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FOR DISCONTINUOUS FUNCTIONS

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Hanoch and Levy [6] presented a general first degree stochastic dominance theorem for the case in which the Von Neumann-Morgenstern utility function $u(x)$ is non-decreasing in x . In contrast to other papers on this subject, Hanoch and Levy did not assume that utility is differentiable or even continuous.¹ Recently, Tesfatsion [7] has shown that the proof of the Hanoch and Levy first degree stochastic dominance theorem was incorrect. Tesfatsion then stated and proved a similar theorem for utility functions that are non-decreasing and continuous. The usefulness of the above theorems extends beyond the rather simple comparison of expected utilities. For example, optimization problems yield first order conditions involving expectations of derivatives of the objective function. In such problems, a kinked pay-off structure makes it desirable to have a theorem that applies to discontinuous functions as well.

Section I contains a proof of several first degree stochastic dominance theorems for functions with discontinuities. Although the original proof of the Hanoch and Levy theorem was deficient, it is shown in section II that the theorem itself is correct subject to a minor modification. An application of these theorems is illustrated in section III in the context of a simple inventory model.

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I. Some First Degree Stochastic Dominance Theorems

Following Fishburn [3] and others, we will make the distinction between strict and non-strict dominance. Let X_1 and X_2 be two random variables defined on an abstract probability space (Ω, \mathcal{F}, P) .

By definition, X_1 stochastically dominates X_2 if $P\{X_1 > x\} \geq P\{X_2 > x\}$ for all x on the real line R , where $\{X_i > x\} \equiv \{\omega \in \Omega : X_i(\omega) > x\}$, $i = 1, 2$. The dominance is said to be strict if $P\{X_1 > x\} > P\{X_2 > x\} \quad \forall x \in R$, and $P\{X_1 > x_0\} > P\{X_2 > x_0\}$ for some $x_0 \in R$.

An equivalent definition of non-strict dominance that will be useful subsequently is given in the following lemma:

Lemma 1. Let X_1 and X_2 be two random variables. A necessary and sufficient condition for X_1 to stochastically dominate X_2 is that $P\{X_1 \geq x\} \geq P\{X_2 \geq x\} \quad \forall x \in R$.

Proof. (a) For necessity let $x \in R$, and consider sets of the form $A_n = \{X_1 > x - \frac{1}{n}\}$ and $B_n = \{X_2 > x - \frac{1}{n}\}$ where n is a positive integer. By hypothesis $P(A_n) \geq P(B_n)$ and by construction $A_n \supset A_{n+1}$ and $B_n \supset B_{n+1}$, $\forall n$. Therefore $P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} P(A_n) \geq \lim_{n \rightarrow \infty} P(B_n) = P(\bigcap_{n=1}^{\infty} B_n)$. Then necessity follows by noting that $\bigcap_{n=1}^{\infty} A_n = \{X_1 \geq x\}$ and $\bigcap_{n=1}^{\infty} B_n = \{X_2 \geq x\}$.

(b) Sufficiency is proved in a similar manner. Let $A_n = \{X_1 \geq x + \frac{1}{n}\}$ and $B_n = \{X_2 \geq x + \frac{1}{n}\}$. $\{A_n\}$ and $\{B_n\}$ are increasing sequences of sets, and by assumption $P(A_n) \geq P(B_n)$, $\forall n$. Thus $P\{X_1 > x\} = P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} P(A_n) \geq \lim_{n \rightarrow \infty} P(B_n) = P(\bigcup_{n=1}^{\infty} B_n) = P\{X_2 > x\}$.

Q.E.D.

The standard definition of the expected value of a random variable X is:

$$EX \equiv \int_{\Omega} X^+(\omega) dP(\omega) - \int_{\Omega} X^-(\omega) dP(\omega)$$

$$\text{where } X^+(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and } X^-(\omega) = \begin{cases} -X(\omega) & \text{if } X(\omega) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

EX is said to exist as long as $\int_{\Omega} X^+ dP$ and $\int_{\Omega} X^- dP$ are not both infinite. These integrals should be interpreted as generalized Lebesgue integrals.

In the next two lemmas, we show that expected values of random variables are ordered by first degree stochastic dominance relationships.

Lemma 2. Let X_1 and X_2 be random variables such that EX_1 and EX_2 exist. If X_1 stochastically dominates X_2 then $EX_1 \geq EX_2$.

Proof. In order to prove this lemma, we make use of a well known result in probability theory that for a non-negative random variable X , $EX = \int_0^{\infty} P\{X > x\} dx$.³ Then $EX_i^+ = \int_0^{\infty} P\{X_i^+ > x\} dx = \int_0^{\infty} P\{X_i > x\} dx$, for $i=1, 2$. Since X_1 stochastically dominates X_2 it follows that $EX_1^+ \geq EX_2^+$. Similarly, $EX_i^- = \int_0^{\infty} P\{X_i^- > x\} dx = \int_0^{\infty} P\{X_i < -x\} dx$, for $i=1, 2$. From Lemma 1 $P\{X_1 < -x\} \leq P\{X_2 < -x\}$ which implies $EX_1^- \leq EX_2^-$. Thus $EX_1 = EX_1^+ - EX_1^- \geq EX_2^+ - EX_2^- = EX_2$.⁴

Q. E. D.

The strict analogue of Lemma 2 is:

Lemma 2'. Let X_1 and X_2 be random variables such that at least one of them has a finite expectation. If X_1 stochastically dominates X_2 in the strict sense, then $EX_1 > EX_2$.

Proof. By hypothesis, there exists an $x_0 \in \mathbb{R}$ such that $P\{X_1 > x_0\} > P\{X_2 > x_0\}$. Since $P\{X > y\}$ is a right continuous function of y , $P\{X_1 > y\} > P\{X_2 > y\}$ on some interval $[x_0, x_0 + \epsilon]$, where $\epsilon > 0$. Assume without loss of generality that $x_0 \geq 0$. Then $[x_0, x_0 + \epsilon] \subset [0, \infty)$ and $EX_1^+ = \int_0^\infty P\{X_1 > x\} dx > \int_0^\infty P\{X_2 > x\} dx = EX_2^+$. From the proof of Lemma 3, $EX_1^- \leq EX_2^-$. Hence $EX_1 > EX_2$.

Q.E.D.

Consider a non-decreasing function u and two random variables X_1 and X_2 . It is not surprising that dominance relationships between these two random variables induce corresponding dominance relationships between the random variables $u \circ X_1$ and $u \circ X_2$.

Lemma 3. Let $u: \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing. If X_1 stochastically dominates X_2 , then $u \circ X_1$ stochastically dominates $u \circ X_2$.

Proof. Since u is non-decreasing, it is Borel measurable, and therefore $u \circ X_1$ and $u \circ X_2$ are random variables. Let $x \in \mathbb{R}$.

case (i) Suppose that $\{z \in \mathbb{R}: u(z) > x\} = \emptyset$. Then $\{u \circ X_1 > x\} = \{u \circ X_2 > x\} = \emptyset$ and therefore $P\{u \circ X_1 > x\} = P\{u \circ X_2 > x\}$.

case (ii) Suppose that $\{z \in R: u(z) > x\} \neq \emptyset$. Let $y = \inf\{z \in R: u(z) > x\}$, where y is possibly $-\infty$. It follows that $(y, +\infty) \subset \{z \in R: u(z) > x\} \subset [y, +\infty)$, so that $\{z \in R: u(z) > x\}$ is either $(y, +\infty)$ or $[y, +\infty)$. Consequently either (a) or (b) must hold:

$$(a) \quad \{u \circ X_i > x\} = \{X_i > y\} \quad \text{for } i = 1, 2.$$

$$(b) \quad \{u \circ X_i > x\} = \{X_i \geq y\} \quad \text{for } i = 1, 2.$$

The result $P\{u \circ X_1 > x\} \geq P\{u \circ X_2 > x\}$ follows by hypothesis if (a) holds, and by Lemma 1 if (b) holds.

Q.E.D.

A strict version of Lemma 3 can be proved under stronger conditions on u .

Lemma 3'. Let $u: R \rightarrow R$ be non-decreasing. Suppose that X_1 stochastically dominates X_2 . If there exists $x_0 \in R$ such that: (i) $u(x) > u(x_0)$ for $x > x_0$ and (ii) $P\{X_1 > x_0\} > P\{X_2 > x_0\}$, then $u \circ X_1$ stochastically dominates $u \circ X_2$ in the strict sense.

Proof. By the monotonicity of u , $X_i \leq x_0 \Rightarrow u \circ X_i \leq u(x_0)$ and by (i) $X_i > x_0 \Rightarrow u \circ X_i > u(x_0)$, for $i = 1, 2$. Hence, $\{u \circ X_i > u(x_0)\} = \{X_i > x_0\}$, $i = 1, 2$. Consequently by (ii) $P\{u \circ X_1 > u(x_0)\} > P\{u \circ X_2 > u(x_0)\}$. Then the conclusion follows from Lemma 3.

Q.E.D.

Immediate application of Lemma 2 and Lemma 3 yields the following theorem:

Theorem 1. Let $u: R \rightarrow R$ be non-decreasing. Suppose X_1 stochastically dominates X_2 and both $E(u \circ X_1)$ and $E(u \circ X_2)$ exist. Then $E(u \circ X_1) \geq E(u \circ X_2)$.

The strict version of theorem 1 follows from Lemma 2' and 3':

Theorem 1'. Let $u: R \rightarrow R$ be non-decreasing. Suppose that X_1 stochastically dominates X_2 . If there exists an $x_0 \in R$ such that:
(i) $u(x) > u(x_0)$ for $x > x_0$, and (ii) $P\{X_1 > x_0\} > P\{X_2 > x_0\}$,
and if $u \circ X_1$ or $u \circ X_2$ has a finite expectation, then $E(u \circ X_1) > E(u \circ X_2)$.

II. The Hanoch and Levy First Degree Stochastic Dominance Theorem

Before we reconsider the original Hanoch and Levy theorem, a brief discussion of integration may be useful. Hanoch and Levy used Lebesgue-Stieltjes integration in their theorem and proof. However, they encountered difficulties when they attempted an integration by parts argument. In order to avoid these difficulties, Tesfatsion shifted to Riemann-Stieltjes integration in her restricted theorem and proof.⁵ However, Riemann-Stieltjes integration may not be appropriate for use in probability theory. One of the reasons for this is that if X is a random variable with distribution function F , and if $u: R \rightarrow R$ is Borel measurable, then $u \circ X$ is a random variable whose distribution function will be denoted by G . In this case, it is desirable to have an integration theory which yields the following equalities:

$$E(u \circ X) = \int_R x dG(x) = \int_R u(x) dF(x) \quad (1)$$

provided that any of these integrals exist. If u and F have a common point of discontinuity, then the Riemann-Stieltjes integral on the right end of (1) does not exist even though $f(x) = x$ may be Riemann-Stieltjes integrable with respect to G . With Lebesgue-

Stieltjes integration, (1) is satisfied. In this section all integrals over the real line are Lebesgue-Stieltjes integrals.

Our modification of Theorem 1 in Hanoch and Levy is:

Theorem 2. Let U be the class of non-decreasing functions $u: R \rightarrow R$, and let F_1 and F_2 be distribution functions. Then (I) and (II) are equivalent:

- (I) $F_1(x) \leq F_2(x)$ for all $x \in R$, and $F_1(x_0) < F_2(x_0)$ for some $x_0 \in R$.
- (II) $\int_R u(x) dF_1(x) \geq \int_R u(x) dF_2(x)$ for any $u \in U$ whenever both integrals exist, and there is a $u^* \in U$ for which this inequality is strict.

Proof. Suppose that (I) holds. For any two distribution functions F_1 and F_2 , there exist random variables X_1 and X_2 defined on a common probability space that have F_1 and F_2 as their respective distribution functions. Then it follows from Theorem 1 that:

$$\int_R u(x) dF_1(x) = E(u \circ X_1) \geq E(u \circ X_2) = \int_R u(x) dF_2(x)$$

whenever the integrals on each end exist.

Next, let $u^*(x) = 1$ if $x \in (x_0, +\infty)$ and zero elsewhere.

Then

$$\int_R u^*(x) dF_1(x) = 1 - F_1(x_0) > 1 - F_2(x_0) = \int_R u^*(x) dF_2(x).$$

Conversely, suppose that (II) holds. For any $y \in R$, let $u_y(x) = 1$ for $x \in (y, +\infty)$, zero elsewhere. Then

$$1 - F_1(y) = \int_R u_y(x) dF_1(x) \geq \int_R u_y(x) dF_2(x) = 1 - F_2(y),$$

and therefore, $F_1(y) \leq F_2(y)$. Since there is a u^* such that $E(u^* \cdot F_1) > E(u^* \cdot X_2)$, it cannot be the case that $F_1(x) = F_2(x)$ for all $x \in R$.

Q.E.D.

This theorem is essentially the same as the Hanoch and Levy theorem, and it does not impose the continuity conditions on u employed by Tesfatsion.⁶

III. An Application

Stochastic dominance theorems are often useful in the comparison of optimizing behavior under two different uncertainty regimes. In such optimization problems, the first order conditions can contain expectations of derivatives of the objective function. These derivatives will be discontinuous if the objective function is kinked. The following example illustrates the application of Theorems 1 and 1' in this type of problem.

Consider an agent who is able to sell units of a homogeneous commodity for p dollars per unit. The quantity demanded at this price is the realization of a continuous random variable x with a density $g(\cdot)$ and a distribution function $G(\cdot)$. It is assumed that the support of $g(\cdot)$ is a finite interval $[a, b]$. The discontinuity in this problem arises because the seller can acquire the commodity for c dollars at the beginning of the period, but the acquisition cost is π dollars per unit obtained subsequent to the realization of the demand x . It is assumed that all demand must be satisfied, and that $\pi > p > c$.⁷

The seller's utility function $u(\cdot)$ is assumed to be twice differentiable with $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$. Then the seller's expected utility as a function of the initial inventory acquisition I is:

$$\int_a^I u(px - cI) g(x) dx + \int_I^b u(px - \pi(x - I) - cI) g(x) dx$$

for $I \in [a, b]$.

The first order condition for determining the optimal inventory can be written as an equation in I :

$$\int_a^b \varphi(I, x) g(x) dx = 0 \quad (2)$$

where

$$\varphi(I, x) = \begin{cases} -c u'(px - cI) & \text{if } x \leq I \\ [\pi - c] u'(px - \pi x + \pi I - cI) & \text{if } x > I \end{cases} \quad (3)$$

It is straightforward to verify that the following inequalities hold for any density function $h(\cdot)$ with support $[a, b]$:

$$\partial/\partial I \left\{ \int_a^b \varphi(I, x) h(x) dx \right\} < 0 \quad \text{for all } I \in (a, b), \quad (4)$$

$$\int_a^b \varphi(a, x) h(x) dx > 0, \quad \text{and}$$

$$\int_a^b \varphi(b, x) h(x) dx < 0.$$

Thus the optimal inventory level for the distribution $G(\cdot)$, denoted by I_G , is uniquely determined by the first order condition (2).

Now consider the effect of a shift in the probability distribution of x on the optimal inventory level. Specifically, suppose that the new distribution function of x is $F(x)$ and that $F(x) \leq G(x) \forall x \in [a, b]$. It is apparent from the concavity of $u(\cdot)$ that $\varphi(I, x)$ defined in (3) is a non-decreasing function of x with a simple discontinuity at $x = I$. Then the implication of Theorem 1 is that:

$$\int_a^b \varphi(I_G, x) f(x) dx \cong \int_a^b \varphi(I_G, x) g(x) dx = 0 \quad (5)$$

where the last equality follows from the first order condition (2).

The optimal inventory for the distribution F denoted I_F , is determined by

$$\int_a^b \varphi(I_F, x) f(x) dx = 0 \quad (6)$$

It is apparent from (5), (6), and the second order condition (4) that $I_G \leq I_F$.

If the seller is risk averse ($u''(\cdot) < 0$), then $\varphi(I, x)$ defined in (3) will be a strictly increasing function of x . Thus it follows from Theorem 1' that $I_G < I_F$ whenever F dominates G in the strict sense.

In this example, the seller's utility is a continuous function of the realized demand x ; the discontinuity appears in the first order condition. In other problems, however, it may be the case that a decision-maker's utility is itself discontinuous as a function of some random variable. For a firm submitting a sealed bid for the right to exploit a mineral lease, the maximum rival bid can be thought of as the realization of a random variable. If the firm's bid exceeds the maximum rival bid, then the firm's profit would generally be significantly larger than would be the case if the lease is lost. Other discontinuities occur because contracts commonly impose penalties if some measure of performance falls short of a specified standard.

Therefore, even though a decision-maker's utility may be a continuous function of monetary gain, utility may not be a continuous function of the realization of a random variable that affects this monetary gain.

In conclusion, the stochastic dominance theorems in this paper may be useful in the analysis of problems in which the objective function is a kinked or discontinuous function of the realization of a random variable.

NOTES

1. For example, Bawa [1] and Hadar and Russell [4, 5] make the differentiability assumption.
2. If the distribution functions of X_1 and X_2 are F_1 and F_2 respectively, then this definition is obviously equivalent to the requirement that $F_1(x) \leq F_2(x) \forall x \in R$.
3. See Feller [2], chapter 5, lemma 6.3.
4. Alternatively, lemma 2 can be proved by showing that if X_1 stochastically dominates X_2 , then there exists a probability space $(\Omega^*, \mathcal{F}^*, \rho^*)$ and two random variables X_1^* and X_2^* that satisfy the following conditions:

(1) X_i has the same distribution function as X_i^*
for $i = 1, 2$.

(2) $X_1^*(\omega^*) \geq X_2^*(\omega^*) \forall \omega^* \in \Omega^*$.

In order to verify (1) and (2), let F_1 and F_2 denote the distribution functions of X_1 and X_2 respectively. Also let Ω^* be the open unit interval, \mathcal{F}^* the collection of Borel subsets of Ω^* , and ρ^* the Lebesgue measure. For $\omega^* \in \Omega^*$, define

$X_i^*(\omega) \equiv \sup\{x \in R : F_i(x) \leq \omega^*\} \quad i = 1, 2$. It follows that

the distribution function of X_i^* is F_i . Furthermore since

X_1 stochastically dominates X_2 , $X_1^*(\omega^*) \geq X_2^*(\omega^*) \forall \omega^* \in \Omega^*$.

5. See [7], Theorem 1*, p. 304.

6. As Tsefatian has noted, Hanoch and Levy were a bit careless in the statement of their theorem. Instead of using (II) in our Theorem 3, their theorem states that

$$\int_{\mathbb{R}} u(x) dF_1(x) - \int_{\mathbb{R}} u(x) dF_2(x) \cong 0$$

for all $u \in U$, with strict inequality holding for some u^* . However, the integrals on the left side may fail to exist. Even if both integrals exist, the left side may be of the form $\infty - \infty$. Condition (II) in our version of the theorem avoids these technical complications.

7. An alternative interpretation of this model is that all acquisitions must be made at the beginning of the period. Then c is the acquisition cost and $\pi - p$ is a penalty incurred for each unit of demand that is not satisfied. This penalty could reflect an expected reduction in future demand.

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