HOW THE SET OF ALLOCATIONS ATTAINABLE THROUGH NASH-EQUILIBRIA DEPENDS ON INITIAL ENDOWMENTS

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How the Set of Allocations Attainable Through Nash-Equilibria Depends on Initial Endowments*

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In his paper, "On Allocations Attainable Through Nash-Equilibria", Hurwicz studies outcome functions[1] whose Nash-equilibrium allocations are Pareto-optimal and individually rational (i.e. making all agents at least as well off as they were at the initial allocation ω). Provided that the class of environments over which these functions are defined is wide enough ("coverage" assumption C) he establishes the following theorems:

Theorem 1: Under a continuity condition of the performance correspondence, every Walrasian allocation with respect to ω is a Nash-equilibrium allocation.

Theorem 2: Under a convexity property of the outcome function in each individual's strategy, every Nash-equilibrium allocation is a Walrasian allocation with respect to ω.

While it might sometimes be desirable to impose on a game the conditions that all players experience a gain in utility, because it ensures or makes it more likely that they would willingly participate in it, arguments can also be advanced against so favoring the initial distribution of resources, and other criteria can be considered: the performance correspondence, required in [2] to be utility-increasing, is here assumed to yield "fair" allocations; several definitions of

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fairness have been proposed in the literature, and we examine them in turn. It is shown that for all of them, exact equivalents of Theorems 1 and 2 hold, where "individually rational" of the above paragraph where they are stated, is replaced by "fair", and "with respect to ω" is replaced by "with respect to I", I being the equal bundle allocation attributing to all n agents $1/n$ of the aggregate endowment of the economy. We refer to those theorems as Theorems 1' and 2'.

In the second section, we examine the role played by the coverage assumption: Hurwicz's theorems apply only to performance functions defined on a "big enough" class of environments including linear economies, (<u>assumption C</u>). Given the nature of linear preferences and the assumption often made in current literature that preference maps should exhibit a non-zero degree of curvature, it is of interest to study how the theorems would be affected if this assumption were relaxed. It is clear, however, that some form of coverage is necessary and we will assume that preferences can be arbitrary flat while maintaining a strictly positive degree of curvature (<u>assumption C'</u>)

It turns out that Theorem 1 can then be replaced by an approximation theorem (Theorem 1"), the proof of which does not require, contrarily to [2], that the Nash-equilibrium correspondence be postulated to be upper-hemi-continuous. The concept of ϵ-Nash-equilibrium", involving ϵ-approximate maximization of each agent's indirect utility against the other players' strategies, taken as given, is introduced in that section and appears in the statement of Theorem 1'.

Theorem 2, which depends on the existence of linear utilities in Hurwicz's paper, can be proved with our weaker coverage assumption.

Apart from a relaxation of the continuity requirement, the concept of ϵ-Nash-equilibrium is used in the last section to study the question whether endowments should be assumed known by the center or not. Indeed, a natural interpretation of the games devised by Hurwicz [3] and Schmeidler [6] is that they guarantee optimality of the outcome with respect to the true preferences even though the knowledge of the preference map of each player is totally decentralized. In fact, nothing need be known about endowments either: initial resources held by individual agents do not explicitly appear in the formulation of those games. It is then interesting to notice that Hurwicz proves his Theorems 1 and 2 by manipulating preferences only, while initial endowments remain fixed, so that the set of individually rational and Pareto-optimal allocations is known as soon as preferences are chosen.

Conversely, one could assume that preferences are known, but not endowments, and characterize the set of outcome functions that would yield individually rational and Pareto-optimal allocations. Under a coverage assumption C'' that takes a somewhat different form, we then establish that the Walrasian mechanism is the only one to satisfy all the requirements.

I. Fair Mechanisms

Definitions

In addition to being efficient, a fair allocation satisfies one of the following conditions:

D1: All individuals should be at least as well-off as if they had $1/n$ of the total endowment.

D2: No agent should be better off with someone else's bundle. (See Foley [1]).

D3: No agent should be better off with the average bundle of the rest of the economy.

D4: There should exist a commodity bundle yielding to each agent as much utility as the considered allocation. (Pazner and Schmeidler [4]).

In this section, outcome functions h are chosen so that the set of Nash-equilibrium allocations they yield $N_h(e)$ be a subset of the set of fair allocation $F_\alpha(e)$, according to the fairness definition D_α, $\alpha = 1, \ldots, 4$. $S(e)$ designates the set of individually rational and optimal allocations. $\underline{I}$ is the equal bundle allocation, and $W_I(e)$ the Walrasian allocation of the economy e with respect to the initial allocation I.

Theorem 1'. D_α The assumptions are the same as in Theorem 1 of [2], except that the condition "$N_h(e) \subseteq S(e)$" is replaced by

$$N_h(e) \subseteq F_\alpha(e).$$

Then $$N_h(e) \supseteq W_I(e)$$

Theorem 2'. D_α The assumptions are the same as in Theorem 2 of [2], except that the condition "$N_h(e) \subseteq S(e)$" is replaced by

$$N_h(e) \subseteq F_\alpha(e).$$

Then $\quad B_{++} \cap N_h(e) \subsetneq W(I)$

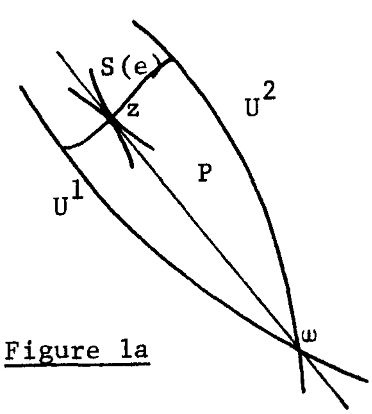
where B_{++} is the interior of the Edgeworth box.

Even though the above theorems are stated for an arbitrary number of agents and commodities, their proofs will be given for only two agents and two commodities. For simplicity we will often use differentiable utility curves and will assume that the agents' maximization problems admit of interior solutions. For a more complete treatment of those issues, see [2]. Finally, whenever a step of a proof is identical to the corresponding step in [2], we have omitted it.

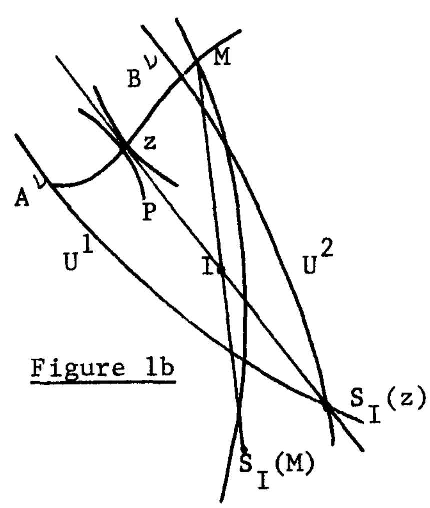
Theorem 1', D1 and Theorem 2', D1 are exactly the same as Theorems 1 and 2 where I replaces ω.

The proof of Theorem 1' D3 and 2' D3 would not be sufficiently different from that of Theorem 1' D2 and 2' D2 to be worth spelling out.

<u>Theorem 1'.D2</u> Given z , a Walrasian allocation relative to I, let us show that it is a Nash-equilibrium allocation.



<u>Figure 1a</u>



<u>Figure 1b</u>

In [2], Hurwicz constructs a sequence of economies e_ν by drawing "flatter and flatter" indifference curves with a slope approaching the slope of the line P joining z to ω. This construction is carried out in such a way that the Pareto-optimal allocations constitute an upward sloping curve, and that $S(e_\nu)$ the set of Pareto-optimal and individually rational points, always contains z, and shrinks to z, until it blows up to P, for the linear economy which is the limit of this sequence (Figure 1a).

A similar sequence is considered here. Define $S_I(z)$ to be the symmetric of z with respect to I, and e_ν, a sequence of economies with flatter and flatter indifference curves the slope of which approaches the slope of zI, such that the set of Pareto-optimal allocations constitute an upward-sloping curve always containing z. z is fair only if both agents prefer z to $S_I(z)$. Observe that the intersections A_ν and B_ν of the Pareto set with the indifference curves going through $S_I(z)$ are the endpoints of a superset of the set of fair allocations $F(e_\nu)$: $F(e_\nu) \subset A_\nu B_\nu$. Otherwise, there would exist a point M outside of $A_\nu B_\nu$ such that its symmetric with respect to I, $S_I(M)$, would yield less utility to agent 2 than M. For this to hold, the indifference curve through M would go to the left of $S_I(M)$. This cannot be done without it crossing the indifference curve through $S_I(z)$ since the positive slope of the Pareto set implies that $S_I(M)$ is to the south-west of $S_I(z)$. It follows that $z \in F(e_\nu) \subset A_\nu B_\nu$. The argument ends as in [2].

Theorem 2'.D2

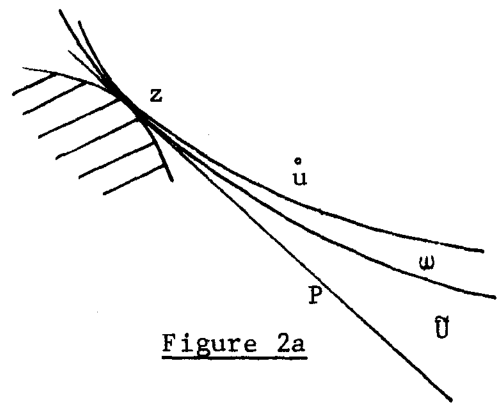


Figure 2a

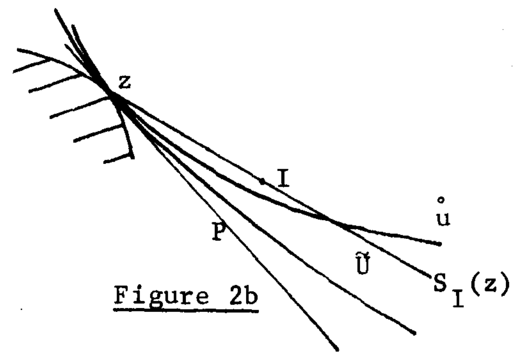


Figure 2b

In order to prove that Nash-equilibrium allocations are Walrasian allocations with respect to I , we proceed as in [2] where $S_I(z)$ replaces ω.

Given z , a Nash-equilibrium allocation, we define $S_I(z)$ as above. If the proposition were false, the common tangency line P to the indifference curves of the 2 agents at z would go below (say) I. This line is taken to be unique without loss of generality. One could then find an economy consisting of agent 2's original preferences and flatter indifferences curves for agent 1, such that z remains a Pareto-optimal and Nash-equilibrium allocation but does not enjoy the property of fairness. This construction is possible because of the convexity of the set of allocations attainable by each agent (shaded area) given the other agent's message. In figure 2, z is not individually rational (resp fair) for an agent with preferences indicated by $\tilde{U}$.

Theorem 1'.D4 . Let z be a Walrasian allocation with respect to I, and p be the slope of zI.

Given an indifference curve U cutting both axes, we define the number $f(U)$ as the smallest distance between two lines of slope p containing the whole of U. (Figure 3).

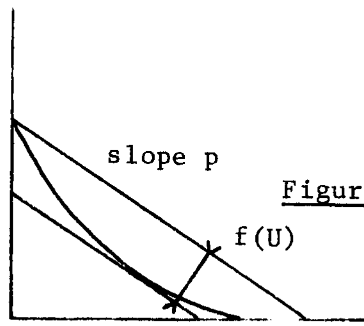


Figure 3

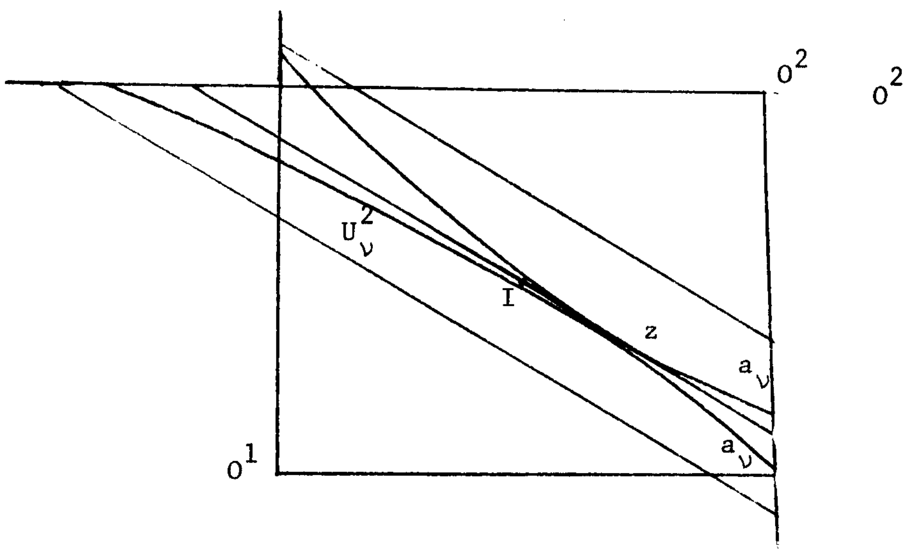
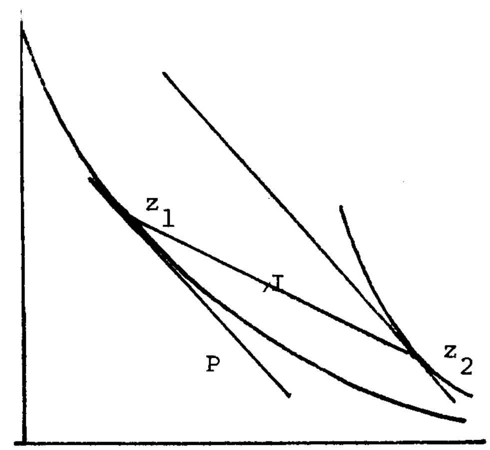


Figure 4

We then choose a sequence of positive numbers a_ν going to 0, and a sequence of economies with flatter and flatter indifference curves such that $f(U^i_\nu)$ be smaller than a_ν , for i=1,2, and for which the set of Pareto-optimal allocations is upward sloping and contains z for all ν . This guarantees that no allocation outside of a strip of width a_ν on either side of z can be fair. $F(e_\nu)$ shrinks to z until it blows up to P for the linear economy which is the limit of this sequence. The proof ends as in [2], (figure 4).

<u>Theorem 2',D4</u> . Given a Nash-equilibrium allocation z, suppose that the proposition were not true. Then the common tangency line P to the indifference curves of the 2 agents at z would go below (say) I. One could then find an economy consisting of the second individual, and of a first individual with flatter indifference curves than the original one, such that z would remain a Pareto-optimal and Nash-equilibrium allocation, but would not be fair any more[2] (figure 5).



<u>Figure 5</u>

It is conceivable that some more general theorems might be proved, encompassing Theorems 1 and 1' on the one hand, Theorems 2 and 2' on the other hand. In each pair of theorems, two requirements are imposed: the "local" condition of Pareto-optimality, and the "global" conditions of individual rationality or fairness. The first requirement is "local" in the sense that it compares the proposed allocation to other allocations in a neighborhood. The second requirement is "global" in the sense that the comparison is made to a reference point (ω or $S_I(z)$) far from it or to a whole indifference curve.

Section II: Approximate Nash-equilibria

In this section we replace Hurwicz's coverage assumption C by the weaker one (C') stated in the introduction. First, we need to define the concept of ε-Nash-equilibrium.

We introduce a numerical representation u^i of the i^{th} agent's preferences, and the associated cardinal indirect utility v^i defined on the space of m-tuples of strategies, by the composition of u^i with the outcome function. Therefore, $v^i(m) = u^i(h^i(m))$, where $h^i(m)$ is the commodity bundle attributed to the i^{th} agent as the result of the strategy choice m. The following definition is given for the case of 2 agents.

Definition: A pair of strategies $(\bar{m}^1, \bar{m}^2)$ is an ε-Nash-equilibrium if: $|v^1(\bar{m}^1, \bar{m}^2) - v^1(\bar{m}^1{}^*, \bar{m}^2)| < \varepsilon$, where $m^1{}^*$ is the 1^{st} agent's best response to $\bar{m}^2$.

$|v^2(\bar{m}^1, \bar{m}^2) - v^2(\bar{m}^1, m^2{}^*)| < \varepsilon$, where $m^2{}^*$ is the 2^{nd} agent's best response to $\bar{m}^1$.

This definition clearly depends on the choice of the numerical representation of the preferences; however, the following theorem, which is the counterpart of Theorem 1 when the coverage assumption C' replaces assumption C, does not.

Assumptions: They are the same as in Theorem 1 of [2] except that the coverage assumption C' replaces C and the performance correspondence is not required to be upper-hemi-continuous.

W(e) is the set of Walrasian allocations of the economy e .

<u>Theorem 1"</u>. Given an allocation z, element of $W(e)$, for every ε positive, there exists an ε-Nash-allocation $z_\varepsilon(e)$ such that $\|z_\varepsilon(e) - z\| < \varepsilon$.

<u>Proof</u> Hurwicz considers a sequence e_ν described in section I, of flatter and flatter economies whose limit is e_L (defined in [2]). The set of Pareto-optimal and individually rational allocation, $S(e_\nu)$, containing the Nash-allocations $N(e_\nu)$, shrinks to a neighborhood of z, until it blows up to the supporting hyperplane at z to the original economy e, (assumed unique for simplicity) at which point the assumption of lower hemi-continuity of the performance correspondence is used to guarantee that z is a Nash-allocation for e_L.

If linear economies are not allowed, the shrinking of $S(e_\nu)$ can still be ensured by the above construction in such a way that z be not away by more than ε from a Nash-allocation z_ν of e_ν, for ν large enough. Let us designate by (m_ν^1, m_ν^2) a pair of Nash-equilibrium strategies in e_ν yielding z_ν as a Nash-allocation. The best response to m_ν^2 available to the first agent in the original economy e, is denoted m_ν^{1*}. It generally differs from m_ν^1. However, because e_ν is "very flat" for ν large enough, the loss in utility A_1 incurs by playing m_ν^1 instead of m_ν^{1*}, measured according to some arbitrary cardinal representation of his true preferences, can be made as close to 0 as we want by the choice of an appropriately large ν.

This is illustrated in figure 6. Since only points south-west of u_ν^2 can be reached as individually rational Nash-allocations in e_ν, no point preferred to $\bar{z}_\nu^1$ by A_1 could qualify as a Nash-allocation in e, where $\bar{z}_\nu^1$ is A_1^1's truly preferred point along u_ν^2.

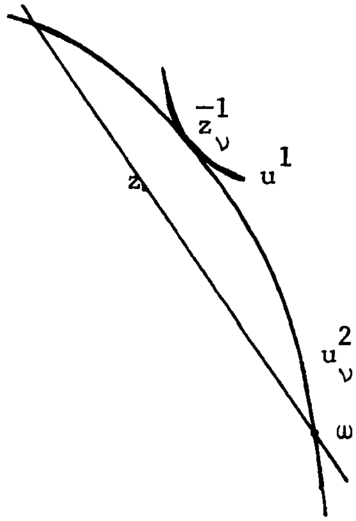


Figure 6

Theorem 2' The assumptions are the same as in Theorem 2 except that C is replaced by C'. The conclusion is identical with that of Theorem 2.

Proof: If this were not the case, for at least one agent, A_i, none of the supporting hyperplanes at z to his indifference curve would contain ω. Hurwicz replaces A_i by an agent whose indifference curve through z is one of these supporting hyperplanes.

Instead, it is easy to find a preference map having the following properties:

1) each indifference hypersurface has a non-zero Gaussian curvature everywhere

2) the indifference curve through z is such that

   a) it does not go through ω,

   b) its supporting hyperplanes at z are a subset of the set of supporting hyperplanes at z of the indifference curve through z of the original agent.

The argument would end as in Hurwicz's paper.

Section III: Manipulation of endowments.

In this section, preferences are initially known. An economy is characterized by an endowment vector ω .

Coverage assumption C". The class of environments over which the performance correspondence is defined contains economies with arbitrarily large endowments ω , and the preference maps may be of the Cobb-Douglas type.

Theorem 1'''. The assumptions are the same as in Theorem 1" of section II, where C' is replaced by C". The conclusions are identical.

Proof: Let z be a Walrasian allocation for the economy $e = (P, \omega)$ where $P = (P^1, P^2)$ are the two agents' preference maps and $\omega = (\omega^1, \omega^2)$ their initial endowments. We propose to establish that for every positive ϵ , z is an ϵ-Nash-allocation. To that effect, we exhibit a pair of messages (m_ν^{1*}, m_ν^{2*}) satisfying the definition stated in Section II.

First we introduce the economy $\bar{e} = (C, \bar{\omega})$ where the preference maps are of the Cobb-Douglas type allowed by the coverage assumption C", and where $\bar{\omega}$ is chosen in such a way that the net trade $t = z - \omega$ is a Walrasian net trade for $\bar{e}$.

Let p be the slope of the common tangency line to A_1 and A_2's indifference curves in the economy $\bar{e}$ at the allocation $\bar{\omega} + t$. Call $E^i(p)$ the expression path of agent i with respect to this price. It is a straight line going through the origin. Let $\Omega^i(p) = \{E^i(p) - t^i\} \cap R^{2+}$.

A sequence of economies $e_\nu = (C, \omega_\nu^1, \omega_\nu^2)$ is defined by

a) preferences are as in $\bar{e}$

b) $\omega_\nu^i \in \Omega^i(p)$ for all i and ν .

c) $\| \omega_\nu^i \| \to \infty$ as $\nu \to \infty$ for all i .

It results that the net trade t is Walrasian for e_ν, no matter what ν is. In addition, the projection Z^i_ν in A_i's commodity space of the set of Pareto-optimal and individually rational allocations for e_ν constitutes a segment colinear with $E^i(p)$ and of decreasing length.

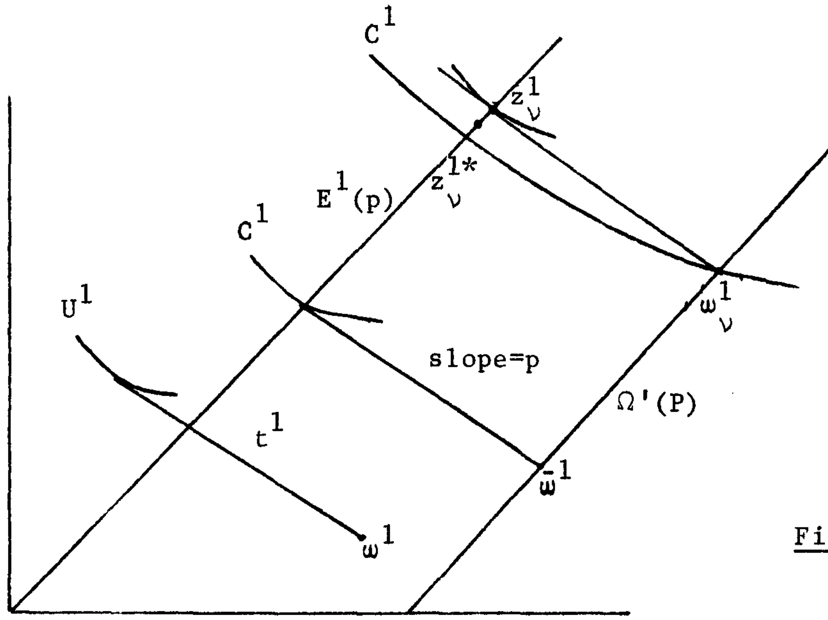


Figure 7

If $z^*_\nu = \omega_\nu + t^*_\nu$ is a Nash-allocation for e_ν corresponding to the pair of messages (m^{1*}_ν, m^{2*}_ν), by taking ν large enough, one can ensure that $\| z^*_\nu - z_\nu \| < \epsilon$, given any ϵ, where $z_\nu = \omega_\nu + t$. Let us call $T(m^{2*}_\nu)$ the set of net trades A_1 can attain when his message ranges over his whole message space. Since m^{1*}_ν is the best response to m^{2*}_ν, the set $\omega^1_\nu + T(m^{2*}_\nu)$ does not intersect the set of allocations preferred by A_1 to z^{1*}_ν.

When ν goes to infinity, one guarantees that if $t^1 \in T(m^{2*}_\nu)$ and satisfies $\| t^1 \| < \| \omega^1 + \omega^2 \|$, then $pt^1 - pz^*_\nu < \epsilon$, for any ϵ. The reason is that in the ball of fixed radius $\| \omega^1 + \omega^2 \|$, the indifference curve through z^{1*}_ν approximates its tangency line at that point with any degree of accuracy as ν increases. By the Nash property, no allocation above that indifference curve should be attainable by the agent.

Where ν is chosen to satisfy both of the above conditions, a pair of messages (m_ν^{1*}, m_ν^{2*}) yielding a Nash-equilibrium-allocation in e_ν, constitutes an ϵ-Nash-equilibrium in the original economy e. This follows from the facts that $\| \omega^i + t_\nu^{i*} - z^i \| < \epsilon$, and that the set of allocations attainable by A_1 does not contain allocations such that $p\bar{z} - pz > \epsilon$. Then m_ν^{1*} is almost, that is up to ϵ, the best response of A_1 to m_ν^{2*}. This completes the proof.

Theorem 2'''. Apart from the fact that the coverage assumption C'' replaces C', the statement is the same as that of Theorem 2'.

Proof. Let z be a Pareto-optimal and individually-rational Nash-allocation for the economy $e = (P, \omega)$ and assume that it were not Walrasian. Then there would be no common supporting hyperplanes to both agents' indifference curves at z that would contain ω. In terms of our 2-person example, this would mean that ω^1 (say) would always be above such a hyperplane p.

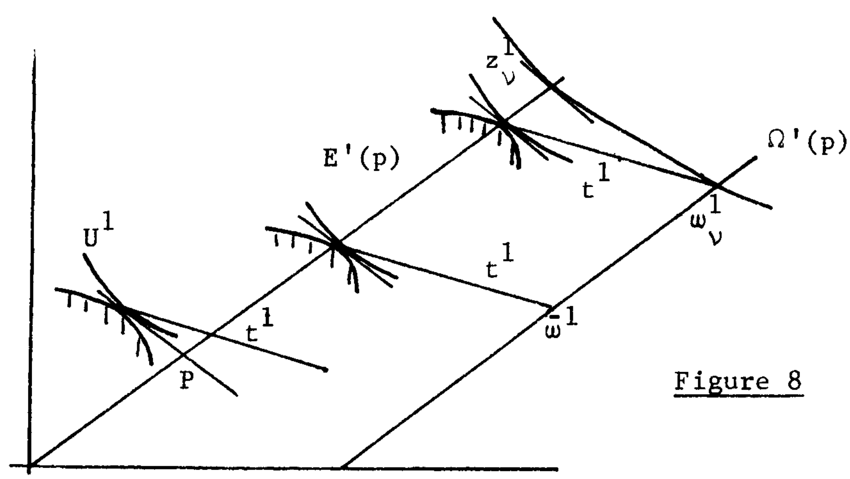


Figure 8

Let us then consider the economy $\bar{e} = (C, \bar{\omega})$ where preferences are of the Cobb-Douglas type C allowed by the coverage assumptions, and $\bar{\omega} = (\bar{\omega}^1, \bar{\omega}^2)$ is drawn so that the marginal rates of substitution of both agents in $\bar{e}$ at the allocation $\bar{\omega} + t$, when $t = z - \omega$, are equal to p. For the first agent only, we define $E^1(p)$ and $\Omega^1(p)$ as in the previous theorem, and we consider the sequence $e_\nu = (C, \omega_\nu)$ with $\omega_\nu = (\omega^1_\nu, \bar{\omega}^2)$, where

a) $\omega^1_\nu \in \Omega^1(p)$ for all ν

b) $\| \omega^1_\nu \| \to \infty$ as $\nu \to \infty$.

To say that z is a Nash-allocation is to say that there exist m^{1*} and m^{2*} such that for no $t^i \in T(m^{j*})$, $(\omega^i + t^i) P_i z^i, j \neq i, \forall i$. Because of the construction of $\bar{e}$, (m^{1*}, m^{2*}) remains a Nash-allocation for $\bar{e}$, and also for e_ν. However, for ν large enough, $z_\nu = \omega_\nu + t$ is not individually rational for e_ν. This contradicts our hypothesis, and completes the proof.

The coverage assumption C" includes a statement about endowments, which parallels the statement about preferences of [2], but also postulates that one "Cobb-Douglas economy" should be covered by the mechanism: this assumption which has no equivalent in [2], is due to the greater difficulty of "flattening" offer curves by manipulation of endowments only.

Instead of using a Cobb-Douglas economy, one could have simply asked for a pair of preference maps satisfying:

a) For every price p, every net trade t, and every ϵ, there exists a ball of radius t in the commodity space, such that the preference map restricted to that ball approximates with ϵ-accuracy (in some appropriate sense) a set of parallel hyperplanes of slope p.

b) Expansion paths are not "thick".

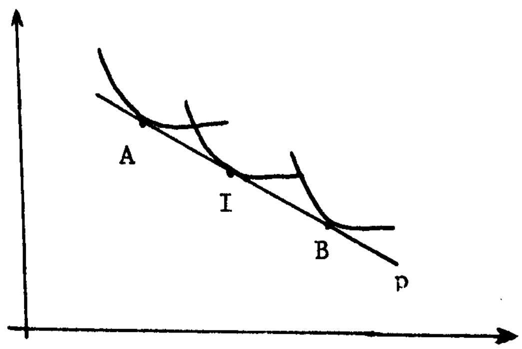
c) No good is an inferior good.

Those conditions are all satisfied by Cobb-Douglas preferences. They prevent the following from occurring: i) $\| z_\nu^* - (\omega_\nu + t) \|$ does not converge to 0 even though ii) $\| pz_\nu^* - p(\omega_\nu + t) \| \to 0$. (I am grateful to M. Richter for discussion of this point.)

Footnotes

1. The terminology and notations are those of M. Reiter in [5] and Hurwicz in [2].

2. As pointed out by Postelwaite (oral communication) these results would not hold for more than 2 people. Indeed, if $n > 2$, it is not in general the case that the Walrasian allocations from the equal bundle allocation satisfy the Pazner-Schmeidler definition of fairness. The following example illustrates that fact.



$n=3$. Sample u-curves are drawn for each agent in such a way as to have a common tangency line p satisfying $IA = IB$. Since they have no point in common, the allocation (I, A, B) is not fair according to D4.

3. As in section I, if we had been interested in fair mechanisms, a proof similar to that presented in section III could have been given, involving manipulation of endowments only.

4. The arguments given in section II and III, would go through for the case of public good and production economies, also considered by Hurwicz.

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