INFORMATIONAL TEMPORARY EQUILIBRIA

by

J. S. Jordan*

Discussion Paper No. 77-89, April 1977

*I am greatly indebted to Professor S. Reiter for several stimulating discussions, and the original description (in [5] of the process studied in this paper.

Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

Informational Temporary Equilibria

J. S. Jordan

Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

Abstract

This paper studies a dynamic process for the formation of expectations and the achievement of equilibrium in stochastic exchange environments. An _informational temporary equilibrium_ is an exchange equilibrium in which agents' expectations are conditioned on their initial information and data generated in previous informational temporary equilibria. An _equilibrium_ is a temporary equilibrium which reveals no further information. This process leads to an equilibrium even when the data observed by agents is insufficient to permit the existence of an _expectations equilibrium_. The relation between equilibrium and expectations equilibrium is explored, and a characterization of those data structures whose equilibria are expectations equilibria for a possibly different data structure is obtained.

1. Introduction

Economic agents faced with uncertainty will attempt to infer the state of the world from observed data. If both the state and the data are generated independently of any economic decisions, a natural presumption is that the statistically correct rule of inference will eventually be learned. Since the term equilibrium refers to a situation which, if achieved, would be indefinitely repeated, the same presumption of learning should apply, in equilibrium, even if the data or the state are influenced by economic decisions. This intuition is one rationale for the recently much studied "expectations equilibrium".[1]

An illustration of this equilibrium concept can be constructed as follows. Consider a pure exchange economy with two possible states of the world, a and b. For the i^{th} agent, state a is characterized by an endowment ω_a^i, and a state dependent utility function u_a^i. State b is analogously characterized by ω_b^i and u_b^i. In each state, the endowment is initially realized, an exchange equilibrium is reached, giving the i^{th} agent a consumption bundle x^i, and then the state dependent utility $u_a(x^i)$ or $u_b^i(x^i)$ is realized. If $\omega_a^i \neq \omega_b^i$, the exogenous data ω_a^i or ω_b^i reveals the state to agent i before trade takes place. If $\omega_a^i = \omega_b^i$, the i^{th} agent is unable to distinguish the state before trading, and attempts to infer the state from the exchange equilibrium price. In

---

[1] References to the recent literature are given in [3].

this example, an expectations equilibrium would consist of a pair of exchange equilibrium prices and net trades $(p_a; y_a^1, \ldots, y_a^i, \ldots, y_a^N) = (p_a, y_a)$ and $(p_b; y_b^1, \ldots, y_b^i, \ldots, y_b^N) = (p_b, y_b)$ with the property that if $p_a \neq p_b$, (p_a, y_a) is an exchange equilibrium for the economy $\{ (\omega_a^i, u_a^i) \}_{i=1}^N$ and (p_b, y_b) is an exchange equilibrium for $\{ (\omega_b^i, u_b^i) \}_{i=1}^N$, since uninformed agents can infer the state from the price. If $p_a = p_b$, uninformed agents remain uninformed, so in this case, (p_a, y_a) and (p_b, y_b) will be exchange equilibria for

$$\{ (\omega_a^i, u_a^i) : \omega_a^i \neq \omega_b^i \} \cup \{ (\omega^i, \lambda u_a^i + (1 - \lambda) u_b^i) : \omega_a^i = \omega_b^i = \omega^i \}$$

and

$$\{ (\omega_b^i, u_b^i) : \omega_a^i \neq \omega_b^i \} \cup \{ (\omega^i, \lambda u_a^i + (1 - \lambda) u_b^i) : \omega_a^i = \omega_b^i = \omega^i \}$$

respectively, where λ is the objective probability of state a. Unfortunately, it is easy to choose λ and the characteristics (ω_a^i, u_a^i) and (ω_b^i, u_b^i) for each i so that no expectations equilibrium exists. That is, the economies $\{ (\omega_a^i, u_a^i) \}_{i=1}^N$ and $\{ (\omega_b^i, u_b^i) \}_{i=1}^N$ may have the same equilibrium price (uniquely), and the second pair of economies mentioned above may have distinct equilibrium prices (uniquely).

Alternatively, suppose that the data observed by the i^{th} uninformed agent is increased to include his own net trade, y^i. Although it may at first seem paradoxical, it is easy to establish the general existence of expectations equilibria for this data structure. First, let (p_a, y_a) and (p_b, y_b) be exchange equilibria for $\{ (\omega_a^i, u_a^i) \}_{i=1}^N$ and $\{ (\omega_b^i, u_b^i) \}_{i=1}^N$. If $(p_a, y_a^i) \neq (p_b, y_b^i)$ for every uninformed agent i, this pair is an expectations equilibrium. Therefore, suppose

$(p_a, y_a^i) = (p_b, y_b^i) = (p, y^i)$ for some i with $\omega_a^i = \omega_b^i = \omega^i$. This implies that y^i maximizes $u_a^i(\omega^i + y'^i)$ and $u_b^i(\omega^i + y'^i)$ subject to $py'^i \leq 0$, so y^i maximizes any convex combination of the two utility functions subject to the same constraint; so we have constructed an expectations equilibrium.

In effect, the expectations equilibrium is constructed by adding information to the economy and showing that the data (p, y^i) preserve this information to the extent that it is relevant. How could this information have been transmitted to uninformed agents? Following [5], we will construct a more dynamic model of the case in which uninformed agents attempt to infer the state from the price. Initially, uninformed agents seek to maximize $\lambda u_a^i + (1 - \lambda) u_b^i$, leading to exchange equilibria (p_{a1}, y_{a1}) and (p_{b1}, y_{b1}) in the respective states. If $p_{a1} \neq p_{b1}$, the state is revealed, and before consummating the trade y_a^i (resp. y_b^i), uninformed agents revise their demands to maximize u_a^i (resp. u_b^i), leading to a final exchange equilibrium (p_{a2}, y_{a2}) (resp. (p_{b2}, y_{b2})). Assuming that agents remember the information generated in the first stage of this process, knowledge of the state will not be lost if $p_{a2} = p_{b2}$. If $p_{a1} = p_{b1}$, uninformed agents do not learn the state and the process stops at the first stage. Let (p_a, y_a), (p_b, y_b) denote the pair of exchange equilibria at which this process stops. If u_a^i and u_b^i are strictly concave and agent i is uninformed, $(p_a, y_a^i) \neq (p_b, y_b^i)$ only if $p_{a1} \neq p_{b1}$.

This implies that the pair (p_a, y_a), (p_b, y_b) constitutes an expectations equilibrium when uninformed agents use the data (p, y^i).

If there are more than two possible states, or if different agents observe different data, the process described above may continue for more than two stages. The intermediate outcomes will be called informational temporary equilibria. If the number of states is finite, the process will stop in a finite number of steps, and the final outcome will be called an equilibrium. This paper analyses the relation between equilibrium and expectations equilibrium. The data observed by the i^{th} agent will be assumed to be represented by a function f^i of the market data (p, y). The first result, Theorem 3.5, states that data structures which generally admit expectations equilibria have the characteristic property that for each i, either f^i is constant or $f^i(p,y) \neq f^i(p',y')$ whenever $(p, y^i) \neq (p', y'^i)$. Such data structures are called admissible. This result generalizes an admissibility characterization obtained in [3] for data structures in which all agents have the same continuous data function. A data structure is called eventually admissible if it generates equilibria which are expectations equilibria for an admissible data structure. A characterization of eventually admissible data structures is given by Theorem 3.10. Section 5 discusses the requirement that temporary equilibrium trades not be consummated. An example is presented to demonstrate that this requirement is essential even to the existence of temporary equilibria. For simplicity, the analysis of sections 2-5 is restricted to the two event case. In section 6, the definitions of temporary equilibrium and equilibrium are extended to the general case, as are the preceding results.

2. The Model

2.1 Static Exchange Environments: There are N agents, indexed by the superscript i, and J commodities, indexed by the subscript j, with $2 \leq N < \infty$ and $2 \leq J < \infty$. The i^{th} agent has a consumption space X^i, an endowment space Ω^i, and a space of net trades $Y^i = X^i - \Omega^i$. Let R_+^J denote the nonnegative orthant of R^J. We will assume that for each i, $\Omega^i = R_+^J \setminus \{0\}$ and $X^i = \text{int}R_+^J$. Let $X^i = \Pi_{i=1}^N X^i$, $\Omega = \{\omega \in \Pi_{i=1}^N \Omega^i : \Sigma_{i=1}^N \omega_j^i > 0 \text{ for each } j\}$, and let $Y = \{ y \in \Pi_{i=1}^N Y^i : \Sigma_{i=1}^N y^i = 0\}$. For each i, let u^i denote the set of continuous, strictly concave, and strictly increasing utility functions $u^i : X^i \to R$, with the additional property that for each $x^i \in X^i$, the closure in R^J of the set $\{x'^i \in X^i : u^i(x'^i) \geq u^i(x^i)\}$ is contained in X^i. Let $U = \Pi_{i=1}^N U^i$.

The space of static exchange environments, E, is defined by $E = \Omega \times U$. Defining $E^i = \Omega^i \times U^i$, and making the obvious identification, we have $E = \Pi_{i=1}^N E^i$. A generic element of E is denoted e, with the identifications $e = (\omega, u)$ and $e = (e^1, \ldots, e^N)$.

2.2 Remarks: The definition of U^i insures that equilibrium allocations will be interior, which facilitates the use of calculus in section 4 below. It will be explained in 4.9 below that the results in this paper would be unaffected if we redefined X^i to be R_+^J, and U^i to be the set of continuous, strictly concave, and strictly increasing functions on R_+^J.

A two-event stochastic environment associates static environments e_a and e_b with the respective states. A stochastic environment has the following interpretation. In state a, an endowment $\omega_a = (\omega_a^1, \ldots, \omega_a^N)$ is realized, and trading ensues. An allocation $x_a = (x_a^1, \ldots, x_a^N)$ is determined as an exchange equilibrium, after which the state-depended utilities $u_a^i(x_a^i)$ are realized. The process for state b is exactly analogous. Some agents, including those for whom $\omega_a^i \neq \omega_b^i$, will initially recognize which state has occurred. These agents will trade to maximize u_a^i and u_b^i in the respective states. Other agents may be unable to discern the state initially, and in the absence of further information, will trade to maximize their expected utility $\lambda u_a^i + (1 - \lambda) u_b^i$ in each state, where λ is the probability of state a. Thus a stochastic environment is described by the two states and their probabilities, and the initial distribution of information. This definition is stated formally below.

<u>2.3 Stochastic Exchange Environments</u>: An <u>information structure</u> is an N-tuple of numbers $\eta = (\eta^1, \ldots, \eta^N)$ where $\eta^i \varepsilon \{0,1\}$ for each i. The i^{th} agent is said to be informed if $\eta^i = 1$, and uninformed otherwise. The set of <u>stochastic environments</u> is defined by $S = \{ s = (\eta, \lambda, e_a, e_b)$: $e_a, e_b \ \varepsilon\ E$, $0<\lambda<1$, and η is an information structure such that for each i, $\eta^i = 1$ if $\omega_a^i \neq \omega_b^i\}$. The number λ is interpreted as the probability of state a.

The (realization) functions $r_a: S \to E$ and $r_b: S \to E$ are defined coordinatewise by:

$$r_a^i(s) = \begin{cases} (\omega_a^i, u_a^i) \text{ if } \eta^i = 1; \text{ and} \\ (\omega_a^i, \lambda u_a^i + (1-\lambda)u_b^i); \text{ if } \eta^i = 0, \end{cases}$$

$$r_b^i(s) = \begin{cases} (\omega_b^i, u_b^i) \text{ if } \eta^i = 1; \text{ and} \\ (\omega_b^i, \lambda u_a^i + (1-\lambda)u_b^i); \text{ if } \eta^i = 0, \end{cases}$$

where $s = (\eta, \lambda, e_a, e_b)$.

2.4 Remarks: The function r_a^i associates with each stochastic environment a static environment consisting of the endowments and expected utility functions realized in state a, given each agent's information. The static environment $r_a(s)$ can be regarded as an "initial realization", in contrast to the "final realization" (ω_a, u_a). We now describe the process by which agents use market data to supplement their initial information.

2.5 Definitions: Let Δ denote the relative interior of the unit simplex in R_+^J, and let $M = \{(p,y) \in \Delta \times \mathbf{Y}: \Sigma_i y^i = 0$ and for each i, $py^i = 0\}$. Define the corespondence μ: $E \twoheadrightarrow M$ by setting $\mu(e)$ equal to the set of competitive equilibrium prices and net trades for e. In the definition of the competitive allocation mechanism presented in [4], M is the competitive message space, and μ is the competitive message process. Hence elements of M will be called messages, and denoted by either (p,y) or m.

For each i, a data function is a function f^i on M to an arbitrary set. A data structure is an N- tuple $\mathfrak{F} = (f^1, \ldots, f^N)$ of data functions.

2.6 Equilibrium:[1] For a stochastic environment $s = (\eta, \lambda, e_a, e_b)$ and a data structure f, a sequence of temporary equilibria is a finite sequence of information structures, $\{\eta_t\}_{t=0}^T$, and message pairs, $\{(m_{at}, m_{bt})\}_{t=1}^T$ with

i) $\eta_0 = \eta$; and for each i and each $t \geq 1$,

$$\eta_t^i = \begin{cases} 1 \text{ if } \eta_{t-1}^i = 1 \text{ or } f^i(m_{at}) \neq f^i(m_{bt}); \text{ and} \\ 0 \text{ otherwise;} \end{cases}$$

---

[1]The concepts defined in this section are specialized to the present context from [5].

ii) for each $t \geq 1$, $m_{at} \varepsilon \mu \cdot r_a(\eta_{t-1}, \lambda, e_a, e_b)$ and

$m_{bt} \varepsilon \mu \cdot r_b(\eta_{t-1}, \lambda, e_a, e_b)$; and

iii) $\eta_T = \eta_{T-1}$.

The pair (m_{at}, m_{bt}) is said to be an equilibrium for (s,f).

2.7 Remarks: The above definitions embody the requirement, discussed in section 5 below, that the trades y_{at} and y_{bt}, $t < T$, are not consummated, and have only an informational influence on the equilibrium trades y_{aT} and y_{bT}. Since there are N agents and only two states, there will always exist a sequence of temporary equilibria with $T \leq N-1$. In stage t, the i^{th} agent uses the observed data $f^i(m_{at})$ or $f^i(m_{bt})$ to supplement his previous information η^i_{t-1}. An implication of (i) is that for each t, $\eta_t \leq \eta_{t+1}$, so no information is ever lost. Condition (ii) states that the messages m_{at} and m_{bt} are competitive equilibria for the respective realized static environments $r_a(\eta_{t-1}, \lambda, e_a, e_b)$ and $r_b(\eta_{t-1}, \lambda, e_a, e_b)$. Condition (iii) is the stationarity condition, which states that the pair (m_{at}, m_{bt}) does not generate additional information for any agent. In particular define the information structure $\hat{\eta}$ by

$$\hat{\eta} = \begin{cases} 1 \text{ if } \eta^i = 1 \text{ or } f^i(m_{aT}) \neq f^i(m_{bT}); \text{ and} \\ 0 \text{ otherwise.} \end{cases}$$

Then (iii) implies that $\hat{\eta} \leq \eta_{T-1} = \eta_T$. If $\hat{\eta} = \eta_{T-1}$, then the equilibrium (m_{aT}, m_{bT}) would be an expectations equilibrium, as defined below.

2.8 Expectations Equilibrium: An expectations equilibrium for a stochastic environment $s = (\eta, \lambda, e_a, e_b)$ and a data structure $\mathfrak{F}$

is a pair $(m_a, m_b) \in M \times M$ such that $m_a \in \mu \cdot r_a(\hat{\eta}, \lambda, e_a, e_b)$ and $m_b \in \mu \cdot r_b(\hat{\eta}, \lambda, e_a, e_b)$, where $\hat{\eta}$ is defined by

$$\hat{\eta}^i = \begin{cases} 1 \text{ if } \eta^i = 1 \text{ or } f^i(m_a) \neq f^i(m_b); \\ 0 \text{ otherwise.} \end{cases}$$

3. Equilibrium and Expectations Equilibrium

This section studies the relation between equilbrium and expectations equilibrium. In [3], it was shown that if all agents have the same continuous data function, the only data functions which generally admit expectations equilibria are 1-1 functions and constant functions. Theorem 3.5 generalizes this result. By contrast, equilibria exist universally. However, the verification of an equilibrium requires the construction of a sequence of temporary equilibria. The expectations equilibrium condition is much more immediate. A natural initial question is: given an equilibrium for $(s,\mathfrak{F})$, does there exist a data structure $\mathfrak{F}'$ such that the equilibrium is an expectations equilibrium for $(s,\mathfrak{F}')$? The following Lemma, which will be used throughout this section, leads to an affirmative answer.

<u>3.1 Lemma</u>: Let $(\lambda,\eta,e_a,e_b) \in S$ and let $\eta' \geq \eta$. Let $(p_a,y_a) \in \mu \cdot r_a(\lambda,\eta', e_a,e_b)$ and $(p_b,y_b) \in \mu \cdot r_b(\lambda,\eta',e_a,e_b)$ and define the information structure η'' coordinatewise by

$$\eta''^{i} = \begin{cases} 0 \text{ if } \eta^{i} = 0 \text{ and } (p_a,y_a^{i}) = (p_b,y_b^{i}); \text{ and} \\ \eta'^{i} \text{ otherwise.} \end{cases}$$

Then $(p_a,y_a) \in \mu\cdot r_a(\lambda,\eta'',e_a,e_b)$ and $(p_b,y_b) \in \mu\cdot r_b(\lambda,\eta'',e_a,e_b)$.

<u>Proof:</u> By the definition of η'', $\eta''^{i} \neq \eta'^{i}$ only if $\eta'^{i} = 1$, $\omega_a^{i} = \omega_b^{i} = \omega^{i}$, and $(p_a,y_a^{i}) = (p_b,y_b^{i}) = (p,y^{i})$, for some ω^{i} and some (p,y^{i}). Thus it suffices to show that under these conditions, y^{i} maximizes $\lambda u_a^{i}(\omega^{i} + y'^{i}) + (1-\lambda)u_b^{i}(\omega^{i} + y'^{i})$ subject to $py'^{i} \leq p\omega^{i}$. But y^{i} maximizes $u_a^{i}(\omega^{i} + y'^{i})$ and $u_b^{i}(\omega^{i} + y'^{i})$ each subject to this constraint, so the proof is complete.

<u>3.2 Remarks</u>: To answer the question posed above, let $(p_{aT}, y_{aT}; p_{bT}, y_{bT}) = (m_{aT}, m_{bT})$ be an equilibrium for a stochastic environment $s = (\lambda, \eta, e_a, e_b)$ and a data structure $\mathfrak{F}$, and let η_T be the final information structure. For each i, let f'^i be a data function such that

i) $f'^i(m_{aT}) \neq f'^i(m_{bT})$ if $\eta_T^i = 1$ and $(p_{aT}, y_{aT}^i) \neq (p_{bT}, y_{bT}^i)$; and

ii) $f'^i(m_{aT}) = f'^i(m_{bT})$ otherwise.

Now apply Lemma 3.1 with $\eta' = \eta_T$, $(p_a, y_a) = m_{aT}$, and $(p_b, y_b) = m_{bT}$; and observe that the information structure η'' given by the Lemma satisfies, for each i,

$$\eta''^i = \begin{cases} 1 \text{ if } \eta^i = 1 \text{ or } f'^i(m_{aT}) \neq f'^i(m_{bT}); \text{ and} \\ 0 \text{ otherwise.} \end{cases}$$

Thus (m_{aT}, m_{bT}) is an expectations equilibrium for $(s, \mathfrak{F}')$, with $\mathfrak{F}' = (f'^1, \ldots, f'^N)$.

The data structure $\mathfrak{F}'$ depends on s as well as on $\mathfrak{F}$. If $(s, \mathfrak{F})$ admits multiple equilibria, $\mathfrak{F}'$ may also vary among these equilibria, so the result that every equilibrium is an expectations equilibrium for some data structure is not very revealing. We now pose the question in a much stronger form: given a data structure $\mathfrak{F}$, does there exist a data structure $\mathfrak{F}'$ such that for every stochastic environment s, every equilibrium for $(s, \mathfrak{F})$ is an expectations equilibrium for $(s, \mathfrak{F}')$?

<u>3.3 Definitions</u>: A data structure $\mathfrak{F}$ is said to be <u>admissible</u> if for each stochastic environment s, there exists an expectations equilibrium for $(s, \mathfrak{F})$. A data structure $\mathfrak{F}$ is said to be <u>eventually admissible</u> if there is a data structure $\mathfrak{F}'$ such that for each stochastic environment s, there is an equilibrium for $(s, \mathfrak{F})$

which is an expectations equilibrium for $(s,\mathfrak{F}')$.

A data function f^i will be said to be trivial if it is a constant function.

3.4 Remarks: The remainder of this section is devoted to the characterization of admissible and eventually admissible information structures. Theorem 3.10(B) below indicates that the characterization of eventual admissibility would be unaffected if, in the above definition, all equilibria for $(s,\mathfrak{F})$ were required to be expectations equilibria for $(s,\mathfrak{F}')$. In the definition of eventual admissibility, the data structure $\mathfrak{F}'$ must be admissible so it is natural to begin with a characterization of admissibility.

3.5 Theorem: A data structure $\mathfrak{F}$ is admissible if an only if for each i, either

i) f^i is trivial; or

ii) $f^i(p,y) \neq f^i(p',y')$ whenever $(p, y^i) \neq (p',y'^i)$.

Proof: Necessity follows from Proposition 4.8 below and the symmetry of the model with respect to agents. To prove sufficiency, let $s = (\lambda,\eta,e_a,e_b)$ be a stochastic environment. If $\eta^i = 0$ for all i, then $r_a(s) = r_b(s)$, so let $\hat{\eta} = \eta$ and let $m_a = m_b$ ε $\mu\cdot r_a(s)$. If $\eta_o^i \neq 0$ for some i, let η' be given coordinatewise by

$$\eta'^i = \begin{cases} 1 \text{ if } \eta'^i = 1 \text{ or } f^i \text{ is not trivial;} \\ 0 \text{ otherwise.} \end{cases}$$

Then $\eta' \geq \eta$. Let $m_a \in \mu \cdot r_a(\eta',\lambda,e_a,e_b)$ and $m_b \in \mu \cdot r_b(\eta',\lambda,e_a,e_b)$, and define $\hat{\eta}$ by

$$\hat{\eta}^i = \begin{cases} 0 \text{ if } \eta'^i = 0 \text{ and } (p_a,y_a^i) = (p_b,y_b^i)\text{; and} \\ \eta'^i \text{ otherwise.} \end{cases}$$

It follows from the hypothesis that

$$\hat{\eta}^i = \begin{cases} 1 \text{ if } \eta'^i = 1 \text{ or } f^i(m_a) \neq f^i(m_b)\text{; and} \\ 0 \text{ otherwise,} \end{cases}$$

for each i. By Lemma 3.1, $m_a \in \mu \cdot r_a(\hat{\eta},\lambda,e_a e_b)$ and $m_b \in \mu \cdot r_b(\hat{\eta},\lambda,e_a,e_b)$, so (m_a,m_b) is an expectations equilibrium for $(s,\mathfrak{F})$.

<u>3.6 Remarks</u>: Theorem 3.5 indicates that the class of admissible data structures is quite narrow. The following Corollary indicates that for the purpose of supporting the expectations equilibria of admissible data structures, only 2N data structures need to be considered.

<u>3.7 Corollary</u>: Suppose that $\mathfrak{F}$ is an admissible data structure, and let $\mathfrak{F}^*$ be a data structure such that for each i,

i) if f^i is trivial then f^{*i} is trivial; and

ii) if f^i is not trivial then f^{*i} is the projection $(p,y) \to (p,y^i)$.

Then for each stochastic environment s, every expectations equilibrium for $(s,\mathfrak{F})$ is also an expectations equilibrium for $(s,\mathfrak{F}^*)$.

<u>Proof</u>: This follows directly from Theorem 3.5 and Lemma 3.1.

<u>3.8 Remarks</u>: It is not true that every expectations equilibrium for $(s,\mathfrak{F}^*)$ is an expectations equilibrium for $(s,\mathfrak{F})$. Suppose that f^i is the identity on M. Then f^{*i} is the projection

$(p,y) \rightarrow (p,y^i)$, so if $N > 2$, f^i reveals additional information which could affect the i^{th} agent's demand.

Corollary 3.7 implies that a data structure $\mathfrak{F}$ is eventually admissible if and only if it admits equilibria which are expectations equilibria for data structures $\mathfrak{F}^*$ such that, for each i, either f^{*i} is trivial or f^{*i} is the projection $(p,y) \rightarrow (p,y^i)$. The characterization of eventual admissibility is also influenced by the fact that, in a sequence of temporary equilibria, if $\eta_t^i = 0$, then the pair (m_{at},m_{bt}) will satisfy the weak axiom for agent i. Whether a data function enables an agent to become informed is therefore determined by whether or not it distinguishes between messages which satisfy the weak axiom for that agent.

<u>3.9 Definition</u>: A pair (p,y), $(p',y') \varepsilon M$ satisfies the weak axiom for agent i if $py'^i \leq 0$ implies $y^i = y'^i$ or $p'y^i > 0$.

<u>3.10 Theorem</u>:

A. A data structure $\mathfrak{F}$ is eventually admissible if and only if for each i, either

i) f^i is trivial; or

ii) $f^i(p,y) \neq f^i(p',y')$ for all (p,y), (p',y') such that $p \neq p'$ and the weak axiom is satisfied for agent i.

B. If $\mathfrak{F}$ is eventually admissible, let $\mathfrak{F}^*$ be a data structure such that for each i,

i) if f^i is trivial then f^{*i} is trivial; and

ii) if f^i is nontrivial then f^{*i} is the projection $(p,y) \rightarrow (p,y^i)$.

Then for each stochastic environment s, every equilibrium for $(s,\mathfrak{F})$ is an expectations equilibrium for $(s,\mathfrak{F}^*)$.

<u>Proof</u>: We will first prove necessity in (A). Let $\mathfrak{F}$ be an admissible data structure. Suppose that $f^1(p_a,y_a) = f^1(p_b,y_b)$ for some (p_a,y_a), $(p_b,y_b) \in M$ such that $p_a \neq p_b$ and the weak axiom is satisfied for agent 1. We will show that f^1 is trivial. Let $e_a,e_b, \in E$ such that

1) $\omega_a^1 = \omega_b^1$; and $\omega_a^i \neq \omega_b^i$ for all $i \geq 2$;

2) (p_a,y_a) is the unique competitive equilibrium for $([\omega_a^1,(\frac{1}{2})u_a^1 + (\frac{1}{2})u_b^1],e_a^2,\ldots,e_a^N)$;

3) (p_b,y_b) is the unique competitive equilibrium for $([\omega_b^1,(\frac{1}{2})u_a^1 + (\frac{1}{2})u_b^1],e_b^2,\ldots,e_b^N)$; and

4) The excess demand at p_a determined by the characteristics (ω^1,u_a^1) is <u>not</u> equal to y_a^1.

Consider the stochastic environment $s = (\lambda,\eta,e_a,e_b)$ with $\lambda = \frac{1}{2}$, $\eta^1 = 0$, and $\eta^i = 1$ for all $i \geq 2$. Since $f^1(p_a,y_a) = f^1(p_b,y_b)$, the pair $[(p_a,y_a), (p_b,y_b)]$ is the unique equilibrium for $(s,\mathfrak{F})$. Since $\mathfrak{F}$ is eventually admissible, this pair must be an expectations equilibrium for some admissible data structure $\mathfrak{F}'$. By (4) above, $(p_a,y_a) \not\in \mu(e_a^1,e_a^2,\ldots,e_a^N)$, so we must have $f'^1(p_a,y_a) = f'^1(p_b,y_b)$. It follows from Theorem 3.5 that f'^1 is trivial. Let (p_a',y_a'), $(p_b',y_b') \in M$ such that $p_a' \neq p_b'$ and the weak axiom is satisfied for agent 1. Let e_a', $e_b' \in E$ satisfy (1) - (4) above for the pair (p_a',y_a'), (p_b',y_b'); and let $s' = (\lambda,\eta, e_a',e_b')$ with $\lambda = \frac{1}{2}$, $\eta^1 = 0$, and $\eta^i = 1$ for all $i \geq 2$. Since f'^1 is trivial, the unique expectations exquilibrium for $(s',\mathfrak{F}')$ is $[(p_a',y_a'), (p_b',y_b')]$. Since $\mathfrak{F}$ is eventually admissible, this must be an equilibrium for $(s',\mathfrak{F})$, so by (4), $f^1(p_a',y_a') = f^1(p_b',y_b')$. This shows that

$f^1(p,y) = f^1(p',y')$ whenever $p \neq p'$ and the weak axiom is satisfied for agent 1. This implies directly that f^1 is trivial, which completes the proof of necessity in (A).

To prove sufficiency in (A), we will show that if $\mathfrak{F}$ satisfies (A.i) or (A.ii) for each i, then for every stochastic environment s, every equilibrium for $(s,\mathfrak{F})$ is an expectations equilibrium for $(s,\mathfrak{F}^*)$, where $\mathfrak{F}^*$ is described in (B). This will also prove (B). Let $\mathfrak{F}$ satisfy (A.i) or (A.ii), let $s = (\lambda,\eta,e_a,e_b) \in S$, and let (m_a,m_b) be an equilibrium for $(s,\mathfrak{F})$. Let η_T be the final information structure corresponding to this equilibrium, as defined in 2.6. In particular, $m_{aT} \in \mu \cdot r_a(\lambda,\eta_T,e_a,e_b)$ and $m_{bT} \in \mu \cdot r_b(\lambda,\eta_T,e_a,e_b)$.

Let $\mathfrak{F}^*$ be a data structure such that for each i, if f^i is trivial then f^{*i} is trivial, and if f^i is nontrivial then f^{*i} is the projection $(p,y) \to (p,y^i)$. For each i, let

$$\hat{\eta}^i = \begin{cases} 0 \text{ if } \eta^i = 0 \text{ and } f^{*i}(m_a) = f^{*i}(m_b)\text{; and} \\ 1 \text{ otherwise.} \end{cases}$$

We have to show that $m_{aT} \in \mu \cdot r_a(\lambda,\hat{\eta},e_a,e_b)$ and $m_{bT} \in \mu \cdot r_b(\lambda,\hat{\eta},e_a,e_b)$. If $\eta^i = 1$, then $\hat{\eta}^i = \eta_T^i = 1$. If $\eta^i = 0$ and f^i is trivial, then f^{*i} is trivial so $\hat{\eta}^i = \eta_T^i = 0$. Finally, suppose $\eta^i = 0$ and f^i satisfies (A.ii). Given any sequence of temporary equilibria for $(s,\mathfrak{F})$, for each t, if $\eta_t^i = 0$, then the pair $(m_{a(t+1)},m_{b(t+1)})$ will satisfy the weak axiom for agent i. Therefore $(p_{aT},y_{aT}^i) \neq (p_{bT},y_{bT}^i)$ only if $\eta_T^i = 1$. Thus

$$\hat{\eta}^i = \begin{cases} 0 \text{ if } \eta^i = 0 \text{ and } (p_{aT},y_{aT}^i) = (p_{bT},y_{bT}^i)\text{; and} \\ \eta_T^i \text{ otherwise} \end{cases}$$

The result now follows from Lemma 3.1.

<u>3.11 Corollary</u>: Let f be a function on M and let $\mathfrak{F}$ be the data structure with $f^i = f$ for all i. Then $\mathfrak{F}$ is admissible if and only if either

i) f is trivial; or

ii) $f(p,y) \neq f(p\ ,y')$ whenever $p \neq p'$, and for some i, either $y^i = y'^i$ or $py'^i \neq 0$.

<u>3.12 Remarks</u>: In the notation of Theorem 3.10(B), every equilibrium for $\mathfrak{F}$ is an expectations equilibrium for $\mathfrak{F}^*$, but the converse is false. For example, consider an environment with one informed agent and one uninformed agent, and let the data structure be the projection $(p,y) \to (p,y^i)$ for each i. The characteristics of each agent can be chosen so that if the uninformed agent remains uninformed, $(p_a,y_a) = (p_b,y_b)$, but if he becomes informed, $p_a \neq p_b$. In this example there are two expectations equilibria, but only the former is an equilibrium.

4. Proof of Necessity in Theorem 3.5

The strongest result stated in the previous section is the necessity assertion in Theorem 3.5. Necessity in Theorem 3.10 followed almost directly. In this section, we will prove that 3.5(i) or (ii) is necessary for admissibility even if S is replaced by a much smaller class of stochastic environments. Specifically, state dependent utility functions will be required to be in the form of a linear function plus a log-linear function, and each agent's initial information will consist of only the information revealed by his endowment. This will emphasize that in the absence of 3.5(i) and (ii), expectations equilibria may fail to exist even for otherwise well-behaved environments.

4.1 Definitions: For each i, let L^i denote the set of utility functions u^i which can be written $u^i(x^i) = \Sigma_{j=1}^{J}(\alpha_j^i x_j^i + \beta_j^i \ln x_j^i)$ for some $\alpha^i \in \mathrm{int}R_+^L$ and $\beta^i \in \mathrm{int}R_+^L$. Thus L^i can be identified with $\mathrm{int}R_+^L \times \mathrm{int}R_+^L$, and elements of L^i will be denoted by the parameters (α^i, β^i) or by the utility function u^i. Let $L = \Pi_{i=1}^N L^i$, let $E^o = \{(\omega,u)\in E\colon\ u \in L\}$, and let $S^o = \{(\eta_o,\lambda,e_a,e_b) \in S\colon\ e_a,e_b \in E^o$, and for each i, $\eta^i = 1$ if and only if $\omega_a^i \neq \omega_b^i\}$. For the remainder of this section, we will be concerned only with stochastic environments in S^o. For such environments, the initial information structure η_o is determined by e_a and e_b, so elements of S^o can be denoted (λ,e_a,e_b).

It should be noted that the individual excess demand function determined by each $(\omega^i, u^i) \in \Omega^i \times L^i$ is smooth and has the gross substitutes property.

4.2 Lemma: Let $\mathfrak{F}$ be an admissible data structure and let $m_a, m_b \in M$ with $f^1(m_a) = f^1(m_b)$. Let $e_a, e_b \in E^o$ such that

i) $\mu(e_a) = m_a$, and $\mu(e_b) = m_b$; and

ii) $\omega^1{}_a = \omega^1{}_b$, and $\omega^i{}_a \neq \omega^i{}_b$ for each $i \geq 2$.

Then for some $0 < \lambda^o < 1$, $f^1 \cdot \mu((\omega^1_a, \lambda^o u^1_a + (1 - \lambda^o) u^1_b), e^2_a, \ldots, e^N_a) = f^1 \cdot \mu((\omega^1_b, \lambda^o u^1_a + (1 - \lambda^o) u^1_b)\ e^2_b, \ldots, e^N_b)$.

If, in addition,

iii) $\mu(e^1_b, e^2_a, \ldots, e^N_a) = m_a$

then $f^1 \mu(e_a, e^2_b, \ldots, e^N_b) = f^1(m_a)$

Proof: For any $0<\lambda<1$, consider the stochastic environment $s = (\lambda, e_a, e_b)$. Since $f^1 \cdot \mu(e_a) = f^1 \cdot \mu(e_b)$. There is no expectations equilibrium with $\hat{\eta}^1 = 1$. Since $\mathfrak{F}$ is admissible, we must have an expectations equilibrium with $\hat{\eta}^1 = 0$, which requires that $f^1 \cdot \mu \cdot r_a(s) = f^1 \cdot \mu \cdot r_b(s)$. The first assertion now follows from the definition of r_a and r_b.

To prove the second assertion, note that for some $\gamma > 1$, $\gamma u^1_a + (1-\gamma) u^1_b \in L^1$. Let $e'^1_a = (\omega^1_a, \gamma u^1_a + (1-\gamma) u^1_b)$, let $e'_a = (e'^1_a, e^2_a, \ldots, e^N_a)$, and let $(p_a, y_a) = m_a$. Since $\omega^1{}_a = \omega^1{}_b$, (i) and (iii) imply that y^1_a maximizes both $u^1_a(\omega^1_a + y^1)$ and $u^1_b(\omega^1_a + y^1)$ subject to $p_a y^1 \leq 0$. Therefore $\mu(e^1_a) = m_a$. The result can now be obtained using the previous paragraph, by replacing e_a with e'_a and setting $\lambda = 1/\gamma$.

4.3 Remarks: As the proof of 4.2 indicates, the phrase "for some $0<\lambda^{o}<1$", can be strengthened to "for any $0<\lambda<1$". The reason for making the weaker assertion is the following. Suppose the concept of expectations equilibrium is generalized by allowing the message associated with each state to be generated randomly. For example, such an equilibrium might associate with state a the message m_a with probability 1/2 and the message m'_a with probability 1/2. Although it may seem improbable, there exist data structures f and stochastic environments $s \varepsilon S^{o}$ such that (s,f) admitts only expectations equilibria of this more general type. For example let

$f^1:(p,y) \rightarrow p$ and let $(\lambda, e_a, e_b) \varepsilon S^{o}$ such that

i) $\omega_a^1 = \omega_b^1$ and $\omega_a^i \neq \omega_b^i$ for all $i>1$;

ii) $f^1 \cdot \mu(e_a) = f^1 \cdot \mu(e_b) = p$;

iii) $f^1 \cdot \mu \cdot r_a(s) \neq f^1 \cdot \mu \cdot r_b(s)$; and

iv) $f^1 \cdot \mu((\omega_a^1, \lambda' u_a^1 + (1-\lambda')u_b^1), e_a^2, \ldots, e_a^N) = f^1 \cdot \mu((\omega_b^1, \lambda' u_a^1 + (1-\lambda')u_b^1), e_b^2, \ldots, e_b^N) = p' \neq p$, for some $0<\lambda'<\lambda$.

With state a, associate the message $\mu((\omega^1{}_a, \lambda' u_a^1 + (1-\lambda')u^1{}_b), e_a^2, \ldots, e_a^N)$ with probability $\gamma = (\lambda'/\lambda)[(1-\lambda)/(1-\lambda')]$ and the message $\mu(e_a)$ with the probability $1-\gamma$. With state b, associate $\mu((\omega^1{}_b, \lambda' u_a^1 + (1-\lambda')u_b^1), e_b^2, \ldots, e_b^N)$ with probability one. Then each of these messages is sustained as an equilibrium when agent 1 conditions his expectations on the price. Although we will not give details here, Lemma 4.2 can be proved for the generalized expectations equilibrium concept, so Theorem 3.5 would be unaffected by this generalization.

The second assertion of 4.2 states that if f^1 gives the same data for m_a and m_b, it must give the same data for yet a third message. The remainder of the proof is devoted to showing that if $f^1(p,y) = f^1(p',y')$ for some (p,y), (p',y') with $(p,y^1) \neq (p',y'^1)$ then this argument can be iterated to the point of implying that f^1 is trivial.

4.4 Definitions: Let $\omega_o^1 \in \Omega^1$, and define the set $P_1 = \{ (m_a,m_b) \in M_a \times M_b$: for all j, $y_{aj}^1 \neq y_{bj}^1$; $p_{a1} > p_{b1}$; and for some $u^1 \in L^1$, $Du^1 (\omega_o^1 + y_a^1)$ is proportional to p_a and $Du^1(\omega_o^1 + y_b^1)$ is proportional to $p_b\}$, where M_a and M_b are identical copies of M.

A P_1 - chain is a finite formal linear combination of elements of P_1, with coefficients in the integers mod 2. If $c = \Sigma_{n=1}^K(m_{an}, m_{bn})$ is a P_1 - chain, the boundary of c, written ∂c, is defined as the formal linear combination $\Sigma_{n=1}^K m_{an} + \Sigma_{n=1}^K m_{bn}$ of elements of the disjoint union, $M_a \cup M_b$. A chain c is said to connect m_a and m_a' if $\partial c = m_a + m_a'$. The purpose of these definitions is to define an equivalence relation $\sim_1$ on M by letting $m \sim_1 m'$ if there is a P_1 - chain connecting the respective copies m_a and m'_a in M_a. Note that $m_a + m_a = 0$, so the zero chain connects each element of M_a with itself.

4.5 Lemma: Let $M_1 = \{(p,y) \in M\colon\ y^1 + \omega_o^1 \in \mathrm{int}R_+^j\}$. Then M_1 is a $\sim_1$ equivalence class.

Proof: First, we will show that P_1 is an open subset of $M_1 \times M_1$. Let $[(p_a,y_a), (p_b,y_b)] \in P_1$, and let $u_o^1 \in L^1$ be the associated utility function. Consider the function

$$g^1: L^1 \times (\mathrm{int}R^j_* - \{\omega^1_o\}) \times (\mathrm{int}R^j_+ - \{\omega^1_o\}) \to (\Delta \times Y^1) \times (\Delta \times Y^1)$$

defined by $g^1(u^1, y^1, y'^1) = [(\overline{D}u^1(\omega^1_o + y^1), y^1), (\overline{D}u^1(\omega^1_o + y'^1), y'^1)]$, where $\overline{D}u^1$ denotes the normalization of the derivative Du^1 to Δ. Since $y^1_{aj} \neq y^1_{bj}$ for all j, it is easily checked that $Dg^1(u^1_a, y^1_a, y^1_b)$ is surjective, so g^1 is an open map on a neighborhood of (u^1_a, y^1_a, y^1_b). It follows directly that P_1 contains a neighborhood of $[(p_a, y_a), (p_b, y_b)]$.

Second, note that the definition of M_1 guarantees that for each $m_a \varepsilon M_1$, there exists $m_b \varepsilon M_1$ such that $(m_a, m_b) \varepsilon P_1$. Since P_1 is open, this implies that each equivalence class in M_1 is open. Third, M_1 is path-connected so there can be only one equivalence class.

<u>4.6 Definitions:</u> Let $\omega_o \varepsilon \Omega$, with ω^1_o as given in 4.4. Let $(p_o, y_o) \varepsilon M_1$ (4.5), and let $L^1_o = \{u^1 \varepsilon L^1 : Du^1(\omega^1_o + y^1_o)$ is proportional to $p_o\}$. Let $P_2 = \{(m, m') \varepsilon M_1 \times M_1$: for each i and each j, $y^i_j \neq y'^i_j$; and $m = \mu(e)$ and $m' = \mu(e')$ for some $e, e' \varepsilon E^o$ with

i) $\omega^1 = \omega'^1 = \omega^1_o$, and $u^1, u'^1 \varepsilon L^1_o$; and

ii) for each $i \geq 2$, $e^i = e'^i$ with $\omega^i \neq \omega^i_o$ $\}$.

A P_2-chain is a finite formal linear combination of elements of P_2, with coefficients in the integers mod 2. If c is a P_2-chain, define ∂c to be the formal linear combination $\Sigma^K_{n=1}(m_n + m'_n)$ of elements of M, written in this way to emphasize that if $m_n = m'_{n'}$, then $m_n + m'_{n'} = 0$. Define the equivalence relation $\sim_2$ on M by setting $m \sim_2 m'$ if there is a P_2-chain c with $\partial c = m + m'$.

4.7 Lemma: Let $M_2 = \{(p,y) \in M_1 : (p,y') \neq (p_o, y_o^1)$ and $Du^1(\omega_o + y^1)$ is proportional to p for some $u^1 \in L_o^1\}$. Then if $J \geq 3$, M_2 is a $\sim_2$ equivalence class. If $J=2$, the sets $\{(p,y) \in M_2 : p_1 > p_{o1}\}$ and $\{(p,y) \in M_2 : p_1 < p_{o1}\}$ are each $\sim_2$ equivalence classes.

Proof: The proof will parallel the proof of 4.5. First, an argument similar to the first step of 4.5 shows that P_2 is a relatively open subset of $M_2 \times M_2$. Second, let $(p,y) \in M_2$, and let $e \in E^o$ such that

i) $\mu(e) = (p,y)$;

ii) $e^1 = (\omega_o^1, u^1)$ for some $u^1 \in L_o^1$; and

iii) For each $i \geq 2$, $\omega^i \gg \omega_o^i$, $\alpha^i = p$, and $\beta_j^i/(\omega_j^i + y_j^i) = p_j$ for each j.

By the definition of M_2, $(p,y^1) \neq (p_o, y_o^1)$, so there is some $u'^1 \in L_o'$ such that if $(p',y') = \mu((\omega_o^1, u'^1), e^2, \ldots, e^N)$ then $(p',y') \in M_2$ and $p' \neq p$. Then for each $i \geq 2$, $y^i \neq y'^i$, we can assume that $y_1^2 \neq y'^2_1$. Let $p'' \in \Delta$ with $p''_1 > p'_1$ and $(p''_2, \ldots, p''_J)$ proportional to $(p'_2, \ldots, p'_J)$. Let y''^1 denote the excess demand at p'' determined by the characteristics (ω_o^1, u'^1) and for each $i > 2$, let y''^i denote the excess demand at p'' determined by e^i. By the gross-substitutes property, for each $i \neq 2$, and $y''^i_j > y''^i_j$ for each $j > 1$. Let $y''^2 = -\Sigma_{i \neq 2} y''^i$. Then for p'' sufficiently near p', $(\omega^2 + y''^2) >> 0$. We will show that for p'' sufficiently near p', characteristics $e'^2 = (\omega'^2, u'^2)$ can be constructed so that $\omega'^2 >> \omega^2$, $\mu((\omega_o^1, u^1), e'^2, e^3, \ldots, e^N) = (p,y)$, and

$\mu((\omega_o^1, u'^1), e'^2, e^3, \ldots, e^N) = (p'', y'')$. For each $j \geq 2$, let

$\rho_j(\omega_j^2) = [1 + (\omega_j^2 + y_j^2)/(\omega_j^2 + y''^2_j)]/[1 + (\omega_j^2 + y_j^2)/(\omega_j^2 + y'^2_j)]$.

Let $k = \min\{\rho_j(\omega_j^2): j \geq 2\}$. Since $y''^2_j < y'^2_j$ for each $j \geq 2$, there is some $\omega'^2_j \geq \omega_j^2$ such that $\rho_j(\omega'^2_j) = k$ for each $j \geq 2$. Let $\omega'^2_1 = \omega_1^2$; and for each $j \geq 2$, let $\beta'^2_j = p_j/(\omega'^2_j + y_j^2)$, and $\alpha'^2_j = \alpha^2_j$. Finally, since $y_1^2 \neq y'^2_1$, it follows that for p'' sufficiently near p', there exist α'^2_1 and β'^2_1 such that $\alpha'^2_1 + \beta'^2_1/(\omega'^2_1 + y_1^2) = p_1$ and $\alpha'^2_1 + \beta'^2_1/(\omega'^2_1 + y''^2_1) = [\alpha'^2_2 + \beta'^2_2/(\omega'^2_2 + y''^2_2)](p''_1/p''_2)$. This completes the construction of e'^2. Then $y''^i_j \neq y'^i_j$ for all i and all $j \neq 1$, and with a further slight increase in p''_1, if necessary, we can assume that e'^2 is chosen so that $y''^i_j \neq y^i_j$ for all $j > 1$. If $y''^i_1 = y^i_1$ for some i, the above process can be repeated to increase p''_2 slightly, so we have shown that there is some $(p'', y'') \in M_2$ such that $[(p,y), (p'',y'')] \in P_2$.

Third, we consider the path components of M_2. Let $(p,y), (p',y') \in M_2$. If $J \geq 3$, let $p(\cdot):[0,1] \to \Delta$ be a path from p to p' with $p(t) \neq p_o$ for all $t \in [0,1]$. Let $e(\cdot):[0,1] \to E^o$ be a path in E^o with the properties that for each t, $p(t)$ is the equilibrium price for $e(t)$, and $e^1(t) \in L_o^1$ for each t. Then $\mu \cdot e(\cdot)$ describes a path in M_2 from (p,y) to (p',y'), so if $J \geq 3$, M_2 is connected and the result follows as in 4.4. If $J = 2$ and $p_1 > p_{o1} > p'_1$, then any path from p to p' in Δ passes through p_o, so there is no path in M_2 from (p,y) to (p',y'). In this case M_2 has two path components: $\{(p,y) \varepsilon M_2 : p_1 < p_o\}$ and $\{(p,y) \varepsilon M_2 : p_1 > p_{o1}\}$. Then each of these is contained in a $\tilde{}_2$ equivalence class. The gross substitutes condition is easily seen to imply that there is no element $[(p,y),(p',y')]$ of P_2 with $p_1 > p_{o1} > p'_1$ or $p_1 < p_{o1} < p'_1$, which completes the proof.

4.8 Proposition: Let $\mathfrak{F}$ be an admissible data structure and let $(m_a,m_b) \varepsilon M$ with $f^1(m_a) = f^1(m_b)$. If $(p_a,y'_a) \neq (p_b,y'_b)$, then f^1 is trivial.

Proof: Let e_a, $e_b \varepsilon E^o$ with $\mu(e_a) = m_a$, $\mu(e_b) = m_b$, $\omega^1_a = \omega^1{}_b$, and $\omega^i_a \neq \omega^i_b$ for $i \neq 1$. Note for reference in the last paragraph that ω^1_a can be chosen arbitrarily large. Now let $0<\lambda^o<1$, $e^* = ((\omega^1_a, \lambda^o u^1_a + (1-\lambda^o)u^1_b), e^2_a,\ldots,e^N_a)$, $e = ((\omega^1_a, \lambda^o u^1_a + (1-\lambda^o)u^1_b), e^2_b,\ldots,e^N_b)$, $(p^*,y^*) = m^* = \mu(e^*)$, and $(p,y) = m = \mu(e)$. By 4.2, $f^1(m^*) = f^1(m')$. Since $(p_a,y^1_a) \neq (p_b,y^1_b)$, u^1_a and $(e^2_b,\ldots,e^N_b)$ can be chosen so that $p^* \neq p$. Now let $\bar{e}$ in E^o such that

i) $\mu(e) = (p,y)$;

ii) $e^1 = (\omega^1_a, u^1)$ for some $u^1 \varepsilon L^1$ for which $Du^1(\omega^1{}_b + y^{*1}_a)$ is proportional to p_a, (e.g. $e^1 = e^{*1}$); and

iii) for each $i \geq 2$, $\omega^i >> \omega^{*i}$, $\alpha^i = p$, and $\beta^i_j/(\omega^i_j + y_j) = p_j$ for each j.

By the second step in the proof of Lemma 4.6, there exists e' with $\omega'^1 = \omega^1_a$, $\omega'^i \neq \omega^i_a$ for each $i \geq 2$, $\mu(e'^1, e^2_a, \ldots, e^N_a) = (p^*, y^*)$, $\mu(e^1, e'^2, \ldots, e'^N) = (p, y)$; and $\mu(e'^1, e'^2, \ldots, e'^N) = (p'', y'')$ with $[(p,y), (p'',y'')] \varepsilon P_2$, $\omega_o = \omega^*_o$ and $(p_o, y_o) = (p^*, y^*)$. Using the information contained in the preceding sentence, and the fact that $f^1(p^*,y^*) = f^1(p',y')$, the second assertion of 4.2 can be applied (with $m_a = (p^*,y^*)$ and $m_b = (p,y)$) to obtain $f^1(p^*,y^*) = f^1(p'',y'')$. Since P_2 is relatively open, we may choose (p'',y'') so that $y''^1_j \neq y^{*1}_j$ for all j. Returning to the notational origin, the above argument shows that we can assume without loss of generality that $(m_a, m_b) \varepsilon P_1$, with $\omega'_o = \omega'_a$ as chosen above.

Now choose m'_a arbitrarily in M. The proof will be completed by showing that $f^1(m'_a) = f^1(m_a)$. We can assume that ω'_a is sufficiently large so that $m'_a \varepsilon M_1$, with $\omega'_o = \omega'_a$ Therefore, let $\Sigma^K_{n=1}(m_{an}, m_{bn})$ be a P_1 - chain connecting m_a and m'_a. Subtracting a chain with boundary zero, if necessary, we can assume that $m_{a1} = m_a$, $m_{b2} = m_{b1}$, $m_{a3} = m_{a2}$; etc. and $m_{an} = m'_a$. Since (m_a, m_b) and $(m_{a1}, m_{b1}) = (m_a, m_{b1})$ are each in P_1, m_b and m_{b1} are in the same $\sim_2$ equivalence class of M_2, with $\omega_o = \omega_a$ and

$(p_o, y_o^1) = (p_a, y_a^1)$. Therefore, let $\Sigma_{n=1}^{K'}(m_n, m_n')$ be a P_2-chain connecting m_b and m_{b1}, with $m_1 = m_b$, $m_2' = m_1'$, $m_3 = m_2$ etc. Applying 4.2 and the definitions of P_1 and P_2, we obtain $f^1(m_a) = f^1(m_b) = f^1(m_1') = f^1(m_2) = \ldots = f^1(m_{b1})$. Let $(\omega_o, u) \in E^o$, with $\omega_o^1 = \omega_a^1$, as before, and $\mu(\omega_o, u) = m_{b1}$. Since (m_a, m_{b1}) and (m_{a2}, m_{b1}) are each in P_1, m_a and m_{a2} are in the same $\sim_2$ component of M_2, with ω_o just defined and $(p_o, y_o) =$ (p_{b1}, y_{b1}). Then $f^1(m_a) = f^1(m_{a2})$, and continuing in this fashion we obtain $f(m_a) = f^1(m_a')$.

This proves that f^1 is constant on the space M_1, which is determined by ω_a^1. But ω_a^1 can be chosen to be arbitrarily large, and M_1 increases to M as ω_a^1 diverges to infinity in each component. This completes the proof.

4.9 Remarks: The result obtained here is not quite a formal generalization of the characterization in [3], because the latter was obtained for log-linear utility functions. I do not know if Proposition 4.8 is valid when L^i is replaced by the set of log-linear functions.

For each $\delta \gg 0$ in R^J, let L_δ^i denote the set of utility functions $u^i: R_+^J \to R$ which can be written $u^i(x^i) = \Sigma_{j=1}^J [\alpha_j^i(x_j^i + \delta_j) + \beta_j^i \ln(x_j^i + \delta_j)]$ for some $\alpha^i \in \mathrm{int} R_+^J$ and $\beta^i \in \mathrm{int} R_+^J$, for each i. Then L_δ^i is a set of continuous, strictly concave, and strictly increasing functions on R_+^J. Since utility functions in L_δ^i have indifference curves which intersect the boundary of R_+^J, $L_\delta^i \cap U^i = \emptyset$. Let $L_\delta = \Pi_{i=1}^N L_\delta^i$, and let $L' = \cup \{L_\delta : \delta \gg 0\}$. The following Lemma, whose straightforward proof is omitted, indicates that Proposition 4.8 can be obtained with L replaced by L'. It follows that the results in section 3 would be unaffected if X^i were redefined to be R^J_+, and U^i were redefined to be the set of continuous, strictly concave, and strictly increasing functions on R^J_+. (The same assertion can be made independently for Proposition 6.4 below.)

<u>4.10 Lemma:</u> Let $\{\omega_k\}_{k=1}^K \subset \Omega$ and $\{u_k\}_{k=1}^K \subset L$. Then there exists $\delta \gg 0$ in R^J such that if

i) $\omega'^i_k = \omega^i_k - \delta$ for each i, k; and

ii) $u'^i_k(x^i) = u^i_k(x^i + \delta)$ for each i, k,

then $\{\omega'_k\}_{k=1}^K \subset \Omega$ and $\{u'_k\}_{k=1}^K \subset L_\delta$; and for each $\gamma^i \in R^K_+$ with $\Sigma_k \gamma^i_k = 1$, $1 \leqq i \leqq N$, and each k, m is a competitive equilibrium for the exchange environment $(\omega^i_k, \Sigma_k \gamma^i_k u^i_k)_{i=1}^N$ if and only if m is a competitive equilibrium for $(\omega'^i_k, \Sigma_k \gamma^i_k u'^i_k)_{i=1}^N$.

5. Pre-equilibrium Trading

The pre-equilibrium trades y_{at} and y_{bt}, $t < T$, in a temporary equilibrium sequence $\{(p_{at}, y_{at}), (p_{bt}, y_{bt})\}$ are not consumated, and have only an informational influence on the final allocations $\omega_a + y_{at}$ and $\omega_b + y_{bt}$. It is natural to consider the alternative process in which the trades proposed at each iteration are made, so that at the t^{th} stage, the i^{th} agents endowments are $\omega^i_{at} = \omega^i_a + \Sigma^{t-1}_{s=1} y^i_{as}$ and $\omega^i_{bt} = \omega^i_b + \Sigma^{t-1}_{s=1} y^i_{bs}$. However, this process introduces a speculative aspect to pre-equilibrium trading. For example, in state a, an informed agent's pre-equilibrium trades would be made with the objective of maximizing $p_{aT}\omega^i_{aT}$ rather than $u^i_{at}(\omega^i_a + y^i_{at})$. The trades of uninformed agents would be analogously influenced. Thus temporary equilibrium trades will be influenced by agents' expectations of eventual equilibrium prices. If $T = 2$, the resulting feedback sequence can be depicted as follows:

$$(p_1, y_1) \to (y_1, \eta_1) \to p_2 \to (p_1, y_1),$$

where the final arrow represents the speculative aspect of this process, and each variable represents a pair of variables, one for each state. The following example shows that the presence of the discontinuous variable η_1 in this feedback can prevent the existence of temporary equilibria.

There are two agents and two commodities. In each state, each agents utility function is of the form $u^i(x^i) = \alpha^i \ln x^i_1 + (1 - \alpha^i)\ln x^i_2$. Agent 1, who is informed, is described by the characteristics

1) $\omega_a^1 = (1,1)$; $\omega_b^1 = (3,1)$; $\alpha_a^1 = \frac{1}{2}$; $\alpha_b^1 = 1/3$.

Agent 2, who is uninformed is described by

2) $\eta_b^2 = 0$; $\omega_a^2 = \omega_b^2 = (1,1)$; $\alpha_a^2 = \frac{1}{2}$; $\alpha_b^2 = 1/3$.

Finally, let λ, the probability of state a, equal $\frac{1}{2}$. Agent 2's data function is the projection $(p,y) \to p$. Since there is only one uninformed agent, we can set $T = 2$, so that we are seeking a temporary equilibrium sequence $\{(p_{at}, y_{at}), (p_{bt}, y_{bt})\}_{t=1}^{2}$. Suppose that p_{a2} and p_{b2} have been determined. Initially, agent 1 knows the state, and thus knows whether p_{a2} or p_{b2} will be the eventual equilibrium. It follows easily that if $p_{a2} \neq p_{b2}$, we must have $p_{a1} \neq p_{b1}$, even if short trades are bounded by requiring $\omega_a^i + y_a^i \in X_1$ and $\omega_b^i + y_b^i \in X^i$ for each i. However, if $p_{a1} \neq p_{b1}$ then agent 2 is informed at the second stage of this process, $(\eta_1^2 = 1)$. Since $\alpha_a^1 = \alpha_a^2$ and $\alpha_b^1 = \alpha_b^2$, we would then have $p_{a2} = p_{b2} = (\frac{1}{2}, \frac{1}{2})$, regardless of the initial trades y_{a1} and y_{b1}. Thus we have established a contradiction $p_{a2} \neq p_{b2} \Rightarrow p_{a2} = p_{b2}$, so it remains to consider the case $p_{a2} = p_{b2}$. However if $p_{a2} = p_{b2}$, then both agents initially know the final equilibrium price, so $p_{a1} = p_{b1} = p_{a2} = p_{b2}$. Then $\eta_1^2 = 0$, and since $p_1 y_1^i = 0$ for each i, stage 1 trades will have no effect on the stage 2 income of either agent in either state. Since $\eta_1^2 = 0$, and $p_{a2} = p_{b2}$, we must have $y_{a2}^2 = y_{b2}^2$. However, it is easily seen that there is no price at which the excess demand determined by (ω_a^1, α_a^1) and (ω_b^1, α_b^1) would be the same, which completes the contradiction.

6. The Many Event Case

Suppose that the class of stochastic environments is enlarged by admitting the possibility of more than two events. If only finitely many events are admitted, the definitions in section 2 have obvious generalizations. If Σ is the set of future states, an information structure can be modelled as an N-tuple of partitions of $\Omega \times \Sigma$. A sequence of temporary equilibria would sucessively refine each agent's partition, and an equilibrium would be reached in a finite number of stages. However, if Σ is an infinite set, examples are easily constructed for which an equilibrium is not reached in any finite number of stages. A temporary equilibrium associates a message with each (ω,σ), so it might seem natural to define an equilibrium as the function which associates with each (ω,σ) the limit of an associated sequence of temporary equilibrium messages. However, the existence of equilibrium would then depend on the existence of this limit, which would not in general be an informational issue. Although the messages associated with each state may not be convergent, for each i, the information sequence $\{\eta^i_t\}^\infty_{t=0}$ (of either partitions or Borel fields) increases to its least upper bound, η^i_*. Consider a function which associates with each (ω,σ) a competitive equilibrium message for utility functions conditioned on the information η^i_* for each i. If the resulting data do not increase any agent's information, we will call this function an equilibrium. **These definitions are stated formally in 6.3. Where appropriate, the notation of section 2 will be recycled.**

6.1. Notation: Given a metric space Z, $\beta(Z)$ denotes the Borel field of subsets of A, and $\mathcal{M}(Z)$ denotes the space of Borel probability measures on Z, endowed with the topology of weak convergence. Unless otherwise noted, all functions will be assumed to be Borel measurable functions taking values in a Borel subset of a complete separable metric space. Given functions h_1 and h_2, h_1 will be said to be h_2-measurable if there exists a function h_3 such that $h_1 = h_3 \cdot h_2$. Given an indexed collection of functions $h_2 : Z \to Z_\alpha$, $\alpha \in A$, the function $\vee_\alpha h_\alpha$ is the function $(h_\alpha)_{\alpha \in A} : Z \to \prod_{\alpha \in A} Z_\alpha$. If $\varphi \in \mathcal{M}(Z)$, and $\mathcal{P}(\cdot)$ is a property of elements of Z, the statement $\varphi\{z : \mathcal{P}(z)\} = 1$ will be written $\mathcal{P}(z)[\varphi]$. If there is a function h_3 such that $h_1 = h_3 \cdot h_2$ $[\varphi]$, h_1 will be said to be h_2 measurable $[\varphi]$.

6.2 Definitions: Let Σ be a compact metric space containing more than one element. For each i, let V^i denote the set of utility functions $v^i : X^i \times \Sigma \to R$ such that

i) v^i is continuous; and

ii) for each $\sigma \in \Sigma$, $v^i(\cdot,\sigma) \in U^i$.

Let $V = \prod_{i=1}^N V^i$. An information structure is an N-tuple $\eta = (\eta^1,\ldots,\eta^N)$ of functions on $\Omega \times \Sigma$ with the property that for each i, the projection $(\omega,\sigma) \to \omega^i$ is η^i-measurable. A stochastic environment s consists of an initial information structure η, a probability measure $\varphi \in \mathcal{M}(\Omega \times \Sigma)$, and an N-tuple of utility functions $v \in V$. The set of stochastic environments is again denoted S. A data structure is an N-tuple $\mathcal{F} = (f^1,\ldots,f^N)$ of functions on M.

A sequence of temporary equilibria for a stochastic environment $s = (\eta, \varphi, v)$ and a data structure $\mathcal{F}$ is a sequence of information structures η_t, $t \geq 0$, and functions $g_t : \Omega \times \Sigma \to M$, $t \geq 1$, such that

i) $\eta_0 = \eta$; and for each i and each $t \geq 1$, $\eta_t^i = \eta_{t-1}^i \vee (f^i \cdot g_t)$; and

ii) for each $t \geq 1$, $g_t(\omega,\sigma) \in \mu([\omega^i, E\{v^i|\eta_{t-1}^i(\omega,\sigma)\}]_{i=1}^N)$ $[\varphi]$, where $E\{v^i|\eta^i(\omega,\sigma)\}$ denotes the utility function $x^i \to E\{v^i(x^i,\cdot)|\eta^i(\omega,\sigma)\}$.

For each i , let $\eta_*^i = \vee_{t=0}^{\infty} \eta_t^i$. Suppose there is a function $g_* : \Omega \times \Sigma \to M$ such that

iii) $g_*(\omega,\sigma) \in \mu([\omega^i, E\{v^i|\eta_*^i(\omega,\sigma)\}]_{i=1}^N)$ $[\varphi]$,

and

iv) for each i , $f^i \cdot g_*$ is η_*^i-measurable $[\varphi]$.

Then g_* is an <u>equilibrium</u> for $(s,\mathcal{F})$.

An <u>expectations equilibrium</u> for $(s,\mathcal{F})$ is a function $g : \Omega \times \Sigma \to M$ such that $g(\omega,\sigma) \in \mu([\omega^i, E\{v^i|\hat{\eta}^i(\omega,\sigma)\}]_{i=1}^N$ $[\varphi]$, where for each i , $\hat{\eta}^i = \eta \vee (f^i \cdot g)$.

<u>6.3 Remarks</u>: The compactness of Σ, together with the continuity of the utility functions v^i, insures the existence of expected utility. Condition (4) is the informational stationarity condition.

6.4 Remarks: A two event stochastic environment $(\eta_o, \lambda, e_a, e_b)$ can of course be identified with a stochastic environment (η, φ, v) by choosing $\sigma_a \neq \sigma_b \in \Sigma$, and defining

i) $\varphi(\{(\omega_a, \sigma_a)\}) = \lambda$ and $\varphi(\{(\omega_b, \sigma_b)\}) = 1-\lambda$;

ii) let $r = d(\sigma_a, \sigma_b)$, where d is the metric on Σ, and for each i, define $v^i : X^i \times \Sigma \rightarrow R$ by
$v^i(x^i, \sigma) = [d(\sigma, \sigma_b)/r]u_a^i(x^i) + [d(\sigma, \sigma_a)/r]u_b^i(x^i)$, for each $(x^i, \sigma) \in X^i \times \Sigma$; and

iii) for each i, let η^i be trivial if $\eta_o^i = 0$, and let η^i be the identity on $\Omega \times \Sigma$ if $\eta_o^i = 1$.

Also, it is easily checked that (m_a, m_b) is an equilibrium for $(\eta_o, \lambda, e_a, e_b)$ as defined in 2.6, if and only if (η, φ, v) has an equilibrium g_*, as defined in 6.2, with $g_*(\omega_a, \sigma_a) = m_a$ and $g_*(\omega_b, \sigma_b) = m_b$. The exactly analogous statement holds for expectations equilibria.

6.5 Definitions: A data structure $\mathcal{F}$ is admissible if for each stochastic environment s, there exists an expectations equilibrium for $(s, \mathcal{F})$. If there exists a data structure $\mathcal{F}'$ such that for each $s \in S$, every equilibrium for $(s, \mathcal{F})$ is an expectations equilibrium for $(s, \mathcal{F}')$, then $\mathcal{F}$ is eventually admissible.

6.6 Remarks: The above definition of admissibility agrees with 3.3. The above definition of eventual admissibility does not directly correspond to 3.3, but Theorem 3.10 B establishes that the above definition, restricted to the two event case, is equivalent to 3.3. The advantage of the above definition in the many event case is that it does not require the existence of equilibrium.

The inclusion of the two event case described in 6.4 indicates that the necessity assertions of Theorems 3.5 and 3.10 extend directly to the model described in 6.2. We now extend the sufficiency assertions, using essentially the same reasoning used in section 3.

6.7 Lemma: Let h_1 and h_2 be functions on $\Omega \times \Sigma$ and let $\varphi \in \mathfrak{M}(\Omega \times \Sigma)$. Suppose that for each (ω,σ) , $(\omega',\sigma') \in \Omega \times \Sigma$, $h_2(\omega,\sigma) = h_2(\omega',\sigma')$ implies $h_1(\omega,\sigma) = h_1(\omega',\sigma')$. Then h_1 is h_2 measurable $[\varphi]$.

Proof: Let K be a compact subset of $\Omega \times \Sigma$ such that h_1 and h_2 are continuous on K . Then there is a continuous function c on $h_1(K)$ such that $h_2|K = c \cdot h_1|K$. By Lusin's Theorem [2 , p.244] , there is a Borel set $C \subset \Omega \times \Sigma$ such that $\varphi(C) = 1$ and C is a countable union of compact sets on which h_1 and h_2 are continuous. Therefore $h_1(C)$ is a Borel set, and there is a Borel measurable function h_3 on $h_1(C)$ such that $h_3|C = h_3 \cdot h_2|C$. This completes the proof.

6.8 Theorem:

A. A data structure $\mathcal{F}$ is admissible if and only if for each i, either

i) f^i is trivial; or

ii) $f^i(p,y) \neq f^i(p',y')$ whenever $(p,y^i) \neq (p',y'^i)$.

B. A data structure $\mathcal{F}$ is eventually admissible if and only if for each i, either

i) f^i is trivial; or

ii) $f^i(p,y) \neq f^i(p',y')$ whenever $p \neq p'$ and the weak axiom is satisfied for agent i.

C. If $\mathcal{F}$ is eventually admissible, let $\mathcal{F}^*$ be a data structure such that for each i,

i) if f^i is trivial then f^{*i} is trivial; and

ii) if f^i is not trivial then f^{*i} is the projection $(p,y) \to (p,y^i)$.

Then for each stochastic environment s, every equilibrium for $(s,\mathcal{F})$ is an expectations equilibrium for $(s,\mathcal{F}^*)$.

Proof: For (A), as noted in 6.5, it only remains to prove sufficiency. Let $\mathcal{F}$ satisfy (A.i) or (A.ii) for each i, and let $s = (\eta,\varphi,v) \in S$. Let η' be the information structure such that for each i, $\eta'^i = \eta^i$ if f^i is trivial, and η'^i is the identity on $\Omega \times \Sigma$ if

f^i is not trivial. Let $g : \Omega \times \Sigma \to M$ be a Borel measurable selection from the correspondence $(\omega,\sigma) \twoheadrightarrow \mu([\omega^i, E\{v^i|\eta'^i(\omega,\sigma)\}]_{i=1}^N)$. For each i , let $\hat{\eta}^i = \eta^i \vee (f^i \cdot g)$. If f^i is trivial, $\hat{\eta}^i = \eta'^i = \eta^i$, so to show that g is an expectations equilibrium for $(s,\mathcal{F})$, it suffices to show that for a.e. (ω,σ) , if $(p,y) = g(\omega,\sigma)$ then y^i maximizes $E\{v^i(\omega^i + y'^i)|\hat{\eta}^i(\omega,\sigma)\}$ subject to $p\,y'^i \le 0$ for each i such that f^i is not trivial. For any i such that f^i is not trivial, let $\pi^i : M \to \Delta \times Y^i$ be the projection and let $g^i = \pi^i \cdot g$. By Lemma 6.7, g^i is $\hat{\eta}^i$ measurable $[\varphi]$. For a.e. (ω,σ) , y^i maximizes $E\{v^i(\omega^i + y'^i,\cdot)|\eta'^i(\omega,\sigma)\}$ subject to $p\,y'^i \le 0$, where $(p,y^i) = g^i(\omega,\sigma)$. Since g^i is $\hat{\eta}^i$ measurable $[\varphi]$, it follows that y^i maximizes $E\{v^i(\omega^i + y^i,\cdot)|\hat{\eta}^i(\omega,\sigma)\}$ subject to $p\,y'^i \le 0$ $[\varphi]$, which proves sufficiency in (A).

For (B) also, only sufficiency remains to be proved. As in the proof of Theorem 3.10, we will establish (B) and (C) by showing that if $\mathcal{F}$ satisfies (B.i) or (B.ii) for each i , and $s = (\eta,\varphi,v) \in S$, then every equilibrium for $(s,\mathcal{F})$ is an expectations equilibrium for $(s,\mathcal{F}^*)$, where $\mathcal{F}^*$ satisfies (C.i) and (C.ii) for each i . Let g_* be an equilibrium for $(s,\mathcal{F})$, and let $g_*^i = \pi^i \cdot g_* = f^{*i} \cdot g_*$ for each i . We first show that for each i such that f^i is not trivial, g_*^i is η_*^i measurable $[\varphi]$, where η_*^i is defined in 6.2. Let $(\omega,\sigma), (\omega',\sigma') \in \Omega \times \Sigma$ with $\eta_*^i(\omega,\sigma) = \eta_*^i(\omega',\sigma')$, and let $(p,y^i) = g_*^i(\omega,\sigma)$ and $(p',y'^i) = g_*^i(\omega,\sigma)$, for any i such that f^i is not trivial. Then $E\{v^i|\eta_*^i(\omega,\sigma)\} = E\{v^i|\eta_*^i(\omega',\sigma')\}$ and $\omega^i = \omega'^i$ so

(p,y^i) and (p',y'^i) satisfy the weak axiom for i. Also, since the common expected utility function is strictly concave, $y^i \neq y'^i$ only if $p \neq p'$. However, (B.ii) and (6.2.iv) imply that $p = p'$, so $g^i_*(\omega,\sigma) = g^i_*(\omega',\sigma')$. Therefore Lemma 6.7 implies that g^i_* is η^i_* measurable $[\varphi]$. Repeating the argument in the first paragraph above, with η^i_* in place of η'^i, completes the proof.

6.9 Remarks: Corollaries 3.7 and 3.11 also extend to the present context.

We now consider the existence of equilibrium. First, it is necessary to assume that data functions are continuous to ensure that the equality $f^i(m) = f^i(m')$ is a closed condition. Otherwise, even if g_t converged pointwise everywhere to g_*, $f^i \cdot g_*$ might distinguish events not distinguished by $f^i \cdot g_t$ for any t. If the support of φ is countable, the sequence $\{g_t\}$ will have a pointwise a.e. convergent subsequence. The first paragraph of the proof of 6.10 below, combined with the continuity of data functions, shows that the pointwise a.e. limit of any subsequence of $\{g_t\}$ is an equilibrium. For the general case, the following result establishes the existence of equilibria for continuous eventually admissible data structures.

6.10 Proposition: Let $\mathcal{F}$ be an eventually admissible data structure such that f^i is continuous for each i. Then for each stochastic environment s, $(s,\mathcal{F})$ has an equilibrium.

Proof: Let $s = (\eta, \varphi, v)$ be a stochastic environment, and let $\{\eta_{t-1}, g_t\}_{t=1}^{\infty}$ be a sequence of temporary equilibria for $(s, \mathcal{F})$. For any i and any $x^i \in X^i$, let $\{x_t^i\}_{t=1}^{\infty}$ be a sequence in X^i converging to x^i. For each t, define the function $v_t^i : \Omega \times \Sigma \to R$ by $v_t^i = E\{v^i(x_t^i, \cdot) | \eta_t^i\}$, and let $v_*^i = E\{v^i(x^i, \cdot) | \eta_*^i\}$. We will show that the sequence $\{v_t^i\}$ contains a subsequence converging to v_*^i pointwise a.e. $[\varphi]$. It suffices to show that $\{v_t^i\}$ converges in measure to v_*^i [2, Theorem D, p. 93]. Let $\tilde{\eta}_*^i$ and $\tilde{\eta}_t^i$ denote the subfields of $\beta(\Omega \times \Sigma)$ generated by η_t^i, for each t, and η_*^i respectively. For any $\gamma > 0$, let $A \in \tilde{\eta}_*^i$ such that

(*) $\varphi(A) > 0$, and for some number c,

$$|v_*^i(\omega, \sigma) - c| \le \gamma \quad \text{for all} \quad (\omega, \sigma) \in A .$$

Let $\delta = \varphi(A)$, and let $k = 2 \sup \{|v^i(x_t^i, \sigma)| : t \ge 1, \sigma \in \Sigma\}$. Since $\eta_*^i = \vee_t \eta_t^i$, there is some t^o and some $B \in \tilde{\eta}_{t^o}^i$ with $\varphi(A \Delta B) < a$, where $a = \gamma^3/2k^2$, and Δ denotes symmetric difference [2, Theorem D, p. 56]. Since $\tilde{\eta}_t^i \subset \tilde{\eta}_{t+1}^i$ for each t, $B \in \tilde{\eta}_t^i$ for all $t \ge t^o$. Let $t' > t^o$ such that for each $t > t'$, $|v^i(x_t^i, \sigma) - v^i(x^i, \sigma)| < b$ for each $\sigma \in \Sigma$, where $b = ka/(\delta + a)$. For any $t > t'$, let $C = \{(\omega, \sigma) \in B : v_t^i(\omega, \sigma) - c > 2\gamma\}$. By the definitions of v_t^i and v_*^i, $\int_C [v_t^i(.) - v_*^i(\cdot)] d\varphi < b\varphi(c) \le b(\delta + a)$.

Since $\int_C [v_t^i(\cdot) - v_*^i(\cdot)] d\varphi = \int_{C \cap A} [\quad] d\varphi + \int_{C \setminus A} [\quad] d\varphi \ge$

$\ge \gamma\varphi(C \cap A) - ka$, $\varphi(C \cap A) < (1/\gamma)[b(\delta + a) + ka] = \gamma^2/k$.

By partitioning the interval $[-k,k]$ into k/γ subintervals of length 2γ, and applying $(v^i_*)^{-1}$, one obtains a collection of at most k/γ subsets of $\Omega \times \Sigma$ which satisfy (*) and whose union has probability one. Therefore, we have shown that for each $t > t'$, $\varphi(\{(\omega,\sigma) : v^i_t(\omega,\sigma) - v^i_*(\omega,\sigma) > 3\gamma\}) < \gamma$. Similarly, $\varphi(\{(\omega,\sigma) : v^i_t(\omega,\sigma) - v^i_*(\omega,\sigma) < -3\gamma\}) < \gamma$, which proves that $\{v^i_t\}$ converges in measure to v^i_*. Thus any subsequence of $\{v^i_t\}$ contains a pointwise a.e. convergent subsequence.

For each t, let $p_t : \Omega \times \Sigma \to \Delta$ be the function obtained by composing g_t and the projection of M onto Δ. Let $p_* : \Omega \times \Sigma \to \Delta$ be a $\vee^{\infty}_{t=1} p_t$ measurable selection from the correspondence $(\omega,\sigma) \twoheadrightarrow \bigcap^{\infty}_{t^o=1} \mathrm{cl}\{p_{t^o}(\omega,\sigma), p_{t^o+1}(\omega,\sigma),\ldots\}$, where cl denotes closure. Define the function $g_* : \Omega \times \Sigma \to M$ by $g_*(\omega,\sigma) = (p,y)$, where $p = p_*(\omega,\sigma)$, for each $1 \le i < N$, y^i is the excess demand at p determined by ω^i and $E\{v^i | \eta^i_*(\omega,\sigma)\}$, and $y^N = -\Sigma_{i<N} y^i$, for each $(\omega,\sigma) \in \Omega \times \Sigma$. The result of the first paragraph above implies that $g_*(\omega,\sigma) \in \mu([\omega^i, E\{v^i | \eta^i_*(\omega,\sigma)\}]^N_{i=1}$ $[\varphi]$. We will now use Lemma 6.7 to show that $f^i \cdot g_*$ is η^i_* measurable $[\varphi]$ for each i. For any i such that f^i is nontrivial, let (ω,σ), $(\omega',\sigma') \in \Omega \times \Sigma$ such that $\eta^i_*(\omega,\sigma) = \eta^i_*(\omega',\sigma')$. Let $(p,y) = g_*(\omega,\sigma)$ and $(p',y') = g_*(\omega',\sigma')$. By 6.8(B), $\vee^{\infty}_{t=1} p_t(\omega,\sigma) = \vee^{\infty}_{t=1} p_t(\omega',\sigma')$, so the definition of g_* implies that $p = p'$. Let $\{t_k\}$ be an increasing sequence of integers such that $\lim_{k\to\infty} p_{t_k}(\omega,\sigma) = p$. Then since $\vee^{\infty}_{t=1} p_t(\omega,\sigma) = \vee^{\infty}_{t=1} p_t(\omega',\sigma')$, the result of the first paragraph

implies that $\lim_{k\to\infty} g_{t_k}(\omega,\sigma) = (p,y)$ and $\lim_{k\to\infty} g_{t_k}(\omega,\sigma) = (p',y')$.

Since f^i is continuous and $f^i \cdot g_t(\omega,\sigma) = f^i \cdot g_t(\omega',\sigma')$ for all t , $f^i(p,y) = f^i(p',y')$, and the result follows from 6.7.

References

[1] P. Billingsly, Convergence of Probability Measures, John Wiley and Sons, Inc., New York, 1968

[2] P. Halmos, Measure Theory, Van Nostrand-Reinhold, New York, 1950

[3] J. Jordan, "Expectations Equilibrium and Informational Efficiency for Stochastic Environments", Center for Economic Research, University of Minnesota, discussion paper 76-71, August 1976.

[4] K. Mount and S. Reiter, "The Informational Size of Message Spaces", J. Econ. Theory 8 (1974), 161-192.

[5] S. Reiter, "On Expectations Equilibrium", August 24, 1976.