ON THE FIRST AND SECOND MOMENTS OF THE TRUNCATED MULTI-NORMAL DISTRIBUTION AND A SIMPLE ESTIMATOR

by

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Abstract

The relations between the first two moments of multivariate normal random variables with some components truncated are found. Simple instrumental variable estimators are available. The analysis extends results of Amemiya, Sickles and Schmidt.

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On the First and Second Moments of the Truncated Multi-Normal Distribution and A Simple Estimator

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1. Introduction

Amemiya (1974) extended Tobin's model (1958) to the Multivariate Regression and Simultaneous Equations models and a simple instrumental variables estimator was proposed. Subsequent works of Sickles and Schmidt (1978) and Lee (1976) extended the models to the case where only some of the dependent variables are truncated. An application of a two equations model is in Sickles, Schmidt and Witte (1979). An example of three equations model with only one dependent variable truncated is a disequilibrium market model in Goldfeld and Quandt (1975) (see Lee (1976)). In Sickles and Schmidt (1978), an instrumental variable estimator for the two equations model was derived. As pointed out in Sickles and Schmidt (1978) it would be desirable to have simple instrumental variables estimators for models with more than two equations. To fill the gap for the above literatures, in this article we derive such simple estimators. The derivation extends the works of Tallis (1961) and Amemiya (1973).

2. The Model and Results

Consider the multiple regression model with n multivariate normal disturbances,

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$$\underset{n\times 1}{y_t^*} = \underset{n\times k}{\alpha}\ \underset{k\times 1}{x_t} + \underset{n\times 1}{\varepsilon_t} \qquad t=1,\ldots,T \qquad (1)$$

where x_t is a $k\times 1$ exogenous variables and $\varepsilon_t \sim$ i.i.d.$(0, \Sigma)$. The observed vectors of dependent variables y_t are truncated as

$$y_t = y_t^* \quad \text{if and only if} \quad y_{G+1t}^* > 0,\ldots,y_{nt}^* > 0;$$

$$y_t \quad \text{is not available, otherwise;}$$

where $y_t^{*\prime} = (y_{1t}^*,\ldots,y_{Gt}^*, y_{G+1t}^*,\ldots,y_{nt}^*)$.

To derive an instrumental variables estimator for the parameters α and Σ, we will investigate the relationships between the first two moments of the truncated distributions of y_t. Before we state and prove the relations, let us introduce some notations. Let

$$\Sigma = \begin{bmatrix} \sigma_{11} \cdots\cdots \sigma_{1G} & \sigma_{1G+1} \cdots\cdots \sigma_{1N} \\ \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots \\ \sigma_{G1} \cdots\cdots \sigma_{GG} & \sigma_{GG+1} \cdots\cdots \sigma_{Gn} \\ \sigma_{G+1\,1} \cdots \sigma_{G+1G} & \\ \vdots \qquad\quad \vdots & \Sigma_{G+1n} \\ \sigma_{n1} \cdots\cdots \sigma_{nG} & \end{bmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Let f be the joint density of $N(0, \Sigma_{G+1n})$; f_q the marginal density of the $(q-G)$th variable of $N(0, \Sigma_{G+1n})$; $f_{(q)}$ the joint conditional density of the remaining $n-G-1$ variables of $N(0, \Sigma_{G+1n})$ given that the $(q-G)$th variable of $N(0, \Sigma_{G+1n})$ is equal to $-\alpha_q x$; f_{qs} the joint marginal density of the $(q-G)$th and $(s-G)$th variables of $N(0, \Sigma_{G+1n})$; $f_{(qs)}$ is the joint conditional density of the remaining $n-G-2$ variables given that the $(q-G)$th and $(s-G)$th variables of $N(0, \Sigma_{G+1n})$ are equal to $-\alpha_q x_t$ and $-\alpha_s x_t$. Furthermore, let $a_{qt} = -\alpha_q x_t$ and

$$p_t = \Pi_{s=G+1}^{n} \int_{a_{st}}^{\infty} f(\lambda)d\lambda \equiv \int_{a_{nt}}^{\infty} \cdots \int_{a_{G+1t}}^{\infty} f(\varepsilon_{G+1},\ldots,\varepsilon_n)d\varepsilon_{G+1},\ldots d\varepsilon_n$$

$$F_{(q)t} = \Pi_{\substack{s=G+1 \\ s \neq q}}^{n} \int_{a_{st}}^{\infty} f_{(q)}(\lambda)d\lambda, \qquad F_{(qs)t} = \Pi_{\substack{r=G+1 \\ r\neq s,q}}^{n} \int_{a_{rt}}^{\infty} f_{(q,s)}(\lambda)d\lambda$$

Denote $S_1 = \{t \,|\, y^*_{G+1t} > 0,\ldots, y^*_{nt} > 0\} \cap \{1,\ldots,T\}$.

The first and second moments of the truncated distribution can be derived as in Tallis (1961):

$$E(\varepsilon_{it}|t \in S_1) = \Sigma_{q=G+1}^{n}\sigma_{iq}f_q(a_{qt})F_{(q)t}/p_t \qquad i=G+1,\ldots,n \tag{2}$$

$$E(\varepsilon_{it}|t \in S_1) = (\sigma_{iG+1},\ldots,\sigma_{in})\Sigma_{G+1n}^{-1}(E(\varepsilon_{G+1t}|t \in S_1),\ldots$$
$$\ldots,E(\varepsilon_{nt}|t \in S_1))' \qquad i=1,\ldots,G \tag{3}$$

$$E(\varepsilon_{it}\varepsilon_{jt}|t \in S_1) = \sigma_{ij} + \Sigma_{q=G+1}^{n}\sigma_{iq}\sigma_{jq}\sigma_{qq}^{-1}a_{qt}f_q(a_{qt})F_{(q)t}/p_t$$
$$+ \Sigma_{q=G+1}^{n}\sigma_{iq}[\Sigma_{s=G+1}^{n}(\sigma_{js}-\sigma_{qs}\sigma_{jq}\sigma_{qq}^{-1})f_{qs}(a_{qt}, a_{st})F_{(qs)t}/p_t] \tag{4}$$
$$i,j=G+1,\ldots,n$$

$$E(\varepsilon_{it}\varepsilon_{jt}|t \in S_1) = \sigma_{ij}-(\sigma_{iG+1},\ldots,\sigma_{in})\Sigma_{G+1n}^{-1}(\sigma_{jG+1},\ldots,\sigma_{jn})'$$
$$+ (\sigma_{iG+1},\ldots,\sigma_{in})\Sigma_{G+1n}^{-1} \begin{pmatrix} E(\varepsilon_{G+1t}^2|t \in S_1)\ldots E(\varepsilon_{G+1t}\varepsilon_{nt}|t \in S_1) \\ \vdots \qquad\qquad \vdots \\ E(\varepsilon_{nt}\varepsilon_{G+1t}|t \in S_1)\ldots E(\varepsilon_{nt}^2|t \in S_1) \end{pmatrix}$$
$$\cdot\ \Sigma_{G+1n}^{-1}(\sigma_{jG+1},\ldots,\sigma_{jn})' \qquad i,j=1,\ldots,G \tag{5}$$

and

$$E(\varepsilon_{it}\varepsilon_{jt}|t \in S_1) = (\sigma_{iG+1},\ldots,\sigma_{in})\Sigma^{-1}_{G+1n}(E(\varepsilon_{jt}\varepsilon_{G+1t}|t \in S_1),\ldots$$
$$\ldots, E(\varepsilon_{jt}\varepsilon_{nt}|t \in S_1))' \qquad i=1,\ldots,G \tag{6}$$
$$j = G+1,\ldots,n$$

With the first and second moments derived in (2) to (6), one can prove the following theorem which generalizes theorem 1 in Amemiya (1974).

<u>THEOREM</u>: Let the density of $\varepsilon' = (\varepsilon_1,\ldots,\varepsilon_n)$ be given by

$$\frac{1}{p}\, g(\varepsilon_1,\ldots,\varepsilon_n) \quad \text{for} \quad a_{G+1} < \varepsilon_{G+1},\ldots,\ a_n < \varepsilon_n;\ 0 \text{ elsewhere,} \qquad (7)$$

where g is the joint density of n-multivariate normal distribution $N(0, \Sigma)$, a_i are constants and $p = \prod_{s=G+1}^{n} \int_{a_s}^{\infty} f(\lambda)d\lambda$. Then

$$\sigma^{i'}E(\varepsilon_i\varepsilon) = 1 + a_i\sigma^{i'}E(\varepsilon) \qquad i=G+1,\ldots,n \qquad (8)$$

and

$$\sigma^{i'}E(\varepsilon_i\varepsilon) = 1, \quad \sigma^{i'}E(\varepsilon) = 0 \qquad i=1,\ldots,G \qquad (9)$$

where $\sigma^{i'}$ is the i^{th} row of Σ^{-1} .

3. <u>Proofs</u>. Denote $\Sigma = [\Sigma_1 \vdots \Sigma_{G+1}] = \begin{pmatrix} \Sigma_{1G} & \vdots \Sigma_* \\ \Sigma_*' & \vdots \Sigma_{G+1n} \end{pmatrix}$ and σ_i' the i^{th} row of Σ.

From (2) and (3), it is obvious that

$$E(\varepsilon) = \Sigma_{G+1}\mu \qquad (10)$$

where $\mu' = (f_{G+1}F_{(G+1)}/p,\ldots,f_nF_{(n)}/p)$. Rewrite (4) in the matrix notation,

$$E(\varepsilon_j\tilde{\varepsilon}) = \tilde{\sigma}_j + \Sigma_{G+1n}D(\sigma_{jq}\sigma_{qq}^{-1}a_q)\mu + \Sigma_{G+1n}H\tilde{\sigma}_j$$
$$- \Sigma_{G+1n}D(\tilde{\sigma}_q{}'h_q\sigma_{qq}^{-1})\tilde{\sigma}_i \qquad j=G+1,\ldots,n \qquad (11)$$

where $\tilde{\varepsilon}' = (\varepsilon_{G+1},\ldots,\varepsilon_n)$; $\tilde{\sigma}_j'$ is the j'th row of Σ_{G+1}; $D(\cdot)$ is a $(n-G)$ diagonal matrix with diagonal elements in the parentheses where $q = G+1,\ldots,n$; H is a $(n-G)$ square matrix with entries $f_{qs}(a_{qs})F_{(qs)}/p$, $q,s = G+1,\ldots,n$ and h_s is the $(s-G)$th column of H, $s = G+1,\ldots,n$. With (11), equations (6) can be rewritten as

$$E(\varepsilon_i\varepsilon_j) = \sigma_{ij} + \tilde{\sigma}_j'D(\sigma_{iq}a_q\sigma_{qq}^{-1})\mu + \tilde{\sigma}_j'H\tilde{\sigma}_i - \tilde{\sigma}_j'D(\tilde{\sigma}_q'h_q\sigma_{qq}^{-1})\tilde{\sigma}_i \qquad i=1,\ldots,G;\; j=G+1,\ldots,n \tag{12}$$

With (11), equation (5) is

$$E(\varepsilon_i\varepsilon_j) = \sigma_{ij} + \tilde{\sigma}_j'D(\sigma_{iq}a_q\sigma_{qq}^{-1})\mu + \tilde{\sigma}_j'H\tilde{\sigma}_i - \tilde{\sigma}_j'D(\tilde{\sigma}_q'h_q\sigma_{qq}^{-1})\tilde{\sigma}_i \qquad i,j=1,\ldots,G \tag{13}$$

It follows from (11), (12) and (13) that

$$E(\varepsilon_i\varepsilon) = \sigma_i + \Sigma_{G+1}D(\sigma_{iq}a_q\sigma_{qq}^{-1})\mu + \Sigma_{G+1}H\tilde{\sigma}_i - \Sigma_{G+1}D(\tilde{\sigma}_q'h_q\sigma_{qq}^{-1})\tilde{\sigma}_i \qquad i=1,\ldots,n \tag{14}$$

It is obvious now that (9) follows from (10) and (14) as $\sigma^{i'}\sigma_i = 1$ and $\sigma^{i'}\Sigma_{G+1} = 0$. Similarly, relations (8) hold as

$$\begin{aligned}
&\sigma^{i'}E(\varepsilon_i\varepsilon) - a_i\sigma^{i'}E(\varepsilon) \\
&= 1 + (a_i\sigma_{ii}\sigma_{ii}^{-1})(f_iF_{(i)}/p) + h_i'\tilde{\sigma}_i - (\tilde{\sigma}_i'h_i\sigma_{ii}^{-1})\sigma_{ii} - \\
&\quad - a_if_iF_{(i)}/p \\
&= 1 \qquad\qquad \text{Q.E.D.}
\end{aligned}$$

An alternative proof which will provide a more clear link between the above theorem and theorem 1 in Amemiya (1974) can be based on limiting arguments.[1] Denote

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[1] I am indebted to a reviewer of this journal for this observation.

$$p^* = \prod_{s=1}^{n} \int_{a_s^*}^{\infty} f(\lambda)d\lambda, \quad F^*_{(q)} = \prod_{\substack{s=1 \\ s\neq q}}^{n} \int_{a_s^*}^{\infty} f_{(q)}(\lambda)d\lambda$$

$$F^*_{(qs)} = \prod_{\substack{r=1 \\ r\neq s,q}}^{n} \int_{a_r^*}^{\infty} f_{(q,s)}(\lambda)d\lambda, \qquad s,q=1,\ldots,n$$

and E^* be the expectation operator based on the density function $h(\varepsilon)$ where $h(\varepsilon) = g(\varepsilon)/p^*$ for $a_s^* < \varepsilon_s$, $s=1,\ldots,n$; $h(\varepsilon) = 0$ otherwise. Amemiya (1974) shows that

$$\sigma^{i'} E^*(\varepsilon_i \varepsilon) = 1 + a_i^* \sigma^{i'} E^*(\varepsilon) \qquad i=1,\ldots,n \qquad (15)$$

By letting $a_i^* = a_i$ for $i = G+1,\ldots,n$ and the remaining a_i^*, $i=1,\ldots,G$ diverge to $-\infty$, it can be shown that (i) p^* converges to p, (ii) $F^*_{(q)}$, $F^*_{(qs)}$ converge to $F_{(q)}$ and $F_{(qs)}$ respectively (iii) $f_q(a_q^*)$ converges to $f_q(a_q)$ for $q = G+1,\ldots,n$; 0 otherwise, (iv) $f_{qs}(a_q^*, a_s^*)$ converges to $f_{qs}(a_q, a_s)$ for $q, s = G+1,\ldots,n$; 0 otherwise. It follows from (i) - (iv) that $E^*(\varepsilon)$ converges to $E(\varepsilon)$ in (10) and $E^*(\varepsilon_i\varepsilon)$ converges to $E(\varepsilon_i\varepsilon)$ in (14) for all $i = 1,\ldots,n$. Hence equations (8) and (9) follow from (15) as a_i^*, $i = 1,\ldots,G$ diverge to $-\infty$.

4. A Consistent Instrumental Variables Estimator

With the relations in (8) and (9), an instrumental variables estimation procedure originated in Amemiya (1974) can be extended to the model (1). It follows from (8) and (9) that

$$y_{it}^2 = \delta_i z_{it} + \eta_{it} \qquad t \in S_1; \quad i = 1,\ldots,n \qquad (16)$$

where $\delta_i = (1/\sigma^{ii})(1, -\sigma^{i1}, -\sigma^{i2}, \ldots, -\sigma^{i(i-1)}, -\sigma^{i(i+1)},\ldots,$

$$-\sigma^{in}, \sigma^{i'}\alpha);$$

$$z'_{it} = (1, y_{it}(y_{1t},\ldots,y_{i-1t}, y_{i+1t},\ldots,y_{nt}, x'_t);$$

σ^{ij} is the (i, j)th element of Σ^{-1} and $\sigma^{i'}$ is the i^{th} row of Σ^{-1}. The instrumental variables method as suggested in Amemiya (1974) can then be applied. This instrumental variable procedure is computationally simple and consistent. The detail is referred to Amemiya (1974).

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