\frac{d}{dt} \bar{F}_x$ exists.

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