MONOTONICITY AND INDEPENDENCE AXIOMS

by

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ABSTRACT

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Let Σ be a family of "choice problems," subsets of R^n representing the payoff vectors (measured in some von Neumann-Morgernstein scales) attainable by a group of n players; a "solution" f on Σ associates to every s in Σ an element $f(s)$ of s .

Solutions satisfying certain axioms are considered; monotonicity axioms specify how $f(s)$ should change when s is subjected to certain geometric transformations while independence axioms require, in similar circumstances, the invariance of $f(s)$. The logical implications between these axioms are established. "Strong monotonicity" is shown to be the strongest axiom and strongly monotonic solutions are characterized.

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Monotonicity and Independence Axioms

A choice problem S is a non-empty subset of R^n representing the payoff vectors attainable by a group of n individuals through some joint actions. Payoffs are measured in some von Neumann-Morgernstern utility scales. A class Σ of such choice problems will be considered. A solution f defined on Σ associates to every set S in Σ a unique element f(S) of S. We are concerned here with solutions satisfying certain monotonicity and independence axioms; assume for instance that T is a superset of S. A requirement of strong monotonicity would be that all agents benefit from the expanded opportunities: f(T) dominates f(S). Two sets S and T can be related in other simple geometric ways. If T is obtained from S by the addition (or the elimination) of points that are all located in a region favorable to a particular agent, one might want the solution to move in his favor (or against him). Such requirements may be seen from a normative viewpoint, as fairness conditions or from a descriptive viewpoint, since no player could reasonably accept a solution that would fail to satisfy them.

In this paper, we provide a characterization of strongly monotonic solutions. This characterization is done through the introduction of a number of monotonicity axioms of increasing weakness, spelled out in Section I. In Section II, the logical implications between these axioms, and various independence axioms, are established. Those independence axioms are related to Nash's Independence of Irrelevant Alternatives [5]. Along with a sequence of axioms specifying movement in the "intuitively correct" direction, we are led to introducing a parallel sequence of "perverse" axioms requiring changes in a counterintuitive direction. We show that all of these axioms are logical consequences of strong monotonicity, thereby revealing the strength of this condition and providing an easy characterization of choice functions satisfying it.

Our work should be seen in the light of the contributions by Kalai and Smorodinsky [2], Kalai [1] and Myerson [4], who have attacked similar problems. Their work is reviewed in Roth [6] and Schmitz [7].

I. The Axioms

In order to specify the class of admissible choice sets Σ, we need to introduce a few definitions.

The following conventions are used for vector inequalities:

$$x \leqq y \iff x_i \leqq y_i \ \forall i \qquad x < y \iff x_i < y_i, \ \forall i.$$

Given an arbitrary subset of R^n, U, the convex hull of U, $V(U)$, is defined by

$$V(U): \{ y : \exists x_1, x_2 \in U, \lambda \in [0,1] \ / \ y = \lambda x_1 + (1-\lambda)x_2 \}$$

The comprehensive convex hull of U, denoted $H(U)$ is in turn defined by

$$H(U) = \{ z : \exists \ y \in V(U) \ / \ z \leqq y \}$$

We will restrict ourselves to the class Σ of all choice problems that are the comprehensive convex hull of non-empty compact subsets of R^n.

Assuming convexity amounts to allowing randomization between available payoff vectors. Comprehensiveness follows from an assumption of free disposal of utility.

We will sometimes consider the subset Σ^* of Σ of choice sets containing at least one strictly positive payoff vector.

The boundary ∂S of S is defined by

$$x \in \partial S \iff \{ x \ / \ x \in S \ ; \ y > x \Rightarrow y \notin S \}$$

In addition to monotonicity and independence axioms, we will always demand of a solution that it picks weakly Pareto-optimal outcomes (WPO).

Weak Pareto-optimality

WPO: $\forall S$, if $x = f(S)$, then $x \in \partial S$.

Strong monotonicity

SM: $\forall S,T$, if $S \supseteq T$, then $f(S) = f(T)$ or $f(S) > f(T)$.

A slightly weaker version of this axiom is used by Kalai in [1]. Our formulation implies that if S contains an outcome that strictly dominates f(T), its solution should also have this property: all agents should benefit from the new opportunities.

Weak monotonicity

WM: $\forall S, T \in \Sigma$, if $S \supseteq T$ and $\sup_{x \in S^1} x_i = \sup_{y \in T^1} y_i$ then

$$\text{either (a): } f(S) = f(T) \text{ or (b): } [f_j(S) > f_j(T),\ \forall j \neq i.]$$

This is a generalization to the n-player case of the monotonicity condition used by Kalai and Smorodinsky [2] in their characterization in Σ^* of the 2-player Raiffa solution [3], (with the inessential difference that in [2] the sups of x_i and y_i are taken for x and y in $S^+ = S \cap R^{+n}$ and $T^+ = T \cap R^{+n}$ respectively). This solution is given by the intersection of the Pareto frontier of S with the line connecting the origin with the point of coordinates $x_i^* = \max_{x \in S} x_i$, $i = 1,2$. The straightforward extension of this solution concept to the n-person case could be similarly characterized with the use of WM as stated above in conjunction with the other axioms of [1], namely weak Pareto-optimality, Invariance with respect to affine transformations, and Symmetry, appropriately reformulated.

Independence of irrelevant alternatives

IIA: $\forall S,T$, if $S \supseteq T$ and $f(S) \in T$, then $f(S) = f(T)$

This axiom was introduced by Nash in [5].

Weak independence of irrelevant alternatives

WIIA: $\forall S$, $f(\{y \,/\, y \leqq f(S)\}) = f(S)$

Twisting

Tw: $\forall S, T$, $\forall i$

if (i) $f(S) \in \partial T$

(ii) $x \in T \backslash S \Rightarrow x_j \leqq f_j(S)$, $\forall j \neq i$ and

(iii) $x \in S$ and $x_j \leqq f_j(S)$, $\forall j \neq i \Rightarrow x \in T$

then either (a): $f_i(T) > f_i(S)$ or (b): $[f_i(T) = f_i(S)$ and $f_j(T) \leqq f_j(S)$ $\forall j \neq i]$.

Notation: Given two sets S and T, and an agent i, we will say that $P_i(S,T)$ iff

$$\{x \,|\, x_i \leqq f_i(s)\} \cap s = \{x \,|\, x_i \leqq f_i(s)\} \cap T .$$

Cutting

C: $\forall S, T$, if $P_i(S,T)$ and $S \supseteq T$, then

either (a): $f_j(T) > f_j(S)$ $\forall j \neq i$ or (b): $[f_j(T) = f_j(S)$ $\forall j \neq i$ and $f_i(T) \leqq f_i(S)]$.

Adding

A: $\forall S, T$ if $P_i(S,T)$, $T \supseteq S$ and $f(s) \in \partial T$, then

either (a): $f_i(T) > f_i(S)$ or (b): $[f_i(T) = f_i(S)$ and $f_j(T) \leqq f_j(S)$ $\forall j \neq i]$.

Domination

D: $\forall S, T$; either (a): $f(S) = f(T)$ or (b): $f(S) > f(T)$ or (c): $f(S) < f(T)$.

As explained in the introduction, in addition to these axioms prescribing sensible changes, we also introduce a parallel set of axioms, four of which demanding changes in counterintuitive directions, and qualified of "perverse."

Perverse weak monotonicity

WM^*: $\forall S,T \in \Sigma$; if $S \supseteq T$ and $\sup_{x \in S} x_i = \sup_{y \in S} y_i$, then

$$\text{either (a): } f(S) = f(T) \quad \text{or} \quad \text{(b): } f_i(S) > f_i(T)$$

Independence of undominating alternatives

IUA: $\forall S,T$; if $S \supseteq T$ and $f(T) \in \partial S$, then $f(S) = f(T)$.

Weak independence of undominating alternatives

WIUA: $\forall S$, if $f(\{y \,/\, y \leqq x\}) = x$ and $x \in \partial S$, then $x = f(S)$.

Perverse Twisting

Tw^*: $\forall S,T$, with $S \cup T \in \Sigma$, $\forall i$

if (i) $f(S) \in \partial T$

(ii) $x \in S \setminus T \Rightarrow x_j \leqq f_j(S)$ $\forall j \neq i$ and

(iii) $x \in T$ and $x_j \leqq f_j(S)$, $\forall j \neq i \Rightarrow x \in S$, then

either (a): $f_i(T) > f_i(S)$ or (b): $[f_i(T) = f_i(S)$ and $f_j(T) \leqq f_j(S)$ $\forall j \neq i]$.

Perverse adding

A^*: $\forall S,T$; if $P_i(S,T)$, $T \supsetneq S$ and $f(s) \in \partial T$, then

either (a): $f_j(T) > f_j(S)$, $\forall j \neq i$ or (b): $[f_j(T) = f_j(S)$, $\forall j \neq i$ and $f_i(T) \leqq f_i(S)]$.

Perverse cutting

C^*: $\forall S,T$; if $P_i(S,T)$ and $S \supseteq T$, then

either (a): $f_i(T) > f_i(S)$ or (b): $[f_i(T) = f_i(S)$ and $f_j(T) \leqq f_j(S)$ $\forall j \neq i]$.

<u>Illustration in the 2 Person Case</u>

Notation: $\alpha_1 \cdots \alpha_k$ denotes the unique choice set whose boundary goes through $\alpha_1, \ldots, \alpha_k$.

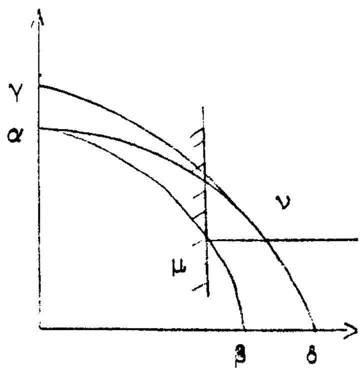


Figure 1

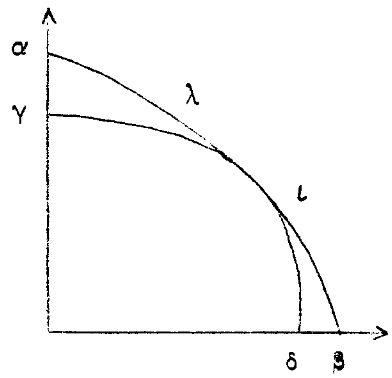


Figure 2

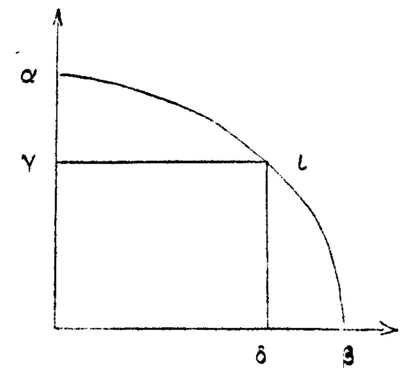


Figure 3

<u>Figure 1</u> SM says that if $\mu = f(\alpha\beta)$, and if opportunities expand to $\gamma\delta$, the new solution ν should dominate the old one. WM states that an expansion from $\alpha\beta$ to $\alpha\delta$, directed in favor of one player should lead to an improvement of his position.

<u>Figure 2</u> IIA requires that the elimination of alternatives from the set $\alpha\beta$, whose solution ι remains an element of the smaller set $\gamma\delta$, should not affect the final outcome. If $\iota = f(\gamma\beta)$, twisting $\gamma\beta$ to $\alpha\delta$ in favor of Player 2 should lead to an improvement of his position (towards λ), according to Tw. Axiom A requires that if $\iota = f(\gamma\delta)$ adding alternatives favoring Player 2, by going from $\gamma\delta$ to $\alpha\delta$, should again make Player 2 better off, and Axiom C demands that eliminating alternatives favorable to Player 2, as a change from $\alpha\beta$ (with $\iota = f(\alpha\beta)$) to $\gamma\beta$ would achieve, should result in a deterioration of his position. For the same transformations, Tw^*, C^*, A^* require a change in the opposite and counterintuitive direction. IUA says that an expansion from $\gamma\delta$ (with $\iota = f(\gamma\delta)$) to $\alpha\beta$, obtained by the addition of alternatives none of which dominates ι, should not result in any change in the solution.

Figure 3 A move from $\alpha\beta$ (with $\iota = f(\alpha\beta)$) to $\gamma\delta$ should not affect the solution according to WIIA. This is considerably weaker than IIA since ι is the unique strongly Pareto-optimal point of $\gamma\delta$. Conversely, if $\iota = f(\gamma\delta)$, WIUA states that ι should also be the solution of $\alpha\beta$.

II. The Logical Implications

In what follows, WPO is always assumed.

SM => IIA

Let S and T satisfy: $S \supseteq T$, $F(S) \in T$. SM => (a) $f(S) = f(T)$ or (b) $f(S) > f(T)$. If (b) held, $f(T)$ would be strictly dominated by an element of T, in violation of WPO. Therefore (a) holds. Q.E.D.

SM => IUA

Let S and T satisfy: $S \supseteq T$, $f(T) \in \partial S$. SM => (a) $f(S) = f(T)$ or (b) $f(S) > f(T)$. If (b) held, $f(T)$ would be strictly dominated by a point of S, in violation of "$f(T) \in \partial S$" and WPO. Therefore (a) holds. Q.E.D.

SM => WM
SM => WM*
IIA => WIIA
IUA => WIUA
D => SM

WM is obtained from SM by a strengthening of the hypothesis and a weakening of the conclusion, whence the first implication. The other four are derived in a similar fashion.

IIA => Tw

Suppose to the contrary, that IIA holds but not Tw. $\exists$ S and T as in the hypothesis of Tw such that "$f_i(T) \leqq f_i(S)$ and [either $f_i(T) \neq f_i(S)$ or $\exists\ j \neq i \ /\ f_j(T) > f_j(S)$]." This last condition can be written as "Either (a_1): $f_i(T) < f_i(S)$ or (a_2): [$f_i(T) \leqq f_i(S)$ and $\exists\ j \neq i \ /\ f_j(T) > f_j(S)$]." Call $R = S \cap T$. If (a_1) holds, then $f(T) \in S$. Otherwise, $f(T) \in T \backslash S$, and

$f_j(T) \leqq f_j(S)\ \forall j \neq i$. Together with (a_1), this yields $f(T) \leqq f(S)$ and $f(T) \in S$ by the comprehensiveness of S, which contradicts $f(T) \in T \backslash S$. If (a_2) holds, $f(T) \in S$ also, since $f_j(T) \leqq f_j(S)\ \forall j \neq i$, implied by $f(T) \in T \backslash S$, is in contradiction with "$\exists\, j \neq i \,/\, f_j(T) > f_j(S)$." $S \supseteq R$. $f(S) \in R$ by construction. Therefore, IIA implies $f(S) = f(R)$. Also $T \supseteq R$, $f(T) \in R$ as we just argued. Again by IIA, $f(T) = f(R)$. However, by hypothesis, $f(S)$ is different from $f(T)$. This contradiction proves the result.

Tw => C

Let S and T be such that $P_i(S,T)$ and $S \supseteq T$. We show that the hypotheses of Tw hold for all $j \neq i$. First, $f(S) \in T$ by $P_i(S,T)$ and WPO. Second, T/S is empty, so that (ii) holds vacuously. Finally, given $j \neq i$, "$x \in S$ and $x_\ell \leqq f_\ell(S)$, $\forall \ell \neq j$" => "$x \in S$ and $x_i \leqq f_i(S)$" => $x \in T$ by $P_i(S,T)$. Applying Tw n-1 times we get "Either $f_j(T) > f_j(S)$ or $[f_j(T) = f_j(S)$ and $f_\ell(T) \leqq f_\ell(S)$, $\forall \ell \neq j$ $\forall j \neq i.]$". The only way these n-1 statements can be simultaneously true is by having "Either $f_j(T) > f_j(S)$, $\forall j \neq i$, or $[f_j(T) = f_j(S)\ \forall j \neq i$ and $f_i(T) \leqq f_i(S)]$," which is precisely C. Q.E.D.

C => A

Suppose, to the contrary, that C holds but not A: $\exists$ S,T,i satisfying $P_i(S,T)$ and $T \supseteq S$ with "$f_i(T) \leqq f_i(S)$ and [either $f_i(T) \neq f_i(S)$ or $\exists\, j \neq i \,/\, f_j(T) > f_j(S)$]." This can be restated as "Either (a_1): $f_i(T) < f_i(S)$ or (b_1): $[f_i(T) \leqq f_i(S)$ and $\exists\, j \neq i \,/\, f_j(T) > f_j(S)]$." Notice that $f_i(T) \leqq f_i(S)$ and $P_i(S,T)$ => $P_i(T,S)$. Therefore, by C, "Either (a_2): $f_j(S) > f_j(T)\ \forall j \neq i$, or (b_2): $[f_j(S) = f_j(T)$, $\forall j \neq i$ and $f_i(S) \leqq f_i(T).]$." Four cases should then be examined: 1/ $(a_1)+(a_2)$ => $f(S) > f(T)$, which is incompatible with WPO, since $f(S) \in T$ by $P_i(S,T)$. 2/ $(a_1)+(b_2)$ => $f_i(T) \geqq f_i(S) > f_i(T)$, a contradiction. 3/ $(b_1)+(a_2)$ => $f_{j_o}(T) > f_{j}(S) > f_{j_o}(T)$ for $j_o \neq i$, a contradiction. Finally, $(b_1)+(b_2)$ => $f_{j_o}(T) > f_{j_o}(S) =$

$f_{jo}(T)$, also a contradiction. This exhausts all the possibilities. It follows that C cannot hold without A holding.

IUA => Tw^*

Suppose to the contrary that IUA holds but not Tw^*. $\exists$ S,T,i, for which the hypotheses of Tw^* hold but not the conclusion: "$f_i(T) \leqq f_i(S)$ and $[f_i(T) \neq f_i(S)$ or $\exists\, j \neq i \,/\, f_j(T) > f_j(S)]$." This can be restated as "Either (a_1): $f_i(T) < f_i(S)$ or (b_1): $[f_i(T) \leqq f_i(S)$ and $\exists\, j \neq i \,/\, f_j(T) > f_j(S)]$." It follows that f(T) is not strictly dominated by any point of S. Since f(T) is WPO for T, if such a point x existed, it would have to belong to S\T. By construction, we would then have "$x_j \leqq f_j(S)$, $\forall j \neq i$." In addition, $f_\ell(T) < x_\ell$, $\forall \ell$. If (a_1) held, we would obtain $f(T) < f(S)$, contradicting the weak Pareto-optimality of f(T), since $f(S) \in T$ by construction. If (b_1) held, for some $j_o \neq i$, we would have $f_{j_o}(T) > f_{j_o}(S) \geqq x_{j_o} > f_{j_o}(T)$, a contradiction. It follows that $f(T) \in \partial S$. Call $R = S \cup T$. $R \supset S$, $f(S) \in \partial R$. By IUA, $f(S) = f(R)$. Also, $R \supset T$. $f(T) \in \partial R$. By IUA, $f(T) = f(R)$. Since $f(S) \neq f(R)$ by hypothesis, a contradiction follows establishing the desired result. Q.E.D.

Tw^* => A^*

Let S and T be such that $P_i(S,T)$ and $T \supseteq S$, and $f(s) \in \partial T$. We show that the hypothesis of Tw^* holds for all $j \neq i$. First, $f(S) \in \partial T$ by assumption. Second, S/T is empty so that (ii) holds vacuously. Finally, given $j \neq i$, "$x \in T$ and $x_\ell \leqq f_\ell(S)$, $\forall \ell \neq j$" => "$x \in T$ and $x_i \leqq f_i(S)$" => $x \in S$ by $P_i(S,T)$. It follows that "$[f_j(T) > f_j(S)$ or $[f_j(T) = f_j(S)$ and $f_\ell(T) \leqq f_\ell(S)$ $\forall \ell \neq j]]$ $\forall j \neq i$". This implies "$f_j(T) > f_j(S)$ $\forall j \neq i$ or $[f_j(T) = f_j(S)$ $\forall j \neq i$ and $f_i(T) \leqq f_i(S)]$."

$A^* \Rightarrow C^*$

Suppose, to the contrary, that A^* holds but not C^*: $\exists$ S,T,i with $P_i(S,T)$, $S \supseteq T$ but "$f_i(T) \leqq f_i(S)$ and [either $f_i(T) \neq f_i(S)$ or $\exists$ j≠i / $f_j(T) > f_j(S)$]." This can be restated as "Either (a_1): $f_i(T) < f_i(S)$ or (b_1): [$f_i(T) \leqq f_i(S)$ and $\exists$ j≠i / $f_j(T) > f_j(S)$]." Notice that $f_i(T) \leqq f_i(S)$ and $P_i(S,T) \Rightarrow P_i(T,S)$. Therefore, by A^*, "Either (a_2): [$f_j(S) > f_j(T)$, ∀j≠i] or (b_2): [$f_j(S) = f_j(T)$, ∀j≠i and $f_i(S) \leqq f_i(T)$]." Four cases should be examined: 1/ (a_1) + (a_2) $\Rightarrow$ $f(T) < f(S)$, which is incompatible with WPO, since $f(S) \in T$ by $P_i(S,T)$. 2/ (a_1) + (b_2) $\Rightarrow$ $f_i(T) < f_i(S) \leqq f_i(T)$, a contradiction. 3/ ($b_1$) + ($a_2$) $\Rightarrow$ $f_{j_o}(T) > f_{j_o}(S) > f_{j_o}(T)$ for some $j_o \neq i$, a contradiction. 4/ Finally, (b_1) + (b_2) $\Rightarrow$ $f_{j_o}(T) > f_{j_o}(S) = f_{j_o}(T)$ for some $j_o \neq i$, also a contradiction. This shows that A^* cannot hold if C^* does not hold. Q.E.D.

IIA $\Rightarrow$ C^*
IUA $\Rightarrow$ A

The hypothesis of C^* is stronger than the hypothesis of IIA, and its conclusion is stronger. The same observation applies to the second implication.

WM $\Rightarrow$ Tw

Suppose, to the contrary, that WM holds, and Tw does not. $\exists$ S,T,i as in the hypothesis of Tw such that "$f_i(T) \leqq f_i(S)$ and [either $f_i(T) \neq f_i(S)$ or [$\exists$ j≠i / $f_j(T) > f_j(S)$]]." This can be restated as "Either [(a_1): $f_i(T) < f_i(S)$] or (b_1): [$f_i(T) \leqq f_i(S)$ and [$\exists$ j≠i / $f_j(T) > f_j(S)$]]." Call $R = S \cap T$. By construction, $T \supset R$ and $\sup_{x \in T} x_j = \sup_{y \in R} y_j$, ∀j≠i. Therefore, by WM applied n-1 times: "[Either $f(T) = f(R)$ or [$f_\ell(T) > f_\ell(R)$, ∀ℓ≠j]], ∀j≠i." By construction of R, "$x \in R$, $x_i \leqq f_i(S)$" ⇔ "$x \in T$, $x_i \leqq f_i(S)$." Since, by hypothesis, $f_i(T) \leq f_i(S)$, it follows that $f(T) \in R$. Two solutions are possible.

(α) If the number of players is two, WM is applied only once and yields "Either (a_2): $f(T) = f(R)$ or (b_2): $f_i(T) > f_i(R)$." We now have $S \supseteq R$, $\sup_{x \in S} x_i = \sup_{y \in R} y_i$. By WM, "Either (a_3): $f(S) = f(R)$ or (b_3): $f_j(S) > f_j(R)$." Four possibilities should be examined: 1) $(a_2) + (a_3) \Rightarrow f(T) = f(S)$ in contradiction with the hypothesis. 2) $(a_2) + (b_3) \Rightarrow f_j(S) > f_j(T)$, which together with (a_1) would yield $f(T) > f(S)$, in contradiction with the fact that $f(S) \in \partial T$; and is directly incompatible with (b_1). 3) $(b_2) + (a_3) \Rightarrow f_i(T) > f_i(S)$ which directly contradicts (a_1) and also (b_1). 4) $(b_2) + (b_3)$ in conjunction with (a_1) give: $f(S) > f(R)$, which violates WPO since $f(S)$, belongs to R as it is a member of both S and R. $(b_2) + (b_3)$ in conjunction with (b_1) give: $f(T) > f(S)$, which violates WPO since $f(T) \in R$ as was established just before (α).

(β) If the number of players is three or more, the application of WM n-1 times yield "Either $f(T) = f(S)$ or $f(T) > f(R)$." Since $f(T) \in R$, WPO demands that $f(T) = f(R)$. We now have $S \supseteq R$, $\sup_{x \in S} x_i = \sup_{y \in R} y_i$. By WM, "Either $f(S) = r(R)$ or $[f_j(S) > f_j(R), \forall j \neq i]$." Equality of $f(S)$ and $f(T)$ $(=f(R))$ is not allowed by hypothesis. If (a_1) holds, $f(S) > f(R)$, which violates WPO since $f(S) \in R$. If (b_1) holds, for some $j_o \neq i$, $f_{j_o}(T) > f_{j_o}(S) > f_{j_o}(R) = f_{j_o}(T)$, also a contradiction. This exhausts all the possibilities. Q.E.D.

$WM^* \Rightarrow Tw^*$

Suppose, to the contrary, that WM^* holds, but not Tw^*: $\exists$ T,S,i such that the hypothesis of Tw^* holds but "$f_i(T) \leqq f_i(S)$ and [either $f_i(T) \neq f_i(S)$ or[$\exists$ $j \neq i$ / $f_j(T) > f_j(S)$]]", which can be written as "Either (a_1): $f_i(T) < f_i(S)$ or (b_1): [$f_i(T) \leqq f_i(S)$ and [$\exists$ $j \neq i$ / $f_j(T) > f_j(S)$]]". Let $R = S \cap T$.

By construction, $S \supseteq R$ and $\sup_{x\in S} x_j = \sup_{y\in R} y_j$, $\forall j\neq i$. By WM^* applied n-1 times: "[Either $f(S) = f(R)$ or $f_j(S) > f_j(R)$] $\forall_j \neq i$". Two situations are possible:

(α). If the number of players is two, one simply obtains "Either (a_2): $f(S) = f(R)$ or (b_2): $f_j(S) > f_j(R)$". Next, we have $T \supseteq R$ and $\sup_{x\in T} x_i = \sup_{y\in R} y_i$. $WM^* \Rightarrow$ "Either (a_3): $f(T) = f(R)$ or (b_3): $f_i(T) > f_i(R)$". Four possibilities arise: 1) $(a_2) + (a_3) \Rightarrow f(S) = f(T)$ in contradiction with the hypothesis. 2) $(a_2) + (b_3) \Rightarrow f_i(T) > f_i(S)$ which is incompatible with both (a_1) and (b_1). 3) $(b_2) + (a_3) \Rightarrow f_j(S) > f_j(T)$, with together with (a_1) yields $f(T) < f(S)$, in contradiction with WPO since $f(S) \in T$; and is in direct contradiction with (b_1). 4) $(b_2) + (b_3)$ in conjunction with (a_1) give $f(S) > f(R)$, in violation of WPO since $f(S) \in R$, as it belongs to both S and T; finally, $(b_2) + (b_3)$ in conjunction with (b_1) give $f(T) > f(R)$, in violation of WPO since $f(T) \in R$ as was established before (α).

(β). If the number of players is three or more, the application of WM^* n-1 times yields "Either (a_2): $f(S)$ or (b_2): $f(S) > f(R)$". Since $f(S) \in R$, if (b_2) held, WPO would be violated. Therefore (a_2) holds. We now have $T \supseteq R$ and $\sup_{x\in T} x_i = \sup_{y\in R} y_i$. $WM^* \Rightarrow$ "Either (a_3): $f(T) = f(R)$ or (b_3): $f_i(T) > f_i(R)$." If (a_3) held, we would have $f(S) = f(R)$, contradicting the hypothesis. But if (b_3) holds, $f_i(T) > f_i(R) = f_i(S)$ in violation of $f_i(T) \leqq f_i(S)$. This exhausts all the possibilities. Q.E.D.

$C + C^* \Rightarrow$ WIIA

Given a set S and its solution $f(S)$, let T_1 be obtained from S by eliminating all points of S whose 1st coordinate is greater than $f_1(S)$, i.e.: $x \in T_1 \Leftrightarrow [x \in S$ and $x_1 \leqq f_1(S)]$. It is the case that $P_1(S,T_1)$. C and C^* apply. Four cases should be examined: 1/ $C(a) + C^*(a) \Rightarrow f(T_1) >$

$f(S)$, a contradiction, since $f(S)$ is WPO for S and also for T_1 since $S \supseteq T_1$ and $f(S) \in T_1$. 2/ $C(a)+C^*(b) \Rightarrow f_j(S) \geqq f_j(T_1) > f_j(S)$, $\forall j \neq i$, a contradiction. 3/ $C(b)+C^*(a) \Rightarrow f_i(S) \geqq f_i(T_1) > f_i(S)$, also a contradiction. 4/ Finally, $C(b)+C^*(b) \Rightarrow f(S) = f(T_1)$.

This truncation procedure is then iterated. At step k, we define T_k by "$x \in T_k \Leftrightarrow x \in T_{k-1}$ and $x_k \leqq f_k(S)$." Applying C and C^* would yield $f(T_k) = f(T_{k-1})$. When $k = n$, we eventually get: $f(T_n) = f(\{y \,/\, y \leqq f(S)\})$ $= f(T_{n-1}) = \ldots = f(S)$, which is the statement of WIIA. Q.E.D.

$A + A^* \Rightarrow$ WIUA

Let x be such that $f(\{y \,/\, y \leqq x\}) = x$. (If there does not exist such an x, WIUA holds vacuously.) Let T be such that $x \in \partial T$. We will reconstruct T from $S = \{y \,/\, y \leqq x\}$ by successive additions. First, let T_1 be defined by: $y \in T_1 \Leftrightarrow [y \in T$ and $y_j \leqq f_j(S) = x_j$, $\forall j \neq i]$. It is the case that $P_1(S,T_1)$ and that $x \in \partial T_1$. Therefore, A and A^* apply. Four cases can occur. 1/ $A(a)+A^*(a) \Rightarrow f(T_1) > f(S)$, a contradiction to WPO since $f(S) \in \partial T_1$. 2/ $A(a)+A^*(b) \Rightarrow f_1(S) \geqq f_1(T_1) > f_1(S)$, a contradiction. 3/ $A(b)+A^*(a)$ $\Rightarrow f_j(S) \geqq f_j(T_1) > f_j(S)$, $\forall j \neq 1$, also a contradiction. 4/ Finally, $A(b)+A^*(b) \Rightarrow f(S) = f(T_1)$, which is the only conclusion consistent with the hypotheses.

The addition procedure is then iterated. At step k, we define T_k as "$y \in T_k \Leftrightarrow y \in T$ and $y_j \leqq f_j(S) = x_j$, $\forall j \notin \{1, \ldots, k\}$." Since $P_k(T_{k-1}, T_k)$ and $x = f(T_{k-1}) \in \partial T_k$, applying A and A^* repeatedly yields $f(T_k) = f(T_{k-1})$. When $k = n$, we eventually get $f(T) = f(T_{n-1}) = \ldots = f(T_1) = f(S) =$ $f(\{y \,/\, y \leqq x\}) = x$, which is the statement of WIUA. Q.E.D.

WIIA + WIUA $\Rightarrow$ D

Suppose to the contrary, that WIIA and WIUA hold, but not D: $\exists$ S and T such that $f(S) \neq f(T)$ and neither $f(S) > f(T)$ nor $f(S) < f(T)$. By WIIA,

$f(S) = f(\{y \mid y \leqq f(S)\})$ and $f(T) = f(\{y \mid y \leqq f(T)\})$. Let R be the convex comprehensive hull of $\{f(S)\} \cup \{f(T)\}$. By hypothesis, $f(S) \in \partial R$ and $f(T) \in \partial R$. In addition $R \supset \{y \mid y \leqq f(S)\}$, and $R \supset \{y \mid y \leqq f(T)\}$. Invoking WIUA twice yields: $f(S) = f(R)$ and $f(T) = f(R)$, which is in contradiction with the hypothesis. Q.E.D.

We now conclude with a characterization of all the solutions satisfying D.

<u>Theorem</u> For any $f(\cdot)$ satisfying D there exists some continuous $g: \mathbb{R} \to \mathbb{R}^n$, strictly increasing in all components, with $\lim_{\alpha \to +\infty} g_i(\alpha) = +\infty$ and $\forall i$ $\lim_{\alpha \to -\infty} g_i(\alpha) = -\infty$, such that $f(S) = \partial S \cap \{g(\alpha) \mid \alpha \in \mathbb{R}\}$ $\forall S \in \Sigma$.

<u>Proof</u> Let $P_\alpha = \{x \in \mathbb{R}^n \mid x_i \leq \alpha \; \forall i\}$. and let $g(\alpha) = f(P_\alpha)$. $g(\cdot)$ must be strictly increasing, by D, because $\partial P_\alpha \cap \partial P_\beta = \emptyset$ if $\alpha > \beta$. Given any S, consider $\gamma = \max_i f_i(S)_j$ then $f(S) \in \partial P_\gamma$ and so $f(S) = f(P_\gamma) = g(\gamma)$ by D. There cannot be two points in $\partial S \cap \{g(\alpha) \mid \alpha \in \mathbb{R}\}$ because no point in ∂S can dominate any other point in ∂S. So $f(S) = \partial S \cap \{g(\alpha) \mid \alpha \in \mathbb{R}\}$.

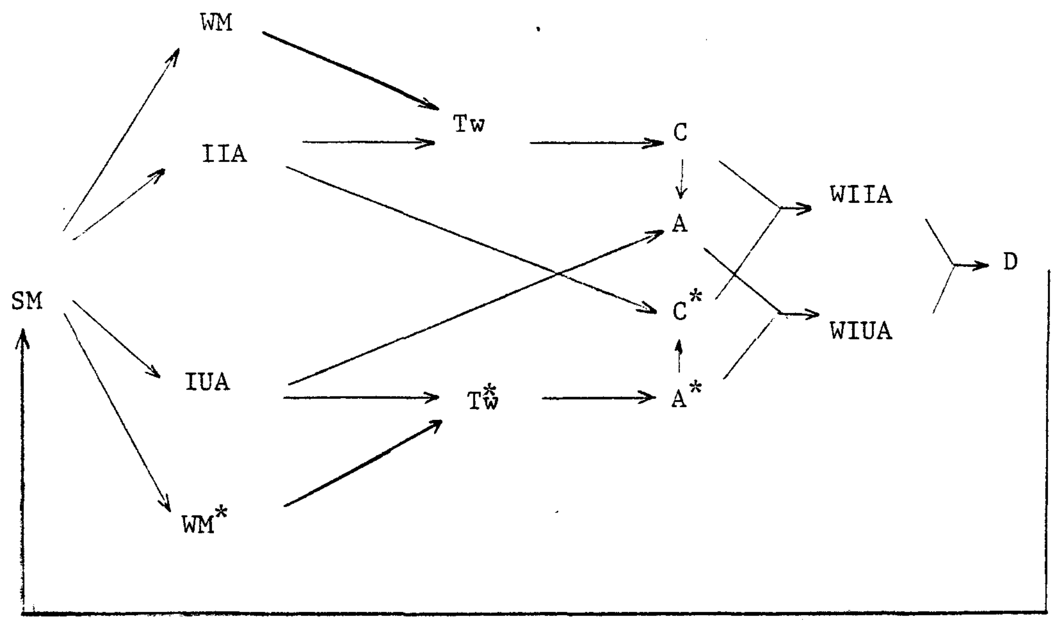
We now show that $g(\cdot)$ is continuous. Suppose that, for some $\alpha \in \mathbb{R}$ and some i, $g_i(\alpha) > \sup_{\beta < \alpha} g_i(\beta)$. Then consider

$$S = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} x_j \leq g_j(\alpha) + 1 \quad \forall j \neq i, \text{ and} \\ x_i \leq (g_i(\alpha) + \sup_{\beta < \alpha} g_i(\beta))/2 \end{array} \right\}$$

Then $g(\beta) \in \operatorname{int}(S)$ for all $\beta < \alpha$, and $g(\beta) \notin S$ for all $\beta \geq \alpha$, because $g(\cdot)$ is increasing. So $f(S) = \partial S \cap \{g(\beta)\} = \emptyset$, a contradiction. So $g_i(\alpha) = \sup_{\beta < \alpha} g_i(\beta)\}$. A similar argument proves that $g_i(\alpha) = \inf_{\beta > \alpha} (g_i(\beta))$. This suffices to show that $g(\cdot)$ is continuous. Q.E.D.

This result should be compared with the one established by Kalai, who obtains a smaller class of solutions due to his imposition of additional axioms, in particular a homogeneity axiom, requiring that if a set is "blown up" from the origin by a factor $k > 0$, so should the solution.

All the logical implications are summarized in the following tableau:



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