

THE MATHEMATICS OF SRAFFA'S MODEL OF  
PRICES, WAGE, AND RATE OF PROFIT\*

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## 1. Introduction

### 1.1 General Background

In Production of Commodities by Means of Commodities, Piero Sraffa investigates the prices, wage, and rate of profit that prevail in a stationary or "self-replacing" economy. His main conclusions are that under certain technical conditions: a) for each wage rate there is a unique rate of profit and set of normalized commodity prices making the value of the output of each productive process equal to the value of its inputs plus profit; b) a higher wage rate is associated with a lower rate of profit and with lower commodity prices relative to the wage rate. Result a) shows that there is an indeterminacy in the price and distribution system even after prices have been normalized. Several authors have tried to explain or evaluate Sraffa's model<sup>1</sup> but disagreements remain, as we see from the recent exchange between Burmeister [3] and Levine [6]. The controversy persists partly because Sraffa's presentation is informal with intuitive arguments taking the place of rigorous proofs. So far, no precise formulation of Sraffa's model has appeared in print. The present paper seeks to fill this gap.

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1. See Levine [5] for a survey of this literature.

Rigorous proofs of Sraffa's results on the Standard commodity and Standard system have already been presented by Burmeister [2] and Miyao [9]. According to Sraffa, however, "the Standard system is a purely auxiliary construction. It should therefore be possible to present the essential elements of the mechanism under consideration without having recourse to it," [10, page 31]. We therefore prove Sraffa's main propositions directly, without reference to the Standard system or commodity. Our proofs, which employ extensions of the Frobenius theorem on nonnegative matrices, are quite different from Sraffa's arguments. Our results are slightly stronger than his since they apply to certain economies with joint production.

We referred above to "Sraffa's model" when in fact he uses two different models to analyze production for subsistence ([10], Chapter 1) and production with a surplus ([10], Chapter 2). Workers' consumption is treated differently in these two situations. This treatment is only implicit since there is no variable representing consumption in either model. In our formulation, the consumption that is necessary for the survival of the workers is treated explicitly. This is necessary for a reasonable definition of a subsistence economy and it permits us to treat the cases of subsistence and surplus in a unified model. Explicit treatment of consumption also allows us to define the "living wage," the "limit below which the wage cannot fall," and to prove that any higher wage will also be a living wage in the sense that goods purchased at the lower wage can also be purchased at the higher wage taking account of the corresponding changes in commodity prices. For a more technical summary of the results to be presented, the reader is directed to Section 1.2 below. Since the formal treatment of these topics is quite lengthy (Sections 2-8) we have omitted other problems discussed by Sraffa: reduction of inputs

to dated quantities of labor, analysis of fixed capital, and the switching of techniques of production.

Sraffa's models of production are similar to stationary "growth" paths of a von Neumann-type economy. Matrices representing quantities of gross inputs and outputs are the same in all time periods. Our notation in dealing with these matrices is presented in Section 2.1. In Section 2.2, the main mathematical tools, the Frobenius theorem and some of its extensions, are presented. We describe our version of Sraffa's economic models in Section 3. In particular, we discuss the consumption set  $\mathcal{C}$ , which makes precise the notion of "necessaries of consumption" that Sraffa refers to; we define the stationary, subsistence and surplus economies and compare our terminology with Sraffa's; we define basic and nonbasic goods and show how they are related to indecomposability in von Neumann models.

In Section 4, we define and discuss a price system that is mathematically similar to the system of exchange-values in Sraffa's model of a subsistence economy. Section 4.2 gives sufficient conditions for the existence and uniqueness (up to multiplication by a positive scalar) of this price system (Theorem 1). In Section 5, we define a price-distribution system, a vector of the prices, wage rate and rate of profit that appear in Sraffa's model of production with a surplus. In Section 5.2, we prove result (a) referred to above: uniqueness of the price-distribution system for each possible wage rate in a "regular" economy (Theorem 2). We also show how the "regularity" conditions, which are used in all of our later theorems, are related to the conditions imposed by Sraffa. In Section 6.1 we prove result (b), i.e., that a rise in the wage rate must be accompanied by a fall in the rate of profit (Theorem 3). We show in Section 6.2 (Theorem 4) that if prices rise with a rise in the wage, they must rise slower than the wage. This result is used in Section 7 (Theorem 5) to show that the notion of a "living wage" makes sense. In Section 8, we return to the characterization of nonbasic goods, and show that a tax on a group of nonbasics will change the prices of the basic goods in the economy where the original normalization of prices is retained. Theorem 6 in Section 8.1 appears to be the strongest statement that can be made: a tax on the nonbasics will leave the prices of the basics relative to the wage rate unchanged.

## 1.2 Technical Summary

Sraffa analyzes stationary economies with fixed gross outputs represented by the  $n \times n$  matrix  $Z$ , fixed gross commodity inputs represented by  $Y$ , also  $n \times n$ , and labor inputs represented by the  $n \times 1$  matrix  $L$ . The columns of these matrices correspond to goods and the rows correspond to production processes, so that the entry in column  $j$  and row  $i$  is the amount of good  $j$  produced or used in process  $i$ . The Leontief economy with constant gross output vector  $x$  and input coefficients matrix  $A$  is identical to the Sraffian economy where  $Z$  is a diagonal matrix with the entries of  $x$  in the diagonal and  $Y$  is the transpose of  $Ax$ . In Sraffa's model, since there is no variation in production, assumptions about technological possibilities (for example, the assumption of constant returns to scale) are unnecessary. The unit of labor is taken to be the total amount of labor supplied.

In order for the economy to be "self-replacing," the output of each good in each period must be at least as large as the amount of the good used as an input. Thus  $\underline{1}(Z - Y) \geq 0$ , where  $\underline{1}$  denotes the row vector with all entries equal to 1. Sraffa distinguishes the case of subsistence, where  $\underline{1}(Z - Y) = 0$ , from "surplus" with  $\underline{1}(Z - Y) > 0$ . These names are sensible only if the "necessaries of consumption" are included among the commodity inputs measured by  $Y$ . In order to make precise the notion of "necessaries of consumption," we define the set  $C$ , where each  $c \in C$  is a vector of goods sufficient for the subsistence of the labor force.

A price-distribution system is a vector of commodity prices  $p$ , a rate of profit  $r$  (the same for all production sectors), and a wage

rate  $w$  satisfying

$$(1 + r) Yp + wL = Zp \quad \text{and} \quad \underline{1} (Z - Y) p = 1 \quad .$$

In the former equation, the value (at solution values  $w$ ,  $\tilde{p}(w)$ ) of inputs plus the profit (at solution value  $\tilde{r}(w)$ ) in each production sector equals the value (at solution values  $\tilde{p}(w)$ ) of output in that sector (see Note 15, in page 18). The latter equation is a price normalization. Sraffa's main result is that in a surplus economy without joint production (each sector produces one output) if every good is basic ( $Z^{-1} Y$  is irreducible), then for each wage rate  $w \in [0, 1]$ , there is a unique rate of profit  $\tilde{r}(w)$  and price vector  $\tilde{p}(w) = (\tilde{p}_1(w), \dots, \tilde{p}_n(w))$  satisfying the above equations. Thus,  $\tilde{p}(w)$ ,  $\tilde{r}(w)$ ,  $w$  form the unique price-distribution system for each  $w \in [0, 1]$ . A generalized version of this result covering cases of joint production in which  $Z^{-1} Y \geq 0$  is stated as Theorem 2 in Section 5.2. Section 4 contains a similar result (Theorem 1) on existence of a price system when wages are not separated from commodity inputs. Theorems 1 and 2 rely on an extension of Mangasarian's Perron-Frobenius theorems [8].

The regularity conditions that are used in the above theorems have other implications as well: a higher wage means a lower rate of profit, i.e.,  $\tilde{r}$  is a decreasing function of  $w$  (Theorem 3). Also,  $\frac{\tilde{p}_i(w)}{w}$ , the price of commodity  $i$  relative to the wage rate  $w$  is nonincreasing as a function of  $w$ . This implies that if the total wage  $w$  (which equals the wage rate since total labor supplied is one unit) is as valuable as a consumption bundle  $c$  ( $w = c \tilde{p}(w)$ ) then any higher wage  $w'$  is at least as valuable as  $c$  ( $w' \geq c \tilde{p}(w')$ ).

## 2. Technical Notation and Preliminary Results

### 2.1 Notation

For  $m \times n$  matrices  $A = (a_{ij})$  and  $B = (b_{ij})$  with real entries,

$A \cong B$  means  $a_{ij} \cong b_{ij}$  for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ .

$A \succeq B$  means  $A \cong B$  and  $A \neq B$ .

$A > B$  means  $a_{ij} > b_{ij}$  for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ .

$A$  is nonnegative if  $A \geq 0$ .  $A$  is strictly positive if  $A > 0$ .

The transpose of a matrix  $A$  is denoted  $A^T$ .

The determinant of a square matrix  $A$  is denoted by  $\det A$ .

$A_j$  will denote the  $j$ th column of the matrix  $A$  unless it is defined otherwise.

If  $A$  is an  $m \times n$  and  $B$  an  $m \times k$  matrix,  $[A B]$  denotes the  $m \times (n + k)$  matrix with the first  $n$  columns equal to the columns of  $A$  and the last  $k$  columns equal to those of  $B$ . That is,  $[A B]_j = A_j$  for  $j = 1, 2, \dots, n$  and  $[A B]_{j+n} = B_j$  for  $j = 1, 2, \dots, k$ .

$I_k$  denotes the  $k \times k$  identity matrix. The subscript will be deleted when the dimension of the matrix is clear from the context.

$\underline{1} = (1 \ 1 \ \dots \ 1)$  is the  $1 \times n$  row vector with all entries equal to 1.

The square matrix  $A$  is reducible if there is a permutation of its rows and columns transforming  $A$  into

$$\tilde{A} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \text{ where } B \text{ and } D \text{ are square matrices. } A \text{ is } \underline{\text{ir-}}$$

reducible if it is not reducible. An irreducible matrix is sometimes called indecomposable.

The matrix  $A = (a_{ij})$  is diagonal if  $a_{ij} = 0$ ,  $\forall i, j$ ,  $i \neq j$ .

A set  $S \subset \mathbb{R}^n$  is called monotone if  $s \in S$  and  $s^1 \geq s$  imply  $s^1 \in S$ .



## 2.2 Technical Results

Let  $A$  be a real  $n \times n$  matrix,  $A \geq 0$ . There is a real scalar  $\lambda \geq 0$  and a vector  $y \geq 0$  with  $Ay = \lambda y$  and  $\lambda \geq |\gamma|$  for any characteristic value  $\gamma$  of  $A$ . (This is Theorem 3 in Gantmacher [4, pg. 66].) We call  $\lambda$  the maximal characteristic value of  $A$ , denoted by  $\rho(A)$ .<sup>1</sup> The Frobenius theorem, Theorem 2 in Gantmacher [4, pg. 53], states that if  $A \geq 0$  is irreducible, then  $\rho(A) > 0$  and the characteristic vector associated with  $\lambda = \rho(A)$  is strictly positive:  $y > 0$ .

According to Theorem 3.1 (i) of Mangasarian [8, pg. 90],<sup>2</sup> if  $A, B$  and  $X$  are real  $n \times n$  matrices with  $A = BX$  and  $X \geq 0$ , then

$$(1) \quad B^T y \geq 0 \Rightarrow A^T y \geq 0 \quad \text{for any } n \times 1 \text{ vector } y.$$

Theorem 4.1 (ii) of Mangasarian [8, pg. 91] states that if  $B$  has rank  $n$  and condition (1) holds, there is a scalar  $\gamma \geq 0$  and a vector  $y$  satisfying

$$(2) \quad A^T y = \gamma B^T y \quad \text{and} \quad B^T y \geq 0;$$

and for any solution to (2),  $\gamma = \rho(X)$ .

Applying these theorems to  $A, B$  and  $X$  where  $X = B^{-1}A \geq 0$ , we get part (a) of our Lemma 1 below. Part (b) of Lemma 1 is a restatement of the first part of Theorem 4.1 (iii) in Mangasarian [8, pg. 92].

Lemma 1 Suppose that  $A$  and  $B$  are real square matrices with  $B$  non-singular and  $B^{-1}A \geq 0$ .

(a) If  $A^T y = \gamma B^T y$  and  $B^T y \geq 0$  for some  $y$  with  $\gamma \geq 0$ , then  $\gamma = \rho(B^{-1}A)$ .

(b) If  $A^T y \leq \nu B^T y$  and  $B^T y > 0$  for some  $y$  with  $\nu \geq 0$ , then  $\nu \geq \rho(B^{-1}A)$ .

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1.  $\rho(A)$  is sometimes called the spectral radius of  $A$ .

2. We thank J. J. Camio who participated in the seminar on Sraffa for useful conversations about Sraffa's book and for bringing this article by Mangasarian to our attention.

Lemma 2 Suppose  $A$  and  $B$  are real nonnegative square matrices with  $B$  nonsingular and  $B^{-1}A \geq 0$  irreducible.

(c) If  $A^T y \leq uB^T y$  and  $B^T y > 0$  for some  $y$  with  $u \geq 0$ , then  $u > \rho(B^{-1}A)$ .

Proof Let  $C \equiv B^{-1}A \geq 0$  and  $D \equiv uI - C^T$ .  $C^T B^T = (B^{-1}A)^T B^T = A^T (B^{-1})^T B^T = A^T$  and since  $C$  is irreducible, by Lemma 5 in Appendix A.1,  $C^T \geq 0$  is also. Letting  $z \equiv B^T y > 0$ , we have  $Dz \geq 0$  since  $Dz = (uI - C^T)B^T y = uB^T y - C^T B^T y = uB^T y - A^T y \geq 0$ . Thus, according to Theorem 4.D.2 in [11, page 392],  $D$  is nonsingular (see Lemma 4, page 21 of this paper).

We know from part (b) of Lemma 1 that  $u \geq \rho(B^{-1}A)$ . If  $u = \rho(B^{-1}A)$  then there is some  $x \neq 0$  with  $B^{-1}Ax = ux$ , or  $(uI - B^{-1}A)x = D^T x = 0$ . But this contradicts the nonsingularity of  $D$ , so  $u > \rho(B^{-1}A)$ .

### 3. The Economy: Notation and Definitions

#### 3.1 Notation

We now describe an economy with  $n$  nonprimary (produced) commodities (also called goods), one primary commodity, labor, and  $n$  production processes.

$Y = (y_j^i) \geq 0$ ,  $i, j = 1, 2, \dots, n$  denotes the matrix of gross nonprimary inputs, where  $y_j^i$  is the quantity of the  $j$ th commodity used in the  $i$ th production process. The superscript refers to the row and subscript to the column an entry appears in.

$L = \begin{pmatrix} 1 \\ \ell^1 \\ \ell^2 \\ \vdots \\ \ell^n \end{pmatrix} \geq 0$  is the  $n \times 1$  column vector of labor inputs, where  $\ell^i$

is the quantity of labor used in the  $i$ th production process,  $i = 1, 2, \dots, n$ .

$Z = (z_j^i) \geq 0$ ,  $i, j = 1, 2, \dots, n$  denotes the matrix of gross nonprimary outputs, where  $z_j^i$  is the quantity of the  $j$ th commodity produced by the  $i$ th process.

$C \subset R_+^n$  denotes the set of aggregate consumption bundles that allow laborers to provide one unit of labor, where  $R_+^n$  stands for the nonnegative orthant of the  $n$ -dimensional Euclidean space.

There is a problem in considering only aggregate consumption and labor supply. Since each worker must eat in order to survive, the distribution of a given aggregate consumption bundle will affect the aggregate labor supply. For example, suppose there are  $m > 1$  workers and that each of them can provide units of labor when (s)he receives the consumption bundle  $c \geq 0$ . If every worker receives  $c$ , then total consumption is  $mc$  and  $m$  units of labor are supplied. However, if the entire consumption bundle  $mc$  is given to one worker and the others receive nothing, only the first worker will survive and aggregate labor supply will be one unit (less than  $m$ ). Sraffa's models deal only with aggregate labor, so we can ignore this problem without creating any internal inconsistencies.

Sraffa himself does not consider the workers' consumption explicitly. In Chapter 1, Section 3 [10, pg. 4], he presents a model of a "subsistence economy" in which labor does not appear. The consumption necessary for the survival of the workers is (implicitly) included in the matrix of commodity inputs. In Sraffa's model of "production with a surplus," (Chapter 2) labor is treated explicitly as a noncommodity input in the production process. The consumption necessary for the workers' survival is no longer included in the commodity input matrix but must form a part of the wage paid for labor. Sraffa notes that these "necessaries [exert] their influence on prices and profits...by setting a limit below which the wage cannot fall." [10, pg. 10]. Thus Sraffa's model of production with a surplus need not have a surplus in fact, since there might not be enough produced to feed the workers.

By treating workers' consumption explicitly, we eliminate confusion over the definition of "surplus" and deal with the cases of surplus and subsistence in a unified model. In Section 7 we analyze the living wage, a wage that is high enough to pay for the "necessaries for survival." We are then able to define precisely the limit below which the wage cannot fall.

### 3.2 Definitions

We now describe our version of Sraffa's models of subsistence and surplus economies. An economy might be thought of as a sequence of input, output and consumption matrices. The von Neumann growth model with consumption (see Malinvaud [7, pg. 236]) has this form. In this paper we consider only stationary economies, i.e., economies with inputs,  $Y$  and  $L$ , and outputs,  $Z$ , that remain the same in all periods. Sraffa calls these economies "self-replacing systems."

Definition 1  $(Y, L, Z, \mathcal{C})$  is a (stationary) economy if  $Y$  and  $Z$  are  $n \times n$  matrices and  $L$  is an  $n \times 1$  column vector,  $Y \geq 0$ ,  $L \geq 0$ ,  $Z \geq 0$

and  $\mathcal{C} \subset \mathbb{R}_+^n$ , and if

- (i)  $\underline{1} (Z - Y) \geq 0$
- (ii)  $\underline{1} L = 1$ , and
- (iii)  $\mathcal{C}$  is monotone.<sup>1</sup>

Note 1 Since this paper deals only with stationary economies, the term "stationary" will usually be omitted below.

Note 2 The assumption (i)  $\underline{1} (Z - Y) \geq 0$  means that at least as much of each good was produced last period as is needed for production inputs in this period. This condition must hold in order for the economy to be self-replacing.

Note 3 Assumption (ii)  $\underline{1} L = 1$  is a normalization. Labor is regarded as homogeneous and we take the total amount of labor supplied in the economy to be one unit.

Note 4 The assumption (iii) that  $\mathcal{C}$  is monotone might be thought of as a consequence of free disposal (although it could be justified in other ways). If  $c \in \mathcal{C}$ , and  $c' \geq c$ , then workers can supply one unit of labor given  $c'$  simply by consuming  $c$  and disposing of  $c' - c \geq 0$ .

Definition 2 An economy  $(Y, L, Z, \mathcal{C})$  is sustainable if for some  $c \in \mathcal{C}$ ,  $\underline{1} (Z - Y) \geq c$ .

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1. See page 4.

Note 5 An economy is thus sustainable if net output is large enough to provide workers with the necessaries for survival so that one unit of labor is supplied.

Definition 3 An economy  $(Y, L, Z, \mathcal{C})$  is called a surplus economy if for some  $c \in \mathcal{C}$ ,  $\underline{1}(Z - Y) \geq c$ .

Definition 4  $(Y, L, Z, \mathcal{C})$  is a subsistence economy if it is sustainable and is not a surplus economy.

Note 6 One might object to definitions 1 and 2 on the grounds that an economy that is not sustainable should not be called stationary. The distinction will be useful, however, when we discuss the relationship between Sraffa's models and ours. Also, while economies that are not sustainable may be less interesting to the economist than those that are, the technical condition of sustainability is not needed to prove our results on prices, wage and rate of profit. It is therefore useful to consider the larger class of economies that we have labeled "stationary."

In the table below, we compare Sraffa's definitions of the economies under consideration with ours.

<u>SRAFFA'S CONDITION</u>		<u>TABLE</u>		<u>OUR CONDITION</u>	
		stationary			$\underline{1}(Z-Y) \geq 0$
self-replacing	$\underline{1}(Z-Y) \geq 0$	sustainable		$\underline{1}(Z-Y) \geq c$ for some $c \in \mathcal{C}$	
subsistence	$\underline{1}(Z-Y) = 0$	subsistence		$\underline{1}(Z-Y) \in \mathcal{C}$ and there is no $c' \in \mathcal{C}$ with $\underline{1}(Z-Y) \geq c'$	
surplus	$\underline{1}(Z-Y) \geq 0$	surplus		$\underline{1}(Z-Y) \geq c$ for some $c \in \mathcal{C}$	

It is easy to see that when  $\mathcal{C} = \mathbb{R}_+^n$ , the corresponding conditions on the left and right-hand sides of the table are identical. Thus Sraffa's models of subsistence and surplus may be viewed as special "sustainable economies" where  $\mathcal{C} = \mathbb{R}_+^n$ , i.e., where workers need not consume any produced commodities

in order to survive. ( $\mathcal{C} = \mathbb{R}_+^n$  if  $0 \in \mathcal{C}$ , since  $\mathcal{C}$  is monotone.) On the other hand, when  $0 \notin \mathcal{C}$  and the necessary consumption is not included in the matrix of commodity inputs  $Y$ , then our conditions for subsistence and surplus correspond more closely than Sraffa's to the way these terms are commonly used.

**Definition 5** Good  $k$ ,  $k = 1, 2, \dots, n$ , is nonbasic in the economy  $(Y, L, Z, \mathcal{C})$  if there exists a diagonal  $n \times n$  matrix  $M = (m_{ij})$  (i.e.,  $m_{ij} = 0 \forall i \neq j$ ) such that  $n > \text{rank } M \cong \text{rank } [ZM \ YM]$ , with  $m_{kk} = 1$ , and  $m_{ii} = 0$  or  $1 \forall i \neq k$ . Otherwise, good  $k$  is basic.

**Note 7**  $(ZM)_j = \begin{cases} Z_j & \text{if } m_{jj} = 1 \\ 0 & \text{if } m_{jj} = 0. \end{cases}$

Thus, the  $j$ th column of  $ZM$  is the same as that of  $Z$  if  $m_{jj} = 1$  and is 0 if  $m_{jj} = 0$ .

**Note 8** If there is only one good in the economy ( $n = 1$ ), that good is basic since there can be no matrix  $M$  with  $n = 1 > \text{rank } M > 0$ .

**Note 9** When  $n \geq 2$ , an example of a nonbasic good is a luxury that is not used in the production of any commodity. Suppose that  $k$  is such a good. Then, since  $k$  is not used as an input, the  $k$ th column of  $Y$  is  $Y_k = 0$ . Letting the

$n \times n$  matrix  $M = (m_{ij})$  be defined by  $m_{ij} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$ , we have  $\text{rank } M = 1 \cong \text{rank } [ZM \ YM]$ . The last inequality holds because  $(ZM)_j = 0$  for  $j \neq k$  and  $(YM)_j = 0$  for  $j \neq k$  and  $(YM)_k = Y_k = 0$ , so  $YM = 0$ .

We now characterize an economy without nonbasic goods.

**Lemma 3** Suppose  $(Y, L, Z, \mathcal{C})$  is an economy with  $Z$  nonsingular. Every good is basic iff  $Z^{-1}Y$  is irreducible. (See page 4 for definition of irreducibility.)

Proof Every good is basic iff  $\text{rank } [ZM \ YM] > \text{rank } M$  for every diagonal  $n \times n$  matrix  $M = (m_{ij})$  with  $m_{ii} = 0$  or  $1$  and  $0 < \text{rank } M < n$ . (See Definition 5.) Suppose  $Z^{-1}Y$  is reducible. Without loss of generality we

may assume  $Z^{-1}Y = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$  where  $D$  is a  $k \times k$  matrix  $0 < k < n$ . Let

$$M = \begin{pmatrix} 0 & 0 \\ 0 & I_k \end{pmatrix}.$$

Since  $Z$  is non-

singular,  $\text{rank } [ZM \ YM] = \text{rank } Z^{-1}[ZM \ YM] = \text{rank } [M \ Z^{-1}YM] =$

$$\text{rank} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_k & 0 & D \end{bmatrix} = k = \text{rank } M \text{ since } Z^{-1}YM = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}. \text{ Hence}$$

not every good is basic. Conversely, if not every good is basic, by permutation of the indices of  $Z$  and  $Y$ , we have  $\text{rank } [ZM \ YM] \leq \text{rank } M$  for some

$$M = \begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}, \quad 0 < k < n. \text{ But then } \text{rank } M = k \geq \text{rank } [ZM \ YM] = \text{rank } [M \ Z^{-1}YM].$$

Let  $Z^{-1}Y = \begin{pmatrix} B & A \\ C & D \end{pmatrix}$  where  $D$  is  $k \times k$ . Then  $Z^{-1}YM = \begin{pmatrix} B & A \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_k \end{pmatrix} = \begin{pmatrix} 0 & A \\ 0 & D \end{pmatrix}$ , and

since  $\text{rank } [M \ Z^{-1}YM] = \text{rank} \begin{pmatrix} 0 & 0 & 0 & A \\ 0 & I_k & 0 & D \end{pmatrix} > k$  unless  $A = 0$ , we see that

$Z^{-1}Y = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$  is reducible.

### 3.3 Regularity Conditions

Throughout the rest of this paper, we refer to the following conditions on an arbitrary economy  $(Y, L, Z, \mathcal{C})$ :

- (A) Every row and every column of  $Z$  has exactly one positive entry. (Since  $Z \geq 0$ , all other entries are 0.)
- (B)  $Z$  is nonsingular and  $Z^{-1}Y \geq 0$ .
- (C)  $Z$  is nonsingular and  $Z^{-1}L > 0$ .
- (D) Every good is basic.

Note 10 Condition (A) means that every good is produced in a positive amount and no process produces more than one good. (There is no joint production.) By Remark 3 in the Appendix, condition (A) may be restated as:  $Z = PD$  where  $P$  is a permutation matrix and  $D$  a nonsingular diagonal matrix.

Note 11 Since  $Y$  and  $Z$  are nonnegative,  $(A) \Rightarrow (B)$ . This is because (A) implies  $Z = PD$  so  $Z$  is nonsingular, and  $Z^{-1} = D^{-1}P^{-1} = D^{-1}P^T \geq 0$  so  $Z^{-1}Y \geq 0$ . Also, if  $L > 0$  then  $(A) \Rightarrow (C)$  since  $Z^{-1} \geq 0$  implies  $Z^{-1}L > 0$ .

## 4. Existence and Uniqueness of a Price System

### 4.1 Definitions

Although Sraffa is primarily concerned with the relationship between wages and profits, he begins by analyzing prices in his subsistence model in which labor, and hence wages, appear only indirectly: workers' consumption rather than labor itself is considered as a production input. (See Chapter 1, [10, pp. 3-5].) Prices are the rates at which goods must be exchanged among production sectors in order for each sector to begin the next period with the same quantities of inputs that it had at the beginning of the current period.

In Definition 6 below, we express this notion of prices in our model, in which both labor and "necessary consumption" appear explicitly. We then compare our "price system" with the prices in Sraffa's model of subsistence. (See Note 12.) In Section 4.2 we prove the existence of a price system and



its uniqueness up to multiplication by a positive scalar under the condition (B) which is weaker than Sraffa's condition of nonjoint production, (A) .

Definition 6 An  $n \times 1$  vector  $p \geq 0$  is called a price system of the economy  $(Y, L, Z, C)$  relative to  $c$  if  $\underline{1}(Z - Y) \geq c$  and

$$(iv) \quad (Y + Lc)p = Zp.$$

$p$  is called a price system of the economy  $(Y, L, Z, C)$  if it is a price system relative to  $c$  for some  $c \geq 0$ .

Note 12  $L$  is an  $n \times 1$  column vector and  $c$  is a  $1 \times n$  row vector so  $Lc$  in equation (iv) is an  $n \times n$  matrix with  $ij$  entry equal to  $L_i c_j$ , the consumption of the  $j$ th good by workers in the  $i$ th production process.

Since Sraffa isolates neither labor nor necessary consumption in his model of a subsistence economy, the consumption matrix  $Lc$  is (implicitly) included with the nonconsumption inputs  $Y$  in a total commodity inputs matrix  $\bar{Y} = Y + Lc$ . Thus in Sraffa's model, equation (iv) becomes  $\bar{Y} p = Zp$ .

Note 13 The inputs for production process  $i$  are the entries of row  $i$  of the matrix  $(Y + Lc)$  and the outputs are the entries of row  $i$  of  $Z$ . Thus equation (iv) states that the value of the inputs for each process equals the value of its outputs when prices are  $p$ .

Note 14 If  $p$  is a price system of  $(Y, L, Z, C)$  then for any scalar  $\alpha > 0$ ,  $\alpha p$  is also a price system. If there is no price system linearly independent of  $p$ , we say that  $p$  is unique "up to multiplication by a positive scalar," abbreviated as unique (umps).

#### 4.2 Results

Sraffa states without proof that prices exist and are unique in his model of subsistence [10, pg. 5]. Since he assumes that there is no joint production (condition (A)), that  $L > 0$ , and that all goods are basic, and since these assumptions imply (B), (C) and (D) (see Note 11), his result is a special case of our Theorem 1 below. It is easy to construct examples of

economies with joint production that satisfy the conditions of Theorem 1.<sup>1</sup>

Thus our assumptions are weaker than Sraffa's. Since our model takes "necessary consumption" explicitly into account and Theorem 1 applies both to surplus economies and to economies that are not even sustainable, we see that the restriction to the subsistence case is unnecessary.

**Theorem 1** If  $(Y, L, Z, C)$  is an economy satisfying conditions (B), (C) and (D), then it has a unique (umps) price system. This price system,  $p$ , is strictly positive ( $p > 0$ ) and relative to  $c = \underline{1}(Z - Y)$ .

**Proof** By Definition 6, if there is a price system relative to  $c$ ,  $c \leq \underline{1}(Z - Y)$ . For any  $c \geq 0$ , we will show that there is a unique (umps) price system relative to  $c$  if  $c = \underline{1}(Z - Y)$  (nonnegative by Definition 1) and no price system relative to  $c$  if  $c < \underline{1}(Z - Y)$ .

Fix  $c$  with  $0 \leq c \leq \underline{1}(Z - Y)$ . Such a  $c$  exists since we can take  $c = 0 \leq \underline{1}(Z - Y)$ . By (B),  $Z^{-1}$  exists and  $Z^{-1}Y \geq 0$ , and by (C),  $Z^{-1}Lc \geq 0$ . Hence,  $Z^{-1}(Y + Lc) = Z^{-1}Y + Z^{-1}Lc$  is nonnegative. By (D) and Lemma 3,  $Z^{-1}Y$  is irreducible so  $Z^{-1}(Y + Lc)$  is the sum of an irreducible matrix and a non-negative matrix, hence is irreducible. We can therefore apply the Frobenius theorem and conclude that  $Z^{-1}(Y + Lc)$  has a characteristic vector  $p > 0$  associated with its maximal characteristic value  $\lambda = \rho[Z^{-1}(Y + Lc)] > 0$ . Thus,  $Z^{-1}(Y + Lc)p = \lambda p$  and

$$(3) \quad (Y + Lc)p = \lambda Z p.$$

---

1. For example,  $(Y, L, Z, C)$  where  $Y = \begin{pmatrix} .5 & 1 \\ .3 & .5 \end{pmatrix}$ ,  $L = \begin{pmatrix} .6 \\ .4 \end{pmatrix}$ ,  $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $C = R_+^2$ .

It is easy to see that this is an economy.  $Z^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ , so  $Z^{-1}Y = \begin{pmatrix} .2 & .5 \\ .3 & .5 \end{pmatrix} > 0$  and (B) and (D) are satisfied.  $Z^{-1}L = \begin{pmatrix} .2 \\ .4 \end{pmatrix} > 0$  so (C) is satisfied. But  $Z$  is not a permutation of a diagonal matrix, so (A) does not hold.

By equation (ii) of Definition 1,  $\underline{1} L = 1$ , and so  $\underline{1} Lc \leq \underline{1}(Z - Y)$  and  $\underline{1}(Y + Lc) \leq \underline{1} Z$ . Thus

$$(4) \quad (Y + Lc)^T \underline{1}^T \leq Z^T \underline{1}^T,$$

and since  $Z \geq 0$  is nonsingular,  $Z^T \underline{1}^T > 0$ .

If  $c = \underline{1}(Z - Y)$ , the condition (4) holds with equality so that Lemma 1 (a) applies letting  $A = Y + Lc$ ,  $B = Z$ ,  $\gamma = 1$  and  $y = \underline{1}^T$ . Thus we have  $\gamma = 1 = \rho[Z^{-1}(Y + Lc)] = \lambda$  and substituting  $\lambda = 1$  into equation (3) we get equation (iv) of Definition 6 so that  $p > 0$  is a price system relative to  $c = \underline{1}(Z - Y)$ .

If  $c \leq \underline{1}(Z - Y)$  then equality cannot hold in condition (4). Hence the conditions of Lemma 2 hold, where  $A = Y + Lc$ ,  $B = Z$ ,  $u = 1$  and  $y = \underline{1}^T$ , so that  $u = 1 > \rho[Z^{-1}(Y + Lc)]$ . But in order for  $p' \neq 0$  to satisfy equation (iv) we must have  $Z^{-1}(Y + Lc)p' = p' \neq 0$  and therefore by definition of the maximal characteristic value,  $\rho[Z^{-1}(Y + Lc)] \geq 1$ . The contradiction proves that there can be no price system  $p'$  relative to  $c \leq \underline{1}(Z - Y)$ .

Finally, the price system  $p$  relative to  $c = \underline{1}(Z - Y)$  is unique (umps), since  $Z^{-1}(Y + Lc)$  cannot have two linearly independent nonnegative characteristic vectors. (See Remark 3 in Gantmacher [4, pg. 63].) Thus vectors of the form  $\alpha p$  where  $\alpha$  is a positive scalar are the only price systems relative to  $c = \underline{1}(Z - Y) \geq 0$  and the only price systems of  $(Y, L, Z, \mathcal{C})$ .

## 5. Existence and Uniqueness of a Price-Distribution System

### 5.1 Definitions

If  $p$  is a price system of  $(Y, L, Z, \mathcal{C})$  and the entire net output is given to the workers, so that the consumption vector is  $c = \underline{1}(Z - Y)$ , then  $cp$ , the value of net output, equals  $\underline{1}(Lc)p$ , the sum of the values of the workers' consumption in the different sectors, and both may be interpreted as a total wage payment. In a surplus economy, however, the workers need not

receive all of the net output. Sraffa uses his second model of price determination to analyze distribution of the surplus between workers and owners of the means of production (capitalists). His price-distribution system (see Definition 7 below) divides the surplus into two parts: a wage that is paid at the rate  $w$  per unit of labor, and a profit that pays for the use of commodity inputs at a rate  $r$  that is the same for all production processes (see Note 15).

"Regular" economies (see Definition 8 below) have a unique price-distribution system for each wage rate  $w \in [0, 1]$  (see Theorem 2). Thus we may study the effect of changes in the wage rate on the rate of profit and prices (see Theorems 3 and 4). In this section we define a price-distribution system and compare it with the price system of Definition 6. We then discuss the relationship between regular economies and the economies defined above (Definitions 2-4) and prove the main theorem about price-distribution systems in regular economies.

Definition 7 Let  $p \geq 0$  be an  $n \times 1$  column vector and let  $r$  and  $w$  be nonnegative real numbers.  $(p, r, w)$  is called a price-distribution system of the economy  $(Y, L, Z, C)$  if

$$(v) \quad (1 + r) Yp + wL = Zp \quad \text{and}$$

$$(vi) \quad \underline{1} (Z - Y)p = 1.$$

Note 15 Equation (v) may be interpreted as follows. In the  $i^{\text{th}}$  production process (row), the inputs are denoted by  $Y^i$  (nonprimary inputs) and  $\ell^i$  (labor inputs) and the outputs are denoted by  $Z^i$ . When prices are  $p$  and the wage rate is  $w$ , the value of the inputs is  $Y^i p + w\ell^i$  and the value of the outputs is  $Z^i p$ .

In Equation (v) we see that for each  $i$ ,  $Y^i p + rY^i p + w\ell^i = Z^i p$  and so, at prices  $p$  and wage  $w$ , the value of the outputs equals the value of the inputs (nonprimary and labor) plus a profit,  $rY^i p$ , received by the owners of the "means of production"  $Y^i$ . For each  $i$ , the ratio

of the profit to the value of the means of production used in that process

is  $r_i = \frac{rY_i^p}{Y_i^p} = r$ , the same for all processes.

Notice that the values of inputs and outputs and the profit are meaningful concepts only after  $p$ ,  $r$ ,  $w$  are determined from the system (v) - (vi). For their determination in an economy  $(Y, L, Z, C)$ : a) a normalization equation is necessary, whence our Definition 7 of a price-distribution system; b) one of the variables in the price-distribution system  $(p, r, w)$  has to be determined exogenously (usually  $w$  or  $r$ ) in order for the others to be determined endogenously by the system (v) - (vi), so that  $p$ ,  $r$ ,  $w$  can neither be determined independently nor simultaneously. This latter statement means that, if the variable we choose to fix exogenously is  $w$ , prices and the rate of profit: a) take on numerical values interdependent of each other; b) take on numerical values dependent on the numerical value of  $w$ , which is decided outside the system.

Note 16 Since the total amount of labor in the economy is one unit,

the total wage is equal to the wage rate:  $\sum_{i=1}^n w l_i = w \underline{L} = w$ . We will therefore use the terms wage and wage rate interchangeably.

Note 17 If  $\underline{1}(Z - Y) = 0$ , there can be no price-distribution system since equation (vi) cannot hold.

Remark 1 If  $(p, 0, 1)$  is a price-distribution system of  $(Y, L, Z, C)$ , then  $p$  is a price system relative to  $c = \underline{1}(Z - Y)$ . Conversely, if  $p$  is a price system normalized with  $cp = 1$ , then  $(p, 0, 1)$  is a price-distribution system.

For, if  $(p, 0, 1)$  is a price-distribution system, substituting  $r = 0$  and  $w = 1$  in (v) we obtain  $Yp + L = Zp$ , so that by equation (vi),

$$(5) \quad (Y + Lc)p = Yp + L[\underline{1}(Z - Y)]p = Yp + L = Zp.$$

Thus equation (iv) in Definition 6 is satisfied so  $p$  is a price system relative to  $c$  and by equation (vi) and the definition of  $c$ ,  $\underline{1}(Z - Y)p = cp = 1$ . For the converse, (v) holds by equation (5) and (vi) holds by definition of  $c$ .

## 5.2 The Main Theorem

Before stating our theorem on the existence of price-distribution systems, we examine the class of economies for which the result holds.

Definition 8 An economy  $(Y, L, Z, \mathcal{C})$  satisfying (B), (C), (D) and  $\underline{1}(Z - Y) \geq 0$  is said to be regular.

We have already discussed the conditions (B), (C) and (D) in Section 4. The condition  $\underline{1}(Z - Y) \geq 0$ , which Sraffa uses to characterize "production with a surplus" is obviously satisfied by any surplus economy (using our Definition 3). This condition is also satisfied by any sustainable economy  $(Y, L, Z, \mathcal{C})$  with  $0 \notin \mathcal{C}$ , i.e., in which some consumption is necessary for the survival of the workers.<sup>1</sup> Thus, a sustainable economy satisfying (B), (C) and (D) with  $0 \notin \mathcal{C}$  is regular.

Note 18 If  $(Y, L, Z, \mathcal{C})$  is regular, then  $\rho(Z^{-1}Y) < 1$ . This can be seen from the proof of Theorem 1. We showed that when (B), (C) and (D) hold and  $c \leq \underline{1}(Z - Y)$ , then  $1 > \rho[Z^{-1}(Y + Lc)]$ . (See the last paragraph of page 14.) Letting  $c = 0 \leq \underline{1}(Z - Y)$ , this means  $\rho(Z^{-1}Y) < 1$ .

We give one last definition and then our main theorem.

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1. In a sustainable economy,  $\underline{1}(Z - Y) \geq c$  for some  $c \in \mathcal{C}$ . If  $0 \notin \mathcal{C}$  then  $c' \geq 0$  for all  $c' \in \mathcal{C}$ , hence  $\underline{1}(Z - Y) \geq c \geq 0$ .

Definition 9 The real number  $\bar{w}$  is called a wage consistent with the economy  $(Y, L, Z, C)$  (or simply consistent wage) if: (a) there is a unique pair  $p, r$  such that  $(p, r, \bar{w})$  is a price-distribution system of  $(Y, L, Z, C)$ , and (b)  $p > 0$ . If  $\bar{w}$  is a consistent wage, the unique  $p$  and  $r$  will be denoted by  $\bar{p}(\bar{w})$  and  $\bar{r}(\bar{w})$  respectively.

Theorem 2 If  $(Y, L, Z, C)$  is a regular economy, then every  $\bar{w} \in [0, 1]$  is a consistent wage for  $(Y, L, Z, C)$ .

Note 19 Theorem 2 illustrates the "degree of freedom" in the price-distribution system. Even with the price normalization (equation (vi)), the wage  $w$  and rate of profit  $r$  are not determined. But if either  $w$  or  $r$  is fixed then the other is determined along with all prices.

The following lemma will be used in the proof of Theorem 2.

Lemma 4 Let  $A \geq 0$  be an  $n \times n$  irreducible matrix and let  $B = I - \beta A$ , where  $\beta$  is a positive real number. Then the following conditions are equivalent.

- (a) There exists an  $n \times 1$  vector  $x \geq 0$  such that  $Bx \geq 0$ .
- (b)  $1 > \beta \rho(A) = \rho(\beta A)$ .
- (c)  $B$  is nonsingular and  $B^{-1} > 0$ .

Proof The lemma comes from Takayama's Theorem 4.D.2 [11, pg. 392].

Proof of Theorem 2 Let  $\lambda \equiv \rho(Z^{-1}Y)$  be the maximal characteristic value of  $Z^{-1}Y$ .

Since the economy is regular,  $\lambda > 0$ , and by Note 18  $\lambda < 1$ . Let  $R \equiv \frac{1-\lambda}{\lambda} > 0$ ,

so that  $\lambda = \frac{1}{1+R}$ .

Case 1:  $\bar{w} = 0$ . Let  $r = R$ . (D) and Lemma 3 imply that  $Z^{-1}Y$  is irreducible, and by (B),  $Z^{-1}Y \geq 0$ . By the Frobenius theorem, the characteristic vector of  $Z^{-1}Y$  associated with  $\lambda$  is  $q > 0$ . Let  $\beta \equiv \underline{1}(Z - Y)q$ . Since  $\underline{1}(Z - Y) \geq 0$ ,  $\beta > 0$ , so we may let  $p \equiv \frac{1}{\beta} q > 0$ .

Since  $Z^{-1}Yq = \lambda q$ , we have  $Z^{-1}Y \beta p = (\frac{1}{1+r}) \beta p$  and by definition of  $p$  and  $\beta$ ,  $\frac{1}{\beta}(Z - Y) \frac{1}{\beta} q = 1$ . Hence  $p$  satisfies both equation (vi) and

$$(v)' \quad (1 + r)Yp + 0 \cdot L = Zp$$

Thus  $(p, r, 0)$  is a price-distribution system of  $(Y, L, Z, C)$ .

To prove uniqueness, suppose that  $(\bar{p}, \bar{r}, 0)$  is a price-distribution system. By equation (v)' we have

$$(6) \quad Z^{-1}Y\bar{p} = \frac{1}{1+\bar{r}} \bar{p}$$

so  $\bar{p}$  is a nonnegative characteristic vector of  $Z^{-1}Y$  and by Remark 3 of Gantmacher [4, pg. 63],  $\bar{p} = \alpha p$ . But since  $\bar{p}$  must satisfy equation (vi),  $\alpha = 1$  and  $\bar{p} = p$ , so by (6),  $\bar{r} = r$  and  $(p, r, 0)$  is unique.

Case 2:  $\bar{w} \in (0, 1]$ . We define  $B(r) = I - (1+r)Z^{-1}Y$  for  $r \geq 0$ .

For any price-distribution system  $(\hat{p}, \hat{r}, \bar{w})$  we have

$$(7) \quad \begin{aligned} (1 + \hat{r}) Y\hat{p} + \bar{w}L &= Z\hat{p} && \Leftrightarrow \\ [Z - (1 + \hat{r}) Y]\hat{p} &= \bar{w}L && \Leftrightarrow \\ [I - (1 + \hat{r})Z^{-1}Y]\hat{p} &= B(\hat{r}) \hat{p} = \bar{w}Z^{-1}L. \end{aligned}$$

By (C),  $Z^{-1}L \geq 0$ , and since  $\bar{w} > 0$ , we have  $B(\hat{r})\hat{p} \geq 0$ . By definition of a price-distribution system, we have  $\hat{p} \geq 0$ . Letting  $A \equiv Z^{-1}Y$ , we see that condition (a) of Lemma 4 is satisfied, since  $A$  is irreducible. Therefore condition (b) of Lemma 4 must hold with  $\beta = 1 + \hat{r}$ ; i.e.,

$$(8) \quad 1 > (1 + \hat{r}) \rho(A) = (1 + \hat{r})\lambda = \frac{1 + \hat{r}}{1 + R}.$$

Thus  $R > \hat{r}$  and we may restrict our search to price-distribution systems  $(p, r, \bar{w})$  with  $r \in [0, R)$ .



For any  $r \in [0, R)$ , by (8), we have  $1 > (1+r)\rho(A)$ , so Lemma 4 implies that  $B(r) = I - (1+r)A$  is nonsingular and that  $M(r) \equiv [B(r)]^{-1} > 0$ . We let  $\Lambda \equiv Z^{-1}L$  and define  $P(r, w) \equiv wM(r)\Lambda$  for  $r \in [0, R)$  and  $w \in (0, 1]$ . Since  $M(r) > 0$  and  $\Lambda \geq 0$ ,  $M(r)\Lambda > 0$ , so since  $w > 0$ ,  $P(r, w) > 0$  everywhere on its domain. From the equations labelled (7) we see that given  $(r, w)$  in the domain of  $P$ ,  $P(r, w) = w[I - (1+r)Z^{-1}Y]^{-1}Z^{-1}L$  is the unique solution  $p$  to equation (v) in the definition of a price-distribution system.

Let the real-valued function  $T$  be defined on the domain of  $P$  by

$$(9) \quad T(r, w) \equiv \underline{1}(Z - Y) P(r, w).$$

$T(r, w) > 0$  since  $P(r, w) > 0$  and, by assumption,  $\underline{1}(Z - Y) \geq 0$ .

We want to show that given  $\bar{w} \in (0, 1]$  there is exactly one  $\bar{r} \in [0, R)$  with  $T(\bar{r}, \bar{w}) = 1$ . The proof will go as follows: for any  $\bar{w} \in (0, 1]$ ,

(a)  $T(r, \bar{w})$  is a continuously differentiable (hence continuous) function of  $r$ ; (b)  $T(0, \bar{w}) \leq 1$ ; (c)  $T(r, \bar{w})$  is a strictly increasing function of  $r$ ; (d)  $T(r, \bar{w}) \rightarrow \infty$  as  $r \rightarrow R$ ; (e) By the Intermediate Value theorem, there is some  $\bar{r}$  with  $T(\bar{r}, \bar{w}) = 1$ . By (c), this  $\bar{r}$  is unique.

Proof of (a). Let  $M(r) = (m_{ij}(r))$ ,  $i, j = 1, 2, \dots, n$ . As before,

$$B(r) = I - (1+r)A = M^{-1}(r). \quad \text{Then } m_{ij}(r) = \frac{C_{ji}(r)}{\det B(r)} \quad \text{where } C_{ji}(r) \text{ is}$$

the  $ji$  cofactor of  $B(r)$ . Both numerator and denominator are polynomials in  $r$  and  $\det B(r) \neq 0$  for  $r \in [0, R)$  where  $M(r)$  is defined. Hence  $m_{ij}(r)$  is continuously differentiable on  $[0, R)$  and since  $T(r, \bar{w}) = \bar{w} \underline{1}(Z - Y) M(r)\Lambda$  is a linear combination of the functions  $m_{ij}(r)$ , it is a continuously differentiable function of  $r \in [0, R)$ .

Proof of (b). For  $(r, w) = (0, 1)$ , equation (v) becomes  $Yp + L = Zp$  or  $(Z - Y)p = L$ . Since  $P(0, 1)$  is the unique solution to this equation and  $\underline{1}L = 1$ , we have  $T(0, 1) = \underline{1}(Z - Y) P(0, 1) = \underline{1}L = 1$ . By definition of  $P$ ,  $P(r, w) = wP(r, 1)$ , so  $T(0, \bar{w}) = \underline{1}(Z - Y) P(0, \bar{w}) = \bar{w} \underline{1}(Z - Y) P(0, 1) = \bar{w} \leq 1$ .

Proof of (c). Let  $0 \leq r_1 < r_2 < R$ . Since  $P(r, \bar{w}) > 0 \forall r \in [0, R)$ , and  $A \geq 0$  is irreducible,  $A P(r_1, \bar{w}) > 0$  and  $(r_1 - r_2) A P(r_1, \bar{w}) < 0$ . By definition of  $B(r_1)$  and  $P(r_1, \bar{w})$ ,

$$B(r_1) P(r_1, \bar{w}) = \bar{w} \Lambda = P(r_1, \bar{w}) - (1 + r_1) A P(r_1, \bar{w}).$$

Hence,

$$\bar{w} \Lambda > P(r_1, \bar{w}) - (1 + r_1) A P(r_1, \bar{w}) + (r_1 - r_2) A P(r_1, \bar{w}) = P(r_1, \bar{w}) - (1 + r_2) A P(r_1, \bar{w})$$

and so  $\bar{w} \Lambda > B(r_2) P(r_1, \bar{w})$ .

Premultiplying both sides of this last inequality by  $M(r_2) > 0$ , we obtain

$$\bar{w} M(r_2) \Lambda = P(r_2, \bar{w}) > M(r_2) B(r_2) P(r_1, \bar{w}) = P(r_1, \bar{w}).$$

Since  $\underline{1}(Z - Y) \geq 0$ , we obtain

$$T(r_2, \bar{w}) = \underline{1}(Z - Y) P(r_2, \bar{w}) > \underline{1}(Z - Y) P(r_1, \bar{w}) = T(r_1, \bar{w}).$$

Proof of (d) Let  $M^i(r)$  denote the  $i$ th row and  $M_j(r)$  the  $j$ th column of the matrix  $M(r)$ . For any  $r \in [0, R)$  and  $i = 1, 2, \dots, n$ ,  $M^i(r) B(r) = e^i$  where  $e^i$  is the row vector with 1 in the  $i$ th place and zeros elsewhere. Suppose  $M^i(r)$  is bounded by a fixed row vector for all  $r \in [0, R)$ . Then we can let  $M^i$  denote the vector of least upper bounds of the functions that are entries of  $M^i(r)$ . Since these functions,  $m_{ij}(r)$ , are strictly increasing in  $r$ , we have  $\lim_{r \uparrow R} M^i(r) = M^i > 0$ . Thus,  $\lim_{r \uparrow R} M^i(r) B(r) = [\lim_{r \uparrow R} M^i(r)] [\lim_{r \uparrow R} B(r)] = M^i B(R) = e^i \geq 0$ . This means that  $[B(R)]^T [M^i]^T \geq 0$ , and since  $[B(R)]^T = I - (1 + R)A^T$  and by Lemma 5 in the Appendix,  $A^T$  is irreducible, we can apply Lemma 4 and conclude that  $1 > (1 + R) \rho(A) = (1 + R)\lambda = 1$ . This contradiction proves that there can be no upper bound for  $M^i(r)$  and since the entries of  $M^i(r)$  are strictly increasing functions of  $r$ , there must be at least one entry  $m_{ik_i}(r) \rightarrow \infty$  as  $r \uparrow R$ .

The  $i$ th entry of  $P(r, \bar{w})$  is  $\bar{w} M^i(r) \Lambda > \bar{w} m_{ik_i}(r) \Lambda_{k_i}$  where  $\Lambda_{k_i}$  is the  $k_i$ th entry of  $\Lambda$  and is strictly positive by (C). Thus we have  $\lim_{r \uparrow R} \bar{w} M^i(r) \Lambda = \infty$  for  $i = 1, 2, \dots, \infty$  and so all the entries of  $P(r, \bar{w})$  go to  $\infty$  as  $r \uparrow R$ . Since  $\underline{1}(Z - Y) \geq 0$ , and  $T(r, \bar{w}) = \underline{1}(Z - Y) P(r, \bar{w})$ , we have  $T(r, \bar{w}) \rightarrow \infty$  as  $r \uparrow R$ . By a symmetric argument, we could show that every column of  $M(r)$  has an unbounded entry and so if  $\underline{1}(Z - Y) > 0$ , we could weaken condition (C) to the assumption that  $\Lambda \geq 0$  and still find  $\lim_{r \uparrow R} T(r, \bar{w}) = \infty$ .

We have proved statements (a) through (d), so (e) holds and given  $\bar{w} \in (0, 1]$ , there is a unique  $\bar{r} \in [0, R)$  with  $T(\bar{r}, \bar{w}) = 1$ . Let  $\bar{p} = P(\bar{r}, \bar{w})$ .  $\bar{p}$  is the unique solution to (v) in Definition 7 given  $\bar{r}$  and  $\bar{w}$ , and solves (vi) since  $\underline{1}(Z - Y)\bar{p} = \underline{1}(Z - Y) P(\bar{r}, \bar{w}) = T(\bar{r}, \bar{w}) = 1$ . Thus  $(\bar{p}, \bar{r}, \bar{w})$  is a price-distribution system. For any other system  $(\hat{p}, \hat{r}, \bar{w})$  we have shown  $\hat{r} \in [0, R)$ , so in order for  $\hat{p}$  to solve (v) we must have  $\hat{p} = P(\hat{r}, \bar{w})$  and in order for  $\hat{p}$  to solve (vi) we must have  $T(\hat{r}, \bar{w}) = 1$ , which occurs only for  $\hat{r} = \bar{r}$ . Thus  $\hat{p} = P(\bar{r}, \bar{w}) = \bar{p}$  and the price-distribution system is unique.

## 6. Changes in the Wage Rate

### 6.1 Effect on Rate of Profit

The second important conclusion Sraffa draws is that a rise in the wage rate  $w$  must be accompanied by a fall in the rate of profit  $r$ . He finds that the relationship between  $r$  and  $w$  is linear when  $w$  is measured in terms of the "Standard Commodity" [10, pg. 22], but notes that the monotonic relationship holds no matter what units  $w$  is measured in [10, pg. 61]. In Theorem 3, we give a direct proof of the monotonicity of this relationship and then derive in Theorem 4 some conclusions about price movements.

Notation We refer below to the regular economy  $(Y, L, Z, \mathcal{C})$ . In the proof of Theorem 2, we defined  $\Lambda = Z^{-1}L$ ,  $P(r, w) = w[I - (1 + r)Z^{-1}Y]^{-1}\Lambda$  and  $T(r, w) = \underline{1}(Z - Y) P(r, w)$  for  $(r, w) \in [0, R) \times (0, 1]$ . By the conclusion of Theorem 2, for each  $w \in [0, 1]$ , there is a unique price-distribution system  $(p, r, w)$  with  $p = \tilde{p}(w)$  and  $r = \tilde{r}(w)$ . Thus  $\tilde{p}$  and  $\tilde{r}$  are well-defined functions on  $[0, 1]$  and by Case 2 in the proof of Theorem 2,  $\tilde{r}(w) < R$  for  $w > 0$ .

**Theorem 3** Let  $(Y, L, Z, \mathcal{C})$  be a regular economy; then

(a) The functions  $\tilde{r}(w)$  and  $\tilde{p}(w)$  are well-defined for  $\forall w \in [0, 1]$  (see above) and  $(\tilde{p}(w), \tilde{r}(w), w)$  is the unique price-distribution system given  $w \in [0, 1]$ .

(b)  $\tilde{r}$  is a strictly decreasing 1 - 1 correspondence from  $[0, 1]$  onto  $[0, R]$ .

(c)  $\hat{w} : [0, R] \rightarrow [0, 1]$ ,  $\hat{w}(r) = w$ , the inverse of  $\tilde{r}$ , is well-defined and continuously differentiable on  $(0, R)$  with  $\hat{w}' \leq 0$ .

**Proof** (a) is a consequence of Theorem 2 and the notation above. (b) If  $1 \cong w_2 > w_1 = 0$ , then  $\tilde{r}(w_1) = \tilde{r}(0) = R$  and  $\tilde{r}(w_2) < R = \tilde{r}(w_1)$ .

If  $1 \cong w_2 > w_1 > 0$ , then  $T(r, w_i)$ ,  $i = 1, 2$ , is defined  $\forall r \in [0, R]$ . By part (c) of the proof of Theorem 2,  $T(r, w)$  is strictly increasing as a function of  $r$  for fixed  $w \in (0, 1]$  and since  $T(r, w) = w T(r, 1)$  with  $T(r, 1) > 0$ , we see that  $T(r, w)$  is strictly increasing as a function of  $w$  for fixed  $r \in [0, R]$ . Hence if  $\tilde{r}(w_2) \cong \tilde{r}(w_1)$ , we have  $1 = T(\tilde{r}(w_1), w_1) < T(\tilde{r}(w_1), w_2) \leq T(\tilde{r}(w_2), w_2)$ , a contradiction since  $\tilde{r}$  is defined with  $T(\tilde{r}(w), w) = 1 \forall w \in (0, 1]$ . Thus when  $w_2 > w_1$ ,  $\tilde{r}(w_2) < \tilde{r}(w_1)$  and  $\tilde{r}$  is strictly decreasing (hence 1 - 1) on  $[0, 1]$ .

To show that  $\tilde{r}$  maps  $[0, 1]$  onto  $[0, R]$ , fix  $r_0 \in [0, R]$ . If  $r_0 = R$ , then  $\tilde{r}(0) = r_0$ . If  $r_0 < R$ ,  $T(r_0, w) > 0$  is well defined for

$w \in (0, 1]$  and letting  $w_0 = \frac{1}{T(r_0, 1)}$  we have  $w_0 T(r_0, 1) = T(r_0, w_0) =$

1, so that  $r_0 = \tilde{r}(w_0)$ .

(c) Since  $\tilde{r} : [0, 1] \rightarrow [0, R]$  is 1 - 1 and onto, the inverse  $\hat{w}$  is defined. Since  $\tilde{r}$  is strictly decreasing, its inverse is too. In the proof of Theorem 2 we showed that the partial derivative of  $T$  with respect to  $r$  is continuous. The partial of  $T$  with respect to  $w$  is  $T_2(r, w) = T(r, 1) > 0$  on  $[0, R] \times (0, 1]$ . Hence by the implicit function theorem (see Theorem 7.6, Apostol [1, page 147]), in any neighborhood of  $r_0 \in (0, R)$  there is a

continuously differentiable function  $\tilde{w}(r)$  with  $T(r, \tilde{w}(r)) = 1$ . But since this equation is solved only by  $\hat{w}(r)$ ,  $\hat{w}(r)$  is continuously differentiable on  $(0, R)$ . Since  $\hat{w}$  is monotonically decreasing,  $\hat{w}'(r) \leq 0$  for  $r \in (0, R)$ .

## 6.2 Effect on Prices

Sraffa mentions another consequence of the conditions used in Theorem 2: "If as a result of a rise in the rate of profits the price falls, its rate of fall cannot exceed the rate of fall of the wage." [10, pg. 38]<sup>1</sup> This statement is formalized and strengthened to strict inequality in (b) of Theorem 4. An alternative formulation is expressed in (c) below: the ratio of the wage  $w$  to any price  $p_i$  falls as the rate of profits rises."

**Theorem 4** Given the regular economy  $(Y, L, Z, C)$ , we define  $\hat{p} : [0, R) \rightarrow \mathbb{R}_+^n$  by  $\hat{p}(r) = P(r, \hat{w}(r))$ , where  $\hat{w}$  the inverse of  $\tilde{r}$ , defined in Theorem 3(c).

Let  $\hat{p}_i(r)$  be the  $i$ th component of the vector  $\hat{p}(r)$ ,  $r \in [0, R)$ . Then

(a)  $\hat{p}$  is continuously differentiable on  $(0, R)$ .

(b)  $\forall r \in (0, R)$ ,  $\frac{r}{w} \frac{d\hat{w}(r)}{dr} < \frac{r}{\hat{p}_i(r)} \frac{d\hat{p}_i(r)}{dr}$  for  $i = 1, 2, \dots, n$ .

(c)  $\frac{d}{dr} \left[ \frac{\hat{w}(r)}{\hat{p}_i(r)} \right] < 0$ ,  $i = 1, 2, \dots, n$ ,  $\forall r \in (0, R)$ .

**Proof** (a) By part (a) of the proof of Theorem 2,  $P(r, w)$  has a continuous partial derivative with respect to  $r$  on  $[0, R) \times (0, 1]$  and since  $P(r, w) = wP(r, 1)$ , it has a continuous partial derivative with respect to  $w$ . Since  $\hat{w}$  is  $C^1$  on  $(0, R)$ , by the chain rule,  $\hat{p}(r)$  is  $C^1$ . (b)

Let  $P_i(r, w)$  denote the  $i$ th component of  $P(r, w)$ . By part (c) of the proof of Theorem 2,  $P_i(r, w) > 0$  and  $\frac{\partial}{\partial r} P_i(r, w) > 0 \forall r \in [0, R)$ . By the chain rule, for  $i = 1, 2, \dots, n$ ,

$$(10) \quad \hat{p}'_i(r) = \frac{\partial}{\partial w} P_i(r, \hat{w}(r)) \hat{w}'(r) + \frac{\partial}{\partial r} P_i(r, \hat{w}(r)).$$

1. Sraffa is referring here to economies satisfying (A), (C), (D) and  $\underline{1}(Z - Y) \geq 0$ .

We simplify notation by omitting the arguments of the functions  $\hat{p}_i$ ,  $P_i$ ,  $\hat{w}$ , and their derivatives. Then, noting that  $\hat{w}(r) > 0$  for  $r \in [0, R)$ , we have

$$(11) \quad \frac{r}{\hat{p}_i} \frac{\partial}{\partial r} P_i > 0 = \frac{r}{\hat{w}} \hat{w}' - \frac{\hat{p}_i}{\hat{w}} \frac{r}{\hat{p}_i} \hat{w}', \quad \forall r \in (0, R).$$

Since  $P_i(r, w) = wP_i(r, 1)$ ,  $\frac{\partial}{\partial w} P_i(r, w) = P_i(r, 1) = \frac{P_i(r, w)}{w}$ . Substituting in (11) gives

$$\frac{r}{\hat{p}_i} \frac{\partial}{\partial r} P_i > \frac{r}{\hat{w}} \hat{w}' - \frac{\partial}{\partial w} P_i \frac{r}{\hat{p}_i} \hat{w}' \quad \Rightarrow$$

$$\frac{r}{\hat{p}_i} \left[ \frac{\partial}{\partial r} P_i + \frac{\partial}{\partial w} P_i \cdot \hat{w}' \right] > \frac{r}{\hat{w}} \hat{w}' \quad \text{and by (10),}$$

$$\frac{r}{\hat{p}_i} \hat{p}'_i > \frac{r}{\hat{w}} \hat{w}', \quad \text{which was to be proved.}$$

$$(c) \quad \frac{d}{dr} \left[ \frac{\hat{w}(r)}{\hat{p}_i(r)} \right] = \frac{\hat{p}_i(r) \hat{w}'(r) - \hat{w}(r) \hat{p}'_i(r)}{[\hat{p}_i(r)]^2}$$

From part (b) above,  $\frac{r}{\hat{w}(r)} \hat{w}'(r) - \frac{r}{\hat{p}_i(r)} \hat{p}'_i(r) < 0 \quad \forall r \in (0, R)$ , and since

$$\frac{1}{r} \hat{w}(r) \hat{p}_i(r) > 0, \quad \forall r \in (0, R),$$

$$\frac{1}{r} \hat{w}(r) \hat{p}_i(r) \left[ \frac{r}{\hat{w}(r)} \hat{w}'(r) - \frac{r}{\hat{p}_i(r)} \hat{p}'_i(r) \right]$$

$$= \hat{p}_i(r) \hat{w}'(r) - \hat{w}(r) \hat{p}'_i(r) < 0 \quad \text{and so}$$

$$\frac{d}{dr} \left[ \frac{\hat{w}(r)}{\hat{p}_i(r)} \right] < 0.$$

## 7. The Consumption Set $\mathcal{C}$ and the Living Wage

We can see from Definition 7 that the question whether  $(p, r, w)$  is a price-distribution system of the economy  $(Y, L, Z, \mathcal{C})$  does not depend on the consumption set  $\mathcal{C}$ . Thus, a price-distribution system may exist even for economies that are not sustainable. If  $(p, r, w)$  is a price-distribution system, in order for workers to purchase the consumption goods necessary for survival there must be some producible  $c \in \mathcal{C}$  with  $w \geq cp$ . (The total wage must not be less than the cost of  $c$ .) One might imagine the anomalous situation in which the wage  $w$  rises but  $cp$ , the "cost of living," rises faster so that at a higher wage workers cannot survive. It will be shown in Theorem 5 that in regular economies such an anomaly cannot occur. This result is suggested by part (b) of Theorem 4: any prices that rise with the wage (as  $r$  falls) are less elastic than the wage with respect to the fall in  $r$ ; i.e., rise slower than the wage.

Definition 10 A consistent wage  $w$  for  $(Y, L, Z, \mathcal{C})$  is called a living wage if for some  $c \in \mathcal{C}$ ,  $\underline{1}(Z - Y) \geq c$  and  $w \geq c\tilde{p}(w)$ .

Note 20 The condition  $w \geq c\tilde{p}(w)$  for  $c \in \mathcal{C}$  does not necessarily imply  $\underline{1}(Z - Y) \geq c$ . Thus workers might be able to afford consumption bundles that they cannot obtain since the net output is not large enough. We present an example of such a situation in Section A.2 of the Appendix.

Remark 2 For the economy  $(Y, L, Z, \mathcal{C})$ :

(a) If there is a living wage then the economy is sustainable, since  $\underline{1}(Z - Y) \geq c$  for some  $c \in \mathcal{C}$ .

(b)  $w = 0$  is a living wage only if  $0 \in \mathcal{C}$ . Since  $c \geq 0 \forall c \in \mathcal{C}$  and  $\tilde{p}(0) > 0$ , if  $0$  is a consistent wage,  $w = 0 \geq c\tilde{p}(0)$  can be satisfied only by  $c = 0$ .

(c) If  $w = 1$  is a consistent wage and the economy is sustainable, then  $w = 1$  is a living wage. Sustainability implies that for some  $c \in \mathcal{C}$ ,  $\underline{1}(Z - Y) \geq c$ . Also, since  $\tilde{r}(1) = 0$  we have by equation (v) in Definition 7

$$1 \cdot L = Z \tilde{p}(1) - Y \tilde{p}(1).$$

Hence  $\underline{1} L = 1 = w = \underline{1}(Z - Y) \tilde{p}(1) \geq c\tilde{p}(1)$

since  $\tilde{p}(1) > 0$ .

We now show that the anomaly referred to at the beginning of this section cannot occur in a regular economy. The set of living wages is an interval.

**Theorem 5** In a regular economy, if  $w$  is a living wage and  $l \cong v > w$ , then  $v$  is a living wage.

**Proof** Since  $w$  is a living wage,  $\exists c \in \mathcal{C}$  with  $l(z - Y) \cong c$  and  $w \cong c \tilde{p}(w)$ . If  $c = 0$ ,  $v > w \cong 0 = c\tilde{p}(v)$  and we are done. If  $c \geq 0$ , then  $c \tilde{p}(w) > 0$ . Let  $r_v = \tilde{r}(v)$  and  $r_w = \tilde{r}(w)$ , so that  $v = \hat{w}(r_v)$  and  $w = \hat{w}(r_w)$ . By (b) of Theorem 3,  $\tilde{r}$  is decreasing, so  $r_w > r_v$ . By (c) of Theorem 4,

$$\frac{\hat{w}(r_w)}{\hat{p}_i(r_w)} \cong \frac{\hat{w}(r_v)}{\hat{p}_i(r_v)} \quad \text{and} \quad \hat{w}(r_w) \hat{p}_i(r_v) \cong \hat{w}(r_v) \hat{p}_i(r_w) \quad \text{for } i = 1, 2, \dots, n.$$

Since  $c \geq 0$ ,  $\hat{w}(r_w) c\hat{p}(r_v) \cong \hat{w}(r_v) c\hat{p}(r_w)$ , and since  $\tilde{p}(v) = \hat{p}(r_v)$  and  $\tilde{p}(w) = \hat{p}(r_w)$ ,  $w c\tilde{p}(v) \cong v c\tilde{p}(w)$ . Since  $w \cong c\tilde{p}(w)$  and  $c\tilde{p}(v) > 0$ , we have  $v c\tilde{p}(w) \cong w c\tilde{p}(v) \cong [c\tilde{p}(w)] c\tilde{p}(v)$  and since  $c\tilde{p}(w) > 0$ ,  $v \cong c\tilde{p}(v)$ , so  $v$  is a living wage.

**Note 21** Since the set of living wages is an interval, the lower end-point of that interval is the "limit below which the wage cannot fall" that Sraffa refers to [10, pg. 10].

## 8. Nonbasic Goods

### 8.1 A Clarification of Sraffa

Sraffa writes in Section 65 "the chief economic implication of the distinction [between basic and nonbasic goods is] that basics have an essential part in the determination of prices and the rate of profits, while nonbasics have none." [10, Pg. 54] His justification is that "a tax on a basic product will affect all prices and cause a fall in the rate of profits that corresponds to a given wage while if imposed on a nonbasic it will have no effect beyond the price of the taxed commodity and those of such other nonbasics as may be linked with it." "Such a tax is best conceived as a tithe, which can be defined independently of prices and has the same effect as would have a fall in the output of the commodity in question all other things (namely the quantities of its means of production and of its companion products)



remaining unchanged." [10, pg. 55].

In Theorem 6 below we analyze the effect of a tithe on a group non-basic goods. Our technique is to compare economies,  $(Y, L, Z, \mathcal{C})$  and  $(Y, L, ZE, \mathcal{C})$ , where the latter has the same inputs as the former and the same outputs of basic goods. The outputs of the nonbasic goods are changed in some proportion that is the same for all production sectors. Prices of basic goods and the relationship between wage and profit rates are affected. It turns out, however, that "the relative prices" of basic goods, i.e., prices relative to the wage rate do not change. In Section 8.2 we show with an example how the tithe on just one nonbasic good will change the price of a basic good. This apparent divergence of our results from Sraffa's previous quotations is due to the normalization we use, i.e., (vi)  $\underline{1}(Z - Y)p = 1$  where  $\underline{1}(Z - Y)$  stands for net output (including nonbasic goods). If equation (vi) is replaced by the normalization  $w = 1$  or if the total value of the basic goods is set equal to 1, Sraffa's conclusion is correct and prices of the basic goods are unaffected by a tithe on the nonbasics.

The assumptions used in Theorem 6 are probably stronger than necessary, but they are weak enough to cover the regular economies that were discussed in Sections 5, 6 and 7.

Definition 11 An economy  $(Y, L, Z, \mathcal{C})$  is  $r$ -invertible if  $[Z - (1+r)Y]$  is nonsingular and  $\underline{1}(Z - Y)[Z - (1+r)Y]^{-1}L > 0$ .

Note 22 If  $(p, r, w)$  is a price-distribution system for  $(Y, L, Z, \mathcal{C})$  and  $[Z - (1+r)Y]$  is nonsingular then  $w > 0$ . This is because  $[Z - (1+r)Y]p = wL$  must hold by equation (v) and  $[Z - (1+r)Y]p = 0$  implies  $p = 0$  when  $[Z - (1+r)Y]$  is nonsingular. Therefore a regular economy is not  $R$ -invertible, where  $\frac{1}{1+R} = \rho(Z^{-1}Y)$ , but is  $r$ -invertible,  $\forall r \in [0, R)$ .

Notation Let  $K \subset \{1, 2, \dots, n\}$  and let  $\mathcal{E}_K$  denote the set of  $n \times n$  nonsingular diagonal matrices  $E = (e_{jj}) \geq 0$  with  $e_{ii} = 1 \forall i \notin K$ . For any  $E \in \mathcal{E}_K$  and  $A$  an  $n \times n$  matrix, the  $j$ th column of  $AE$  is  $(AE)_j = e_{jj} A_j =$

$\begin{cases} A_j & \text{if } j \notin K \\ e_{jj} A_j & \text{if } j \in K \end{cases}$ . Thus postmultiplication by  $E \in \mathcal{E}_K$  leaves the  $j$ th column

unchanged if  $j \notin K$  and multiplies it by a scalar  $e_{jj} > 0$  if  $j \in K$ .

Theorem 6 Suppose  $(Y, L, Z, \mathcal{C})$  is an economy with a group of nonbasic goods  $K \neq \{1, 2, \dots, n\}$ , and a price-distribution system  $(p, r, w)$ . Suppose also that  $Z$  and  $[Z - (1+r)Y]$  are nonsingular. For any  $E \in \mathcal{E}_K$ , if  $(Y, L, ZE, \mathcal{C})$  is  $r$ -invertible, then it has a unique price-distribution system of the form  $(\bar{p}, r, \bar{w})$  and  $\frac{1}{\bar{w}} \bar{p}_i = \frac{1}{w} p_i$  for all basic goods  $i \notin K$ .

Note 23 By definition of  $\mathcal{E}_K$ , the columns of  $ZE$  and  $Z$  associated with basic goods are equal, while the  $j$ th column of  $ZE$  is a scalar multiple of the  $j$ th column of  $Z$  when  $j$  is a nonbasic. When these scalar multiples are less than one,  $ZE$  may be thought of as the gross output matrix after tithes on the nonbasic goods have been extracted.

Proof of Theorem 6 If  $K = \emptyset$  then  $\mathcal{E}_K$  contains only  $I_n$ , the identity and the theorem is trivial:  $(\bar{p}, r, \bar{w}) = (p, r, w)$ . We therefore assume  $K \neq \emptyset$ . Without loss of generality we may relabel goods making  $K = \{k+1, k+2, \dots, n\}$ . Since  $K \neq \emptyset$  and  $K \neq \{1, 2, \dots, n\}$ ,  $0 < k < n$ . Since  $Z$  is nonsingular,

Lemma 3 applies and we may write  $(1+r)Z^{-1}Y = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$ , where  $B$  is a  $k \times k$  matrix. Consider  $E \in \mathcal{E}_K$  with  $(Y, L, ZE, \mathcal{C})$   $r$ -invertible. Both  $E$  and  $[ZE - (1+r)Y]$  are nonsingular, so we may define  $q(E) \equiv [ZE - (1+r)Y]^{-1} L = [I - (1+r)E^{-1}Z^{-1}Y]^{-1} E^{-1}Z^{-1} L$ .  $q(E)$  is then the unique solution to

$$(1+r)Yq + L = ZEq.$$

We partition  $q$  and  $Z^{-1}L$  as follows:  $q(E) = \begin{pmatrix} q_1(E) \\ q_2(E) \end{pmatrix}$  and  $Z^{-1}L = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}$

where  $q_1(E)$  and  $\Lambda_1$  and  $k \times 1$  vectors. By definition of  $\mathcal{E}_K$ ,

$E = \begin{pmatrix} I_k & 0 \\ 0 & U \end{pmatrix}$  where  $U \geq 0$  is a diagonal  $(n-k) \times (n-k)$  matrix.  $U$  is

nonsingular since  $E$  is. Now we have

$$(1+r)E^{-1}Z^{-1}Y = \begin{pmatrix} I_k & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} B & 0 \\ U^{-1}C & U^{-1}D \end{pmatrix}$$

$$\text{and } \begin{pmatrix} q_1(E) \\ q_2(E) \end{pmatrix} = \begin{pmatrix} I_k - B & \cdot & 0 \\ \cdot & \cdot & \cdot \\ -U^{-1}C & I_{n-k} & -U^{-1}D \end{pmatrix}^{-1} \begin{pmatrix} I_k & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}$$

so that

$$(12) \quad \begin{pmatrix} q_1(E) \\ q_2(E) \end{pmatrix} = \begin{pmatrix} (I_k - B)^{-1} \Lambda_1 \\ X \Lambda_1 + (I_{n-k} - U^{-1}D)^{-1} U^{-1} \Lambda_2 \end{pmatrix}, \text{ where}$$

$X$  is an  $(n - k) \times k$  matrix.

Since  $(Y, L, ZE, \mathcal{C})$  is  $r$ -invertible,  $\gamma \equiv \underline{1}(ZE - Y) q(E) > 0$ . Also since  $(p, r, w)$  is a price-distribution system for  $(Y, L, Z, \mathcal{C})$ , by Note 22,  $w > 0$ . Hence  $(1 + r)Y \left(\frac{p}{w}\right) + L = Z \left(\frac{p}{w}\right)$ . Since  $I_n \in \mathcal{E}_K$  and  $[Z - (1+r)Y]$  is invertible,  $q(I_n)$  is well-defined; and since  $ZE = Z \Leftrightarrow E = Z^{-1}Z = I$  we know that  $q(I)$  is the unique solution to  $(1 + r)Yq + L = Zq$ . Therefore  $\frac{1}{w}p = q(I)$ . Let  $\bar{p} = \frac{1}{\gamma} q(E)$  and  $\bar{w} = \frac{1}{\gamma} > 0$ . Then  $(\bar{p}, r, \bar{w})$  is a price-distribution system for  $(Y, L, ZE, \mathcal{C})$  since

$$(1 + r)Y\bar{p} + \bar{w}L = (1 + r)Y \frac{1}{\gamma} q(E) + \frac{1}{\gamma}L = \frac{1}{\gamma} ZE q(E) = ZE\bar{p}, \text{ and}$$

$$\underline{1}(ZE - Y)\bar{p} = \underline{1}(ZE - Y) \frac{1}{\gamma} q(E) = 1.$$

If  $(\hat{p}, r, \hat{w})$  is a price-distribution system for  $(Y, L, ZE, \mathcal{C})$ , by Note 22,  $\hat{w} > 0$ .  $\frac{1}{\hat{w}} \hat{p}$  is then a solution to  $(1 + r)Yq + L = ZE q$  and  $\frac{1}{\hat{w}} \hat{p} = q(E) = \frac{1}{w} p$ . Also  $\underline{1}(ZE - Y)\hat{p} = 1 = \underline{1}(ZE - Y) \frac{1}{\gamma} q(E) = \underline{1}(ZE - Y) \bar{w} \left(\frac{1}{\hat{w}} \hat{p}\right)$ , so  $\hat{w} = \bar{w}$  and  $\hat{p} = \bar{p}$  and  $(\bar{p}, r, \bar{w})$  is unique.

We have shown that  $\frac{1}{w}p = q(I)$  and by definition  $\frac{1}{w}p = q(E)$ . By equation (12),  $q_1(E)$  does not depend on  $E$ :  $q_1(E) = (I_k - B)^{-1} \Lambda_1 = q_1(I)$ . Thus since  $q_1(E)$  denotes the first  $k$  terms of  $q(E)$ , we have

$$\frac{1}{w}p_i = \frac{1}{w}p_i \quad \text{for } i = 1, 2, \dots, k, \text{ or for } \forall i \in K.$$

8.2 An Example of a Tithe on a Nonbasic Good

We consider the economy  $(Y, L, Z, \mathcal{C})$  where  $Y = \begin{pmatrix} .1 & 0 \\ .2 & .1 \end{pmatrix}$   $L = \begin{pmatrix} .8 \\ .2 \end{pmatrix}$  and  $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . We do not have to specify  $\mathcal{C}$  for the purpose at hand. It easily checked that this economy does in fact satisfy the conditions of Definition 1. Also one can check that the economy satisfies (A) and (C) but not (D) since good 1 is basic while good 2 is nonbasic. With a little more effort it can be seen that the economy has a price-distribution system  $(p, r, w)$  where

$$p = w \begin{pmatrix} \frac{16}{17} \\ \frac{164}{289} \end{pmatrix} \quad r = .5 \quad \text{and}$$

$$w = \frac{289}{338} = \frac{17^2}{2(13)^2}.$$

We extract a "tithe" of  $\frac{1}{4}$  of the output of the nonbasic good 2.

This is accomplished by considering the economy  $(Y, L, ZE, \mathcal{C})$  where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix} \text{ and so } ZE = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix}. \text{ This new economy has a}$$

price-distribution system  $(p^E, r, w^E)$  where  $r = .5$ , as before, but

$$p^E = w^E \begin{pmatrix} \frac{16}{17} \\ \frac{41}{51} \end{pmatrix} \quad \text{and} \quad w^E = \frac{204}{241} = \frac{3 \cdot 4 \cdot 17}{241}.$$

$w^E > w$ , so the price of the basic good when the tithe is imposed is greater than in the original economy:  $p_1^E = \frac{16}{17} w^E > \frac{16}{17} w = p_1$ .

On the other hand the conclusion of Theorem 6 still holds since the relative prices of the basic good with and without the tithe are equal:

$$\frac{1}{w^E} \cdot p^E = \frac{16}{17} = \frac{1}{w} \cdot p.$$

APPENDIX

A.1 Notes on Irreducibility

A square matrix  $P$  is called a permutation matrix if exactly one entry in each row and each column of  $P$  equals 1 and all other entries equal zero. If  $P$  is a permutation matrix, then  $P^T$  is also. Let

$P = (p_{ij})$ ,  $i, j = 1, 2, \dots, n$ , be a permutation matrix. Letting  $PP^T = (q_{ij})$

we have  $q_{ij} = \sum_{k=1}^n p_{ik} p_{jk} = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j \neq i \end{cases}$  since  $p_{ik} = 1$  for only one  $k$  and

$p_{jk} = 1$  if and only if  $j = i$ . Thus  $PP^T = I_n$  and  $P^T = P^{-1}$ . By the Remark of Takayama [11, pg. 369] we see that if  $P$  and  $Q$  are permutation matrices then  $PQ$  is one also.

According to Gantmacher [4, pg. 51], an  $n \times n$  matrix  $X$  is reducible iff there is a permutation matrix  $P$  with  $P^{-1}XP = \begin{pmatrix} B & O \\ C & E \end{pmatrix}$  where  $B$  and  $E$  are square matrices. From this we see that if  $Q$  is a permutation matrix and  $Y$  is reducible then  $X = Q^{-1}YQ$  is reducible. This is because  $QXQ^{-1} = Y$  and permutation matrix  $P$ ,  $P^{-1}YP = \begin{pmatrix} B & O \\ C & E \end{pmatrix} = P^{-1}(QXQ^{-1})P = (P^{-1}Q)X(Q^{-1}P) = (Q^T P)^{-1}X(Q^T P)$ , since  $Q^T = Q^{-1}$ . The last equation implies that  $X$  is reducible since  $Q^T P$  is a permutation matrix.

Lemma 5 An  $n \times n$  matrix  $X$  is irreducible iff  $X^T$  is irreducible.

Proof Suppose  $X^T$  is reducible. There is a permutation matrix  $P$  with

$P^{-1}X^T P = \begin{pmatrix} B & O \\ C & E \end{pmatrix}$ , where  $B$  is  $k \times k$  and  $O$  is  $k \times (n - k)$  with all

entries equal to zero. Let  $A^T \equiv P^{-1}X^T P$  so that  $A = (a_{ij}) = \begin{pmatrix} B^T & C^T \\ O^T & E^T \end{pmatrix}$

where  $O^T$  is  $(n - k) \times k$ . Since the entries of  $O^T$  are all zero,

$$(13) \quad a_{ij} = 0 \quad \text{for } i = k + 1, k + 2, \dots, n \\ \text{and } j = 1, 2, \dots, k.$$

$$\text{Let } M = (m_{ij}) = \begin{pmatrix} 0 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix} \text{ for } n \times n; \quad m_{ij} = \begin{cases} 1 & \text{for } i = n + 1 - j \\ 0 & \text{otherwise} \end{cases}$$

Furthermore,  $M^{-1} = M^T = M$ , so

$$M^{-1}AM = MAM = \begin{pmatrix} 0 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{n1} & a_{n2} & \dots & a_{nn} \\ a_{(n-1)1} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix} \begin{pmatrix} 0 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{nn} & \dots & a_{nk} & a_{n(k-1)} & \dots & a_{n1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(k+1)n} & \dots & a_{(k+1)k} & a_{(k+1)(k-1)} & \dots & a_{(k+1)1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{1k} & a_{1(k-1)} & \dots & a_{11} \end{pmatrix}$$

and since the  $(n - k) \times k$  matrix in the upper right-hand corner is

$$\begin{pmatrix} a_{nk} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{(k+1)k} & \dots & a_{(k+1)1} \end{pmatrix} = 0 \quad \text{by (13),}$$

we have shown that  $M^{-1}AM = \begin{pmatrix} H & O^T \\ F & G \end{pmatrix}$  where  $H$  is  $(n-k) \times (n-k)$ , so

$A$  is reducible.

Since  $P^{-1}X^T P = A^T$ , we have  $X^T = PA^T P^{-1}$  and  $X = [PA^T P^{-1}]^T = (P^{-1})^T A P^T = (P^T)^{-1} AP^T$ . Since  $A$  is reducible,  $X$  is, so we have shown  $X$  irreducible  $\Rightarrow X^T$  irreducible. If  $X^T$  is irreducible,  $(X^T)^T = X$  is.

**Remark 3** An  $n \times n$  matrix  $Z = (z_{ij}) \geq 0$  has exactly one strictly positive entry in each row and column iff  $Z = PD$  where  $P = (p_{ij})$  is an  $n \times n$  permutation matrix and  $D = (d_{ij})$  is a diagonal matrix of full rank.

Proof  $\Rightarrow$ : Let  $p_{ij} = \begin{cases} 1 & \text{if } z_{ij} > 0 \\ 0 & \text{if } z_{ij} = 0 \end{cases}$ . Then  $P = (p_{ij})$  is a permutation

matrix. Let  $d_{ij} = \begin{cases} z_{ik_i} > 0 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$  where  $z_{ik_i}$  is the unique positive

entry in the  $i$ th row of  $Z$ . Then  $D = (d_{ij})$  is a nonsingular diagonal matrix.

Letting  $PD = (a_{ij})$ , we have

$$a_{ij} = \sum_{h=1}^n p_{ih} d_{hj} = p_{ij} d_{jj} = \begin{cases} z_{ij} & \text{if } z_{ij} > 0 \\ 0 & \text{if } z_{ij} = 0 \end{cases} = z_{ij}.$$

Thus  $PD = Z$ .

$\Leftarrow$ : If  $Z = PD$ , then in the above notation,  $z_{ij} = p_{ij} d_{jj}$ . For each  $j$ ,  $d_{jj} > 0$  and, since  $P$  is a permutation matrix, there is exactly one  $i$  with  $p_{ij} = 1$ , hence  $z_{ij} > 0$ . (For all  $k \neq i$ ,  $p_{kj} = 0$ , hence  $z_{kj} = 0$ .) For each  $i$ , there is exactly one  $j$  with  $p_{ij} = 1$  hence  $z_{ij} > 0$ . (For

all  $k \neq j$ ,  $p_{ij} = 0$ , hence  $z_{ij} = 0$ .) QED.

Condition (A) is thus equivalent to

(A')  $Z = PD$  where  $P$  is a permutation matrix and  $D$  is a nonsingular diagonal matrix.

## A.2 On the Definition of the Living Wage

There are economies for which  $w \geq c\bar{p}(w)$  for some  $c \in \mathcal{C}$  but  $w$  is not a living wage (that is,  $1(Z - Y) \not\geq c$ ).

Example Consider the Economy  $(Y, L, Z, \mathcal{C})$  where

$$Y = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad L = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and  $\mathcal{C} = \{c \mid c \geq (\frac{5}{2}, 0, 0)\}$ . (Note  $1L = 1$ .) This economy has a

price-distribution system  $(p, r, w)$  where  $p = \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$   $r = \frac{1}{2}$   $w = \frac{1}{2}$

as we see from

$$\begin{aligned} \text{(v)} \quad (1+r)Yp + wL &= Zp && \Leftrightarrow \\ \frac{3}{2} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} && \Leftrightarrow \\ \frac{1}{6} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \frac{1}{6} \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}. \end{aligned}$$



and

$$(vi) \quad \underline{1}(Z - Y)p = \underline{1} \begin{pmatrix} 4 & -2 & 0 \\ 0 & 4 & -2 \\ -2 & 0 & 4 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{6} (2 \ 2 \ 2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1.$$

$w \cong \hat{c}\tilde{p}(w)$  for  $\hat{c} = (\frac{5}{2}, 0, 0) \in \mathcal{C}$ , since  $\tilde{p}(w) = p$  and  $\frac{1}{2} > (\frac{5}{2}, 0, 0) \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$

$\frac{5}{12}$ . However,  $w$  is not a living wage since no  $c \in \mathcal{C}$  can be produced:

$$\underline{1}(Z - Y) = (2 \ 2 \ 2) \not\cong (\frac{5}{2}, 0, 0) \cong c \in \mathcal{C} \text{ since } \frac{5}{2} > 2.$$

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