INVESTMENT AND PRODUCTION IN THE ABSENCE OF CONTINGENT MARKETS I - III

by

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Section I

I. Introduction

A large and growing literature has recently developed concerning the firm's choice of a production plan and the investor's choice of a portfolio in the absence of contingent securities markets. Much of the literature is devoted to establishing conditions under which the firm can choose a production plan with the unanimous approval of its shareholders. Except under these highly restrictive conditions, I am not aware of any proposed decision mechanism which generally achieves (constrained) Pareto optimal production and investment plans. In response to this fundamental lacuna, an axiomatic approach to the study of optimal decision mechanisms is obviously desirable.

The present paper, which is the first in a series on optimal investment and production decision mechanisms, is devoted to the institution of majority control. Section 2 describes the model which will be used in this and subsequent papers. Since most of the results are in the nature of impossibility theorems, they will be presented in thé simplest nontrivial model. This simplicity causes the negative results to be as definitive as possible and also clarifies the difficulties which lead to them.

There are two periods, present and future, and in the future there are several equiprobable states. There is a single commodity in the present and in each future state. The present commodity is used as an input in the production of the future commodity by a single firm whose technology has constant returns to scale. Thus the alternative production plans can be summarized as a set of activities, where each activity consists of a vector of state-dependent outputs obtained from one unit of input. There are several potential investors, each of whom is described by an endowment of the commodity in the present and in each future state, and a state-independent utility function. Each investor is assumed to know that the future states are equiprobable, so investors are homogeneous with respect to beliefs, but not with respect to endowments and utility functions. The decision problem is to select a single activity and each investor's investment. An investment consists of an amount of the input provided by the investor, which entitles him to the corresponding amount of output in each state determined by the activity. Put somewhat differently, one "share" in the firm is defined as entitling an investor, in each state, to the output derived from one unit of the input. Since there are constant returns to scale, this definition is unambiguous. In return for each share, an investor contributes one unit of the present commodity. An investment-production plan is Pareto optimal if there is no feasible alternative plan which is strictly preferred by some investor and not less preferred by any other investor. A mechanism is said to be efficient if it selects Pareto optimal plans.

Gevers [2] and Hart [3] have observed that corporate majority rule, in which each investor has one vote per share, is subject to the same Condorcet paradox which arises in the general social decision problem. That is, given any activity y chosen by the firm, there may exist an activity y' which is preferred by a coalition of investors with more than 50% of the shares. However, as Hart [3, p.62] has emphasized, this is not a definitive objection to majority control since it ignores the informational and organizational impediments to the formation of majority coalitions. In order for the activity y to be effectively opposed, a majority coalition of investors must locate one another and agree to support an alternative activity y'. Since the Condorcet paradox depends on the existence of another potential majority coalition which opposes y', such an agreement is extremely problematic.

If the majority coalition consists of a single investor, these informational and organizational difficulties vanish. Accordingly, in section 3, we will refine the concept of majority control to the concept of a "controlling interest". Given a mechanism for selecting investment-production plans, a controlling interest is defined as a fraction, strictly between 0.5 and 1.0, such that if a single investor owns more than that fraction of the total stock, the chosen activity must be his most preferred activity. For example, if a corporate charter specifies the election of directors in such a way that an investor who owns 90% of the total stock can nominate and elect the entire board, then .9 could be interpreted as a controlling interest. However, it will be proved in section 3 that even this extreme refinement of majority control is logically inconsistent with Pareto optimality.

2. The Model

2.1 Definitions: There are I investors, indexed by the superscript i, and S future states, indexed by the subscript s. We will assume that $I \geq 2$ and $S \geq 2$. The future state is generated by an objective probability distribution which assigns probability $1/S$ to each state. There is a single current commodity, indexed by the subscript 0, and a single commodity in each future state. For each i, the i^{th} household has a consumption space $X^i = R^{S+1}_{++}$ [1], and an endowment space $\Omega^i = R^{S+1}_{++}$. Let U^i denote the set of (state-independent) utility functions $u^i\colon R_{++} \to R$ such that

i) u^i is C^2 and for each $r > 0$, $Du^i(r) > 0$ and $D^2u^i(r) < 0$; and

ii) $\lim_{r \to 0} Du^i(r) = \infty$ and $\lim_{r \to \infty} Du^i(r) = 0$.

Given $u^i \in U^i$, the i^{th} investor's preferences on X^i are represented by the expected utility function $v^i\colon X^i \to R$ defined by $v^i(x^i) = u^i(x^i_0) + \Sigma^S_{s=1}(1/S)u^i(x^i_s)$. Thus for each i, $\Omega^i \times U^i$ is the set of characteristics for the i^{th} investor. Let $\Omega = \Pi^I_{i=1}\Omega^i$, with generic element $\omega = (\omega^1,\ldots,\omega^I)$, and let $U = \Pi_i U^i$, with generic element $u = (u^1,\ldots,u^I)$.

Let $\mathcal{Y}$ denote the collection of compact convex sets $Y \subset R^S_+$ with the property that if $y \in Y$ and $0 \leq y' \leq y$ then $y' \in Y$. Finally, the set of environments is the set $E = \Pi_i(\Omega^i \times U^i) \times \mathcal{Y}$ or $\Omega \times U \times \mathcal{Y}$, with generic element $e = ((\omega^i,u^i)_i, Y)$ or (ω,u,Y).

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[1] The nonnegative orthant of R^N is denoted R^N_+, and the strictly positive orthant is denoted R^N_{++}, for each N.

2.2 Remarks: A set $Y \varepsilon \mathcal{Y}$ is interpreted as a set of technologically feasible production activities. Each activity $y = (y_1,\ldots,y_s,\ldots,y_S) \varepsilon Y$ is described by the amount of output that will be realized in each future state per unit of current input. That is, if r units of the current commodity are invested in the activity y, then in each state s, ry_s units of the commodity will be produced.

2.3 Investment-Production Plans: For each i, let $\Theta^i = R_+$, and let $\Theta = \Pi_i \Theta^i$, with generic element $\theta = (\theta^1,\ldots,\theta^I)$. The space of investment production plans is $\Theta \times R^S_+$, with generic element (θ, y). A plan (θ,y) is feasible for an environment (ω,u,Y) if

i) $\omega^i + \theta^i(-1,y) \varepsilon X^i$ for each i; and

ii) $y \varepsilon Y$.

A plan (θ,y) is Pareto optimal for an environment (ω,u,Y) if (θ,y) is feasible and there is no feasible plan (θ', y') such that $v^i(\omega^i + \theta'^i(-1,y')) \geq v^i(\omega^i + \theta^i(-1,y))$ for all i, with strict inequality holding for some i, where for each i, v^i is derived from u^i as in 2.1 above.

2.4 Remarks: These definitions embody the two fundamental constraints which characterize the investment-production decision problem. First, if the i^{th} investor invests θ^i units of the current commodity in the activity y, his current consumption is $\omega^i_0 - \theta^i$ and his future consumption in each state s is $\omega^i_s + \theta^i y_s$.

Since there are no contingent markets, each agent's future consumption must equal his endowment plus his share of the output. Secondly, only one activity can be chosen, so that all investors must invest in the same activity. This technological constraint causes the activity to have a public good aspect. In the absence of this second constraint, each investor could independently invest in his own most preferred activity, and the model would formally reduce to the trivial case of household production. We also assume that the current commodity is not redistributed among the investors. This exclusion of "sidepayments" means that for any investment-production plan (θ, y), the i^{th} investor's "net trade" $\theta^i(-1,y)$ can be interpreted as a stock market transaction, where the price of one share (one unit of θ^i) in terms of the present commodity is one.

The convexity of the activity set and the concavity of the expected utility functions leads immediately to the first-order necessary conditions for optimality stated in 2.5 below. Since returns to scale are constant, once the activity is chosen each investor's level of investment should maximize his expected utility. This fact, which is reflected in 2.5(i), makes it natural to study investors' "indirect utility functions" on the space of activities. These are defined in 2.6 and some properties of them are stated in Lemma 2.7.

2.5 Necessary Conditions for Optimality: If (θ, y) is a Pareto optimal plan for (ω, u, Y) then

i) for each i, $Dv^i(\omega^i + \theta^i(-1,y))(-1,y) \leq 0$, with equality holding if $\theta^i > 0$; and

ii) there exist nonnegative numbers λ^i, $1 \leq i \leq I$, not all zero such that

$$[\Sigma_i \lambda^i \theta^i Dv^i(\omega^i + \theta^i(-1,y))](-1,y') \leq 0$$

for all $y' \varepsilon Y$.

2.6 Definitions: For each i, given $(\omega^i, u^i) \varepsilon \Omega^i \times U^i$, define $\hat{v}^i$: $R^S_+ \to R$ by $\hat{v}^i(y) = \max\{v^i(\omega^i + \theta^i(-1, y)): \theta^i \geq 0\}$, and define $\hat{\theta}^i$: $R^S_+ \to \Theta^i$ by $v^i(\omega^i + \hat{\theta}^i(y)(-1,y)) = \hat{v}^i(y)$ (recall that u^i, and thus v^i, is strictly concave).

An activity $y \varepsilon R^S_+$ is Pareto optimal for an environment (ω, u, Y) if $y \varepsilon Y$ and there is no $y' \varepsilon Y$ with $\hat{v}^i(y') \geq \hat{v}^i(y)$ for all i, with strict inequality holding for some i. It follows that a plan (θ, y) is Pareto optimal for (ω, u, Y) if and only if the activity y is Pareto optimal for (ω, u, Y) and $\theta^i = \hat{\theta}^i(y)$ for each i.

<u>2.7 Lemma</u>: For each i and each $(\omega^i, u^i) \in \Omega^i \times U^i$,

i) $\{y \in R^S_+ : \hat{\theta}^i(y) > 0\} = \{y: Dv^i(\omega^i)(-1,y) > 0\}$;

ii) $\hat{\theta}^i$ and $\hat{v}^i$ are C^1 on $\{y: \hat{\theta}^i(y) > 0\}$; and

iii) $\hat{v}^i$ is quasi-concave, and

$\hat{v}^i | \{y \in R^S_+ : \hat{\theta}^i(y) > 0\}$ is strictly quasi-concave; and

iv) for any convex set $C \subset R^S_+$ and any $y^* \in C$ with $\hat{\theta}^i(y^*) > 0$, y^* maximizes $\hat{v}^i$ on C if and only if $D\hat{v}^i(y^*)(y-y^*) \leq 0$ for all $y \in C$.

<u>Proof</u>: Assertion (i) follows from the strict concavity of the function $\theta^i \to v^i(\omega^i + \theta^i(-1,y))$. Assertion (ii) can be verified by computation using the fact that, when it is positive, $\hat{\theta}^i(y)$ is defined implicitly by the equation $[Dv^i(\omega^i + \hat{\theta}^i(y)(-1,y))](-1,y) = 0$. To establish (iii), let $y, y' \in R^S_+$ with $\hat{v}^i(y) \geq \hat{v}^i(y')$. Let $\theta^i = \hat{\theta}^i(y)$, let $\theta'^i = \hat{\theta}^i(y')$, let $0 < \lambda < 1$, and let $y_\lambda = \lambda y + (1-\lambda)y'$. If $\theta'^i = 0$, then $\hat{v}^i(y^o) \geq v^i(\omega^i) = \hat{v}^i(y')$ for all $y^o \in R^S_+$, so $\hat{v}^i(y_\lambda) \geq \hat{v}^i(y')$. If $\theta'^i > 0$, then $\hat{v}^i(y') > v^i(\omega^i)$ by the definition of θ'^i and the strict concavity of v^i. Thus if $\theta'^i > 0$ then $\theta^i > 0$, and $\hat{v}^i(y_\lambda) \geq$

$v^i(\omega^i + \theta^i\theta'^i(\lambda\theta'^i + (1-\lambda)\theta^i)^{-1}(-1,y_\lambda)) = v^i(\omega^i + \lambda\theta'^i(\lambda\theta'^i + (1-\lambda)\theta^i)^{-1}[\theta^i(-1,y)]$

$+ [1 - \lambda\theta'^i(\lambda\theta'^i + (1-\lambda)\theta^i)^{-1}[\theta'^i(-1,y')]) > \lambda\theta'^i(\lambda\theta'^i + (1-\lambda)\theta^i)^{-1}\hat{v}^i(y)$

$+ [1 - \lambda\theta'^i(\lambda\theta'^i + (1-\lambda)\theta^i)^{-1}]\hat{v}^i(y') > \hat{v}^i(y')$. To prove (iv), let

$C \subseteq R^S_+$ be a convex set and let $y^* \varepsilon C$ with $\hat{\theta}^i(y^*) > 0$. Suppose that y^* maximizes $\hat{v}^i$ on C. Since C is convex, for any $y \varepsilon C$, $y^* + t(y-y^*) \varepsilon C$ for all $0 < t < 1$. By (ii), $\hat{v}^i$ is C^1 at y^*, so $D\hat{v}^i(y^*)(y-y^*) \leq 0$. Conversely, suppose that for some $y \varepsilon C$, $\hat{v}^i(y) > \hat{v}^i(y^*)$. Then $D\hat{v}^i(y^*)(y-y^*) = \lim_{t \to 0} t^{-1} [\hat{v}^i(y^*+t(y-y^*)) - \hat{v}^i(y^*)]$, but for each t, $t^{-1}[\hat{v}^i(y^* + t(y-y^*)) - \hat{v}^i(y^*)] > \hat{\theta}^i(y^*)[t\hat{\theta}^i(y^*) + (1-t)\hat{\theta}^i(y)]^{-1}[\hat{v}^i(y) - \hat{v}^i(y^*)]$, by the inequality used above to prove (iii). This implies that $D\hat{v}^i(y^*)(y-y^*) > 0$, which completes the proof of (iv).

2.8 Remarks: As Lemma 2.7 indicates, the function $\hat{v}^i$ is well behaved on the set of activities which the i^{th} investor would choose to invest in. Unfortunately, the complementary set: $\{y \varepsilon R^S_+ : Dv^i(\omega^i)(-1,y) \leq 0\}$ constitutes a thick indifference class, so that $\hat{v}^i$ does not have the local nonsatiation property. This causes the necessary conditions 2.5(i) and (ii) to be insufficient, and leads to other pathologies as well.

These difficulties are easily described graphically. Figure 1 depicts an environment with two states, two investors, and an activity set $Y = \{(y_1,y_2) \varepsilon R^2_+ : y_1^2 + y_2^2 \leq 1\}$. The activity y^1 maximizes $\hat{v}^1$ and y^2 maximizes $\hat{v}^2$. For this environment, the set of Pareto optimal activities consists of the points y^1 and y^2 together with the dashed interval, not including its endpoints. Thus even if the utility functions u^1 and u^2 and the activity set Y are smooth, the set of Pareto optimal activities may be not closed,

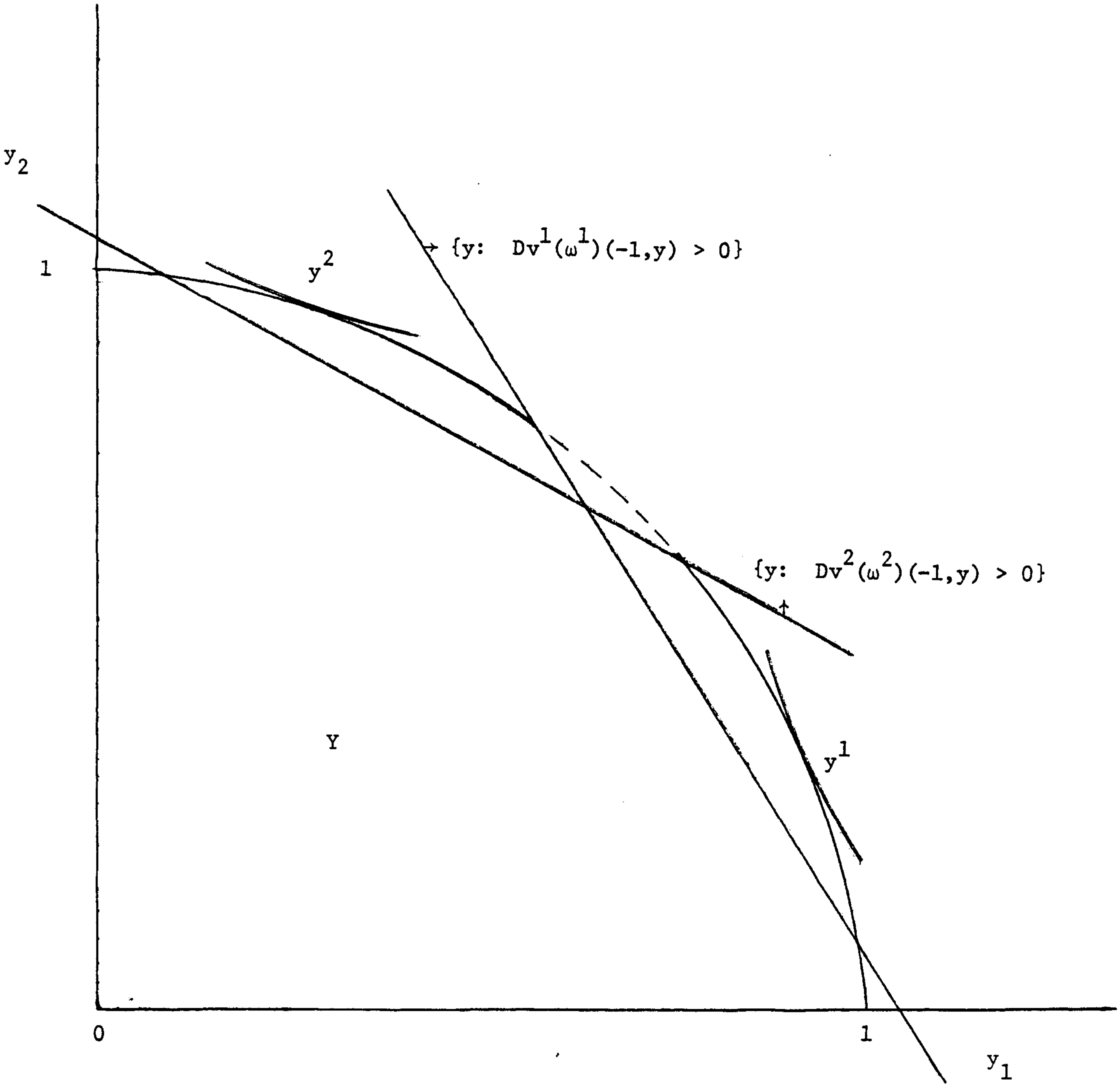

Figure 1

not connected, and not a manifold. It should also be noted that for every activity y with $Dv^1(\omega^1)(-1,y) \leq 0$ and $Dv^2(\omega^2)(-1,y) \leq 0$, the investment-production plan $(0,0,y)$ satisfies 2.5 (i) and (ii). If investor 1's endowment in state 2, ω^1_2, is increased, the hyperplane $Dv^1(\omega^1)(-1,y) = 0$ will rotate clockwise, reducing the Pareto set to y^1 (which will be relocated northwest of the previous y^1) and y^2. If ω^1_1 is increased also, the Pareto set can be reduced to y^2 alone. Since analogous variations can be made in investor 2's endowment, the Pareto correspondence on Ω to Y, given (u^1, u^2, Y), admits no continuous selection. In analysing mechanisms for choosing Pareto optimal investment-production plans, it is therefore logically inconsistent to assume that the chosen plan varies continuously with the environment. This limits the tools of analysis which can be applied to this problem. In particular, it is not possible to use the standard message space dimension-counting approach to study the information required to decentralize optimal decisions (see [4] for a more precise discussion of this point in a more abstract setting).

The possible nonconnectedness of the Pareto set is crucial to our main result, and its role is described more explicitly in 3.6 below. The thick indifference classes which give rise to this pathology would not be present if θ^i were permitted to be negative, i.e., if short sales were permitted. However, if θ^i can be negative, the indirect utility function $\hat{v}^i$ can fail to be quasi-concave, which can also cause the Pareto set to be nonconnected. Thus, although I have not attempted to reproduce any of the results in the case of short sales, the same pathology is present in this case as well.

3. The Controlling Interest Paradox

The discussions of the majority rule paradox by Gevers and Hart do not apply directly to the present model. Gevers presents two examples, one of which is for the case of price uncertainty [2, pp. 189-190], and the other requires different investors to have different subjective probabilities [2, pp. 188-189]. Hart's analysis requires sidepayments [3, pp. 61-62]. In order to compare this paradox with the controlling interest paradox we will now construct an example in the context of the present model.

<u>3.1 The Majority Rule Paradox</u>: Let $S = I = 3$, let $Y^\circ = \{y \in R^3_+ :$ $y_s \geq 1$ for each s and $\Sigma_s y_s = 6\}$, and let $Y = \{y \in R^3_+ :$ $y \leq y^\circ$ for some $y^\circ \in Y^\circ\}$. Then Y° is the set of technologically efficient activities in Y. The three investors will have identical utility functions, and their endowments will differ only by a permutation of states. Let $\omega^1 = (2, 3, 11, 11)$, and let u^1 be any utility function with

1) $Du^1(1) = 11\ 1/3$,
2) $Du^1(4) = 11$,
3) $Du^1(7) = 6$,
4) $Du^1(12) = 5$, and
5) $Du^1(15) = 4.5$.

Let $u^2 = u^3 = u^1$, let $\omega^2 = (2, 11, 3, 11)$ and let $\omega^3 = (2, 11, 11, 3)$.

We will show that for any $y \in Y°$ there exists $y' \in Y°$ and investors i and j with $[\hat{\theta}^i(y) + \hat{\theta}^j(y)]/[\hat{\theta}^1(y) + \hat{\theta}^2(y) + \hat{\theta}^3(y)] > 1/2$ such that $\hat{v}^i(y') > \hat{v}^i(y)$ and $\hat{v}^j(y') > \hat{v}^j(y)$. In other words, if each investor is permitted his most preferred level of investment, for every technologically efficient activity there exists another activity which is preferred by at least two investors who hold a majority of the shares.

First, note that for each $y \in Y°$, $(1/3)(y_1)Du^1(\omega_1^1) + (1/3)(y_2)Du^1(\omega_2^1) + (1/3)(y_3)Du^1(\omega_3^1) > (1/3)(y_1)(11) + (1/3)(y_2)(5) + (1/3)(y_3)(5) \geq 12 > Du^1(\omega_0^1)$ so by 2.7(i), $\hat{\theta}^1(y) > 0$ for all $y \in Y°$. Similarly, $\hat{\theta}^i(y) > 0$ for all $y \in Y°$, $i = 2, 3$. Let $y^1 = (4, 1, 1)$, $y^2 = (1, 4, 1)$, and $y^3 = (1, 1, 4)$. Then (1-5) imply that $\hat{\theta}^i(y^1) = \hat{\theta}^i(y^2) = \hat{\theta}^i(y^3) = 1$ for each i. Finally, (1, 3, and 4) imply that $3D\hat{v}^1(y^1) = (6, 5, 5)$, so y^1 maximizes v^1 on Y by 2.7(iv). Similarly, y^i maximizes $\hat{v}^i$ on Y for $i = 2, 3$. Now let $y \in Y°$. If $y = y^1$, let $y^* = (1, 2.5, 2.5)$. Then for $i = 2, 3$, $D\hat{v}^i(y)(y^*-y) = D\hat{v}^i(y)(-3, 1.5, 1.5) = 1.5 > 0$, so there is some y' in the line segment $[y, y^*]$ with $D\hat{v}^i(y') > D\hat{v}^i(y)$ for $i = 2, 3$. Also $[\hat{\theta}^2(y) + \hat{\theta}^3(y)]/(\Sigma_i\hat{\theta}^i(y)) = 2/3$, so y' is preferred by a share majority. The cases $y = y^i$, $i = 2, 3$ are analogous. Next, suppose that $y \neq y^i$ for each i. Then $1 \leq y_s < 4$ for each s. We will show that there is some $y' \in Y°$ which is preferred by investor 1 and either investor 2 or investor 3.

If $y_2 \geq y_3$, then for any $\varepsilon > 0$ and $\delta > 0$, $D\hat{v}^1(y)(0, -\delta, \delta) \geq 0$, $D\hat{v}^3(0, -\delta, \delta) > 0$, and $D\hat{v}^1(y)(\varepsilon, -\varepsilon, 0) > 0$. Then we can choose δ and ε sufficiently small so that if $y^* = (y_1 + \varepsilon, y_2 - \varepsilon - \delta, y_3 + \delta)$ then $y^* \varepsilon Y°$ and $D\hat{v}^i(y)(y^* - y) > 0$ for $i = 1, 3$. Thus there is some y' in the line segment $[y, y^*]$ which is preferred by investors 1 and 3. Similarly, if $y_2 \leq y_3$, there is some y' preferred by 1 and 2. Repeating this argument for investors 2 and 3, we have that for each $y \varepsilon Y° \setminus \{y^1, y^2, y^3\}$ and each investor i there exists $y' \varepsilon Y°$ and $j \neq i$ such that y' is preferred to y by both i and j. Since $\hat{\theta}^i(y) > 0$ for all i and all $y \varepsilon Y°$, this, together with the previous treatment of the case $y \varepsilon \{y^1, y^2, y^3\}$, implies that every technologically efficient activity is opposed by a share majority.

3.2 Remarks: It should be emphasized that the majority rule paradox depends on each investor's ability to purchase his most preferred number of shares at a price of one unit of the present commodity per share. For example, suppose that investor 1 has historically been the owner and sole investor, so that the activity is y^1, when investors 2 and 3 arrive and request shares. Investor 1, in order to retain control, could refuse to sell shares or could sell at a price sufficiently high so that investors 2 and 3 would demand less than a majority interest. Since Pareto optimality requires that

θ^i maximize $v^i(\omega^i + \theta^i(-1, y))$ for each i, both strategems are inconsistent with Pareto optimality. Alternatively, investor 1 could charge a lump sum fee for the right to purchase shares at one unit of the present commodity per share. It may (but will not generally) be possible to set this fee so that investors 2 and 3 will each be willing to pay but, after paying, will demand less than a majority interest. This fee is a sidepayment which redistributes the endowment of the present commodity. With respect to the new endowments, the resulting investment-production plan will be Pareto optimal. However, Hart [3] has shown that if sidepayments are permitted, the majority rule paradox becomes even more pervasive than if sidepayments are excluded. Hence the paradox can only be avoided at the expense of Pareto optimality.

We will now refine the concept of majority rule to the concept of a controlling interest and show that this is also inconsistent with Pareto optimality.

3.3 Definitions: Given an activity set Y, let c: $\Omega \times U \twoheadrightarrow \ni \times Y$. The correspondence c should be interpreted as associating with each (ω, u) the plans which arise as outcomes of an unspecified decision mechanism. If there is a fraction $(1/2) < \delta < 1$ such that

(*) for each (ω, u), each $(\theta, y) \in c(\omega, u)$, and each i, if $\theta^i > 0$ and $(\theta^i/\Sigma_j\theta^j) \geq \delta$ then y maximizes $\hat{v}^i$ on Y, then δ is a <u>controlling interest</u>. The correspondence c is <u>efficient</u> if for each (ω, u) and each $(\theta, y) \varepsilon c(\omega, u)$, (θ, y) is Pareto optimal for $(\omega, u; Y)$. The definition of a controlling interest excludes the trivial case $\delta = 1$, which would be a controlling interest for every efficient correspondence.

<u>3.4 Remarks</u>: It should be emphasized that the efficiency and controlling interest properties do not presuppose any particular decision mechanism. That is, c could be the core correspondence of a cooperative game form, the Nash equilibrium correspondence of a noncooperative game form, the performance correspondence of a decentralized decision mechanism; etc. Since investor behavior cannot be discussed at this level of abstraction, these properties are not behavioral. Unfortuantely, there are activity sets for which no efficient correspondence admits a controlling interest. Thus the concept of majority control, even in this extremely weak form, is antithetical to the concept of Pareto optimality.

<u>3.5 The Controlling Interest Paradox:</u>

Let $S = 8$, $I = 4$, and $Y° = \{y \varepsilon R^8_+ \colon\ y_1 + y_3 = 8,\ y_2 + y_4 = 8,$ $y_5 + (8-y_4)^4 = 8^4,\ y_6 + (8-y_1)^4 = 8^4,\ y_7 + (8 - y_2)^4 = 8^4$, and $y_8 + (8 - y_3)^4 = 8^4\}$. Let $Y = \{y \varepsilon R^8_+ \colon\ y \leq y° \text{ for some } y° \varepsilon Y°\}$. Let $c\colon\ \Omega \times U \twoheadrightarrow \Theta \times Y$ and suppose that c admits a controlling interest. Then c is not efficient.

Proof: To verify the convexity of Y, let Y' be defined by replacing the equalities in the definition of $Y°$ by weak inequalities. Then Y' is convex and $Y = \{y \in R^8_+ : y \leq y' \text{ for some } y' \in Y'\}$. The set $Y°$ contains all technologically efficient activities. Thus for any (ω, u), each Pareto optimal activity y is determined by its first two coordinates (y_1, y_2). The set $Y°$, with these coordinates, is depicted in Figure 2.

The theorem will be proved by showing that for each $1/2 < \delta < 1$ there exists $(\omega, u) \in \Omega \times U$ such that if (θ, y) is Pareto optimal for $(\omega, u; Y)$, then there is some i satisfying

a) $\theta^i > 0$ and $(\theta^i/\Sigma_j \theta^j) \geq \delta$, and

b) y does not maximize $\hat{v}^i$ on Y.

The characteristics of the four investors will differ only by a permutation of the states.

Let $n > 8^4$ and $0 < \varepsilon < 8^{-4}$, and, dropping the superscript i, let investor 1's endowment satisfy

(1) $$n^2 + \varepsilon = \omega_0 < \omega_5, \quad \omega_5 + 8^4(n^2 + \varepsilon) < \omega_2, \quad \omega_2 + 8(n^2 + \varepsilon) < \omega_1,$$
$$\omega_1 + 8(n^2 + \varepsilon) < \omega_3 = \omega_4 = \omega_6 = \omega_7 = \omega_8.$$

A C^1 function $g\colon R_{++} \to R$ is admissible if

(i) $g(x) > 0$ for all x,

(ii) $Dg(x) < 0$ for all x, and

(iii) $\lim_{x\to 0} g(x) = \infty$ and $\lim_{x\to\infty} g(x) = 0$.

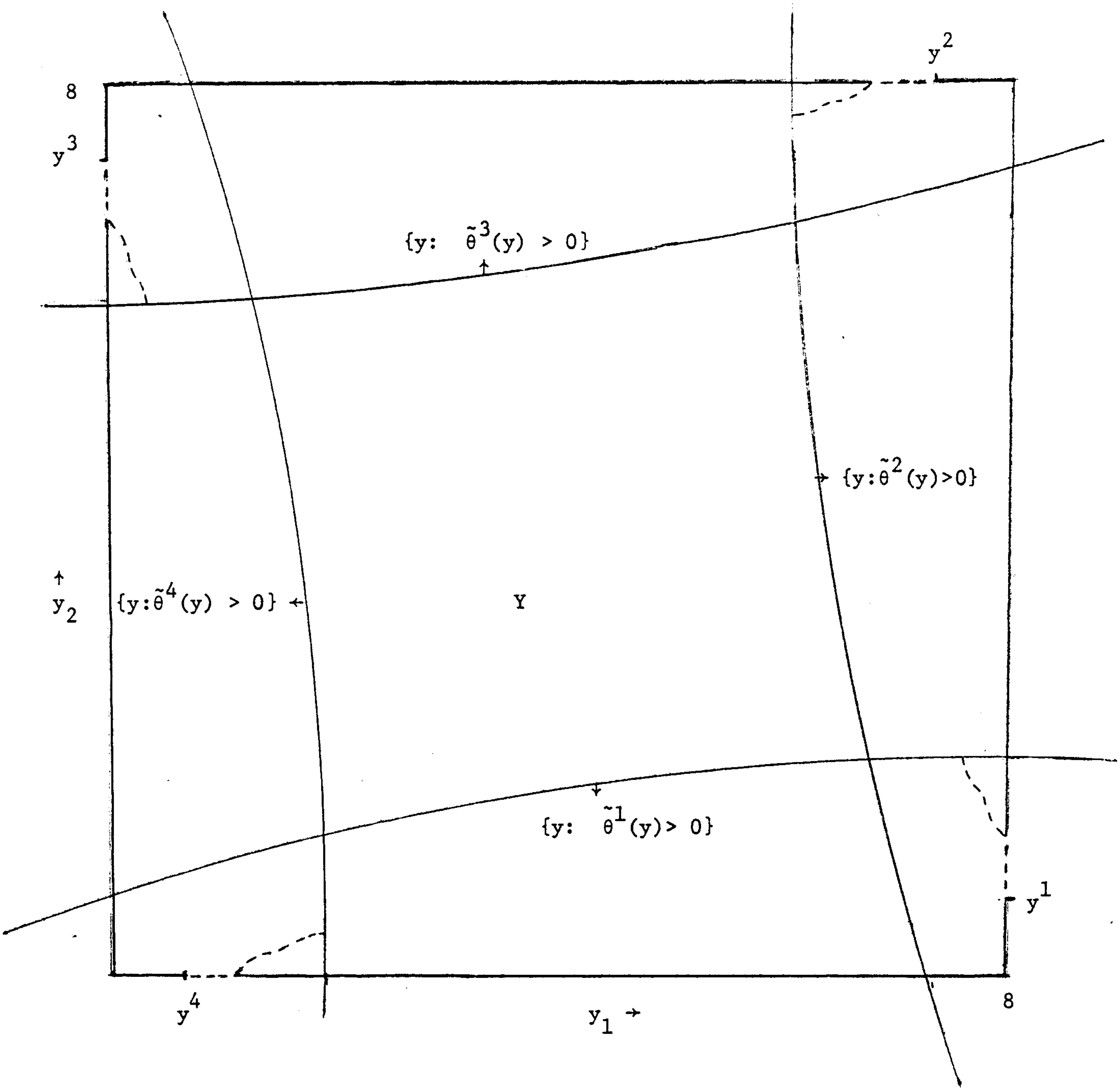

Figure 2 1/

1/ The set $\{y: \tilde{\theta}^i(y) > 0\}$ is not a half-space because of the nonlinear relation between y_4 and y_5; etc.

By Definition 2.1, a function g is admissible if and only if it is the derivative of a utility function. We will specify investor 1's utility function by imposing requirements on its derivative g. Let g be a C^1 function on R_{++} to R satisfying (i-iii) for $0 < x < \omega_3$, and

2) $\quad 8\, g(\omega_0 - n^2) = (2/n) g(\omega_1 + 2n) + 8^4\, g(\omega_5 + 8^4 n^2)$,

3) $\quad 8\, g(\omega_0 - n) = 8g(\omega_1 + 8n) + (2/n) g(\omega_2 + 2) + (8^4 - (2/n)^4) g(\omega_5 + n[8^4 - (2/n)^4])$,

4) $\quad 8\, g(\omega_0) = 8g(\omega_1) + g(\omega_2) + (8^4 - 1) g(\omega_5)$, and

5) $\quad g(\omega_5) = n^4 + \varepsilon$

6) $\quad g(\omega_5 + n[8^4 - (2/n)^4]) = n^4$

7) $\quad g(\omega_5 + 8^4 n^2) = n^4 - \varepsilon$

8) $\quad g(\omega_2) = 32(n + \varepsilon/n^3)$

9) $\quad g(\omega_2 + 2) = 32n$

10) $\quad g(\omega_1) = 32n - \varepsilon$

11) $\quad g(\omega_1 + 2n) = 32n - 2\varepsilon$

12) $\quad g(\omega_1 + 8n) = \varepsilon$

Substituting (5-12) into (2-4) shows that these values are consistent with (ii).

Since $\omega_3 > \omega_1 + 8(n^2 + \varepsilon)$, we can choose $g(\omega_3)$ arbitrarily small. For simplicity, we will temporarily make the inadmissible assumption

13) $\quad g(x) = 0$ for all $x \geq \omega_3$.

Then g is the derivative of a C^2 concave function $\tilde{u}$: $R_{++} \to R$. The optimal investment in an activity y for an investor with the "utility function" $\tilde{u}$ is independent of y_s for $s = 3, 4, 6, 7, 8$. More precisely, let $\tilde{\theta}^1$: $R_+^8 \to R_+$ and $\tilde{v}^1$: $R_+^8 \to R$ be defined by replacing u^1 with $\tilde{u}^1$ in the definitions of $\hat{\theta}^1$ and $\hat{v}^1$ (2.6). Let $S^* = 3$, let $\omega^{*1} = (\omega_0^1, \omega_1^1, \omega_2^1, \omega_5^1)$, and let u^{*1} be a utility function whose derivative equals g on the interval $(0, \omega_1 + 8(n^2 + \varepsilon))$. Then for each $y \in Y$,

14) $\quad \tilde{\theta}^1(y) = \hat{\theta}^{*1}(y_1, y_2, y_5)$ and $\tilde{v}^1(y) = \hat{v}^{*1}(y_1, y_2, y_5)$.

Denoting efficient activities by their first two coordinates and using (14), (2) implies that $\tilde{\theta}^1(2/n,0) = n^2$; (3) implies that $\tilde{\theta}^1(8, 2/n) = n$; and (4) implies that $\tilde{\theta}^1(y) = 0$ for all $y \in Y°$ with $y_2 \geq 1$. Also, by 2.7(iv), since $g(\omega_2 + 2) = 4(2/n)^3 Du^1(\omega_5 + n[8^4 - (2/n)^4])$, the activity $y^1 = (8, 2/n)$ maximizes $\tilde{v}^1$ on Y. Let $\tilde{u}^2 = \tilde{u}^3 = \tilde{u}^4 = \tilde{u}^1$, let ω^2 be obtained from ω^1 by permuting the states from (1, 2,...,8) to (2, 3, 4, 1, 6, 7, 8, 5) let ω^3 be obtained by the permutation (3, 4, 1, 2, 7, 8, 5, 6), and let ω^4 be obtained by the permutation (4, 1, 2, 3, 8, 5, 6, 7). For each $i > 1$, the i^{th} investor's most preferred activity can be obtained from y^1 by the same permutations, so that $y^2 = (8 - (2/n), 8)$, $y^3 = (0, 8 - (2/n))$, and $y^4 = (2/n, 0)$.

The set of activities which are weakly optimal is represented by the four dashed segments in Figure 2. Let Σ denote the segment containing y^1, which is depicted in Figure 3. Note that $\tilde{\theta}^1(y) > 0$ and $\tilde{\theta}^2(y) > 0$ imply that $y_2 \leq 1$ and $y_1 \geq 7$, which imply that $\tilde{\theta}^3(y) = \tilde{\theta}^4(y) = 0$. To further characterize Σ, we need to further specify g. Let $g(\omega_5 + 8^4(n^2 + \varepsilon)) = n^4 - 2\varepsilon$, and on the interval $[\omega_2, \omega_2 + 2(n + \varepsilon/n)]$, let g be specified by

15) $$g(\omega_2 + \theta^1(2/n)) = 4(2/n)^3 g(\omega_5 + \theta^1[8^4 - (2/n)^4]) \quad \text{for each}$$

$$0 \leq \theta^1 \leq n^2 + \varepsilon .$$

Note that (15) is consistent with (5-9). By 2.7 (iv), (15) implies that for each $(y_1, y_2) \in Y°$, $\tilde{v}^1(y_1, 2/n) \geq \tilde{v}^1(y_1, y_2)$. This implies that for each $y \in \Sigma$, $y_2 \geq (2/n)$. Applying the appropriate permutation of (15) to investor 2 yields $y_1 \geq 8 - (2/n)$ for each $y \in \Sigma$. We will now study $\tilde{\theta}^1(y)$, which will be abbriviated $\tilde{\theta}^1$, for $y \in \Sigma$. Let $A = \{y \in Y°\colon\ 8 - (2/n) \leq y_1 \leq 8$ and $2/n \leq y_2 \leq 2/\sqrt{n}\ \}$. To see that $\tilde{\theta}^1 < n + 1$ for $y \in A$, observe that if $\tilde{\theta}^1 \geq n + 1$, then

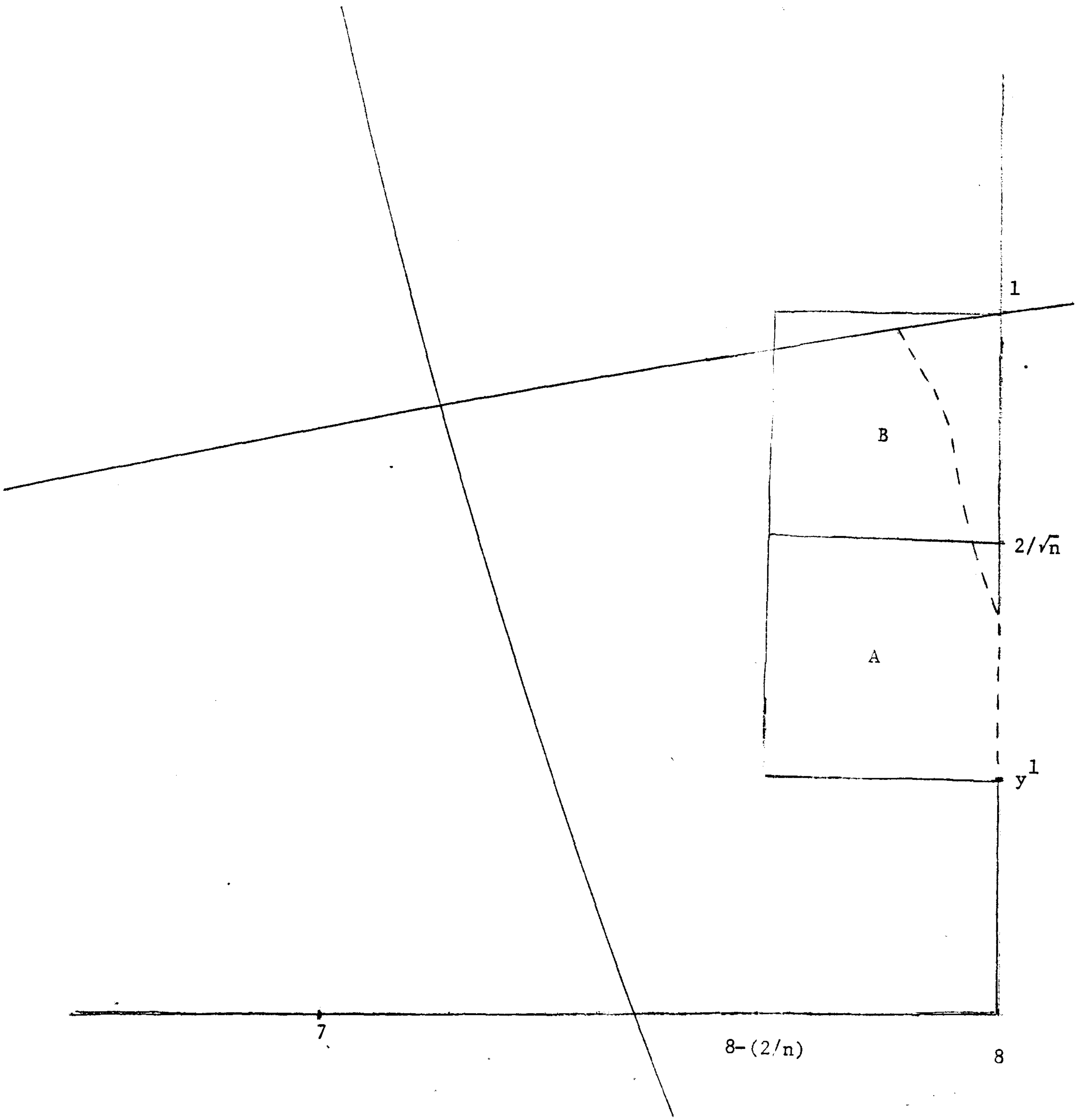

Figure 3

$$y_1 g(\omega_1 + \tilde{\theta}^1 y_1) + y_2 g(\omega_2 + \tilde{\theta}^1 y_2) + (8^4 - y_2{}^4) g(\omega_5 + \tilde{\theta}^1 [8^4 - y_2{}^4]) <$$

$$y_1 g(\omega_1 + 8n) + y_2 g(\omega_2 + 2) + (8^4 - y_2{}^4) g(\omega_5 + n[8^4 - (2/n)^4]) \leq$$

$$8 g(\omega_1 + 8n) + (2/n) g(\omega_2 + 2) + (8^4 - (2/n)^4)\; g(\omega_5 + n[8^4 - (2/n)^4]),$$

where the first inequality follows from the monotonicity of g and the second follows from (15). The first-order necessary condition for $\tilde{\theta}^1$ implies that $8g(\omega_0 - \tilde{\theta}^1)$ is equal to the first sum in the above inequalities. But this puts the monotonicity of g in conflict with (3), so we must have $\tilde{\theta}^1 < n + 1$ for $y \in A$. Now let $B = \{y \in Y^\circ\colon\ 8 - (2/n) \leq y_1 \leq 8$ and $2/\sqrt{n} \leq y_2 \leq 1\}$, and let

(16) $\qquad g(\omega_o - 1/n) = g(\omega_0 - n) - \varepsilon.$

Then for any $\theta^1 \geq 0$ and any $y \in B$,

$$y_1 g(\omega_1 + \theta^1 y_1) + y_2 g(\omega_2 + \theta^1 y_2) + (8 - y_2{}^4) g(\omega_5 + \theta^1 (8 - y_2{}^4)) < g(\omega_0 - 1/n)$$

so $\hat{\theta}^1 < 1/n$ for each $y \in B$. Applying the appropriate permutation of (15) to investor 2 yields $\tilde{\theta}^2 > n^{3/2}$ for $y \in A$ and $\tilde{\theta}^2 > 1$ for $y \in B$. Since $\Sigma \subset A \cup B$, this proves that for each $y \in \Sigma$, $\tilde{\theta}^2/\tilde{\theta}^1 > n^{3/2}/(n+1)$. The other segments are symmetric, so we have proved that for each Pareto optimal plan (θ, y), there is some i with $(\theta^i/\Sigma_{j\neq i}\theta^j) > n^{3/2}/(n+1)$ and $\max\{|y^i_s - y_s|\colon\ 1 \leq s \leq 8\} > 7$.

Finally, let $0 < \alpha < \varepsilon$, and suppose that

(13') $\qquad g(x) = \alpha/(1+x-\omega_3)$ for each $x \geq \omega_3$.

Then there is an admissible g which satisfies (1-12, 13', 15, 16). A straightforward upper semi-continuity argument shows that for α sufficiently small, for each Pareto optimal plan (θ, y) there is some i with $(\theta^i/\Sigma_{j\neq i}\theta^j) > n^{1/3}$ and $\max\{|y_s^i - y_s|: 1 \leq s \leq 8\} > 6$. Hence if δ is a controlling interest for c, we can choose n with $n^{1/3}/(n^{1/3}+1) > \delta$ to prove that c is not efficient.

3.6 Remarks: A still further generalization of majority control is the requirement that an investor with "nearly" 100% of the shares have his activity preferences "nearly" maximized. Since utility magnitudes cannot be compared across environments, the near maximization of preferences must be formalized in terms of the distance between activities rather than the difference between utility levels. Accordingly, given an activity set Y, a correspondence $c: \Omega \times U \twoheadrightarrow \Theta \times Y$ is investment-responsive if for each $\varepsilon > 0$ there exists $\delta < 1$ such that

*') for each (ω, u), each $(\theta, y) \in c(\omega, u)$, and each i, if $\theta^i > 0$ and $(\theta^i/\Sigma_j\theta^j) \geq \delta$ then $||y-y^i|| < \varepsilon$, where y^i maximizes $\hat{v}^i$ on Y.

If Y is the activity set defined in 3.5, the proof shows that for each $\delta < 1$ there exists (ω, u) such that for each Pareto optimal plan (θ, y) there is some i with $\theta^i/\Sigma_j\theta^j > \delta$ and $||y-y^i|| > 6$. Hence investment-responsiveness is also inconsistent with Pareto optimality.

The controlling interest paradox relies on two "pathologies" which are not essential to the majority rule paradox. First, an investor may optimaly invest less in a more preferred activity. In the example constructed in the proof, when the activity changes from y^1 to y^2, θ^2 drops from n^2 to n, and $\theta^2/\Sigma_{i\neq 2}\theta^i$ drops from n to $1/n$. However, this phenomenon alone does not cause difficulties. Suppose that the set of weakly optimal activities contains a path from y^1 to y^2. Since the functions $\hat{\theta}^i$ are continuous, it follows that for each $1/2 < \delta < 1$ there is some y on the path such that the weakly optimal plan (θ, y) satisfies $1-\delta < (\theta^2/\Sigma_j\theta^j)$. Such a plan satisfies (*) vacuously. Thus the nonconnectedness of the Pareto set is also required. To emphasize the rôle of this property, we state the following straightforward proposition.

<u>3.7 Proposition</u>: Given an activity set Y, let $E^\circ = \{(\omega, u) \in \Omega \times U:$ the set of activities in Y which are Pareto optimal for $(\omega, u; Y)$ is pathwise connected$\}$. There exists a correspondence c: $E^\circ \twoheadrightarrow \Theta \times Y$ such that for each $(\omega, u) \in E^\circ$ and each $(\theta, y) \in c(\omega, u)$

i) (θ, y) is Pareto optimal for $(\omega, u; Y)$; and

ii) for each i, if $(\theta^i/\Sigma_j\theta^j) > 1/2$ then y maximizes $\hat{v}^i$ on Y.

3.8 Remarks: The Proposition states that if investor characteristics are restricted to E°, efficiency permits a majority of the shares to constitute a controlling interest. It is easy to show that E° contains all (ω, u) such that for each i, $\hat{\theta}^i(y) > 0$ for every Pareto optimal activity y, so the environment constructed in 3.1 to illustrate the majority rule paradox is covered by Proposition 3.7.

4. Conclusion

The controlling interest paradox states that Pareto optimality cannot be generally acheived by any decision mechanism which permits an investor to control the firm if he owns a sufficiently large percentage of the equity. However, the acquisition of a controlling interest is not the only way in which an investor's influence can increase with his relative shareholding. For example, in [1], Drèze has described a concept of "stockholders' equilibrium" in which the production plan is chosen to maximize value according to an average of investors' marginal rates of substitution, where each investor's marginal rate of substitution is weighted by his relative shareholding. In the context of the present model, this choice correspondence can be described as follows: Given an activity set Y, let $c\colon \Omega \times U \twoheadrightarrow \Theta \times Y$ be defined by $c(\omega, u) = \{(\theta, y)\colon y \in Y$, and

i) $\theta^i = \hat{\theta}^i(y)$ for each i; and

ii) there is some $\lambda \in R^I_+$ with $\Sigma_i \lambda^i = 1$, $\theta^i = \lambda^i(\Sigma_j \theta^j)$ for each i, and $(\Sigma_i \lambda^i q^i)(-r, y') \leq (\Sigma_i \lambda^i q^i)(-\Sigma_i \theta^i, y)$ for each $r \in R_+$ and each $y' \in Y$, where $q^i = (Du^i(\omega^i_0 - \theta^i), (1/S)Du^i(\omega^i_1 + \theta^i y_1), \ldots, (1/S)Du^i(\omega^i_S + \theta^i y_S))$ for each $i.\}$.

For most activity sets, c does not admit a controlling interest, nor is it investment-responsive. However, c also fails to be efficient. For example, consider the environment depicted in Figure 4 with $I = S = 2$, $Y = \{y \varepsilon R^2_+ :\ y_1^2 + y_2^2 \leq 2\}$, $u^1(x) = u^2(x) = \ln x$ for each $x \varepsilon R_{++}$, $\omega^1 = (4, 3, 6)$, and $\omega^2 = (4, 6, 3)$. Then the plan $(0, 0; 1, 1)$ is a stockholder's equilibrium (with $\lambda^1 = \lambda^2 = 1/2$) but this plan is clearly not Pareto optimal. This suggests that the controlling interest paradox may be merely one implication of a much more general conflict between efficiency and responsiveness to relative shareholdings.

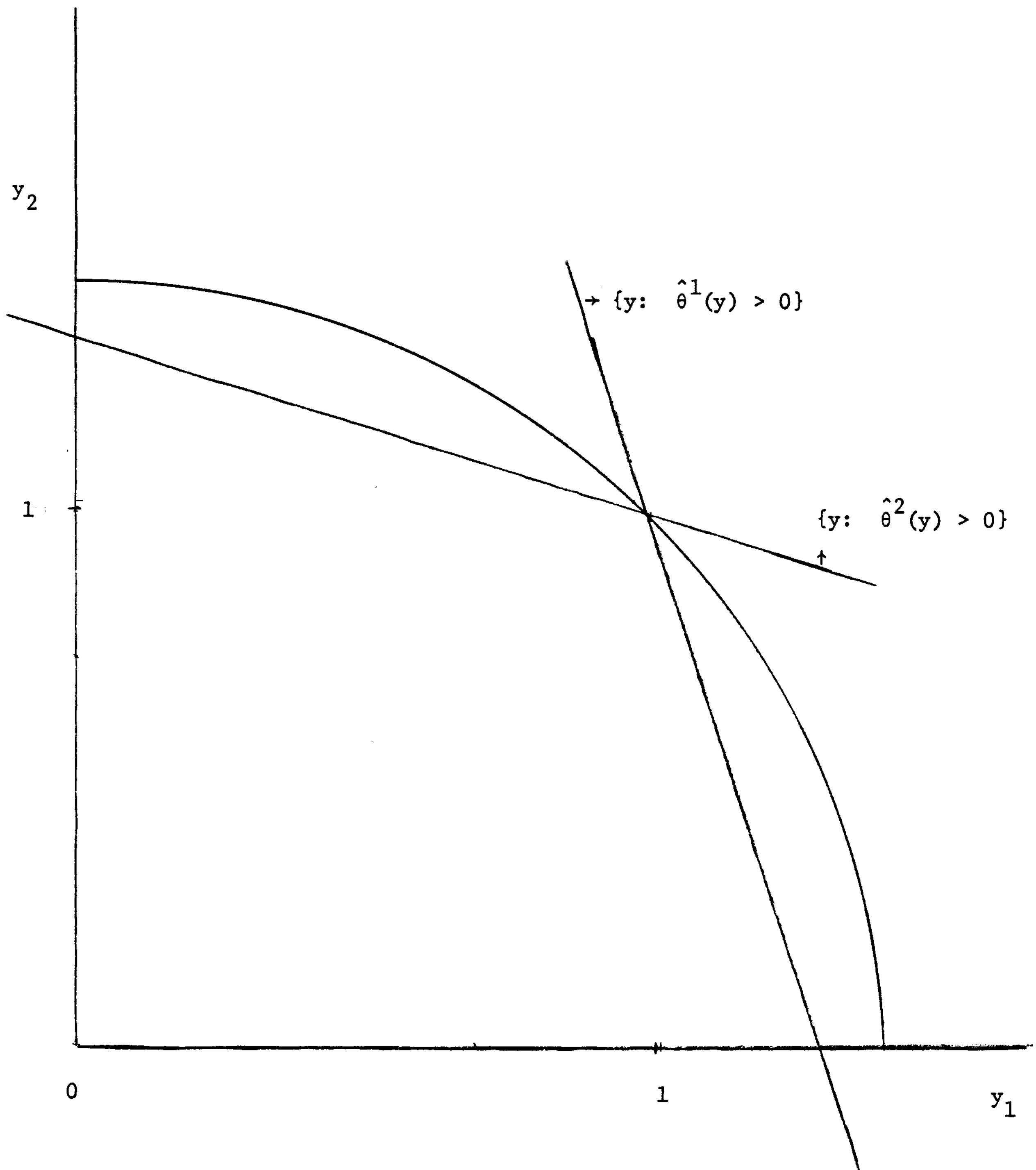

Figure 4

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<u>Section II</u>

1. Introduction

This paper continues the study of mechanisms for choosing a firm's production plan and investors' levels of investment in the absence of contingent markets. It was shown in [1] that a (constrained) Pareto optimal decision cannot be ensured by any mechanism which is sufficiently sensitive to relative shareholdings so that an investor whose optimal investment constitutes, say, 99% of the equity can determine the production activity. In other words, even if coalition formation is sufficiently costly to eliminate the usual majority rule paradox, share voting is not generally consistent with Pareto optimality. A natural alternative, and the subject of this paper, is an institution in which decisions are made by an impartial manager.

In most theoretical discussions of the firm, the manager is merely an interpretive device. A firm decision rule associates with each economic environment a decision or set of decisions, and the manager is functionary who implements these decisions. However, this generally presumes that the manager has perfect knowledge of the environment. The manager may naturally be assumed to know the firm's technological possibilities, but it is less natural to assume that he knows the preferences of the investors, particularly if the absence of contingent markets prevents these preferences from being revealed by market prices. This knowledge of investor preferences could be acquired by direct communication with investors, but only if the investors are willing to communicate their true preferences. The point of departure of this paper is the requirement that investors be given at least a naive incentive to communicate the information required for an optimal decision.

It is useful to draw an analogy with the Walrasian auctioneer process. The auctioneer announces a price p, and each trader announces a trade z, subject to $pz \leq 0$. A trader's incentive to announce his true preference maximizing trade is his belief that he will make the trade he announces. This belief is naive because if the announced trades do not balance, they will not be allowed and the auctioneer will change the price. A managerial decision mechanism is defined in this paper as an abstract analogue of the Walrasian auctioneer process.

There are two properties which are naturally desireable for managerial decision mechanisms. First, the outcome of the mechanism should be Pareto optimal. Second, the mechanism should treat the investors symmetrically. That is, the mechansim should not discriminate between investors by name, but only by those characteristics of investors, such as preferences and wealth, which are relevant to the decision problem.

The main result of this paper is that these two properties are incompatible. Specifically, a nonpathological economic environment is constructed in which no symmetric managerial decision mechanism can meet the optimality requirement. This result is Theorem 3.3. Section 3.6 gives an example of a nonsymmetric mechanism which achieves optimal outcomes.

2. The Model

The model of firm and investor characteristics is a special case of the model developed in [1]. Since all of the definitions and results stated in this section are stated more generally and discussed in [1 , Sections 1 and 2], exposition will be minimized.

<u>2.1 Definitions</u>: There are two periods, present and future, and in the future there are two possible states. Each state has probability 1/2, which is known by all investors. In the present and in each future state there is a single commodity. The firm uses the present commodity as an input in the production of the commodity in each future state. The firm has constant returns to scale, but the output per unit of input may differ between the two states. Specifically, let $Y = \{y \in R^2_+ \colon\ y_1^2 + y_2^2 \leq 25\}$.[1/] The set of feasible production activities is the set Y in the sense that if the firm chooses the activity $y \in Y$ and r units of the present commodity are devoted to production, the output in state s is ry_s, $s = 1, 2$.

There are I investors, indexed by the superscript i. For each i, let U^i denote the set of (state-independent) utility functions $u^i\colon\ R_{++} \to R$ which satisfy

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[1/] $R^n_+ = \{x \in R^n\colon\ x_j \geq 0$ for all $j\}$, and $R^n_{++} = \{x \in R^n\colon\ x_j > 0$ for all $j\}$.

i) u^i is C^2 and for each $r > 0$, $Du^i(r) > 0$

and $D^2u^i(r) < 0$; and

ii) $\lim_{r\to 0} Du^i(r) = \infty$ and $\lim_{r\to\infty} Du^i(r) = 0$

A consumption bundle for the i^{th} investor is a triple $x = (x_0, x_1, x_2) \in R^3_{++}$ of consumption in the present, in future state 1, and in future state 2 respectively. Given $u^i \in U^i$, the i^{th} investor's preferences on R^3_{++} are represented by the expected utility function v^i: $R^3_{++} \to R$ defined by $v^i(x) = u^i(x_0) + (1/2)u^i(x_1) +$ $+ (1/2)u^i(x_2)$. The i^{th} investor has an initial endowment $\omega^i = (\omega^i_0, \omega^i_1, \omega^i_2) \in R^3_{++}$. For each i, let $\Omega^i = R^3_{++}$ denote the set of initial endowments for investor i. Thus the i^{th} investor is completely described by a pair $(\omega^i, u^i) \in \Omega^i \times U^i$. Since the technology and the probabilities of the states are fixed, the set of possible environments is the set $E = \Pi_i(\Omega^i \times U^i)$, with generic element $(\omega^i, u^i)_i$.

<u>2.2 Investment-Production Plans</u>: For each i, let $\Theta^i = R_+$ be the set of possible investment levels for the i^{th} investor, and let $\Theta = \Pi_i\Theta^i$. The set of investment-production plans is $\Theta \times Y$, with generic element $(\theta, y) = (\theta^1,\ldots,\theta^I; y)$. A plan (θ, y) is feasible for an environment $(\omega^i, u^i)_i$ if $\omega^i + \theta^i(-1, y) \in R^3_{++}$ for each i, and the plan is <u>Pareto optimal</u> if there is no other feasible plan (θ', y') with $v^i(\omega^i + \theta'^i(-1,y')) \geq v^i(\omega^i + \theta^i(-1, y))$ for all i, with strict inequality holding for some i.

In other words, a plan $(\theta^1,\ldots,\theta^I; y)$ specifies an investment θ^i for the i^{th} investor in the activity y, which results in the consumption bundle $\omega^i + \theta^i(-1, y) = (\omega_0^i - \theta^i, \omega_1^i + \theta^i y_1, \omega_2^i + \theta^i y_2)$. The definitions of feasibility and Pareto optimality for plans refer to the feasibility and Pareto optimality of the resulting consumption bundles.

Since the firm has constant returns to scale, Pareto optimality clearly requires that each investor's level of investment maximize his expected utility, given the activity. This makes it natural to study investors' "indirect utilities" on Y. These are defined below, and some properties of them are stated in Lemma 2.4, which is essentially Lemma 2.7 of [1].

2.3 Definitions: For each i, given $(\omega^i, u^i) \in \Omega^i \times U^i$, define $\hat{\theta}^i\colon R_+^2 \to R_+$ by letting $\hat{\theta}^i(y)$ maximize $v^i(\omega^i + \theta^i(-1,y))$ subject to $\theta^i \geq 0$, for each $y \in R_+^2$. Since u^i is strictly concave, v^i is strictly concave, so $\hat{\theta}^i(y)$ is unique for each $y \in R_+^2$. Also, let $\hat{v}^i\colon R_+^2 \to R$ be the maximized value of v^i, that is, $\hat{v}^i(y) = v^i(\omega^i + \hat{\theta}^i(y)(-1, y))$ for each $y \in R_+^2$. It follows that a plan $(\theta, y) \in \Theta \times Y$ is Pareto optimal for an environment $(\omega^i, u^i)_i$ if and only if it is feasible, $\theta^i = \hat{\theta}^i(y)$ for each i, and there is no other activity $y' \in Y$ with $\hat{v}^i(y') \geq \hat{v}^i(y)$ for all i, with strict inequality holding for some i.

2.4 Lemma: For each i and each $(\omega^i, u^i) \in \Omega^i \times U^i$,

i) $\{y \in R_+^2\colon \hat{\theta}^i(y) > 0\} = \{y\colon Dv^i(\omega^i) \cdot (-1,y) > 0\}$;

ii) $\hat{\theta}^i$ and $\hat{v}^i$ are C^1 on $\{y \in R_+^2\colon \hat{\theta}^i(y) > 0\}$;

iii) $\hat{v}^i$ is quasi-concave, and $\hat{v}^i|\{y \in R_+^2\colon \hat{\theta}^i(y) > 0\}$ is strictly quasi-concave; and

iv) for any $y^* \in Y$ with $\hat{\theta}^i(y^*) > 0$, y^* maximizes $\hat{v}^i$ on Y if and only if $D\hat{v}^i(y^*) \cdot (y - y^*) \leq 0$ for all $y \in Y$.

3. Managerial Decision Mechanisms

3.1 Definitions: A managerial decision mechanism is characterized by a message set M^o for the manager, and for each i, a message set M^i, a constraint correspondence $c^i\colon M^o \twoheadrightarrow M^i$, and an outcome function $h^i\colon M^o \times M^i \to R^3$; and a managerial communication correspondence $\alpha\colon \Pi_i M^i \twoheadrightarrow M^o$. Given the mechanism, for each i define the correspondence $d^i\colon \Omega^i \times U^i \times M^o \twoheadrightarrow M^i$ by $d^i(\omega^i, u^i; m^o) = \{\hat{m}^i \in M^i\colon \hat{m}^i$ maximizes $v^i(\omega^i + z^i)$ subject to $z^i = h^i(m^o, m^i)$ for some $m^i \in c^i(m^o)\}$. A message $(m^o; m^1,\ldots,m^I)$ is an equilibrium for $(\omega^i, u^i)_i$ if

*) $m^o \in \alpha(m^1,\ldots,m^I)$ and $m^i \in d^i(\omega^i, u^i; m^o)$ for each i.

A managerial decision mechanism is efficient if for each environment $(\omega^i,u^i)_i$ there is an equilibrium message and moreover, for each equilibrium message $(m^o; m^1,\ldots,m^I)$ there is a Pareto optimal plan (θ,y) with $\theta^i(-1,y) = h^i(m^o,m^i)$ for each i. A managerial decision mechanism is symmetric if for each $i, j \in \{1,\ldots,I\}$, $M^i = M^j$, $c^i = c^j$, and $h^i = h^j$.

3.2 Remarks: If a mechanism is efficient, the equilibrium outcomes $h^i(m^o, m^i)$, $1 \leq i \leq I$ are attainable through investment in a feasible activity. For the sake of abstraction and generality, $h^i(m^o, m^i)$ is permitted to be any "net trade" in R^3 out of equilibrium. Thus the inducements or threats which the manager can use to elicit the appropriate information are not bound by feasibility constraints. Since the set of possible characteristics, $\Omega^i \times U^i$ is the same for each investor i, symmetry is a natural requirement. However, the following theorem states that symmetry and efficiency are incompatible.

3.3 Theorem: There is no symmetric efficient managerial decision mechanism.

Proof: The theorem follows from Lemmas 3.4 and 3.5 below.

3.4 Lemma: Let Y^o denote the set of technologically efficient activities, $Y^o = \{y \in Y: y_1^2 + y_2^2 = 25\}$. Suppose there exist (ω^1, u^1), (ω^2, u^2) such that

i) $\hat{y}_1^1 < \hat{y}_1^2$, where $\hat{y}^i$ maximizes $\hat{v}^i$ on Y for each i; and

ii) for each $y \in Y^o$ with $\hat{y}_1^1 \leq y_1 \leq \hat{y}_1^2$, $\hat{\theta}^1(y) = \hat{\theta}^2(y)$.

Then there does not exist a symmetric efficient managerial decision mechansim.

Proof: For each $i > 2$, let $(\omega^i, u^i) = (\omega^2, u^2)$. For each Pareto optimal plan (θ, y), $y \in Y^o$ and $\theta^i = \hat{\theta}^i(y)$ for each i. Also, it follows from Lemma 2.4 (iii) that $\hat{y}_1^1 \leq y_1 \leq \hat{y}_1^2$, so by assumption (ii), $\theta^i = \hat{\theta}^1(y)$ for all i.

Suppose by way of contradiction that a symmetric efficient managerial decision mechanism exists. Then there is an equilibrium $(m^o; m^1,\ldots,m^I)$ for $(\omega^i, u^i)_{i=1}^I$ with an associated Pareto optimal outcome (θ, y). Then either $y \neq \hat{y}^1$ or $y \neq \hat{y}^2$, so suppose $y \neq \hat{y}^1$. Let $(\omega'^i, u'^i) = (\omega^1, u^1)$ for each $i > 1$. Since (θ, y) is not Pareto optimal for $(\omega^1, u^1; \omega'^2, u'^2;\ldots;\omega'^I, u'^I)$, $(m^o, m^1,\ldots,m^I)$ cannot be an equilibrium for this environment, so for some $i \geq 2$ there is some $m'^{i} \in c^{i}(m^{o})$ with $v^{1}(\omega^{1} + h^{i}(m^{o}, m'^{i}))$ $> v^{1}(\omega^{1} + \theta^{i}(-1, y))$. But $\theta^{i} = \theta^{1}$, and since the mechanism is symmetric, $M^{i} = M^{1}$, $c^{i}(m^{o}) = c^{1}(m^{o})$, and $h^{i}(m^{o}, m'^{i}) =$ $h^{1}(m^{o}, m'^{i})$. This contradicts the fact that $m^{1} \in d^{1}(\omega^{1}, u^{1}; m^{o})$. The case $y \neq \hat{y}^{2}$ is analogous.

Lemma 3.5: There exist (ω^{1}, u^{1}) and (ω^{2}, u^{2}) with the properties described in the hypothesis of 3.4.

Proof: We will first construct (ω^{1}, u^{1}). Let $\omega^{1} = (2, 10, 4)$. To construct u^{1}, First note that any C^{1} function $g\colon R_{++} \to R_{++}$ which satisfies

a) $Dg(r) < 0$ for all $r \in R_{++}$; and

b) $\lim_{r \to 0} g(r) = \infty$ and $\lim_{r \to \infty} g(r) = 0$

is the derivative of a utility function.

Let g be any such function which satisfies

1) $g(1) = 12.5$; and

2) for each $3 \leq y_{1} \leq 4$, $g(\omega_{1}^{1} + y_{1}) = 9/y_{1}$; and

3) for each $3 \leq y_{2} \leq 4$, $g(\omega_{2}^{2} + y_{2}) = 16/y_{2}$.

Let u^{1} be a utility function with $Du^{1} = g$. Then (1-3) imply that for each $y \in Y$ with $3 \leq y_{1} \leq 4$ and $3 \leq y_{2} \leq 4$, $Du^{1}(\omega_{0}^{1} - 1) = (1/2)y_{1}Du^{1}(\omega_{1}^{1} + y_{1}) + (1/2)y_{2}Du^{1}(\omega_{2}^{1} + y_{2})$, which implies that $\hat{\theta}^{1}(y) = 1$. Also $(3/4)Du^{1}(\omega_{2}^{1} + 4) = Du^{1}(\omega_{1}^{1} + 3)$

which by Lemma 2.4 (iv), implies that $\hat{y}^1 = (3, 4)$. Now let $u^2 = u^1$, $\omega_0^2 = \omega_0^1$, $\omega_1^2 = \omega_2^1$, and $\omega_2^2 = \omega_1^1$. Then permuting the states in (2) and (3), we have $\hat{y}^2 = (4, 3)$, and for each $y \in Y^o$ with $3 \leq y_1 \leq 4$, $\hat{\theta}^2(y) = 1 = \hat{\theta}^1(y)$. This completes the construction.

3.6 Remarks: It is also possible to construct (ω^1, u^1) and (ω^2, u^2) satisfying 3.4 (i, ii) with $\omega^1 = \omega^2$. This implies that the Theorem can be strengthened by requiring investors to have identical endowments and supposing that the manager knows the common endowment, so that only the utility functions u^i are not known by the manager in advance.

Nonsymmetric efficient managerial decision mechanisms are easily constructed. For example, let $I = 2$ and let $M^o = \{(p^1,p^2) \in R_+^3 \times R_+^3$: $\Sigma_{s=0}^3 p_s^1 = \Sigma_{s=0}^3 p_s^2 = 1\}$, and $M^1 = M^2 = R^3$. For each $(p^1, p^2) \in M^o$, let $c^1(p^1, p^2) = \{z^1 \in M^1\colon\ p^1 z^1 \leq 0\}$ and $c^2(p^1,p^2) = \{z^2 \in M^2\colon\ p^2 z^2 \leq 0\}$, and let h^1 and h^2 be defined by $h^1(p^1, p^2; z^1) = z^1$ and $h^2(p^1, p^2; z^2) = z^2$. Thus each investor chooses a "trade" z subject to a budget constraint, but the prices may differ between investors. The manager's decision correspondence is defined by $\alpha(z^1, z^2) = \{(p^1, p^2)$:

a) if there is some $(\theta^1, \theta^2, y) \in \Theta \times Y$ with $z^1 = \theta^1(-1, y)$ and $z^2 = \theta^2(-1, y)$ then either

i) $p^1(-1, y) \leq 0$ for all $y \in Y$; or

ii) $p^2(-1,y) \leq 0$ for all $y \in Y$,

with (i) holding if $\theta^2 = 0$ and (ii) holding if $\theta^1 = 0$; and

b) p^1 maximizes $p'^1 z^1$ and p^2 maximizes $p'^2 z^2$ subject to $(p'^1, p'^2) \varepsilon M^o$ otherwise}.

To characterize the plans corresponding to the equilibria of this mechanism, for $i = 1,2$, let y^i maximize $\hat{v}^i(y)$ on Y. Then it can be verified that the set of equilibrium plans contains

i) $(\theta^1(y^1), \theta^2(y^2), y^1)$ if $\theta^1(y^1) > 0$; and

ii) $(\theta^1(y^2), \theta^2(y^2), y^2)$ if $\theta^2(y^2) > 0$; and

iii) $(0, 0, y)$ for all $y \varepsilon Y$ if $\hat{\theta}(y^1) = \hat{\theta}^2(y^2) = 0$,

and no other plans.

This mechanism is symmetric in the sense that permutation of investor superscripts in the environment leads only to the permutation of investor superscripts in the equilibria, but nonsymmetric in that the investors may face different prices, so that $c^1 \neq c^2$. This property is also exhibited by the Lindahl mechanism for allocation in environments with public goods. Since the production activity is in the nature of a public good, it is perhaps not surprising that this sort of asymmetry is required.

Reference

[1] J. Jordan, "Investment and Production in the Absence of Contingent Markets I: The Controlling Interest Paradox", July 1978.

Section III

1. Introduction

This paper continues the study of the choice of a production plan and each investor's level of investment in the absence of contingent markets. In [6], it was shown that Pareto optimal decisions cannot be achieved an impartial manager who communicates with each investor symmetrically, given that investors communicate so as to maximize their preferences. This paper studies the case in which decisions are made by the investors jointly in the manner of a noncooperative game.

Section 3 introduces two games whose Nash equilibria are Pareto optimal. In each of these games, the production activity is chosen by a manager. However, in contrast to the managerial decision mechanisms modelled in [6], in this case the manager is an investor who is elected by a plurality vote, and the activity is chosen to maximize the manager's own preferences. The controlling interest paradox [5, 3.5] implies that a manager cannot generally be elected in this fashion if investors vote with their shares, even though noncooperative voting behavior excludes the usual majority rule paradox. The first voting mechanism, described in Section 3.3, gives each investor a single vote. The second mechanism, described in Section 3.5, is one which investors vote with their shares for a board of directors, which then elects a manager, each director having a single vote. Although the second mechanism is richer in institutional detail than the first, the equilibrium outcomes are the same. For each shareholder there is an equilibrium in which he is elected manager.

These voting games achieve Pareto optimal outcomes, but only at the cost of assuming that investors are strategically naive. For example, each investor announces the production activity he would choose as manager, but assumes that his choice of activity has no influence on the votes he receives. Noncooperative games with sophisticated behavior are studied in Section 4. The main result of that section, Corollary 4.13, states that if investors are treated symmetrically and there are more than two future states, such games cannot generally achieve Pareto optimal outcomes.

2. The Model

The model of investor characteristics and technological possibilities has been described in [5], so exposition will be minimized here.

2.1 Definitions: There are two periods, present and future, and in the future there are S equiprobable states indexed by the subscript s. There is a single commodity in the present and in each future state, and the present commodity is used as an input in the production of the future commodity in each state. Production is subject to constant returns to scale, and the technological possibilities are described by a compact convex set $Y \subset R^S_+$ 1/ with $0 \in Y$, and if $y \in Y$ and $0 \leq y' \leq y$ then $y' \in Y$. Elements of Y are called activities. If the activity y is chosen and r units of the present commodity are allocated to production, the output in state s is ry_s.

There are I investors, indexed by the superscripts i and j, and the i^{th} investor has a state-independent utility function u^i: $R_{++} \to R$. For each i, let U^i denote the set of utility functions u^i such that

i) u^i is C^2 and for each $r > 0$, $Du^i(r) > 0$ and $D^2u^i(r) < 0$; and

---

[1] The nonnegative orthant of R^n is denoted R^n_+, and the strictly positive orthant is denoted R^n_{++}.

ii) $\lim_{r\to 0} Du^i(r) = \infty$ and $\lim_{r\to\infty} Du^i(r) = 0$.

Given $u^i \in U^i$, the i^{th} investor's preferences on R^{S+1}_{++} are determined by the expected utility function v^i: $R^{S+1}_{++} \to R$ defined by $v^i(x) = u^i(x_0) + \Sigma^S_{s=1}(1/S)u^i(x_s)$ for each $(x_0, x_1,\dots,x_S) \in R^{S+1}_{++}$.

Each investor also has an endowment of the commodity in the present and in each future state. For each i, let $\Omega^i = R^{S+1}_{++}$ denote the set of possible endowments for the i^{th} investor, with generic element $\omega^i = (\omega^i_0, \omega^i_1,\dots,\omega^i_S)$. Since the activity set Y is fixed, the set of environments is the set of investor characteristics $\Pi_i(\Omega^i \times U^i)$, with generic element $(\omega^i, u^i)_i$.

<u>2.2 Investment-Production Plans</u>: For each i, let $\Theta^i = R_+$, with generic element θ^i, and let $\Theta = \Pi_i\Theta^i$, with generic element $\theta = (\theta^1,\dots,\theta^I)$. The set of investment-production plans is the set $\Theta \times Y$. If a plan $(\theta^1,\dots,\theta^I; y)$ is chosen, the i^{th} investor gives up θ^i units of the present commodity and receives $\theta^i y_s$ units of the commodity in state s. Thus the i^{th} investor's consumption is described by the vector $\omega^i + \theta^i(-1,y) =$
$= (\omega^i_0 - \theta^i,\ \omega^i_1 + \theta^i y_1,\dots,\ \omega^i_S + \theta^i y^i_S)$. Given an environment $(\omega^i, u^i)_i$, a plan (θ, y) is <u>feasible</u> if $\omega^i + \theta^i(-1,y) \in R^{S+1}_{++}$ for each i. A feasible plan (θ, y) is <u>Pareto optimal</u> if there is no other feasible plan (θ', y') with $v^i(\omega^i + \theta'^i(-1,y')) \geq v^i(\omega^i + \theta^i(-1,y))$ for all i, with strict inequality holding for at least one i.

2.3 Remarks: The i^{th} investor's level of investment θ^i can be interpreted as a shareholding. Suppose that a share is defined as entitling the holder to the output resulting from one unit of input. Since there are constant returns to scale, this definition is unambiguous, given the activity. Also, given the activity, Pareto optimality requires that each investor's level of investment maximize his expected utility. Since we have excluded sidepayments of the present commodity between investors, this implies that an investor gives up one unit of the present commodity per share. We now define an investor's expected utility maximizing level of investment and the resulting "indirect utility" of activities.

2.4 Definitions: For each i, given $u^i \varepsilon U^i$, define $\hat{v}^i$: $\Omega^i \times R^S_+ \to R$ by $\hat{v}^i(\omega^i, y) = \max\{v^i(\omega^i + \theta^i(-1,y)): \theta^i \geq 0\}$, and define $\hat{\theta}^i$: $\Omega^i \times R^S_+ \to \mho^i$ by $v^i(\omega^i + \hat{\theta}^i(\omega^i,y)(-1,y)) = \hat{v}^i(\omega^i,y)$. Since u^i is strictly concave, v^i is strictly concave, so $\hat{\theta}^i$ is well-defined.

An activity $y \varepsilon Y$ is Pareto optimal for an environment $(\omega^i, u^i)_i$ if there is no other activity $y' \varepsilon Y$ with $\hat{v}^i(\omega^i, y') \geq \hat{v}^i(\omega^i,y)$ for all i, with strict inequality holding for at least one i. It follows that a plan (θ, y) is Pareto optimal if and only if y is Pareto optimal and $\theta^i = \hat{\theta}^i(\omega^i, y)$ for each i.

2.5 Lemma: For each i and each $u^i \varepsilon U^i$,

i) $\{y \varepsilon R^S_+: \hat{\theta}^i(\omega^i,y) > 0\} = \{y: Dv^i(\omega^i)(-1,y) > 0\}$;

ii) on $\{(\omega^i,y): \hat{\theta}^i(\omega^i,y) > 0\}$, $\hat{\theta}^i$ is C^1 and $\hat{v}^i$ is C^2;

iii) for each $\omega^i \in \Omega^i$, $\hat{v}^i(\omega^i, \cdot)$ is quasi-concave, and $\hat{v}^i(\omega^i, \cdot)|\{y \in R^S_+: \hat{\theta}^i(\omega^i, y) > 0\}$ is strictly quasi-concave; and

iv) for each $\omega^i \in \Omega^i$ and any $y^* \in Y$ with $\hat{\theta}^i(\omega^i, y^*) > 0$, y^* maximizes $\hat{v}^i(\omega^i, \cdot)$ on Y if and only if $D\hat{y}^i(\omega^i, y^*)(y-y^*) \leq 0$ for all $y \in Y$, where Dy denotes the derivative of $\hat{v}^i$ with respect to y.

Proof: Assertions (i, iii, and iv) follow from [5, Lemma 2.7 (i, iii, and iv) respectively]. To prove (ii), observe that, when it is positive, $\hat{\theta}^i(\omega^i, y)$ is defined implicitly by

$$(*) \qquad [Dv^i(\omega^i + \hat{\theta}^i(\omega^i, y)(-1,y))](-1,y) = 0.$$

It follows from the Implicit Function Theorem that $\hat{\theta}^i$ is C^1 on the set $\{(\omega^i, y): \hat{\theta}^i(\omega^i, y) > 0\}$. Using (*), one obtains $D\hat{v}^i(\omega^i, y) = (Du^i(\omega^i_0 - \hat{\theta}^i(\omega^i, y)), Du^i(\omega^i_1 + \hat{\theta}^i(\omega^i, y)y_1), \ldots, Du^i(\omega^i_S + \hat{\theta}^i(\omega^i, y)y_S;$ $\hat{\theta}^i(\omega^i, y)Du^i(\omega^i_1 + \hat{\theta}^i(\omega^i, y)y^i_1), \ldots, \hat{\theta}^i(\omega^i, y)Du^i(\omega^i_S + \hat{\theta}^i(\omega^i, y)y^i_S))$. Since $\hat{\theta}^i$ is C^1 it follows that $\hat{v}^i$ is C^2 on $\{(\omega^i, y): \hat{\theta}^i(\omega^i, y) > 0\}$.

3. Efficient Game Forms

3.1 Definitions: A game form is characterized by a strategy set M^i for each i, and an outcome function $h: \Pi_i M^i \to \Theta \times Y$. Let $M = \Pi_i M^i$, let $m \in M$, and let $(\theta, y) = h(m)$. The strategy I-tuple m is an (Nash) equilibrium for an environment $(\omega^i, u^i)_i$ if for each i, $v^i(\omega^i + \theta^i(-1, y)) \geq v^i(\omega^i + \theta'^i(-1,y'))$ for all $(\theta', y') \in h(m^1,\ldots,m^{i-1}, M^i, m^{i+1},\ldots,m^I)$, where $(\theta, y) = h(m)$. If m is an equilibrium, $h(m)$ is called an equilibrium outcome. A game form is said to be efficient if for each $(\omega^i, u^i)_i \in \Pi_i(\Omega^i \times U^i)$,

i) there is an equilibrium for $(\omega^i, u^i)_i$; and

ii) every equilibrium outcome for $(\omega^i, u^i)_i$ is Pareto optimal.

3.2 Remarks: A game form determines for each environment $(\omega^i, u^i)_i$ a noncooperative game, with the outcome function $h'\colon M \to (R \cup \{-\infty\})^I$ defined by $h'(m) = (v'^i[\omega^i + \theta^i(-1,y)])_i$, where $(\theta,y) = h(m)$ and

$$v'^i(\omega^i + \theta^i(-1,y)) = \begin{cases} v^i(\omega^i+\theta^i(-1,y)) & \text{if} \quad \omega^i+\theta^i(-1,y) \in R^{S+1}_{++}, \\ -\infty & \text{otherwise,} \end{cases}$$

for each i. The term game form is taken from [1].

There are many efficient game forms, including analogues of many of those constructed by Hurwicz and Schmeidler [4]. Perhaps the simplest is one in which the activity is chosen by a single investor who is elected manager by a pluarlity vote. Each investor chooses his own investment in the selected activity.

3.3 A Simple Voting Game Form: For each i, the i^{th} investor's strategy set is $M^i = \Theta^i \times Y \times \{1,\ldots,I\}$. The outcome function $h: M \to \Theta \times Y$ is defined as follows: for each $m = [(\theta^i, y^i, j^i)_{i=1}^I] \in M$,

a) (no positive investors) if $\theta^i = 0$ for all i, then $h(m) = (\theta^1,\ldots,\theta^I;0)$;

b) (no ties) if there is some i^* with $\theta^{i^*} > 0$ and $\#\{i: j^i = i^*\} > \#\{i: j^i = i'\}$ for all $i' \neq i$ with $\theta^{i'} > 0$, then $h(m) = (\theta^1,\ldots,\theta^I;y^{i^*})$; and

c) (ties) $h(m) = (\theta^1,\ldots,\theta^I;0)$ otherwise.

In case (b), investor i^* will be called the manager.

3.4 Remarks: The interpretation of case (c) is that a tie vote prevents the firm from acting, so the input $\Sigma_i \theta^i$ contributed by the investors yields no output in any state.

The equilibrium outcomes of this mechanism are easily characterized. Let $m = [(\theta^i, y^i, j^i)_{i=1}^I]$ be an equilibrium, and suppose that m satisfies the hypothesis of case (a). Then by case (b), any investor i can become manager by setting $\theta^i > 0$ and voting for himself (setting $j^i = i$). Therefore, if m is an equilibrium which satisfies the hypothesis of case (a), we must have $\hat{\theta}^i(\omega^i, y) = 0$ for all i and all $y \in Y$. If m satisfies the hypothesis of case (b), then y^{i^*} is the most preferred activity of the manager i^*, and $\theta^i = \hat{\theta}^i(\omega^i, y^{i^*})$ for each i. Finally, case (c) cannot occur as an equilibrium, since each investor i with $\theta^i > 0$ would prefer to set $\theta^i = 0$.

Conversely, first suppose that for each i, $\hat{\theta}^i(\omega^i, y) = 0$ for all $y \in Y$. Then $[(0,0,i)_{i=1}^I]$ is an equilibrium, with the (unique) associated outcome $(0,\ldots,0;0)$. Otherwise, let i^* such that $\theta^{i^*}(\omega^{i^*},y) > 0$ for some $y \in Y$, and let y^{i^*} maximize $\hat{v}^{i^*}(\omega^{i^*},y)$ subject to $y \in Y$. Then $[(\hat{\theta}^i(\omega^i,y^{i^*}),y^{i^*},i^*)_{i=1}^I]$ is an equilibrium, with the outcome $(\hat{\theta}^1(\omega^1,y^{i^*}),\ldots,\hat{\theta}^I(\omega^I,y^{i^*});y^{i^*})$. Therefore this game form is efficient.

This game form gives each shareholder a single vote, regardless of his shareholding, although it could easily be modified to count only the votes of investors with positive shareholdings without affecting the set of equilibrium outcomes. The controlling interest paradox [5, 3.5] implies that efficiency will be lost if the game form is modified to give each investor one vote per share. However, the shareholders of an incorporated firm commonly vote their shares not for a manager, but for a board of directors. The manager, or principal executive officer of a corporation is elected by the board of directors, each director having one vote. "The board of directors acts independently of the shareholders, and a contract or agreement or understanding whereby a director subordinates his will and judgement to the dictates of a shareholder or group of shareholders is generally held to be contrary to public policy and illegal." [10, p. 791].

This two-stage election process can be easily formalized as an efficient game form. The definitions are stated formally in section 3.5 below. If there is only one positive investor, that investor is the manager. If there are two or more positive investors, the firm is constituted as a corporation with two directors. The two directors are elected from among the positive investors by cumulative share voting. The directors vote for a manager, each director having one vote. Tie votes, either for the directors or the manager, result in the zero activity.

3.5 <u>A Corporate Voting Game Form</u>: For each i, let $M^i = \Theta^i \times Y \times \{1,\ldots,I\}^3$. The outcome function $h\colon M \to \Theta \times Y$ is defined as follows. For each $m = [(\theta^i, y^i, j_1^i, j_2^i, j_3^i)_{i=1}^I] \in M$,

a) (no positive investors) if $\theta^i = 0$ for all i, then $h(m) = (\theta^1,\ldots,\theta^I;0)$;

b) (one positive investor) if there is one and only one i with $\theta^i > 0$, then $h(m) = (\theta^1,\ldots,\theta^I;\ y^i)$;

c) (at least two positive investors, no ties) if there exist $i_1 \neq i_2$, with $\theta^{i_1} > 0$, $\theta^{i_2} > 0$, and

$$\Big(\sum_{\{i:j_1^i = i_1\}} \theta^i + \sum_{\{i:j_2^i = i_1\}} \theta^i\Big) \geq \Big(\sum_{\{i:j_1^i = i_2\}} \theta^i + \sum_{\{i:j_2^i = i_2\}} \theta^i\Big) >$$

$$> \Big(\sum_{\{i:j_1^i = i'\}} \theta^i + \sum_{\{i:j_2^i = i'\}} \theta^i\Big) \quad \text{for all } i' \neq i_1, i_2 \text{ with}$$

$\theta^{i'} > 0$, and there exists i^*, possibly equal to i_1 or i_2, with $\theta^{i^*} > 0$ and $j_3^{i_1} = j_3^{i_2} = i^*$, then $h(m) = (\theta^1,\ldots,\theta^I;\ y^{i^*})$; and

d) (ties) $h(m) = (\theta^1,\ldots,\theta^I;\ 0)$ otherwise.

<u>3.6 Remarks</u>: This game form is more elaborate and richer in interpretation than the one defined in 3.3, but the two have exactly the same set of equilibrium outcomes for each $(\omega,u) \in \Omega \times U$. In the corporate voting game form, an investor i with $\theta_i/\Sigma_j\theta_j > 2/3$ can elect himself director and can also choose the second director. However, since he cannot choose the second director's vote, he cannot choose the manager.

In each of these game forms, the Nash equilibrium concept entails somewhat implausible investor behavior. Each investor i assumes that his announced activity y^i has no influence on the votes he receives from other investors. Furthermore, suppose there are two investors, so that the two game forms are virtually identical, and suppose that investor 1 votes for himself as manager regardless of investor 2's vote. If investor 2 behaves according to the Nash concept, he will also always vote for investor 1 in order to avoid a tie, so investor 1 will always be elected manager. Thus a sophisticated investor can advantageously violate the assumption of Nash behavior. The next section is devoted to game forms which cannot be manipulated in this fashion.

4. Nonmanipulable Game Forms

For the purpose of generality, investor utility functions will be assumed in this section to be fixed and identical. Thus investors are homogeneous with respect to tastes and, as before, beliefs, and differ only by their endowments. This implicitly permits the game forms studied in this section to be designed according to investor utility functions, as well as beliefs and the activity set. Since the main results of this section, Theorem 4.12 and Corollary 4.14, assert the nonexistence of certain game forms, this increases the generality of these results. We will also assume for convenience that the activity set satisfies certain regularity conditions.

4.1 Definitions: Let $f\colon R^S_+ \to R$ be C^2, strictly convex, and strictly increasing, with

i) $D^2 f(y)$ positive definite for each $y \in R^S_{++}$;

ii) $\frac{\partial}{\partial y_s} f(y_1, \ldots, y_{s-1}, 0, y_{s+1}, \ldots, y_S) = 0$ for each s and each $y \in R^S_+$; and

iii) $f(y) = 1$ for some $y \in R^S_{++}$.

Let $Y = \{y \in R^S_+\colon\ f(y) \leq 1\}$.

For each i, let $u^i \in U^i$ with $u^i = u^j$ for each i, j. Let $\Omega = \Pi_i \Omega^i$, with generic element $\omega = (\omega^i)_i$. Since utility functions are fixed, Ω is the space of investor characteristics.

Given a game form (M, h), a reaction correspondence for investor i, for each i, is a correspondence $g^i\colon M \twoheadrightarrow M^i$. A strategy i-tuple

m is a quasi-equilibrium for an I-tuple of reaction correspondences $(g^i)_i$ if $m^i \in g^i(m)$ for each i; and if $(\theta,y) = h(m)$ then (θ,y) is a quasi-equilibrium outcome. For each i, define the correspondence $g^{*i}\colon\ \Omega^i \times M \twoheadrightarrow M^i$ by $m^{*i} \in g^{*i}(\omega^i,m)$ if

$v^i(\omega^i + \theta^i(-1,y)) \geq v^i(\omega^i + \theta'^i(-1,y'))$ for all

$(\theta',y') \in h(m^1,\ldots,m^{i-1},M^i, m^{i+1},\ldots,m^I)$, where $(\theta,y) = h(m^1,\ldots,m^{i-1}, m^{*i}, m^{i+1},\ldots,m^I)$. It follows that m is an equilibrium for an environment ω if and only if m is a quasi-equilibrium for the reaction correspondences $[g^{*i}(\omega^i,\cdot)]_i$.

For each i, given $\omega^i \in \Omega^i$, and subsets $A \subset \Theta \times Y$ and $B \subset \Theta \times Y$, say that A is as good as B for each $(\theta,y) \in A$ and each $(\theta', y') \in B$, $v^i(\omega^i + \theta^i(-1,y)) \geq v^i(\omega^i + \theta'^i(-1,y'))$. A game form (M,h) is nonmanipulable if for each i, each $\omega \in \Omega$, and each reaction correspondence g^i, the set of equilibrium outcomes for ω is as good as the set of quasi-equilibrium outcomes for $(g^{*1}(\omega^1,\cdot),\ldots,g^{*i-1}(\omega^{i-1},\cdot), g^i, g^{*i+1}(\omega^{i+1},\cdot),\ldots,g^{*I}(\omega^I,\cdot))$.

4.2 Remarks: A game form is nonmanipulable if there is no profile of endowments such that some investor, by changing his reaction correspondence, can influence the set of equilibrium outcomes to his advantage. In choosing an alternative reaction correspondence, an investor is not restricted to choose one which exhibits Nash behavior for some endowment. An alternative approach to strategic sophistication was introduced by Hurwicz [3], in a general context.

In the present context, suppose that the relation between environments and equilibrium outcomes is a function $c\colon\ \Omega \to \Theta \times Y$. The function c is said to be incentive-compatible if for each environment ω, no investor i prefers the outcome $c(\omega^1,\ldots,\omega^{i-1}, \omega'^i, \omega^{i+1},\ldots,\omega^I)$ to $c(\omega)$. That is, the i^{th} investor could not obtain a preferable outcome by behaving as though his endowment were ω'^i instead of ω^i. We now examine the relation between incentive-compatibility and nonmonipulability.

4.3 Definitions: A choice function is a function $c\colon\ \Omega \to \Theta \times Y$. A choice function c is efficient if for each $\omega \in \Omega$, $c(\omega)$ is Pareto optimal for $(\omega^i, u^i)_i$. A choice function c determines a game form (Ω, c). With respect to this game form, if for each $\omega \in \Omega$, ω is an equilibrium for ω, then c is incentive-compatible. If for each $\omega \in \Omega$ and each permutation $\sigma\colon\ \{1,\ldots,I\} \to \{1,\ldots,I\}$, if $(\theta,y) = c(\omega)$ then $[(\theta^{\sigma(i)})_i, y] = c[(\omega^{\sigma(i)})_i]$, then c is symmetric.

A game form (M, h) is symmetric if for each $\omega \in \Omega$, each permutation σ; and each equilibrium outcome (θ,y) for ω, $[(\theta^{\sigma(i)})_i, y]$ is an equilibrium outcome for $(\omega^{\sigma(i)})_i$.

4.4 Proposition: Suppose that (M,h) is an efficient nonmanipulable game form. Then there is an efficient incentive-compatible choice function c such that for each $\omega \in \Omega$, $c(\omega)$ is an equilibrium outcome for ω. If (M,h) is symmetric then c can be chosen to be symmetric.

Proof: Let $c\colon \Omega \to \Theta \times Y$ such that for each ω, $c(\omega)$ is an equilibrium outcome for ω. Since (M,h) is efficient, c is efficient. If (M,h) is symmetric choose c to be symmetric. We need to show that c is incentive-compatible. Suppose by way of contradiction that for some $\omega \in \Omega$ and some i, say $i = 1$, there is $\omega'^1 \in \Omega^1$ such that if $(\theta,y) = c(\omega)$ and $(\theta',y') = c(\omega'^1, \omega^2,\ldots,\omega^I)$ then $v^1(\omega^1 + \theta'^1(-1,y')) > v^i(\omega^1 + \theta(-1,y))$. Let m' be an equilibrium for $(\omega'^1,\omega^2,\ldots,\omega^I)$ with $h(m') = (\theta',y')$. Let (θ^*,y^*) be an equilibrium outcome for ω such that if (θ^o,y^o) is any other equilibrium outcome for ω then $v^1(\omega^1+\theta^{*1}(-1,y^*)) \geq v^1(\omega^1+\theta^{o1}(-1,y^o))$. Let m^* be an equilibrium for ω such that $h(m^*) = (\theta^*,y^*)$. If $v^1(\omega^1 + \theta^{*1}(-1,y^*)) \geq v^1(\omega^1+\theta'^1(-1,y'))$ then define the reaction correspondence $g^1\colon M \to M^1$ by

$$g^1[(m^i)_i] = \begin{cases} m^{*1} & \text{if } m^i = m^{*i} \text{ for all } i > 1; \text{ and} \\ M^1/\{m^1\} & \text{otherwise.} \end{cases}$$

Then (θ^*,y^*) is the unique quasi-equilibrium outcome for $(g^1, g^{*2}(\omega^2,\cdot),\ldots,g^{*I}(\omega^I,\cdot))$, contradicting the nonmanipulability of (M,h). If $v^1(\omega^1 + \theta'^1(-1,y')) > v^1(\omega^1 + \theta^{*1}(-1,y^*))$ then define the reaction correspondence $g^1\colon M \to M^1$ by

$$g^1[(m^i)_i] \begin{cases} m'^1 & \text{if } m^i = m'^i \text{ for each } i > 1; \text{ and} \\ M^1/\{m^1\} & \text{otherwise.} \end{cases}$$

Then (θ',y') is the unique quasi-equilibrium outcome for $(g^1, g^{*2}(\omega^2,\cdot),\ldots,g^{*I}(\omega^I,\cdot))$, which completes the contradiction.

4.5 Remarks: For classical exchange environments with variable preferences and endowments, Hurwicz [3] established the nonexistence of efficient incentive-compatible choice functions which give each agent a consumption bundle at least as desireable as his endowment. The case of fixed preferences has been treated by Postelthwaite [7]. In the present context, since we have excluded sidepayments, Pareto optimality implies that each investor is made at least as well off as at his endowment. The approach taken by Hurwicz and generalized by Ledyard [6] is to show that each agent can appropriate all of the gains from trade by, in effect, threatening to withdraw from trading otherwise. In the present problem such threats have no significance. Each investor would prefer to be the only investor since Pareto optimality would then require the activity to be his most preferred.

The present problem has some aspects of the social choice problem studied by Gibbard [1] and Satterthwaite [9]. In [1], a game form is said to be nonmanipulable (Gibbard uses the term "straightforward") if in each environment each player has a dominant strategy. A straightforward game form yields an incentive-compatible choice function and vice-versa. It is remarked in 4.15 below that our concept of nonmanipulability is not quite as closely related to incentive-compatibility. The major difference between the investment-production choice problem and the social choice problem considered in [1] and [9] is that the "unlimited domain" assumption of the latter is far from satisfied in the present problem. This influences both the analysis and the results.

One choice function which is efficient and incentive-compatible is a form of serial dictatorship. Investor 1, say, is offered his most preferred activity. If he prefers not to invest in any activity, investor 2 is then given his choice, and so on. We now define serial dictatorship formally.

<u>4.6 Definition</u>: For each i, let $\hat{y}^i$: $\Omega^i \to Y$ such that for each $\omega^i \in \Omega^i$, $\hat{y}^i(\omega^i)$ maximizes $\hat{v}^i(\omega^i,y)$ subject to $y \in Y$. A choice function c is <u>serially dictatorial</u> if, renumbering investors if necessary, for each $\omega \in \Omega$, if $(\theta,y) = c(\omega)$ then $\theta^i = \hat{\theta}^i(y)$ for each i and

$$
y = \begin{cases}
\hat{y}^1(\omega^1) \quad \text{if} \quad \hat{\theta}^1(\omega^1,\hat{y}^1(\omega^1)) > 0 \\
\vdots \\
\hat{y}^i(\omega^i) \quad \text{if} \quad \hat{\theta}^i(\omega^i,\hat{y}^i(\omega^i)) > 0 \quad \text{and} \\
\qquad\qquad \hat{\theta}^j(\omega^j,\hat{y}^j(\omega^j)) = 0 \quad \text{for all} \quad j < i \\
\vdots \\
\hat{y}^I(\omega^I) \quad \text{if} \quad \hat{\theta}^I(\omega^I, \hat{y}^I(\omega^I)) > 0 \quad \text{and} \\
\qquad\qquad \hat{\theta}^j(\omega^j, \hat{y}^j(\omega^j)) = 0 \quad \text{for all} \quad j < I.
\end{cases}
$$

<u>4.7 Remarks</u>: Proposition 4.8 below states that if there are two investors and more than two states, serial dictatorship is the only efficient incentive-compatible choice function. The proof is rather long, and is divided into seven steps. The theme of the proof is that incentive-compatibility, together with efficiency, requires more regularity of c than can be satisfied by any other choice function. This regularity is developed in steps 1-3, and used in steps 4 and 5 to show that every choice made by c must be maximal for at least one of the two agents, irrespective of the number of states. The main tool in developing and exploiting this regularity is a kind of "envelope theorem" whose usefulness in the analysis of incentives has been emphasized by Mirlees [8], and many subsequent authors. Steps 6 and 7 conclude the proof by showing that if there are more than two states, the order of dictatorship must be invariant on Ω.

<u>4.8 Proposition</u>: Let $I = 2$ and suppose that c is an efficient incentive-compatible choice function. If $S \geq 3$ then c is serially dictatorial.

<u>Proof</u>: For each $i = 1,2$, define $c^i\colon\ \Omega \to R_+^{S+1}$ by $c^i(\omega^i) = \theta^i(-1,y)$, where $(\theta, y) = c(\omega)$, and define $w^i\colon\ \Omega \to R$ by $w^i(\omega) = v^i(\omega^i + c^i(\omega))$.

<u>Step 1</u>: For each $\omega^2 \varepsilon \Omega^2$, the function $w^1(\cdot,\omega^2)$ is continuous.

Let $\omega^{o1} \varepsilon \Omega^1$ and let $\{\omega_n^1\}_{n=1}^{\infty}$ be a sequence in Ω^1 converging to ω^{o1}. Since c is incentive-compatible,

$$v^1(\omega^{o1} + c^1(\omega_n^1, \omega^2)) - v^1(\omega_n^1 + c^1(\omega_n^1,\omega^2)) \leq$$

$$v^1(\omega^{o1} + c^1(\omega^{o1},\omega^2)) - v^1(\omega_n^1 + c^1(\omega_n^1,\omega^2)) =$$

$$w^1(\omega^{o1},\omega^2) - w^1(\omega_n^1,\omega^2) \leq v^1(\omega^{o1} + c^1(\omega^{o1},\omega^2)) -$$

$$v^1(\omega_n^1 + c^1(\omega^{o1},\omega^2))$$

for each n. Since v^1 is continuous, the last difference converges to 0. Since c is efficient and Y is compact, the set $\{\omega_n^1 + c^1(\omega_n^1,\omega^2)\colon\ n=1,2,\ldots,\} \cup \{\omega^{o1} + c^1(\omega_n^1,\omega^2)\colon\ n = 1,2,\ldots\}$ has compact closure in R_{++}^{S+1}. Hence, since v^1 is continuous, the first difference converges to 0, so $w^1(\omega_n^1,\omega^2) \to w^1(\omega^{o1},\omega^2)$.

<u>Step 2</u>: Let $\omega^o \varepsilon \Omega$ with $w^1(\omega^o) > v^1(\omega^{o1})$. Then $c(\cdot,\omega^{o2})$ is continuous on a neighborhood of ω^{o1}.

Since c is efficient, $c(\omega) = (\hat{\theta}^1(\omega^1,c_y(\omega)), \hat{\theta}^2(\omega^2,c_y(\omega)), c_y(\omega))$ for each $\omega \varepsilon \Omega$, where c_y is the activity coordinate of c. Since $\hat{\theta}^i$ is continuous for each i, it suffices to prove that $c_y(\cdot,\omega^{o2})$ is continuous near ω^{o1}. By step 1, $w^1(\cdot,\omega^{o2})$ is continuous so $w^1(\omega^1,\omega^{o2}) > v^1(\omega^1)$ for all ω^1 sufficiently near ω^{o1}. Hence $\hat{\theta}^1(\omega^1,c_y(\omega^1,\omega^{o2})) > 0$ for ω^1 near ω^{o1}. Lemma 2.5(iii) implies that for ω^1 near ω^{o1}, $\hat{v}^1(\omega^1,\cdot)$ is strictly quasi-concave on the set $\{y \varepsilon Y\colon\ \hat{v}^1(\omega^1,y) \geq w^1(\omega^1,\omega^{o2})\}$. Since c is efficient, it follows that for ω^1 near ω^{o1}, $c_y(\omega^1,\omega^{o2})$ is the unique maximizer of $\hat{v}^2(\omega^{o2},y)$ subject to $y \in Y$ and $\hat{v}^1(\omega^1,y) \geq w^1(\omega^1,\omega^{o2})$. The continuity of $c_y(\cdot,\omega^{o2})$ near ω^{o1} now follows from Step 1 and the Maximum Theorem.

<u>Step 3</u>: Let $\omega^o \in \Omega$ such that $v^1(\omega^{o1}) < w^1(\omega^o) < \max\{\hat{v}^1(\omega^{o1},y): y \in Y\}$ and $v^2(\omega^{o2}) < w^2(\omega^o) < \max\{\hat{v}^2(\omega^{o2},y): y \in Y\}$. Then there is an open neighborhood N of ω^{o1} in Ω^1 such that the function $g\colon N \to Y$, defined by letting $g(\omega^1)$ maximize $\hat{v}^2(\omega^{o2},y)$ subject to $y \in Y$ and $\hat{v}^1(\omega^1,y) \geq w^1(\omega^o)$ for each $\omega^1 \in N$, is C^1.

Define the (possibly empty valued) correspondence $g^o\colon \Omega^1 \twoheadrightarrow Y$ by $g^o(\omega^1) = \{y\colon y$ maximizes $\hat{v}^2(\omega^{o2},y^1)$ subject to $y^1 \in Y$ and $\hat{v}^1(\omega^1,y^1) \geq w^1(\omega^o)$. Using Step 1 and the Maximum Theorem as they were used to prove Step 2, one can show the existence of a neighborhood N^o of ω^{o1} in Ω^1 such that $g^o|N^o$ is a continuous function. Since c is efficient, $c_y(\omega^o)= g^o(\omega^{o1})$. Since $g^o|N^o$ is continuous, there is an open neighborhood N of ω^{o1} in N^o such that for each $\omega^1 \in N$, $v^1(\omega^1) < \hat{v}^1(\omega^1, g^o(\omega^1)) = w^1(\omega^o) < \max\{\hat{v}^1(\omega^1,y)\colon y \in Y\}$ and $v^2(\omega^{o2}) < \hat{v}^2(\omega^{o2}, g^o(\omega^1)) < \max\{\hat{v}^2(\omega^{o2},y)\colon y \in Y\}$. Let $g=g^o|N$. Then $\hat{\theta}^1(\omega^1, g(\omega^1)) > 0$ and $\hat{\theta}^2(\omega^{o2}, g(\omega^1)) > 0$ for each $\omega^1 \in N$, so Lemma 2.5 (ii) implies that for each $\omega^1 \in N$, $\hat{v}^1$ is C^2 at $(\omega^1,g(\omega^1))$ and $\hat{v}^2(\omega^{o2},\cdot)$ is C^2 at $g(\omega^1)$. For any $\omega^1 \in N$, let $y = g(\omega^1)$ and let $r = w^1(\omega^o)$. Then there exist Lagrange multipliers $\lambda > 0$ and $\gamma > 0$ such that

$$D_y \hat{v}^2(\omega^{o2},y) + \lambda D_y \hat{v}^1(\omega^1,y) - \gamma Df(y) = 0$$

$$\hat{v}^1(\omega^1,y) - r = 0$$

$$f(y) - 1 = 0,$$

where D_y denotes the derivative with respect to y. Using the quasi-concavity of $\hat{v}^2(\omega^{o2},\cdot)$ and $\hat{v}^1(\omega^1,\cdot)$ and the fact that $D^2f(y)$ is positive definite, implicit differentiation shows that g is C^1 on N.

Step 4: Let $\omega^o \in \Omega$ with $w^i(\omega^o) < \max\{\hat{v}^i(\omega^{oi},y): y \in Y\}$ for each i, and define the path $\rho:[0,\infty) \to \Omega^1$ by $\rho(t) = \omega^{o1}+(1-e^{-t})c^1(\omega^o)$. Then $c_y(\rho(t), \omega^{o2}) = c_y(\omega^o)$ and $w^i(\rho(t), \omega^{o2}) = w^i(\omega^o)$ for each $t \geq 0$ and each i.

Consider the differential equation

$$(*) \qquad \sigma(0) = \omega^{o1}, \text{ and } \dot{\sigma}(t) = c^1(\sigma(t), \omega^{o2}).$$

By step 2 and the Peano Existence Theorem there is some $\varepsilon > 0$ such that a C^1 solution to (*) exists on $[0,\varepsilon)$, so let $\sigma: [0,\varepsilon) \to \Omega^1$ be such a solution. Let $0 \leq t < t' < \varepsilon$. Since c is incentive-compatible,

$$[v^1(\sigma(t') + c^1(\sigma(t),\omega^{o2})) - v^1(\sigma(t) + c^1(\sigma(t),\omega^{o2}))](t'-t)^{-1} \leq$$

$$\leq [v^1(\sigma(t') + c^1(\sigma(t'),\omega^{o2})) - v^1(\sigma(t) + c^1(\sigma(t),\omega^{o2}))](t'-t)^{-1} \leq$$

$$\leq [v^1(\sigma(t') + c^1(\sigma(t'),\omega^{o2})) - v^1(\sigma(t) + c^1(\sigma(t'),\omega^{o2}))](t'-t)^{-1}.$$

The difference quotient in the middle is

$$[w^1(\sigma(t'), \omega^{o2}) - w^1(\sigma(t), \omega^{o2})](t'-t)^{-1},$$

so using the continuity of $c^1(\sigma(\cdot),\omega^{o2})$ at t (Step 2), we have that $w^1(\sigma(\cdot),\omega^{o2})$ is C^1 at t and $\frac{d}{dt} w^1(\sigma(t),\omega^{o2}) = Dv^1(\sigma(t)+c^1(\sigma(t),\omega^{o2}))\dot{\sigma}(t)$. But $\dot{\sigma}(t) = c^1(\sigma(t),\omega^{o2})$, so since c is efficient,

$\frac{d}{dt} w^1(\sigma(t),\omega^{o2}) = \hat{\theta}^1(\sigma(t),c_y(\sigma(t),\omega^{o2}))[Dv^1(\sigma(t) + c^1(\sigma(t),\omega^{o2}))](-1,c_y(\sigma(t),\omega^{o2}))=$ $= 0$. Thus $w^1(\sigma(\cdot),\omega^{o2})$ is constant on $[0,\varepsilon)$.

Since c is efficient $w^i(\omega^o) > v^i(\omega^{oi})$ for each i, so ω^o satisfies the hypothesis of step 3. Let g: $N \to Y$ be the function given by Step 3, and let $0 < \delta < \varepsilon$ such that $\sigma(t) \in N$ for all $0 \leq t < \delta$. Since c is efficient, σ is a solution to the differential equation

(**) $\sigma(0) = \omega^{o1}$, and $\dot{\sigma}(t) = \hat{\theta}^1(\sigma(t), g(\sigma(t)))(-1,g(\sigma(t)))$ for all $t \in [0,\delta)$.

By Step 3, Lemma 2.5 (ii), and the Picard Uniqueness Theorem, the solution to (**) is unique. It follows that $\sigma(t) = \omega^o + (1-e^{-t})c^1(\omega^o) = \rho(t)$ for each $0 \leq t < \delta$. Since $w^1(\cdot,\omega^{o2})$ is continuous by Step 1, $w^1(\rho(\delta),\omega^{o2}) = w^1(\omega^o)$. Since c is efficient $c_y(\rho(\cdot),\omega^{o2}) = g(\rho(\cdot)) \equiv c_y(\omega^o)$ on $[0,\delta)$. By Step 2, $c_y(\cdot,\omega^{o2})$ is continuous at $\rho(\delta)$, so $c_y(\rho(\delta),\omega^{o2}) = c_y(\omega^o)$, and thus $w^2(\rho(\delta),\omega^{o2}) = w^2(\omega^o)$. Therefore the endowment $(\rho(\delta),\omega^{o2})$ satisfies the hypothesis of Step 3, so by the above argument, there is some $\delta' > \delta$ such that ρ is the unique solution to (*) on $[0,\delta')$. Thus ρ is the unique solution to (*) on an open and closed subset of R_+ so ρ is the unique solution on R_+. Also, $c_y(\rho(\cdot),\omega^{o2})$ and $w^i(\rho(\cdot),\omega^{o2})$, $i=1,2$, are constant on an open and closed subset of R_+, which completes the proof of step 4.

<u>Step 5</u>: There is no $\omega^o \in \Omega$ with $w^i(\omega^o) < \max\{\hat{v}^i(\omega^{o1},y)\colon y \in Y\}$ for each i.

Suppose, by way of contradiction, that such an ω^o exists. Then ω^o satisfies the hypothesis of Step 4, so let ρ be the path given by Step 4 and let $\bar{\omega}^1 = \omega^{o1} + c^1(\omega^o) = \lim_{t\to\infty} \rho(t)$. Since $w^1(\rho(t),\omega^{o2}) = w^1(\omega^o)$ for each t, Step 1 implies that $w^1(\bar{\omega}^1,\omega^{o2}) = w^1(\omega^o) = v^1(\bar{\omega}^1)$. Since c is efficient, this implies that $c_y(\bar{\omega}^1,\omega^{o2})$ maximizes $\hat{v}^2(\omega^{o2},y)$ subject to $y \in Y$. For each $0 \leq t < \infty$, let $\rho^2(t) = \omega^{o2} + (1-e^{-t})c^2(\omega^o)$, and let $\bar{\omega}^2 = \omega^{o2} + c^2(\omega^o) = \lim_{t\to\infty} \rho^2(t)$. Let $Y^* = \{y \in Y\colon Dv^1(\bar{\omega}^1)(-1,y) \leq 0\}$, and for each t, let $\tilde{w}^2(t) = \max\{\hat{v}^2(\rho^2(t),y)\colon y \in Y\}$. Since c is efficient, for each t, if $c_y(\bar{\omega}^1,\rho^2(t)) \in Y^*$ then $w^2(\bar{\omega}^1,\rho(t)) = \tilde{w}^2(t)$. Also, $c_y(\bar{\omega}^1,\rho^2(0)) \in Y^*$. Since $\tilde{w}^2(t) - \sup\{\hat{v}^2(\rho^2(t), y)\colon y \in Y/Y^*\} > 0$ for all t, $c_y(\bar{\omega}^1,\rho^2(\cdot))$ cannot leave Y^* without making $w^2(\bar{\omega}^1,\rho(\cdot))$ discontinuous. By Step 1, applied to investor 2, $w^2(\bar{\omega}^1,\rho(t)) = \tilde{w}^2(t)$ for all t and $w^2(\bar{\omega}^1,\bar{\omega}^2) = \max\{\hat{v}^2(\bar{\omega}^2,y)\colon y \in Y\}$. However, the property assumed for ω^o is symmetric with respect to the investors, so $w^1(\bar{\omega}^1,\bar{\omega}^2) = \max\{\hat{v}^1(\bar{\omega}^1,y)\colon y \in Y\}$. This contradicts the Pareto optimality of $c(\omega^o)$, and completes Step 5.

Step 6: Let $\omega^o \in \Omega$ with $w^1(\omega^o) = \max\{\hat{v}^1(\omega^{o1},y): y \in Y\}$ and $w^2(\omega^o) < \max\{\hat{v}^2(\omega^{o2},y): y \in Y\}$. Let $K = \{\omega^2 \in \Omega^2: w^1(\omega^{o1},\omega^2) = w^1(\omega^o)\}$. If $S \geq 3$ then $K = \Omega^2$.

Let $P^2 = \{\omega^2 \in \Omega^2: Dv^2(\omega^2)(-1,y) > 0$ for some $y \in Y\}$. Since v^2 is C^2, P^2 is open. We now show that P^2 is connected. Let $\omega^2, \omega'^2 \in P^2$ and let $y, y' \in Y$ with $Dv^2(\omega^2)(-1,y) > 0$, $Dv^2(\omega'^2)(-1,y') > 0$, and $\omega_0^2 \geq \omega'^2_0$. Let $\bar{\omega}^2$ be defined by $\bar{\omega}_0^2 = \omega_0^2$ and $\bar{\omega}_s^2 = \min\{\omega_s^2, \omega'^2_s\}$ for each s. Then, using the fact that $D^2u^2(r) < 0$ for all $r > 0$, every ω''^2 in the line segment $[\omega^2, \bar{\omega}^2]$ satisfies $Dv^2(\omega''^2)(-1,y) > 0$, and every ω''^2 in the line segment $[\bar{\omega}^2, \omega'^2]$ satisfies $Dv^2(\omega''^2)(-1,y') > 0$, so P^2 is connected. Now let $y^o = c_y(\omega^o)$ and define the C^1 function $g: \{(\omega^2,\theta^2,\lambda) \in P^2 \times R \times R: \omega^2 + \theta^2(-1,y^o) \in R_{++}^{S+1}\} \to R^{S+1}$ by $g(\omega^2,\theta^2,\lambda) = Dv^2(\omega^2 + \theta^2(-1,y^o)) - \lambda(1,Df(y^o))$. Since $D^2v^2(x^2)$ is negative definite for all $x^2 \in R_{++}^{S+1}$, g is a submersion, so by [5,Th2.8], $g^{-1}(0)$ is a 2-dimensional submanifold of the domain of g. Hence the set $P^o = \{\omega^2 \in P^2:$ for some $(\theta^2,\lambda) \in R^2$, $g(\omega^2,\theta,\lambda) = 0\}$ is a 2-dimensional submanifold of the S+1-dimensional manifold P^2.

Let $\omega^2 \in \Omega^2$. If $\omega^2 \notin P^2$, $\max\{v^2(\omega^2,y): y \in Y\} = v^2(\omega^2)$, so $\omega^2 \in K$ by the efficiency of c. If $\omega^2 \in P^2$, then since $\omega^{o2} \in P^2$ and P^2 is open and connected, there is a C^∞ path $\sigma^o: [0, 1] \to P^2$ with $\sigma^o(0) = \omega^{o2}$ and $\sigma^o(1) = \omega^2$. Let

δ: $(0, 1) \to \text{int } R_+$ such that $\lim_{t\to 0} \delta(t) = \lim_{t\to 1} \delta(t) = 0$ and consider the Whitney open set of C^∞ functions $\sigma: (0,1) \to P^2$ with $||\sigma^o(t) - \sigma(t)|| < \delta(t)$ for all t. By the Transversal Density Theorem, [5, Cor. 4.12], there is some σ in this set which intersects P^o transversally. But P^o is 2-dimensional, and since $S \geq 3$, P^2 is at least 4-dimensional. Hence $\sigma(t) \notin P^o$ for all $t \in (0, 1)$. If $\omega^2 \in P^o$, then $\max \{\hat{v}^2(\omega^2,y): y \in Y\} = \hat{v}^2(\omega^2,y^o)$, so $\omega^2 \in K$ by the efficiency of c. If $\omega^2 \notin P^o$, extending σ to its limits at 0 and 1 gives a continuous path from ω^{o2} to ω^2 such that for each $0 \leq t \leq 1$, $\hat{v}^2(\sigma(t), y^o) < \max \{\hat{v}^2(\sigma(t),y): y \in Y\}$. Hence by Step 5 and step 1, applied to investor 2, $\sigma(t) \in K$ for all $0 \leq t \leq 1$, so $\omega^2 \in K$. Hence $K = \Omega^2$.

<u>Step 7</u>: If $S \geq 3$, c is serially dictatorial.

Let $L^1 = \{\omega \in \Omega: w^1(\omega) = \max \{\hat{v}^1(\omega^1,y): y \in Y\}\}$ and let $L^2 = \{\omega \in \Omega: w^2(\omega) = \max \{\hat{v}^2(\omega^2,y): y \in Y\}\}$. By Step 5, $\Omega = L^1 \cup L^2$. Suppose there is some $\omega^o \in L^1/L^2$. Let $P^1 = \{\omega^1:$ for some $y \in Y$, $Dv^1(\omega^1)(-1,y) > 0\}$. As in the proof of step 6, P^1 is open and connected. Since $\omega^o \notin L^2$, the efficiency of c implies that $\omega^{o1} \in P^1$. Step 6 implies that $\{\omega^{o1}\} \times \Omega^2 \subset L^1$. Let $A = \{\omega^1 \in P^1: \{\omega^1\} \times \Omega^2 \subset L^1\}$. By Step 1, A is closed in P^1, and by Steps 1 and 6, A is open in P^1. Therefore $A = P^1$ so $\Omega^1 = L^1$, which completes the proof.

4.9 Remarks: If $I > 2$ or $S = 2$, the result no longer holds. First suppose that $S = 2$. In this case the technologically efficient set of activities is a one-dimensional curve such as that depicted in Figure 1. Consider the mechanism in which each agent is asked to announce his most preferred activity and level of investment in that activity. If the announced θ^i's are both positive or both zero, the activity which is furthest northwest is chosen. Otherwise the activity announced by the positive investor is chosen. Each investor is then given his most preferred level of investment in the chosen activity, which may of course differ from his announced θ^i if his announced activity is not chosen. Using the quasi-concavity of the functions $v^i(\omega^i,\cdot)$, the incentive-compatibility of the resulting choice function is easily established, irrespective of the number of investors. Alternatively, the "median" announced activity could be chosen. If $I = 2$, the efficient incentive-compatible choice functions can be completely characterized as follows.

4.10 Proposition: Let $I = S = 2$, and let c be a choice function and let c_y denote the activity coordinate of c. Then c is efficient and incentive-compatible if and only if for each $\omega \in \Omega$, if $(\theta,y) = c(\omega)$ then $\theta^i = \hat{\theta}^i(\omega^i,y)$ for each i, and either

i) c is serially dictatorial; or

ii) for each ω

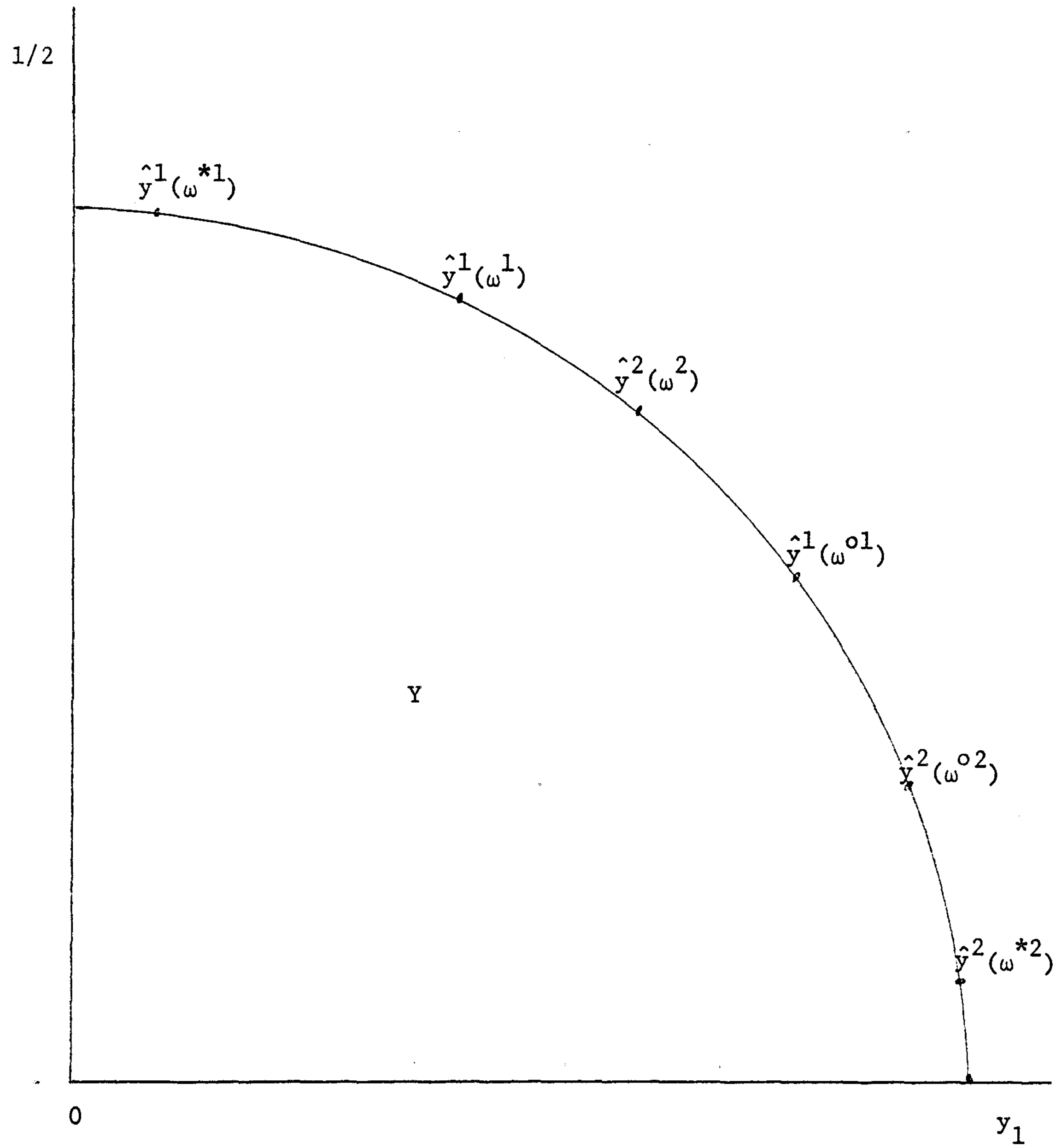
1/2
$\hat{y}^1(\omega^{*1})$
$\hat{y}^1(\omega^1)$
$\hat{y}^2(\omega^2)$
$\hat{y}^1(\omega^{o1})$
Y
$\hat{y}^2(\omega^{o2})$
$\hat{y}^2(\omega^{*2})$
0
y_1

Figure 1

$$(*) \quad c_y(\omega) = \begin{cases} \hat{y}^1(\omega^1) & \text{if } \hat{\theta}^1(\omega^1, \hat{y}^1(\omega^1)) > 0 \text{ and either } \hat{\theta}^2(\omega^2,\hat{y}^2(\omega^2)) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{or} \\ & \hat{y}_s^1(\omega^1) \leq \hat{y}_s^2(\omega^2); \text{ and} \\ \hat{y}^2(\omega^2) & \text{if } \hat{\theta}^2(\omega^2, \hat{y}^2(\omega^2)) > 0 \text{ and either } \hat{\theta}^1(\omega^1,\hat{y}^1(\omega^1)) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{or} \\ & \hat{y}_s^2(\omega^2) \leq \hat{y}_s^1(\omega^1), \end{cases}$$

for $s = 1$; or

iii) for each ω, $c_y(\omega)$ satisfies (*) for $s = 2$.

<u>Proof</u>: Sufficiency is straightforward, so we will only prove necessity. As a preliminary step, let $Y^o = \{y \in Y: y_s > 0 \text{ for } s = 1,2, \text{ and } f(y) = 1\}$. Then 4.1 (ii) implies that for each $\omega \in \Omega$, if (θ, y) is Pareto optimal for (ω^i, u^i) and $\theta^i > 0$ for some i then $y \in Y^o$. For each i, let $P^i = \{\omega^i \in \Omega^i: \hat{\theta}^i(\omega^i,y) > 0$ for some $y \in Y^o\}$. Note that if $y \in Y^o$ and $\omega^i \in P^i$ with $\hat{y}^i(\omega^i) = y$ then Lemma 2.5 (iv) implies that $(Du^i[\omega_s^i + \hat{\theta}^i(\omega^i,y)y_s])_{s=1,2}$ is proportional to $Df(y)$. Using this fact, given $\omega^i, \omega'^i \in P^i$ with $\hat{y}^i(\omega^i) = y$ and $\hat{y}^i(\omega'^i) = y'$, it is straightforward to construct a continuous path ρ: $[0, 1] \to P^i$ with $\rho(0) = \omega^i$, $\rho(1) = \omega'^i$, and for each $t \in [0, 1]$, $|y-\hat{y}^i(\rho(t))| \leq |y-y'|$ and $|y'-\hat{y}^i(\rho(t))| \leq |y'-y|$. Such a path will be called a direct path from ω^i to ω'^i.

Since c is efficient, for each $\omega \in \Omega$, if $(\theta,y) = c(\omega)$ then $\theta^i = \hat{\theta}^i(\omega^i, y)$ for each i. We need only be concerned with endowments in $P^1 \times P^2$, since efficiency implies that if $\omega^1 \in P^1$ and $\omega^2 \notin P^2$ then $c_y(\omega) = \hat{y}^1(\omega^1)$, and if $\omega^1 \notin P^1$ and $\omega^2 \in P^2$ then $c_y(\omega) = \hat{y}^2(\omega^2)$, and if $\omega^1 \notin P^1$ and $\omega^2 \notin P^2$ then $c_y(\omega)$ is irrelevant since $\hat{\theta}^1(\omega^1,y) = \hat{\theta}^2(\omega^2,y) = 0$ for all $y \in Y$.

Let $\omega^o \varepsilon P^1 \times P^2$ with $\hat{y}_1^1(\omega^{o1}) < \hat{y}_1^2(\omega^{o2})$. By step 5 of 4.8, $c_y(\omega^o) = \hat{y}^1(\omega^{o1})$ for some i, so suppose that $c_y(\omega^o) = \hat{y}^1(\omega^{o1})$. We will show that for each $\omega \varepsilon P^1 \times P^2$ with $\hat{y}_1^1(\omega^1) < \hat{y}_1^2(\omega^2)$, $c_y(\omega) = \hat{y}^1(\omega^1)$. Let $\omega^{*1} \varepsilon P^1$ with $\hat{y}_1^1(\omega^{*1}) < \min\{y_1^1(\omega^{o1}), \hat{y}_1^1(\omega^1)\}$ and let $\omega^{*2} \varepsilon P^2$ with $\hat{y}_1^2(\omega^{*2}) > \min\{\hat{y}_1^2(\omega^{o2}), \hat{y}_1^2(\omega^2)\}$(see Figure 1). Let

$$\left.\begin{array}{l}\rho^{*1}\\ \rho^{1}\\ \rho^{*2}\\ \rho^{2}\end{array}\right. \quad \text{be a direct path from} \quad \begin{array}{l}\omega^{o1} \text{ to } \omega^{*1}\\ \omega^{*1} \text{ to } \omega^{1}\\ \omega^{o2} \text{ to } \omega^{*2}\\ \omega^{*2} \text{ to } \omega^{2}\ .\end{array}$$

Using ρ^{*1}, it follows from Step 5 and Step 1 of 4.8 that $c_y(\omega^{*1}, \omega^{o2}) = \hat{y}^1(\omega^{*1})$; using ρ^{*2} it follows that $c_y(\omega^{*1}, \omega^{*2}) = \hat{y}^1(\omega^{*1})$; using ρ^1 it follows that $c_y(\omega^1, \omega^{*2}) = \hat{y}^1(\omega^1)$; and using ρ^2, it follows that $c_y(\omega) = \hat{y}^1(\omega^1)$, which was to be shown. Maintaining the assumption that $c_y(\omega^o) = \hat{y}^1(\omega^{o1})$, we now consider two cases.

a) There is some $\omega^{\#} \varepsilon P^1 \times P^2$ with $\hat{y}_1^1(\omega^{\#1}) > \hat{y}_1^2(\omega^{\#2})$ and $c_y(\omega^{\#}) = \hat{y}^1(\omega^{\#1})$. In this case one shows as above that $c_y(\omega) = \hat{y}^1(\omega)$ for all $\omega \varepsilon P^1 \times P^2$ with $\hat{y}_1^1(\omega^1) > \hat{y}_1^2(\omega^2)$, which implies that c is serially dictatorial.

b) There is some $\omega^{\#} \varepsilon P^1 \times P^2$ with $\hat{y}_1^1(\omega^{\#1}) > \hat{y}_1^2(\omega^{\#2})$ and $c_y(\omega^{\#}) = \hat{y}^2(\omega^{\#2})$. In this case one shows as above that $c_y(\omega) = \hat{y}^2(\omega)$ for all $\omega \varepsilon P^1 \times P^2$ with $\hat{y}_1^1(\omega^1) > \hat{y}_1^2(\omega^2)$, which implies that c satisfies 4.10 (ii).

The proof of necessity is completed by permuting the indices for investors and states in the previous paragraph.

4.11 Remarks: An essential feature of serial dictatorship mechanisms is that the order of dictatorship, or hierarchy, is invariant. For more than two investors, there exist efficient, incentive-compatible choice functions with changing hierarchies. For example, if $I = 3$, consider the mechanism in which investor 1 is asked to announce his characteristic, and then serial dictatorship is applied with the hierarchy, in descending order, (2, 3, 1) if $Dv^1(\omega^1)(-1,y) \leq 0$ for all $y \in Y$, and (3, 2, 1) otherwise. Thus whether investor 2 or 3 is first dictator depends on whether investor 1 would invest in any feasibile activity. Investor 1 is required to invest in accordance with his announced characteristic, so he is unable to take advantage of his influence. This mechanism, like serial dictatorship and unlike the mechanisms defined in 4.9, is asymmetric in that the permutation of agent superscripts in the environment can change the chosen activity. Theorem 4.12 derives the non-existence of symmetric, efficient incentive-compatible choice functions as an easy consequence of Proposition 4.8 if $S \geq 3$.

4.12 Theorem: If $S \geq 3$, there does not exist an efficient, symmetric incentive-compatible choice function.

Proof: Let c be an efficient incentive-compatible choice function, and for each $i > 2$, let $\omega^i \in \Omega^i$ such that $Dv^i(\omega^i)(-1,y) \leq 0$

for all $y \varepsilon Y$. Then $c^i(\cdot, \cdot, \omega^3, \ldots, \omega^I) \equiv 0$ for each $i > 2$, so $c(\cdot, \cdot, \omega^3, \ldots, \omega^I)$ is an efficient incentive-compatible choice function for the environment consisting of the first two investors alone. If $S \geq 3$, Proposition 4.8 implies that c cannot be symmetric.

4.13 Remarks: Proposition 4.4 leads immediatley to the following Corollary for game forms.

4.14 Corollary: If $S \geq 3$, there does not exist an efficient, symmetric nonmanipulable game form.

4.15 Remarks: Suppose that $S = I = 2$. Then the choice function described in 4.10 (i-ii) are efficient, symmetric and incentive compatible, but the game forms they determine are not efficient. For example, suppose that c satisfies 4.10 (i), so that (Ω, c) can be thought of as a game form in which each agent announces an investment, which he receives, and an activity; and the activity further northwest is chosen. Suppose that both agents announce the same activity y, but y is northwest of each of their most preferred activities. Since neither agent, by changing his announcement, can move the chosen activity southeast, this is an equilibrium whose outcome is not Pareto optimal. However, by modifying this game form slightly, one obtains an efficient, symmetric nonmanipulable game form whose equilibrium outcomes are given by the choice function c.

4.16 Proposition: If $S = 2$, there exists an efficient, symmetric nonmanipulable game form.

Proof: For each i, let $M^i = R_+ \times Y$ and define the outcome function h as follows. For each $m = (\theta^i, y^i)_i \in M$,

a) if $\theta^i = 0$ for all i, then $h(m) = (\theta^1,\ldots,\theta^I; 0)$;

b) if $y_1^i = y_1^j$ for some $i \neq j$ with $\theta^i > 0$ and $\theta^j > 0$ then $h(m) = (\theta^1,\ldots,\theta^I; 0)$; and

c) otherwise, $h(m) = (\theta^1,\ldots,\theta^I; y^j)$ where $\theta^j > 0$ and $y_1^j < y_1^i$ for all $i \neq j$ with $\theta^i > 0$.

It is easily verified that (M,h) is efficient, symmetric, and nonmanipulable.

4.17 Remarks: The game form constructed in the proof of 4.16 avoids the source of inefficiency mentioned above by forcing different investors to announce different activities. This implies that the investor whose announced activity is chosen can unilaterally move the activity in either direction.

Nonmanipulability is an extremely desireable property, but it may be much stronger than is necessary to ensure Pareto optimal equilibrium outcomes. For example, if the number of investors is large, the knowledge required to manipulate a game form may not be available to any investor. It would be useful to investigate concepts of nonmanipulability which presume that investors have incomplete knowledge of each other's characteristics.

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