A CLASS OF SOLUTION TO BARGAINING PROBLEMS

by

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Abstract

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In his classic paper on the bargaining problem, Nash characterized the unique solution to satisfy a list of four axioms. A generalization of one of them, his independence of irrelevant alternatives axiom, is proposed. In conjunction to the other three, it is shown to yield a whole class of new solutions. Examples are provided.

The most general reformulation of the axiom is incompatible with single-valuedness of solutions; solution correspondences are then defined and an existence result is provided.

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In [6], Roth obtained a new solution to the classic bargaining problem through a reformulation of the axiom of independence of irrelevant alternatives that appears in Nash's original treatment [4]. A more general reformulation of this axiom is proposed here and is shown to yield a large class of new solutions, including the Nash and Roth solutions as particular cases. This reformulation is motivated in Section I, while an existence theorem and additional examples are provided in Section II. In Section III, it is shown that the most general reformulation of the independence axiom is not compatible with single-valuedness of solutions, and an existence result is provided for solution correspondences.

Section I

A two-person "bargaining problem" consists of a convex and compact subset of R^2, denoted S, representing the utility vectors (measured in some von-Neumann-Morgernstern utility scales) available to the two players. Σ is the class of all such bargaining problems. A <u>solution in Σ'</u>, a subset of Σ, associates to every S in Σ' a unique point of S; this point represents the agreement reached by the players. Alternatively, one can think of an impartial arbitrator whose role is to select a point of S, and impose it on the players.[1]

Nash characterized such a solution by first specifying an outcome d, sometimes called a "status quo", and demanding that a solution satisfy four axioms: 1: Pareto-optimality, 2: symmetry, 3: scale invariance and 4: independence of alternative other than d. (These axioms are stated below in a more precise way.) Given $d = (d_x, d_y)$, and calling Σ_d the subclass of Σ consisting of all the bargaining problems S such that d belong to S and be strictly dominated by at least one point of S, Nash showed that the only solution in Σ_d to satisfy his four axioms, was given by the point (x^*, y^*) of S maximizing $(x-d_x)(y-d_y)$. This solution is denoted $f_d(\cdot)$.

Defining now the "point of minimal expectations" as $m(S) = (a_y(S), b_x(S))$ where $a(S)$ (resp. $b(S)$) is the Pareto-optimal point of S the most favorable to Player 1 (resp. Player 2), Roth [6] redefined axiom 4 as 4' : independence of alternatives other than $m(S)$, and established the existence of a unique solution in Σ satisfying Axioms 1, 2, 3 and 4' : it is the Nash solution with $m(S)$ as status quo if $m(S)$ is not Pareto-optimal, and $m(S)$ otherwise.

Roth also showed that no solution existed that would satisfy Axioms 1, 2, 3 and 4", with 4" : independence of alternatives other than $i(S) = (a_x(S), b_y(S))$. The "ideal point" $i(S)$ appears in the construction of a solution proposed by Raiffa [5] and characterized axiomatically by Kalai and Smorodinsky [3].

It is the purpose of this paper to investigate the existence of solutions satsifying Axioms 1, 2, 3 as well as a generalized axiom of independence of irrelevant alternatives, following up the idea of Roth's paper. To that effect, we consider a class of functions $g : \Sigma \to R^2$ associating to each bargaining problem S a point $g(S)$ of the payoff space. It is with respect to $g(S)$ that the independence condition will be formulated. Our motivation is that the geometry of S is not relevant in all of its details; only a few essential features matter. It is those essential features that $g(\cdot)$ is meant to capture. $g(S)$ does not have to be the outcome that would achieve if the players failed to reach an agreement, even though such an interpretation may sometimes be pertinent. But in general, $g(S)$ may not even be feasible (for instence $i(S)$ is feasible only in the trivial cases where it is the unique Pareto-optimal point and $m(S)$ is not feasible if the segment connecting $A(S)$ to $B(S)$ constitutes the South-West boundary of S); it is rather a "reference" point to which the players or the arbitrator find it natural to compare any proposed solution. This reference point summarizes the main features of S and leads to a simplification of the bargaining process.

How exactly is the reference point chosen is more difficult to explain and this is why a whole class of functions will be examined here. To further motivate the search for additional solutions, examples will now be provided to show how the existing solutions often select counterintuitive outcomes.

In what follows $co\{M_1,M_2,...,M_n\}$ denote the convex hull of the n points $M_1,M_2,...,M_n$. Such a set is a well defined bargaining problem.

Let $0 = (0,0)$, $A = (1,0)$, $B = (0,1)$, and let $d = (0,0)$. The Nash solution for $S_1 = co\{0,A,B\}$ is $C = (1/2, 1/2)$. It was observed by Luce and Raiffa [5] that $S_2 = co\{0,A,C,D\}$ with $D = (0,1/2)$ has the same Nash solution as S_1 , which is somewhat disturbing since S_2 is obtained by cutting off from S_1 a region of points that are all favorable to Player 2. Player 2's bargaining position deteriorates in the process and one should expect the final outcome to reflect this change. No such move is prescribed by the

Nash solution, no matter how d is chosen.

Roth's solution is more appealing since it would prescribe C as the solution of S_1 and $E = (3/4, 1/4)$ as the solution of S_2. Each coordinate of $m(S)$ represents the maximal utility achievable by the corresponding player when the other player receives his preferred outcome. Assuming that Pareto-optimality is preserved, it is the minimum that each player can expect, and embodies a rather pessimistic outlook on the bargaining process. This is why Roth refers to $m(S)$ as the "point of minimal expectations".

On the other hand, it seems that a minimum of cooperation among the players would allow them to randomize between their best points $a(S)$ and $b(S)$. The middle $M(S)$ of the segment joining $a(S)$ and $b(S)$ could be called the "point of minimal compromise". As a reference point, it is somewhat more optimistic than $m(S)$. It will be shown later that there exists a solution associated with this reference point.

The solutions associated with $m(S)$ and $M(S)$ are not very satisfactory from the viewpoint of continuity (examples of perverse discontinuities when the Hausdorff topology is used, are provided in the Appendix.) In addition, they would select the same outcome C for both S_1 and $S_3 = \text{co}\{A,B,E,F\}$ where $E = (0,-1)$ and $F = (1,-1)$, even though Player 2 seems to be in a much better position in S_1 than in S_3. Indeed all the solutions so far proposed are irresponsive to the geometry of S in its South, West and Southwest regions, although in actual bargaining, the potential losses of the players are often as relevant as their potential gains. For that reason, we will examine the existence of solutions having as reference point $D(S)$, the intersection of the diagonals of the smallest rectangle with sides parallel to the axes including S.

Such a solution will be shown to exist, and to yield the same outcome for $S_4 = co\{A,B,D\}$ and $S_5 = co\{A,B,G\}$ with $G = (1/2,0)$ in spite of the fact that the set S_5 is clearly more favorable to Player 1. The solutions associated with $m(\cdot)$, $M(\cdot)$, $D(\cdot)$ sometimes depend on "accidental" features of the boundary of S and are invariant under a number of geometric transformations performed on S that modify the shape of S in ways that one feels ought to be taken into account. As a final example, we will therefore take the center of gravity $G(S)$ of S as reference point, since $G(S)$ exhibits the right kind of responsiveness. In the next section, the general existence of a solution associated with a reference function g is investigated.

Section II

We are given the function $g: \Sigma \to R^2$.

$\Sigma_g \subseteq \Sigma$ is defined by

"$S \varepsilon \Sigma_g \Leftrightarrow \exists x \varepsilon S$ such that $x > g(S)$" .

A g-solution in Σ_g is a function $f_g(\cdot)$ associating to each bargaining problem in Σ_g a unique point of S , $f_g(S)$. $P(S)$ denotes the set of Pareto-optimal points of S .

$$P(S) \equiv \{z = (x,y) \varepsilon R^2 | z \varepsilon S \text{ , } z' \geqq z \text{ , } z' \neq z \Rightarrow z' \notin S\}$$

Δ is the 45° line.

$T(S)$ denotes the symmetric of S with respect to Δ .

$$T(S) = \{(x,y) \varepsilon R^2 \mid (y,x) \varepsilon S\}$$

$A(\cdot) : R^2 \to R^2$ is a positive affine transformation iff

$\exists\ a_1, a_2 > 0,\ b_1, b_2 \geqq 0$ such that

$$A(x,y) = (a_1 x + b_1 \text{ , } a_2 y + b_2) \quad \forall (x,y) \varepsilon R^2 \text{ .}$$

The following axioms are imposed on $f_g(\cdot)$

<u>Axiom 1</u> - Pareto-optimality : $f_g(S) \varepsilon P(S)$

<u>Axiom 2</u> - Symmetry : $[T(S) = S , T(g(S)) = g(S)] \Rightarrow T(f_g(S)) = f_g(S)$.

<u>Axiom 3</u> - Scale invariance : $f_g(A(S)) = A(f_g(S))$ for every positive affine transformation A .

<u>Axiom 4</u> - Independence of alternatives other than $g(S)$.

$[S \subset S' , g(S) = g(S') , f_g(S') \varepsilon S] \Rightarrow f_g(S') = f_g(S)$

In addition, we will demand that g satisfy two properties

<u>Property 1</u> - Scale invariance: $g(A(S)) = A(g(S))$ for every positive affine transformation A .

Let $L(S)$ be the unique line of slope -1 supporting S at a Pareto-optimal point, and let $I(S) = L(S) \cap \Delta$.

<u>Property 2</u> - Invariance under symmetrization of almost symmetric bargaining problems:

$[T(g(S)) = g(S) , I(S) \varepsilon S] \Rightarrow [\exists S' \supseteq S$ such that $T(S') = S' , I(S') = I(S)$ and $g(S') = g(S)]$.

P1 is easy to interpret. P2 states that bargaining problems exhibiting enough symmetry $(g(S) , I(S) \varepsilon S)$ can be replaced by symmetric bargaining problems with the same essential features. It is clear that such conditions are needed in order for a characterization to be possible since they guarantee the relevance of Axioms 3 and 4.

As an illustration of this comment, consider the following example: $g(S)$ is defined to be the middle of the segment joining $m(S)$ and $A(S)$. Such a function could result from a non-symmetric assessment of the bargaining situation by the two players. In such conditions, it is unreasonable to expect that the solution should enjoy any symmetry property. Indeed, since for a symmetric game, $g(S)$ belongs to Δ only if $b(S) = a(S)$ (which implies that $P(S)$ is a singleton), the hypotheses under which the symmetry axiom applies are never satisfied in non-trivial games, and therefore the symmetry axiom is mostly irrelevant. Multiplicity of solutions f_g should then be expected. It is easy to check that the Nash solution with $g(S)$ as status quo is one of them. A general analysis of the case where the symmetry axiom is not imposed is provided by Brito, Buencriastiani and Intriligator [1].

Proposition 1 If g satisfies Properties 1 and 2, there exists a unique g-solution in Σ_g satisfying Axioms 1-4: it is the Nash solution with $g(S)$ as status quo.

Proof: It follows the same lines as Nash's proof:

Given $S \varepsilon \Sigma_g$ and $M = g(S)$, determine the unique point M^* of S whose coordinates (x^*, y^*) maximize the product $(x-g_x(S))(y-g_y(S))$ for all (x,y) in S. An affine transformation of S placing M and M^* on Δ is then performed, yielding a new bargaining problem S' such that $g(S') = g(A(S)) = A(g(S)) = A(M^*)$ by P1. Because affine transformations preserve equalities of slopes, $I(S') = A(M^*)$. P2 is then invoked to yield a symmetric set S'' whose solution has to be $A(M^*)$ by Axioms 1 and 2. By Axiom 4, $A(M^*)$ is the solution of S', and Axiom 3 requires that if a solution exists, it should be M^*. The sufficiency is easily established. QED

In view of Proposition 1, it remains to check whether the different functions $g(\cdot)$ proposed in the first section satisfy P1 and P2. This is the object of the followings Lemmas. Making sure that P1 holds is in general a simple task. The verification of P2 requires a constructive argument. Starting from a set S satisfying the hypotheses of P2, we exhibit a set $S'(S)$ having the characteristics stated in the conclusion of P2.

Lemma 1: The reference function $g(S) = d \ \forall S$ satisfies P1 and P2.

Proof: P1 holds trivially. That P2 holds follows from the work of Nash.

Given a set S as in P2, Nash showed how to construct a symmetric rectangle $S'(S)$ having the desired properties; an alternative construction is now proposed that will be helpful in the next two lemmas.

We define $S^* = co\{S \cup T(S)\}$ and we show that S^* satisfies the conclusion of P2. S^* is clearly convex, compact and symmetric. Next we check that $I(S^*) = I(S)$; $M \varepsilon S \Rightarrow M$ is below $L(S)$. By symmetry, $M \varepsilon T(S) \Rightarrow M$ is below $L(S)$. By convexity, $M \varepsilon S^* \Rightarrow M$ is below $L(S)$. Since $S^* \supseteq S$, $I(S) \varepsilon S^*$. The last two conclusions together imply that $L(S) = L(S^*)$, which, because of the summetry of S^* yield $I(S^*) = I(S)$. The last property of S^* , $g(S^*) = g(S)$, is trivially satisfied.

<u>Lemma 2</u>: The reference function $g(S) = D(S)\ \forall S$ satisfies P1 and P2.

<u>Proof</u>: P1 is easily verified. Given S satisfying the hypotheses of P2, we show that the same set S^* as in Lemma 1 satisfies the conclusion of P2. All the properties have been established in Lemma 1 except $D(S^*) = D(S)$. Let $R(S)$ be the smallest rectangle with sides parallel to the axes containing S . x_1 and x_2 , with $x_1 \leqq x_2$ are the abcissa of the vertical sides; y_1 and y_2 with $y_1 \leqq y_2$ are the ordinates of the horizontal sides. Therefore, there exist M_1 , M_2 , N_1 , N_2 S with $x_{M_1} = x_1$, $x_{M_2} = x_2$, $y_{N_1} = y_1$, $y_{N_2} = y_2$ such that $M \varepsilon S \Rightarrow x_1 \leqq x_M \leqq x_2$ and $y_1 \leqq y_M \leqq y_2$. Let $M' \varepsilon S^*$, and let $z_{M'}$ be either $x_{M'}$ or $y_{M'}$. It is clear that $z_1 = \min\{x_1, y_1\} \leqq z_{M'} \leqq z_2 = \max\{x_2, y_2\}$. In addition, if $z_1 = x_1$ (say), then $x_{M_1} = y_{T(M_1)} = z_1$ so that for some point of S^* , each of the above inequalities on the left becomes an equality (similarly if $z_1 = y_1$ and for the inequalities on the right). Therefore the vertical sides of $R(S^*)$ have abcissa z_1 , z_2 and the horizontal sides have ordinates z_1 , z_2 . Then $D(S^*)$ has for coordinates $(z_1 + z_2)/2$, $(z_1 + z_2)/2$ and $D(S^*) = D(S)$. Q.E.D.

<u>Lemma 3</u>: The reference function $g(S) = m(S)\ \forall S$ satisfies P1 and P2.

<u>Proof</u>: P1 is easy to check. To establish that P2 holds, we take S as in the hypothesis of P2 and we show that $S^* = co\{S \cup T(S)\}$ satisfies the conclusion of P2. The first two conditions have already been established in Lemma 1. It remains to show that $m(S^*) = m(S)$. We denote $a(S) = (x_1, m)$ and $b(S) = (m, y_2)$. This implies that $m(S) = (m, m)$. Then

(1) $$M = (x, y) \varepsilon S \Rightarrow \begin{cases} x \leqq x_1 & \text{and } x < x_1 \text{ if } y > m \\ y \leqq y_2 & \text{and } y < y_2 \text{ if } x > m \end{cases}$$

(2) $$M' = (x', y) \varepsilon T(S) \Rightarrow \begin{cases} x' \leqq y_2 & \text{and } x' < y_2 \text{ if } y' > m' \\ y' \leqq x_1 & \text{and } y < x_1 \text{ if } x' > m \end{cases}$$

Let $z = \max\{x_1, y_2\}$, $A' = (z, m)$ $B' = (m, z)$. We show that $A', B' \varepsilon S^*$ (this follows from the construction), and that $a(S^*) = A'$, $b(S^*) = B'$, which can be written as

(3) $$M'' = (x'', y'') \varepsilon S^* \Rightarrow \begin{cases} x'' \leqq z & \text{and } x' < z \text{ if } y'' > m \\ y'' \leqq z & \text{and } y'' < z \text{ if } x'' > m \end{cases}$$

If $M'' \varepsilon S$, (3) follows from (1) after replacing x_1 and x_2 by z, which constitutes a weakening of the inequalities. If $M'' \varepsilon T(S)$, (2) is used to show that (3) holds. It remains to investigate the case $M'' = \lambda M + (1-\lambda)M'$ with $M \varepsilon S$ and $M' \varepsilon T(S)$. In that case, (1) and (2) yield

(4) $$x'' \leqq \lambda x_1 + (1-\lambda) y_2 \leqq z$$

$y'' = \lambda y + (1-\lambda)y' > m$ necessitates either $y > m$ or $y' > m$ or both. If $y > m$, $x < x'$ by (1) and by ( ) $x'' < z$. The other three cases are established in the same way. The second line of (3) would be proved in an identical way. It follows that $a(S^*) = A'$ and $b(S^*) = B'$. Then $m(S^*) = (m, m) = m(S)$. This complete the proof.

It should be noted that S^* is the smallest superset of S allowing to show that P2 holds in Lemmas 1, 2, 3.

Lemma 4: The reference function $g(S) = M(S) \forall S$ satisfies P1 and P2.

Proof: P1 is easy to check. Given S as in the hypothesis of P2, we construct $S'(S)$ satisfying the conclusions of P2 as follows (see Figure 1):

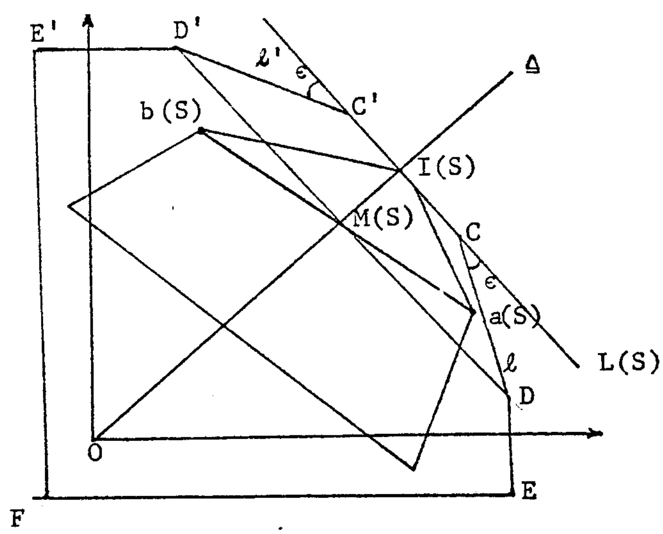

Figure 1

Given S, find C on $L(S)$ such that neither C nor $T(C)$ belongs to S. The existence of such a point C is guaranteed by the boundedness of S. Consider a line ℓ going through C and making an angle of $\varepsilon > 0$ with $L(S)$, and its symmetric $\ell' = T(\ell)$. For ε small enough, neither ℓ nor ℓ' will intersect S. ℓ and ℓ' meet a straight line parallel to $L(S)$ and going through $M(S)$ in D and D', of coordinates (a, b) and (b, a). The convex hull S' of $C, C', D, D', E: (a, -k), E': (-k, a)$ and $F: (-k, -k)$ for k large enough has all the properties required: indeed $S' \supset S$, $D = a(S')$, $D' = b(S')$, $M(S') = M(S)$, $T(S') = S'$ and $I(S) \varepsilon P(S')$.

Lemma 5: The reference function $g(S) = G(S)\ \forall S$ satisfies P1 and P2.

Proof: 1) we first prove that $G(S)$ satisfies P1. If S is a segment, $G(S)$ is the middle of the segment; given a positive affine transformation A, $A(S)$ is also a segment, whose middle is clearly its center of gravity as well as the image under A of $G(S)$. If S is a non-degenerate two dimensional area, $G(S)$ of coordinates (G_x, G_y) is defined by:

$$(6) \qquad \iint_S (x - G_x)\,dxdy = 0 \quad ; \quad \iint_S (y - G_y)\,dxdy = 0$$

Let A be a positive affine transformation: $A(x, y) = (a_1x + b_1, a_2y + b_2)$ with $a_1, a_2 > 0$. The center of gravity of $A(S)$, $G(A(S))$ of coordinates (G_x^1, G_y^1) is defined by:

$$(7) \qquad \iint_{A(S)} (x - G_x^1)\,dxdy = 0 \quad ; \quad \iint_{A(S)} (y - G_y^1)\,dxdy = 0$$

Performing the change of variables $\tilde{x} = a_1x + b_1$, $\tilde{y} = a_2y + b_2$ in (7), and using (6), yields:

$$G_x^1 = a_1G_x + b_1 \qquad G_y^1 = a_2G_y + b_2$$

Therefore $G(A(S)) = A(G(S))$ and P1 holds.

2) To establish that P2 holds, we first assert that given any compact convex set S, and two parallel lines L_1 and L_2 on either side of S and tangent to S, if d designates the distance between them, and if d_1 and d_2 are the distances of $G(S)$ to L_1 and L_2 respectively, then $d_1 \geqq d/3$ and $d_2 \geqq d/3$ (Figure 2).

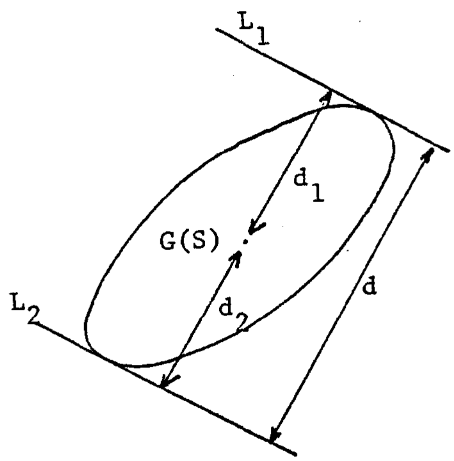

Figure 2

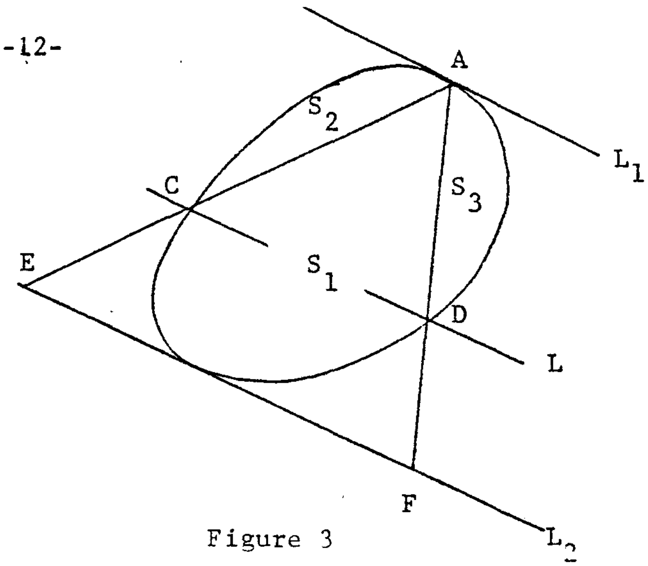

Figure 3

This results from the observation that the center of gravity of a triangle is the intersection of its medians, (point located at 2/3 of the way from each vertex to the middle of the opposite side), and the following construction: given S , L_1 and L_2 as above, let $A \in S \cap L_1$, and let L be a line parallel to L_1 and L_2 , at a distance $d/3$ of L_2 . If L intersects the boundary of S in only one point, S is a segment, and $G(S)$ is at equal distance of L_1 and L_2 . Otherwise L intersects the boundary of S in two (and only two) distinct points C and D . The lines AC and AD cut L_2 in E and F , and divide S into three subsets: S_1 , intersection of S with the triangle AEF , S_2 and S_3 being two remaining pieces between L_1 and L . The center of gravity of AEF is on L . It follows that $G(S_1)$ is between L_1 and L , since S_1 is obtained from AEF by cutting off pieces between L_2 and L . $G(S_2)$ and $G(S_3)$ are also between L_1 and L since S_2 and S_3 are convex sets having that property. $G(S)$ is a weighted average of $G(S_1)$, $G(S_2)$ and $G(S_3)$ and is also between L_1 and L . Notice that $G(S)$ will belong to L is and only if S is a triangle having a side in L_2 (Figure 3).

Next, we consider a set S as in the hypothesis of P2: $T(G(S)) = G(S)$ and $I(S) \varepsilon S$. Let L_1 and L_2 be as in the preceding paragraph with $L_1 = L(S)$. Call $J = L_2 \cap \Delta$. According to what we just saw, if $G(S)$ is at a distance $d/3$ of either L_1 or L_2, S is a triangle with a side in L_2 or in L_1 respectively. Because $G(S)$ belongs to Δ and $I(S)$ belongs to S, S itself is symmetric and the conclusion of P_2 trivially follows with $S'(S) = S$.

Otherwise, $G(S)$ is at a distance at least equal to $d/3 + \varepsilon$ of both L_1 and L_2 for some positive ε. Since S is bounded, there is a positive number c such that the extremities (M_1,M_2) and (N_1,N_2) of two symmetric segments of middle $I(S)$ and J and length $2c$ do not belong to S. Because of the convexity of S, there is $\alpha_0 > 0$ such that any line ℓ_1 through M_1 making with L_1 an angle $\alpha \leqq \alpha_0$, and its symmetric ℓ_2 do not intersect S. Similarly, there is $\beta_0 > 0$ such that any line ℓ' through N_1 making with L_2 an angle $\beta \leqq \beta_0$, and its symmetric ℓ'_2 do not intersect S. ℓ_1 and ℓ' intersect in P_1; ℓ_2 and ℓ'_2 intersect in P_2. Let G_1 be the center of gravity of $S_1 = co\{M_1,M_2,N_1,N_2\}$ and m_1 be the moment of S_1 with respect to a line parallel to L_1 through $G(S)$. Let G_2 and m_2 be defined in a similary way for $S_2 = co\{M_1,N_1, P_1\} \cup co\{M_2,N_2,P_2\}$. For every (α, β), G_1 and G_2 belong to Δ. Given any d^* such that $d/3 \leqq d^* \leqq 2d/3$, α and β can be chosen so that G_2 will be at a distance d^* of L_1. If $G_1 = G(S)$, G_2 is chosen so that $G_2 = G(S)$. Otherwise, one has to pick α^* and β^* such that G_1 and G_2 be on either side of $G(S)$ and $m_1 + m_2 = 0$. (This might require the choice of small α and β if m_1 is large). This completes the construction of the set $S' = S_1 \cup S_2$ satisfying the conclusion of P_2 (see figure 4).

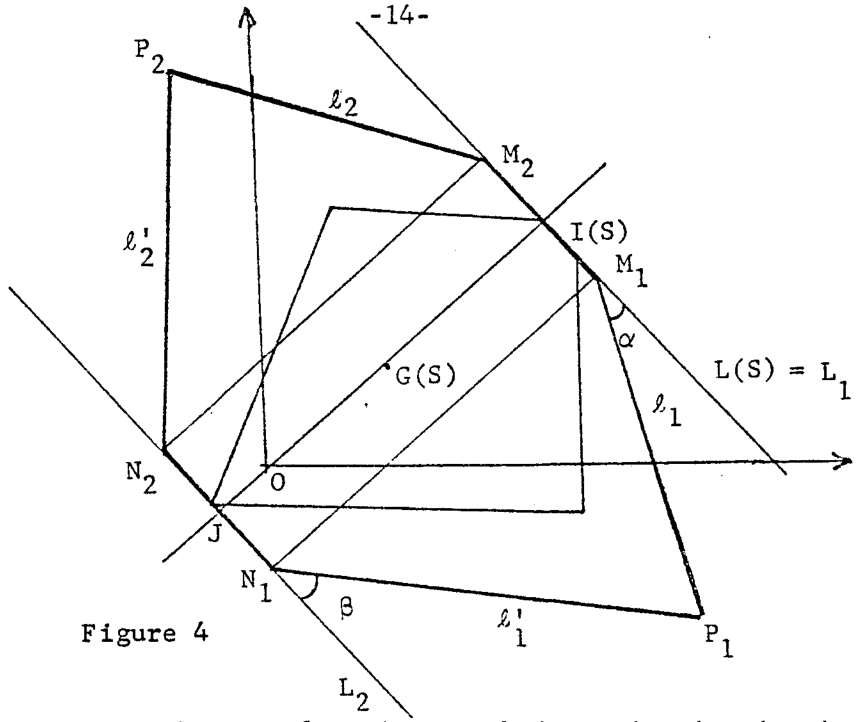

Figure 4

Remark: Another reference function worth investigating is given by $B(S)$, the center of gravity of the boundary of S. An objection to a solution often takes the form of a comparison to other feasible alternatives. Since comparisons to alternatives on the boundary of S are more convincing, $B(S)$ arises as a natural point of reference.

Unfortunately, $B(S)$ does not satisfy P1. The following counterexample is a proof of that fact. Let $A = (1,0)$ and $B_n = (0,1/n)$ and define $S_n = \text{co}\{0,A,B_n\}$. S_n is obtained from S, by the affine transformation $A_n(x , y) = (x , y/n)$. In order for $B(S)$ to satisfy P1, it should be the case that $B(S_n) = A_n(B(S_1))$. Since A_n leaves abcissa invariant, $B(S_n)$ and $B(S_1)$ should have the same abcissa. However, it is clear that $B_x(S_1)$ is not equal to $\frac{1}{2}$ and that $B_x(S_n) \to \frac{1}{2}$ as $n \to \infty$. Q.E.D.

Section III Multivalued Solutions

As proved in [6], the Roth solution can actually be extended to the case of games for which $m(S)$ belongs to the boundary of S, since it is then the unique Pareto-optimal of S. This is not true of the other examples of function g examined in Section II for which Σ_g is nevertheless a large class of games. Unfortunately, even though $i(\cdot)$ satisfies P1 and

P2 (it is immediate to verify P1; P2 is shown to hold by using $S^* = co\{S \cup T(S)\}$ as in lemmas 1, 2 and 3), Σ_i is empty, and it is not possible to do away with the requirement that $S \varepsilon \Sigma_g$ for the proposition to hold. As mentioned in the introduction, Roth provided an example of a set $S \notin \Sigma_i$ establishing the inconsistency of Axiom 1-4 when $g(\cdot) = i(\cdot)$. As will become clear in the proof of proposition 2, what causes problems is the possibility that there exist several optimal points M of S where S has a line of support with a slope equal to the negative of the slope of the line connecting M to i(S). The Roth example is based on precisely this kind of non-uniqueness. The natural way to extend the result of Part I is therefore to allow for multi-valued solutions. Such a solution narrows down the set of optimal points to a few among which the final outcome is eventually chosen. Bargaining or arbitration procedures often involve several successive steps, and what follows can be seen as an attempt to formalize the "first step".

Given a reference function G, a multivalued g-solution is a correspondence associating to every S a non-empty subset of S.

$$S \varepsilon \Sigma \xrightarrow{f_g} f_g(S) \subset S .$$

Since Axioms 1-4 were previously written for functions, they are now restated for correspondences, the main difference being that the transformations T and A should be understood to operate on the <u>set</u> $f_g(S)$.

<u>Axiom 1'</u> - Pareto-optimality: $x \varepsilon f_g(S) \Rightarrow x \varepsilon P(S)$

<u>Axiom 2'</u> - Symmetry: $[T(S) = S \;,\; T(g(S)) = g(S)] \Rightarrow T(f_g(S)) = f_g(S)$ and $\Delta \cap S \neq \emptyset$

<u>Axiom 3'</u> - Scale invariance: $f_g(A(S)) = A(f_g(S))$ for every positive affine transformation A.

<u>Axiom 4'</u> - Independence of alternatives other than g(S).

$$[S \subset S' \;,\; g(S) = g(S') \;,\; x \varepsilon f_g(S') \;,\; x \varepsilon S] \Rightarrow x \varepsilon f_g(S) .$$

Proposition 2: There exists a multi-valued solution satisfying Axioms 1'-4'. It associates to every S in Σ the set $E(S)$ of Pareto-optimal points M of S where S has a line of support of slope different from 0 and $-\infty$ and equal to the negative of the slope of the line connecting $i(S)$ and M, if $i(S) \notin S$, or the unique point $i(S)$, if $i(S) \in S$.

Proof: It is done in several steps.

Step 1: $\forall S \in \Sigma \setminus \Sigma_i$, $E(S) \neq \emptyset$.

Given $\alpha \in [0,\pi/2]$, let L_α be a line through $i(S)$ with a slope equal to $tg\alpha$. If $\alpha \in]0,\pi/2[$, by convexity of S, $L_\alpha \cap S$ is a singleton, denoted $M(\alpha)$. Define $M(0) = b(S)$ and $M(\pi/2) = a(S)$. Let T_α be the set of lines of support of S at $M(\alpha)$ and let $\beta(\alpha) = \{\beta \in R \mid \exists T \in T_\alpha$ with slope equal to $-tg\beta\}$. It is clear that $\beta(\cdot)$ is a non-empty correspondence. By compactness of S,

(1) $\forall \alpha \in [0,\pi/2]$, $\beta(\alpha)$ is a non-empty closed subinterval of $[0,\pi/2]$, and by convexity of S,

(2) $\forall \alpha, \alpha_2 \in [0,\pi/2]$, $\alpha_1 \leqq \alpha_2$, $\beta_1 \in \beta(\alpha_1)$, $\beta_2 \in \beta(\alpha_2) \Rightarrow \beta_1 \leqq \beta_2$

Finally, let M_0, M_1 in R^2 be such that

(3) $x_{M_1} < x_{M_0}$, $y_{M_1} < y_{M_0}$

To every set S in $\Sigma \Sigma_i$ can be associated a unique list $(M_0, M_1, \beta(\cdot))$ satisfying (1), (2), (3) and such that:

(4) $i(S) = M_0$, $M_1 \in P(S)$ and $\beta(\cdot)$ be the correspondence defined after the statement of step 1.

Conversely, given a list $(M_0, M_1, \beta(\cdot))$ satisfying (1), (2), (3) does there always exist S in $\Sigma \setminus \Sigma_i$ such that (4) holds? First, notice that if $(M_0, M_1, \beta(\cdot))$ defines a set

S in $\Sigma \backslash \Sigma_i$, and M_1' satisfies (3), $(M_0 , M_1' , \beta(\cdot))$ also defines a set S' in $\Sigma \backslash \Sigma_i$ (S and S' will in fact be related by a homethetic transformation of center M_0 and ratio $||M_0M_1||/||M_0M_1'||$.)

Next, we argue that if $(M_o , M_1 , \beta(\cdot))$ defines a set S in $\Sigma \backslash \Sigma_i$, the correspondence β has a fixed point different from 0 and $\pi/2$. Suppose not. Then either (i): $\forall\alpha\varepsilon]0,\pi/2[$, $\forall\beta\varepsilon\beta(\alpha)$, $\beta < \alpha$ or (ii): $\forall\alpha\varepsilon]0,\pi/2[$, $\forall\beta\varepsilon\beta(\alpha)$, $\beta > \alpha$. Assume (i) holds and define $\beta^*(\alpha) = \alpha\ \forall\alpha\varepsilon[0,\pi/2]$. Let α_1 be the unique (by (3)) angle such that L_{α_1} goes through M_1 . (3) also indicates that $\alpha_1\varepsilon]0,\pi/2[$. Let α vary from α_1 to $\pi/2$, and let $M(\alpha)$ (resp $M^*(\alpha)$) be the locus of M , (resp M^*) given by $(M_0,M_1,\beta(\cdot)))$ (resp$(M_0,M_1,\beta^*(\cdot)))$. By (i) , it is clear that $||M_0M(\alpha)|| \geqq ||M_0M^*(\alpha)||\ \forall\alpha\varepsilon[\alpha_0\ \pi/2[$. However the locus of $M^*(\alpha)$ is a section of a hyperbola with asymptotes going through M_0 and parallel to the axes. Since $M^*(\alpha)$ gets infinitely far from M_0 as α tends to $\pi/2$, so will $M(\alpha)$, which violates the compactness of S . If (ii) held, we would consider α in $]0,\alpha]$ and go through the same reasoning. Therefore $\beta(\cdot)$ has a fixed point different from 0 and $\pi/2$, which is equivalent to saying that $E(S) \neq \emptyset$. This proves step 1.

Step 2: If f_i exist, $\forall S\varepsilon\Sigma\backslash\Sigma_i$, $\forall M\varepsilon E(S)$, $M\varepsilon f_i(S)$.

Let $S\varepsilon\Sigma\backslash\Sigma_i$, and let $M\varepsilon E(S)$. Such an M exists by Step 1. That $M\varepsilon f_i(S)$ follows from the fact that i(S) satisfies Properties P1 and P2 of Section II. Consider an affine transformation placing M and i(S) on Δ . By P1, $A(i(S)) = i(A(S))$. Since equalities of slopes are preserved by affine transformations, A(S) has a line of support at A(M) with a slope equal to the negative of the slope of the line connecting $A(M)$ to $A(I(S))$, namely -1 . Consider $S^{*} = co\{A(S) \cup T(A(S))\}$. S^{*} is symmetric;by Axioms 1' and 2' $\Rightarrow A(M) \varepsilon f_i(S^{*})$. In addition, $A(S) \subset S^{*}$, $i(A(S)) = i(S^{*})$. Therefore, by Axiom 4', $A(M) \varepsilon f_i(A(S))$. Invoking Axiom 3' again indicates that $M \varepsilon f_i(S)$.

<u>Step 3</u>: The correspondence f_i^{*} defined by

$$\forall S \varepsilon \Sigma \qquad f_i^{*}(S) = E(M) \quad \text{if} \quad i(S) \notin S$$
$$f_i^{*}(S) = i(S) \quad \text{if} \quad i(S) \in S$$

is a solution satisfying Axioms 1'-4' with $g = i$. When $S \varepsilon \Sigma \setminus \Sigma_i$, We have shown in step 1 that $f_i(S) \supseteq E(M) \neq \emptyset$. When $S \varepsilon \Sigma_i$, $P(S) = \{i(S)\}$ and it has to be the case that $f_i(S) = i(S)$.

The sufficiency is clear. This proves the Proposition.

In the final paragraph, we construct all multivalued solutions. First, we claim that if f' is a solution, $\forall S \varepsilon \Sigma$, $f^{*}(S) \subseteq f'(S) \subseteq P(S)$, where $P(\cdot)$ is the Pareto-correspondence. The first inclusion follows from the necessary conditions of Proposition 2, the second from Axiom 1'. Next, let h be an arbitrary subcorrespondence of $\tilde{P}$, where $\tilde{P}(S) = P(S) \cup \emptyset$, $\forall S \varepsilon \Sigma$. We now construct the smallest solution containning h . If $h(S) = \emptyset \; \forall S$, we clearly obtain f^{*} . If $\exists S_0 \varepsilon \Sigma$ such that $h(S_0) \neq \emptyset$, let $M_0 \varepsilon h(S_0)$ and let $\Sigma(M_0 , S_0) \subseteq \Sigma$ be the class of bargaining problems that can be obtained from S_0 by a positive affine transformation, or are (maybe symmetric, not necessarily proper) subsets of S , with M_0 on their boundary. Since $M_0 \varepsilon f'(S_0)$, if $S' \varepsilon \Sigma(M_0 , S_0)$, either the affine transformed of M_0 , or its symmetric or M_0 itself belongs to $f'(S')$, by Axiom 2', 3' or 4'. Carrying out this procedure recursively from any pair (M',S') where M' has been shown to belong to $f'(S')$, and from any initial pair (M_0,S_0) yields the smallest correspondence f_h such that $h(S) \subseteq (f^* \cup f_h)(S)\ \forall S \in \Sigma$. It is clear that $f^* \cup f_h$ is a solution. Conversely, let f' be a solution; f' can be written as $f^* \cup \tilde{f}$ for some $\tilde{f}$, subcorrespondence of $\tilde{P}$, and the above procedure yields f' as the solution associated with $h = \tilde{f}$. This construction shows that there is a one to one correspondence between solutions and subcorrespondences of $\tilde{P}$.

1/ The term "bargaining problem" is not used here with exactly the same meaning it has in Nash's paper, mainly because no threat point has as yet been chosen. The term "arbitration problem" might be more appropriate. However, since the situation studied by Nash is covered by the present paper, an altogether new term should be used. Not having found any, I have kept the term "bargaining problem". I am grateful to R. Rosenthal for bringing to my attention this terminological confusion.

<u>Appendix</u>: Continuity Properties of the Solutions

Using the Hausdorff topology, we will consider sequences of games $\{S_n, n = 1, \ldots\}$ and assume that $S_n \to S$ as $n \to \infty$. What can be said of $f_g(S_n)$ as $n \to \infty$? The convergence of $f_g(S_n)$ to $f_g(S)$ depends on each function $g(\cdot)$.

$a(S)$ and $b(S)$ are not continuous functions of S. It follows that neither $m(S)$ nor $M(S)$ are continuous functions and the solutions f_m and f_M are not continuous either. The following examples illustrate this negative result.

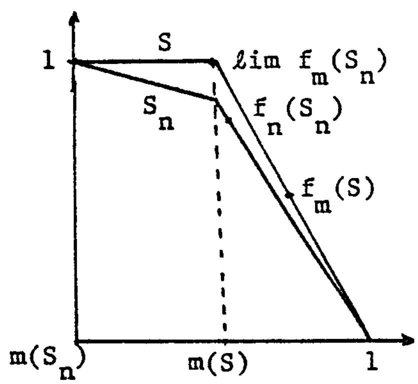

Figure 5

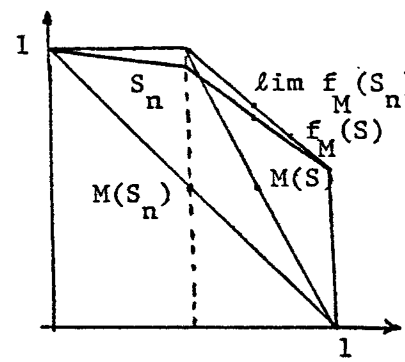

Figure 6

S_n is the convex hull of $(0, 0), (1, 0), (0, 1)$ and $(1/2, 1 - \varepsilon_n)$ with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. $m(S_n) = (0, 0) \forall n$ and $f_m(S_n) \to (1/2, 1)$. However, defining $S = \ell im\ S_n$, we have $m(S) = (1/2, 0)$ and $f_m(S) = (3/4, 1/2)$ (Figure 5).

S_n is the convex hull of $(0, 0), (1, 0), (0, 1), (1/2, 1 - \varepsilon_n)$, $(1, 1/2)$. $M(S_n) = (1/2, 1/2)$ and $f_M(S_n) \to (3/4, 3/4)$. However, defining S as the limit of S_n, $M(S) = (3/4, 1/2)$ and $f_M(S) = (7/8, 5/8)$ (Figure 6).

On the other hand $G(S)$ is a continuous function of S, and so is $f_G(S)$.

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