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Leonid Hurwicz and Marcel K. Richter

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Center for Economic Research

Department of Economics

University of Minnesota

Minneapolis, Minnesota 55455

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I. Introduction.

Ever since Antonelli noted ([2], [3]) the "integrability" (symmetry) conditions necessarily obeyed by an indirect demand function derived from maximizing a utility function, and ever since Volterra emphasized ([37], [38]) their importance to Pareto's attempt [23] to construct utility from consumer purchase data -- these conditions have retained a technical character eluding intuitive motivation. It is our purpose here to show that an axiom of Ville ([35], [36]) provides an intuitively appealing equivalent of these symmetry conditions. In doing this, with the help of our integrability theorem of [16], we will extend Ville's result and, we hope, clarify his very important contribution to axiomatic consumer theory.[1]

From the dual versions ([27], Theorems 16 and 18) of an extension of Hurwicz and Uzawa's Theorems 1 and 2 [17], we know, roughly speaking, that the following two conditions together are equivalent to utility-rationality[2] of a given C^1 competitive inverse demand function satisfying the budget identity: negative semi-definiteness of the Antonelli matrix ((II.5) below), and symmetry of the Antonelli matrix. From duality theorems ([27], Theorems 20 and 12(b)) applied to a recent result of Kihlstrom, Mas-Colell, and Sonnenschein ([19], Theorems 1 and 2), we know that the first (negative semi-definiteness) condition is equivalent to a weak version of the intuitively appealing Weak Axiom of Revealed Demand Preference.[3] What about the second condition, the symmetry of the Antonelli matrix? A natural conjecture might be that the Strong Axiom of Revealed Preference characterizes symmetry. But that is not the case: as we show in Section V, the Strong Axiom is too strong. To get an equivalent of just symmetry, we need an axiom much weaker than the Strong Axiom; in fact, as a moment's reflection will show (cf. Section V), the symmetry, or "integrability," condition does not even imply the Weak Axiom. The Ville Axiom will turn out to be precisely what is needed to characterize symmetry.

Finally, by the Duality Metatheorem of [27], our extension of Ville's theorem dualizes to assert the equivalence of the symmetry of the Slutsky matrix and a dual version of the Ville Axiom.

II. Integrability.

Let the "commodity space" X be $R^n_>$,[1] and let the family $\mathcal{B}$ of competitive "budgets" be represented by $R^n_>$, so that, if $p \in R^{n-1}_>$ & $m \in R^1_>$, then (p,m) represents the budget set

$$\text{(II.1)} \qquad \{x \in R^n_> : p^1 x^1 + \ldots + p^{n-1} x^{n-1} + x^n \leq m\}.$$

Let $\beta: X \to \mathcal{B}$ be a C^1 "indirect demand function," or "budgeter" ([27], Part II.A), satisfying the budget identity:

$$\text{(II.2)} \qquad \forall x_{x \in X} \beta^1(x) x^1 + \ldots + \beta^{n-1}(x) x^{n-1} + x^n = \beta^n(x).$$

A. The Antonelli Matrix

For each $i, j = 1,\ldots,n-1$, and for each $x \in X$, let

$$\text{(II.3)} \qquad A^{ij}(x) = \frac{\partial \beta^i}{\partial x_j}(x) - \beta^j(x) \frac{\partial \beta^i}{\partial x_n} \quad ,$$

or writing $\beta^i_k(x) = \dfrac{\partial \beta^i(x)}{\partial x_k}$,

$$\text{(II.4)} \qquad A^{ij}(x) = \beta^i_j(x) - \beta^j(x) \beta^i_n(x) \quad .$$

For each $x \in X$, we define the Antonelli matrix

$$\text{(II.5)} \qquad A(x) = \begin{bmatrix} A^{11}(x) & \ldots & A^{1,n-1}(x) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A^{n-1,1}(x) & \ldots & A^{n-1,n-1}(x) \end{bmatrix} \quad .$$

The Negative Semi-Definiteness Budgeter Axiom (NSDBA) on a set $S \subseteq X$ asserts that, for each $x \in S$, the matrix $A(x)$ is negative semi-definite.

<u>The Symmetry Budgeter Axiom (SBA)</u> on a set $S \subseteq X$ asserts that, for $x \in S$, the matrix $A(x)$ is symmetric. (This was Antonelli's condition ([3], p. 347 (21b).)

We have already observed in the Introduction that in order for β to be rational it is necessary and sufficient that both the NSDBA and the SBA hold on X. We have also observed there that the NSDBA holds on X if and only if a weak form (the Weak Weak Axiom of Revealed Demand Preference) of the Weak Axiom of Revealed Demand Preference holds. In Section IV below we will characterize the SBA by a differential revealed preference axiom, but we will first note here several equivalent forms of the SBA.

B. <u>Integrability.</u>

<u>Proposition 1</u>. Let $\beta\colon X \to R^n$ be C^k on $X = R^n_>$, with $k \geqq 1$, and let S be a nonempty open subset of X. The following five conditions are equivalent:

a) the SBA holds on S;

b) there is a neighborhood $N_1 \times N_2 \subseteq R^{n-1} \times R^n$ of any $x \in S$ such that the partial differential equation system:

$$\text{(II.6)}\qquad \begin{aligned} \frac{\partial x_n}{\partial x_1} &= -\beta^1(x_1,\ldots,x_n) \\ &\ \ \vdots \\ \frac{\partial x_n}{\partial x_{n-1}} &= -\beta^{n-1}(x_1,\ldots,x_n) \end{aligned}$$

has a C^k unique solution φ on $N_1 \times N_2$, with $x_n = \varphi(x_1,\ldots,x_{n-1};\ \bar{x}_1,\ldots,\bar{x}_n)$, through any $(\bar{x}_1,\ldots,\bar{x}_n) \in N_2$;

c) there is a neighborhood N of any $\bar{x} \epsilon S$ such that the partial differential equation system (II.7) below has a C^k Solution U for all $x \epsilon N$, such that $U_n > 0$ on N:[1]

$$\text{(II.7)} \qquad \begin{aligned} \frac{U_1(x)}{U_n(x)} &= -\beta^1(x) \\ &\vdots \\ \frac{U_{n-1}(x)}{U_n(x)} &= -\beta^{n-1}(x) \ ; \end{aligned}$$

d) the C^k 1-form ω defined on the tangent spaces $T_{\bar{x}}X$ of X at each $\bar{x} \epsilon X$ by:

$$\text{(II.8)} \qquad \forall y_{y \epsilon T_{\bar{x}}X} \omega_{\bar{x}}(y) = \beta^1(\bar{x}) y_1 + \ldots + \beta^{n-1}(\bar{x}) y_{n-1} + y_n$$

has a positive integrating factor on some neighborhood $N(\bar{x})$ of each $\bar{x} \epsilon S$; that is, there is a C^k function $U: N(\bar{x}) \rightarrow R^1$ and a C^{k-1} function $\lambda: N(\bar{x}) \rightarrow R^1$ with $\lambda > 0$, such that:

$$\text{(II.9)} \qquad \forall x_{x \epsilon N(\bar{x})} \omega_x = \lambda(x) d_x U \ .$$

e) for every $\bar{x} \epsilon S$, there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ and a C^k function $U: N(\bar{x}) \rightarrow R^1$ with $U_n > 0$ on $N(\bar{x})$, and a C^{k-1} function $\lambda: N(\bar{x}) \rightarrow R^1$ with $\lambda > 0$ on $N(\bar{x})$, such that, for every $x \epsilon N(\bar{x})$, every positive real r, and every C^1 curve $\lambda: [0, r] \rightarrow N(\bar{x})$:[2]

$$\text{(II.10)} \qquad \forall t_{t \epsilon [0,r]} \beta^1(\gamma(t) \dot{\gamma}^1(t) + \ldots + \beta^{n-1}(\gamma(t)) \dot{\gamma}^{n-1}(t) + \dot{\gamma}^n(t)$$

$$= \lambda(\gamma(t)) \frac{dU(\gamma(t))}{dt} \ .$$

<u>Proof</u>. That (a) and (b) are equivalent is the famous theorem of Frobenius (cf. [8], p. 308(10.9.4) and p. 310(10.9.5)); the equivalence of those with (c) and (d) is indicated in [14], pp. 117-120 (although stated there only in terms of C^1 functions, the C^k property follows easily); and (e) is simply a restatement of (d) that avoids mention of differential forms.

Thus, in establishing the local "integrability" of a system of partial differential equations such as (II.6), it will be sufficient to prove any of the conditions (a) through (e).

III. The Differential Axioms of Revealed Preference.

A. Paths.

We define a <u>commodity path</u> to be a function $x(\cdot) = (x^1(\cdot),\ldots,x^n(\cdot))$: $[0,r] \to X$ for any real $r > 0$. We define a <u>budget path</u> to be a function $c(\cdot) = (p^1(\cdot),\ldots,p^{n-1}(\cdot),m(\cdot)):[0,r] \to \mathcal{B}$ for any real $r > 0$. By a <u>path</u> we shall mean a function $(x(\cdot),c(\cdot)):[0,r] \to X \times \mathcal{B}$ where $x(\cdot)$ is a commodity path and $c(\cdot)$ is a budget path. Note that, through β, each commodity path $x(\cdot)$ determines a path $(x(\cdot),\beta(x(\cdot)))$.

If for times $\tau_1 < \tau_2$ we observe that

$$\text{(III.1)} \qquad \beta^1(\tau_2)x^1(\tau_2) + \ldots + \beta^{n-1}(\tau_2)x^{n-1}(\tau_2) + x^n(\tau_2) > \beta^1(\tau_2)x^1(\tau_1) + \ldots + \beta^{n-1}(\tau_2)x^{n-1}(\tau_1) + x^n(\tau_1),$$

or, writing $P(\cdot) = (\beta^1(\cdot),\ldots,\beta^{n-1}(\cdot),1)$,

$$\text{(III.2)} \qquad P(\tau_2)\cdot x(\tau_2) > P(\tau_2)\cdot x(\tau_1),$$

then $x(\tau_2)$ is directly revealed preferred to[1] $x(\tau_1)$ in the Samuelson sense ([28]; [32]; cf. [25]), and so we may write $x(\tau_2)Sx(\tau_1)$. Consequently, according to Houthakker's Strong Axiom of Revealed Demand Preference ([15]; cf. [25], [27]), we should never have $x(\tau_1)$ even indirectly revealed preferred to $x(\tau_2)$; i.e., we must never have $x(\tau_1)Hx(\tau_2)$, where H is the transitive closure of the relation S on X.

We can rewrite (III.2) as

$$\text{(III.3)} \qquad P(\tau_2)\cdot(x(\tau_2) - x(\tau_1)) > 0,$$

or as

(III.4) $P(\tau_2) \cdot \Delta x > 0,$

which suggests, as a natural differential analogue:[1]

(III.5) $P(\tau) \cdot \dot{x}(\tau) > 0,$

if $x(\cdot)$ is differentiable.

Then a natural differential analogue of the Strong Axiom of Revealed Demand Preference would be the condition that there exist no quantity path $x(\cdot)$, making a "cycle" starting and ending at the same $x \epsilon X$, on which (III.5) holds for all times τ. This would say that we cannot continually be moving toward commodity bundles revealed preferred in the sense of (III.5), eventually returning to the starting commodity bundle.[2] We will shortly formulate this rigorously as an axiom, and then we will show that it provides an equivalent of the symmetry axioms of Integrability Theory, thereby bridging the gap between Revealed Preference Theory and Integrability Theory.

We should observe here that, while we have followed Ville in spirit, Ville's formulation [35] (cf. [36]) is rather sketchy and not always easy to follow. Because he uses preference teminology rather than revealed preference terminology, he might at first glance appear to employ circular reasoning, assuming a preference exists in order to prove a preference exists. Nevertheless, it is clear to us what he had in mind, and we think we have captured the intentions of his brilliant contribution.

We can also give another motivation for an axiom ruling out commodity cycles on which (III.5) holds throughout. Suppose for the moment, that, for each time τ, the commodity bundle $x(\tau)$ is derived from maximizing a C^1 "utility" function U subject to the budget constraint given by $\beta(\tau)$. Then by standard methods:

$$\frac{dU(x(\tau))}{d\tau} = \sum_{i=1}^{n} U_i\,(x(\tau))\,\dot{x}^i(\tau)$$

$$= \lambda(\tau)(P(\tau)\cdot\dot{x}(\tau)), \tag{III.6}$$

where $\lambda(\tau) = U_n(x(\tau))$. We see from (III.6) that, when a C^1 function U exists and $U_n > 0$, then (III.5) implies that

$$\frac{dU(x(\tau))}{d\tau} > 0,$$

and so utility increases as we move slightly in the direction $\dot{x}(\tau)$. (Cf. Figure 1.) It is therefore natural to call $\dot{x}(\tau)$ a <u>direction of increasing revealed preference</u> at $x(\tau)$ when (III.5) holds (cf. Allen [1], pp. 203-205; Georgescu-Roegen [10], p. 552; Katzner [18], p. 118).[1]

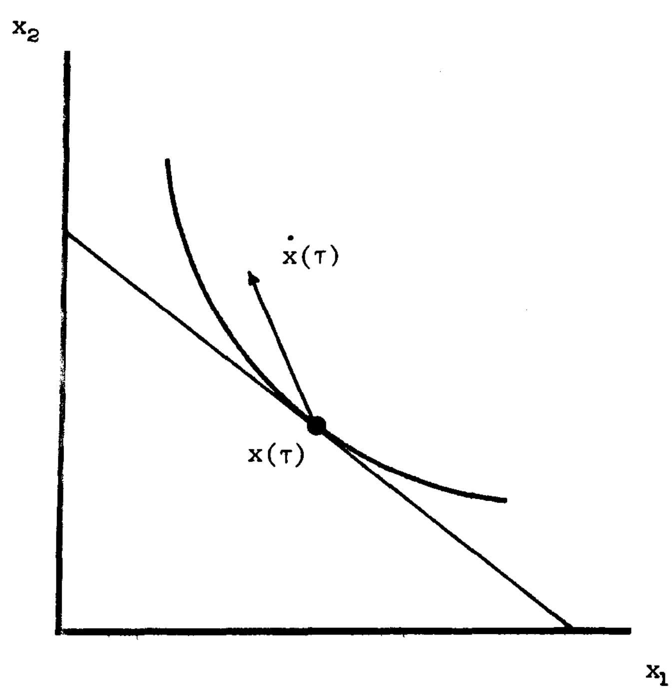

Figure 1

It is now apparent why the existence of such a utility function prohibits "commodity cycles" which move always in a direction of increasing revealed preference. For from (III.6) we have

$$
\begin{aligned}
U(x(b)) - U(x(a)) &= \int_a^b \frac{dU(x(\tau))}{d\tau}\, d\tau \\
&= \int_a^b \lambda(\tau)\,(P(\tau)\cdot\dot{x}(\tau))\,d\tau.
\end{aligned}
\tag{III.7}
$$

For a commodity cycle we would have $x(a) = x(b)$, so the left hand term of (III.7) vanishes; but the right hand term is positive by our hypotheses. This contradiction shows that:

(III.8) the existence of a C^1 utility function with $U_n > 0$ implies the nonexistence of C^1 commodity cycles satisfying (III.5) for all τ.

Our aim is to follow Ville and prove a converse: if such cycles are not possible, then the symmetry integrability conditions are satisfied.

B. <u>The Ville Axioms</u>.

We formalize the discussion of Section III.A as follows: Let S be a subset of X, and let $x(\cdot): [0,r] \to X$ be a C^k commodity path for some $k \geq 1$ with $k \leq \infty$, and some real $r > 0$. With the motivation and notation of Section III.A, we shall say that $x(\cdot)$ is a <u>C^k positive Ville (commodity) cycle on S</u>, if:

$$
\begin{aligned}
&x(0) = x(r) \quad \text{and} \\
&P(\tau)\cdot\dot{x}(\tau) > 0 \quad \text{for all } \tau\in[0,r].
\end{aligned}
\tag{III.9}
$$

We say that the budgeter β satisfies the <u>C^k Ville Axiom</u> on an open set $S \subseteq X$ if there does not exist a C^k Ville cycle on S.

In Section IV we show that the C^∞ Ville Axiom characterizes the set of budgeters β satisfying the symmetry conditions (SBA) of Proposition 1.

IV. The Ville Connection.

As suggested in Section III, we will show that the integrability conditions of Proposition 1 are equivalent to the absence of Ville cycles.

Theorem 1. As in Section I, let the commodity space $X = R^n_>$, let the budget space $\mathcal{B} = R^n_{\geqq}$, and let $\beta\colon X \to \mathcal{B}$ be a C^1 budgeter. Let S be an open subset of X on which β never vanishes. Then the SBA holds on S if S contains no C^∞ Ville cycles, and only if S contains no continuous, piecewise C^1 Ville cycles.

Remark. The relationship between Theorem 1 and Ville's result is discussed in Section VII.

Proof of Theorem 1.

a) Suppose that there exists a C^1 Ville cycle $x(\cdot)\colon [0,r] \to S$ for some open subset S of X. We must show that the SBA fails to hold on S. If, on the contrary, the SBA were to hold throughout S then the equivalence of conditions (a) and (e) of Proposition 1 would give locally, in a neighborhood of each point in S, a positive function λ and a C^1 function U on S such that (e) held. Then by Debreu's argument in [7], pp. 608-610, we could take the C^1 function U to be globally defined on X. So for the Ville cycle $x(\cdot)$:

$$\int_0^r \frac{1}{\lambda(x(t)}\,(\beta^1(x(t)\dot{x}^1(t) +\ldots+ \beta^{n-1}(x(t))\dot{x}^{n-1}(t) + \dot{x}^n(t))$$

(IV.1)

$$= \int_0^r \frac{dU(x(t))}{dt}\;dt,$$

which on the one hand would be zero since $x(r) = x(0)$, and on the other hand would be positive by the definition of a Ville cycle (III.8). This contradiction shows that the SBA fails on S.

b) Appealing again to Debreu's argument ([7], pp. 608-610), local integrability implies global integrability, so to prove integrability on an open set $S \subseteq X$, it suffices to prove integrability in a neighborhood of each $x \in S$. Let $x \in S$ and suppose that the SBA fails at x, hence it fails on every neighborhood of x. We must show that X contains a Ville cycle. By the equivalence of conditions (a) and (d) of Proposition 1, the failure of SBA implies the absence of a positive integrating factor for the 1-form $\omega = P \cdot dx$ (as defined explicitly in (II.8)) on each neighborhood of x. By the Ville-type integrability theorem in Hurwicz and Richter [16] (restated in Section IX below), it follows that every neighborhood of x contains a C^{∞} positive ω-cycle, i.e., a C^{∞} Ville cycle. Q.E.D.

V. Revealed Preference and the Ville Axioms.

As we have seen in Section III.A, the Strong Axiom of Revealed Demand Preference (SARDP) is close in spirit to the Ville Axioms. In this sense it is suggestive to refer to the SARDP as "the Ville-Houthakker Strong Axiom." But suggesting that the SARDP and the Ville Axioms are the same obscures the very important distinctions between them.

Certainly, under differentiability the SARDP implies the Ville Axioms. (In fact even much weaker forms of the SARDP imply the Ville Axioms. Cf. [27], Theorem 21.) But the SARDP implies much more than the integrability condition; it implies the negative semi-definiteness condition as well.[1]

Yet the Ville Axioms do not even imply the Weak Axiom of Revealed Demand Preference -- a fortiori not the SARDP -- as the following Example shows.

Example. Let the budgeter β: $R^2_> \to R^2_>$ be defined by:

$$(\beta^1(x_1, x_2), \beta^2(x_1, x_2)) = \left(\frac{x_1}{x_2}, \frac{(1 + (x_1)^2}{x_2} \right). \tag{V.1}$$

So β is C^∞. Since n = 2, the system (II.6) becomes a single ordinary differential equation, so the integrability symmetry conditions are trivially satisfied and no Ville cycles exist. Indeed the solutions of (II.6) are the level curves of the function f: $R^2_> \to R^1$ given by

$$f(x_1, x_2) = x_1{}^2 + x_2{}^2 . \tag{V.2}$$

Cf. Figure 2.

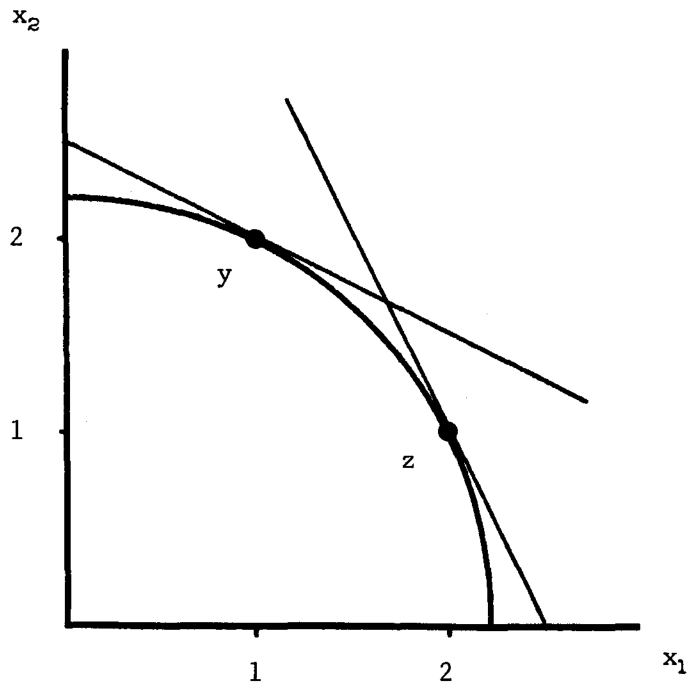

Figure 2

Now it is clear from elementary calculations or from Figure 2 that $y = (1,2)$ is directly revealed preferred to $z = (2,1)$, and that z is directly revealed preferred to y:

$$\text{(V.3)} \qquad \begin{gathered} \beta^1(1,2)1 + 2 = 2.5 > 2 = \beta^1(1,2)2 + 1 \\ \text{and} \\ \beta^1(2,1)2 + 1 = 5 > 4 = \beta^1(2,1)1 + 2, \end{gathered}$$

a contradiction of the Weak Axiom of Revealed Demand Preference (indeed, even a contradiction of the Weak Weak Weak Axiom ([27], Part III.C.1(c))).

If they don't even imply the Weak Axiom, what then do the Ville Axioms imply? As we have shown, nothing more nor less than the mathematical integrability conditions, the SBA.

VI. Duality

If, instead of starting with a budgeter $\beta\colon X \to \mathcal{B}$ we had started with a single valued demand $\xi\colon \mathcal{B} \to X$, then according to the Duality Metatheorem of [27] we would get a theorem dual to Theorem 1, asserting the equivalence of the symmetry of the Slutsky matrix and the absence of Ville (budget) cycles. For details, see [27].

VII. Comparison With Ville's Method

Although it does not give as strong a result as Theorem 1, we have included in Appendix B (Section X) an outline of Ville's method of proof. Here we will sketch the difference in results.

Our method, based on our Theorem of [16], does not require that β be more than C^k, in order to obtain the SBA and hence a C^k utility U. Ville's method on the other hand, is based on the theorem of Darboux (cf. Section X), which is usually stated in terms which would require β to be C^∞. By carefully keeping track of the order of differentiability in the proof of the Darboux theorem, one can state it (as we have done in Section X) for C^k 1-forms; but then its application to our present problem requires the strong hypothesis that $k \geqq 2n + 1$.

It should also be noted that the conclusion of Theorem 1 is stronger than would follow from Ville's application of Darboux's Theorem. In particular, Theorem 1 asserts that, if x has a neighborhood N_1 containing no Ville cycles, then the SBA holds on N_1. But application of the Darboux theorem would only allow the assertion that there is a nonempty subset S of N_1 (not necessarily containing x) on which the SBA holds.

Finally, Ville did not assert the "only if" part of Theorem 1.

VIII. Historical Remark

In seeking to circumvent the difficulties with Ville's method, it became apparent that Carathéodory's accessibility theorem ([4], §4) could be used to give a better result. This led to an examination of the manner in which Carathéodory had applied his theorem to prove the existence of entropy in his foundational work on thermodynamics. More recent modifications ([34], [39]) of Carathéodory's approach employing the Kelvin-Planck Second Law of Thermodynamics suggested that a still better theorem could be obtained by replacing the Kelvin-Planck Axiom with the Ville Axiom, yielding thereby an improved formulation of the Second Law of Thermodynamics. Cf. [16].

It is interesting that Ville's contribution was also somewhat ahead of the thermodynamics literature of its time. Much the same method (described in Section X below, and based on Darboux's Theorem) that Ville used in his 1946 paper was employed later in Landsberg's Thermodynamics of 1961 to yield a proof ([20], pp. 51-53) of Carathéodory's accessibility theorem which is then used to prove the existence of entropy, assuming Carathéodory's axiom ([4], pp. 55-56).

It should be noted that already in the older literature there are ideas concerning the relationship of consumer theory to thermodynamics. Pareto ([24], p. 543) observed that utility could be obtained as an integral, as in thermodynamics entropy was obtainable as an integral. Samuelson ([28], p. 70) noted a general analogy between utility theory and thermodynamics. Davis ([6], Chapter 8, Section 5) remarked on Pareto's observation, and was followed by Lisman [21], who noted in particular the similarity of the budget identity to the First Law of Thermodynamics, but remained dubious about establishing analogies between utility theory and the Second Law of Thermodynamics. (See [16] for an indication of how the Ville Axioms establish such analogies.) Georgescu-Roegen ([11], p. 17) also noted the similarity.

IX. Appendix A (Integrability Theorem).

For the reader's convenience, we restate here the integrability theorem proved in [16].

Let M be a C^r n-manifold ($1 \leqq r \leqq \infty$) and let ω be a C^k 1-form ($1 \leqq k \leqq r$) on M which never vanishes. We seek conditions on ω that it admit, for a set $N \subseteq M$, a <u>C^k integrating pair (φ,λ) on N</u>, more specifically a never-vanishing function (integrating factor) $\lambda: N \to R^1$ and a C^k function $\varphi: N \to R^1$ such that $\lambda\omega = d\varphi$ on N; it follows that such λ would be C^{k-1} and could be chosen either positive or negative.

For any $N \subseteq M$, by a <u>C^r positive ω-cycle in N</u> we mean a C^r function γ: $[0,T] \to N$, for some real $T > 0$, such that $\gamma(0) = \gamma(T)$ and for all $t \in [0,T]$, $\omega_{\gamma(t)}(\dot{\gamma}) > 0$.

<u>Theorem</u>. Let $\bar{y} \in M$. There is a neighborhood of $\bar{y}$ on which ω admits a C^k integrating factor if some neighborhood of $\bar{y}$ contains no C^r positive ω-cycle and only if some neighborhood of $\bar{y}$ contains no continuous, piecewise C^1 positive ω-cycle.

X. Appendix B (Ville's Method).

We may write the 1-form (II.8) as:

(X.1) $$\omega = z^1 dy^1 + \ldots + z^{n-1} dy^{n-1} + dy^n ,$$

where $(y^1, \ldots, y^n)$ is a local coordinate chart. Assuming that ω does not admit a positive integrating factor, we want to construct a Ville cycle. If all the function z^i, y^i were functionally independent locally, so we could vary them locally at will, then it would be easy to construct a Ville cycle (cf. (X.14-X.17)). But of course we are not free to vary these variables arbitrarily: for example, the z^i, representing the β^i values, are <u>dependent on</u> the y^i values. Nevertheless, a normal form theorem of Darboux for 1-forms asserts that ω has a different representation in terms of variables that <u>are</u> functionally independent; and then Ville showed how to use this theorem to obtain Ville cycles[1] when ω does not admit a positive integrating factor.

We first state Darboux's normal form theorem for 1-forms.

<u>Darboux's Theorem.</u>[2]

Let M be a C^k m-dimensional manifold[3] and suppose that S is a coordinate neighborhood on M. Let ω be a C^k 1-form on S such that ω never vanishes on S. By standard results,[4] ω has a representation on S of the form:

(X.2) $$\forall a_{a \in U} \forall v_{v \in T_a S} \omega_a(v) = \sum_{i=1}^{r} X^i(x^1(a), \ldots, x^r(a)) d_a x^i(v) ,$$

for some positive integer $r \leqq m$, where the x^i are real valued C^k functions on S and are independent on S,[5] and the X^i are real valued C^k functions on $(x^1, \ldots, x^r)(S) \subseteq R^r$. Suppose that $k \geqq 2r+1$. Then there is a C^{k+1-r} coordinate chart $(y^1, \ldots, y^m)$ on S such that <u>either</u>:

i) for some p with $2p+1 \leqq r$, ω has a representation on some nonempty open subset $V \subseteq S$:

(X.3) $$\forall a_{a \in V} \forall v_{v \in T_a V} \omega_a(v) = Z^1(y^1(a), \ldots, y^r(a)) d_a y^1(v) + \ldots$$
$$+ Z^p(y^1(a), \ldots, y^r(a)) d_a y^p(v) + d_a y^{p+1}(v)$$

for some C^{k+2-2r} functions $Z^i : (y^1, \ldots, y^r)(V) \to R^1$, where $y^1, \ldots, y^{p+1}, Z^1(y^1(\cdot), \ldots, y^r(\cdot)), \ldots, Z^p(y^1(\cdot), \ldots, y^r(\cdot))$ are functionally independent on V; and indeed, if we define $F^i(\cdot) = Z^i(y^1(\cdot), \ldots, y^r(\cdot))$ for $i = 1, \ldots, p$, then for all $a \in V$, the matrix

$$\text{(X.4)} \qquad \begin{bmatrix} \left.\frac{\partial F^1}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial F^1}{\partial x^r}\right|_a \\ \vdots & & \vdots \\ \left.\frac{\partial F^p}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial F^p}{\partial x^r}\right|_a \\ \left.\frac{\partial y^1}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial y^1}{\partial x^r}\right|_a \\ \vdots & & \vdots \\ \left.\frac{\partial y^{p+1}}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial y^{p+1}}{\partial x^r}\right|_a \end{bmatrix}$$

has rank $2p+1$;

<u>or</u>

ii) for some p with $2p \leqq r$, ω has a representation on some nonempty subset $V \subseteq S$:

$$\text{(X.5)} \qquad \forall a_{a\in V} \forall v_{v \in T_a V} \omega_a(v) = Z^1(y^1(a), \ldots, y^r(a))\, d_a y^1(v) + \ldots + Z^p(y^1(a), \ldots, y^r(a))\, d_a y^p(v)$$

for some C^{k+2-2r} functions $Z^i: (y^1, \ldots, y^r)(V) \to R^1$, where $y^1, \ldots, y^p, Z^1(y^1(\cdot), \ldots, y^r(\cdot)), \ldots, Z^p(y^1(\cdot), \ldots, y^r(\cdot))$ are functionally independent on V; and, indeed, for all $a \in V$, the matrix

$$\text{(X.6)} \qquad \begin{bmatrix} \left.\frac{\partial F^1}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial F^1}{\partial x^r}\right|_a \\ \vdots & & \vdots \\ \left.\frac{\partial F^p}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial F^p}{\partial x^r}\right|_a \\ \left.\frac{\partial y^1}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial y^1}{\partial x^r}\right|_a \\ \vdots & & \vdots \\ \left.\frac{\partial y^p}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial y^p}{\partial x^r}\right|_a \end{bmatrix}$$

has rank $2p$. (End of Darboux's Theorem.)

Remark. In cases (i) or (ii) we have, respectively:

$$\text{i}') \quad \omega = \sum_{i=1}^{p} z^i dy^i + dy^{p+1} \quad \text{, or}$$

$$\text{ii}') \quad \omega = \sum_{i=1}^{p} z^i dy^i \quad ,$$

where the z^i, y^i are functionally independent variables on a neighborhood in R^{2p+1} or R^{2p}, respectively.

Use of Darboux's Theorem to construct Ville-cycles when integrability conditions fail to hold.

Let ω be the C^k 1-form defined on X by (II.8), with $\beta: X \to \mathcal{B}$ the indirect demand function as in Section II. Let N be a nonempty open subset of X, such that ω fails to have a positive integrating factor on every nonempty open subset of N; and assume that $k \geqq 2n + 1$. We will use Darboux's Theorem to show that N contains a C^{k+2-2n} Ville cycle.[1] (Taking the contrapositive, this will show that the absence of Ville cycles on N implies the existence of a nonempty open subset on which ω has a positive integrating factor. Cf. the second remark in Section VII.)

By Darboux's Theorem there is a C^{k+1-n} coordinate chart $y = (y^1, \ldots, y^n)$ on N such that either:

i) for some p with $2p+1 \leqq n$, ω has a representation on some nonempty open subset V of N, of the form (X.3), with $r = n$, such that (X.4), with $r = n$, has rank $2p+1$ on V; we may thus rewrite (X.3) for this case as:

$$\forall a_{a \in V} \forall v_{v \in T_a V} \omega_a(v) = \sum_{i=1}^{p} F^i(a) d_a F^{p+1}(v) + d_a F^{2p+1}(v) \tag{X.7}$$

where:

$$F^i(a) = \begin{cases} Z^i(y^1(a), \ldots, y^n(a)), & i = 1, \ldots, p \\ y^{i-p}(a) \quad , & i = p+1, \ldots, 2p+1 \end{cases} \tag{X.8}$$

so $F: V \to R^{2p+1}$;

or ii) for some p with $2p \leqq n$, ω has a representation on some nonempty open subset V of N, of the form (X.5), with $r = n$, such that (X.6) has rank 2p on V; we may thus rewrite (X.5) for this case as:

$$\forall a_{a \in V} \forall v_{v \in T_a V} \omega_a(v) = \sum_{i=1}^{p} F^i(a) d_a F^{p+1}(v), \tag{X.9}$$

where

$$F^i(a) = \begin{cases} Z^i(y^1(a), \ldots, y^n(a)), & i = 1, \ldots, p \\ y^{i-p}(a) \quad , & i = p+1, \ldots, 2p \end{cases} \tag{X.10}$$

so $F: V \to R^{2p}$.

In case (ii), the rank condition implies that F is C^{k+2-2n} and bijective on a nonempty open subset V^* of V, and the image of V^* under F is an open subset W of R^{2p}. Note that $p \neq 0$ since ω is not the zero form. If $p = 1$, then clearly, (X.5) (with $r = n$) shows that

$$\frac{1}{F^1(\cdot)} = \frac{1}{Z^1(y^1(\cdot),\ldots,y^n(\cdot))}$$

is an integrating factor for ω on V (the denominator is never zero since ω is never the zero 1-form (cf. (II.8)). So either $1/F^1(\cdot)$ or its negative is a positive integrating factor on V, contradicting the assumption on N.

So we may assume $p \geq 2$. Let $w = (z_1,\ldots,z_p,y_1,\ldots,y_p) \in W \subseteq R^{2p}$ and let δ_1,δ_2 be a curve $\sigma\colon [0,2\pi] \to R^{2p}$

(X.11)
$$\begin{aligned}
\sigma^1(t) &= z_1 - \delta_1 \sin(t)\\
\sigma^2(t) &= z_2 + \delta_2 \sin(t)\\
\sigma^i(t) &= z_i \qquad (i = 3,\ldots,p)\\
\sigma^{p+1}(t) &= y_1 + \delta_1 \cos(t)\\
\sigma^{p+2}(t) &= y_2 - \delta_2 \cos(t)\\
\sigma^i(t) &= y_i \qquad (i = p+3,\ldots,2p)
\end{aligned}$$

Since W is open, for all small enough δ_1,δ_2, σ will take all its values in W; in fact we also choose δ_1,δ_2 so that $\delta_1 z_1 = \delta_2 z_2$. Now we can define a curve $\gamma\colon [0,2\pi] \to N$ by:

(X.12)
$$\forall t_{t\in[0,2\pi]}\, \gamma(t) = F^{-1}(\sigma(t)),$$

and a simple calculation shows that, for all $t\in[0,2\pi]$:

(X.13)
$$\begin{aligned}
&\beta^1(\gamma(t))\dot{\gamma}^1(t) + \ldots + \beta^{n-1}(\gamma(t))\dot{\gamma}^{n-1}(t) + \dot{\gamma}^n(t)\\
&= \omega_{\gamma(t)}(\dot{\gamma}(t))\\
&= \sum_{i=1}^{p} F^i(\gamma(t))\, d_a F^i(\dot{\gamma}(t))\\
&= \sum_{i=1}^{p} F^i(\gamma(t))\frac{dF^i(\gamma(t))}{dt}\\
&= \sum_{i=1}^{p} \sigma^i(t)\frac{d\sigma^{p+i}(t)}{dt}\\
&= -(z_1 - \delta_1 \sin(t))\delta_1 \sin(t) + (z_2 + \delta_2 \sin(t))\delta_2 \sin(t)\\
&= ((\delta_1)^2 + (\delta_2)^2)\sin^2(t),
\end{aligned}$$

which is positive except for multiples of π, where it vanishes. Thus in case (ii) (arbitrarily small) positive Ville cycles can be constructed in N.[1]

In case (i), if $p = 0$, then (X.3) shows that 1 is a positive integrating factor for ω on N, contradicting the definition of N; thus we may assume that $p \geqq 1$.

The rank condition for case (i) implies (by the Inverse Function Theorem) that F is C^{k+2-2n} and bijective on a nonempty open subset V^* of V, and the image of V^* under F is an open subset W of R^{2p+1}. Let $w = (z_1, \dots, z_p, y_1, \dots, y_{p+1}) \in W \subseteq R^{2p+1}$; since W is open, we can assume that all components of w are nonzero.

If $p \geqq 2$, we define $\sigma: [0,2\pi] \to R^{2p+1}$ as in (X.11) together with:

$$\sigma^{2p+1}(t) = y_{p+1} \quad . \tag{X.14}$$

Then, with $\gamma: [0,2\pi] \to N$ defined as in (X.12), the same sort of calculation as in (X.13) shows that, for all $t \in [0,2\pi]$, we again have

$$\begin{aligned} \beta^1(\gamma(t))\dot{\gamma}^1(t) + \dots + \beta^{n-1}(\gamma(t))\dot{\gamma}^{n-1}(t) + \gamma^n(t) \\ = ((\delta_1)^2 + (\delta_2)^2)\sin^2(t) \end{aligned} \tag{X.15}$$

and we thus have positive Ville quantity cycles[2] for all small enough δ_1, δ_2.

It only remains to obtain a Ville cycle for the case $p = 1$. Then we define $\sigma: [0,2\pi] \to R^3$ by:

$$\begin{aligned} \sigma^1(t) &= z_1 - \sin(t) \\ \sigma^2(t) &= y_1 + \delta_1 \cos(t) \\ \sigma^3(t) &= y_2 - \delta_2 \cos(t), \end{aligned} \tag{X.16}$$

and since (z_1, y_1, y_2) is in the open set W, for all small enough δ_1, δ_2, σ will take its value in W. Again defining $\gamma: [0,2\pi] \to N$ by (X.12), calculations similar to our earlier ones show that, for all $t \in [0,2\pi]$:

$$\begin{aligned} \beta^1(\gamma(t))\dot{\gamma}^1(t) + \dots + \beta^{n-1}(\gamma(t))\dot{\gamma}^{n-1}(t) + \dot{\gamma}^n(t) \\ = -(z_1 - \sin(t))\delta_1 \sin(t) + \delta_2 \sin(t) \\ = \delta_1 \sin^2(t) + \sin(t)(\delta_2 - \delta_1 z_1) \; ; \end{aligned} \tag{X.17}$$

So, as we pick $\delta_1 > 0$ and $\delta_2 = \delta_1 z_1$, then we again obtain (arbitrarily small) Ville commodity cycles in N, as was to be shown under the special assumption that $k \geqq 2n + 1$.

FOOTNOTES

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Page 1, n. 1. Indeed, we shall see that this contribution was essentially a mathematical one, as applicable to thermodynamics as to consumer choice.

Page 1, n. 2. I.e., derivability from utility maximization. Cf. [27], Part II.B.1; [26].

Page 1, n. 3. This generalizes Samuelson's result in [28], or a dual version of Samuelson's result in [29] and in [31], pp. 112-113.

Page 3, n. 1. We define $R^n_> = \{x \epsilon R^n : x > 0\}$ and $R^n_\geqq = \{x \epsilon R^n : x \geqq 0\}$. For $x \epsilon R^n$, $x > 0$ means that each component of x is strictly positive, and $x \geqq 0$ means that each component is nonnegative.

Page 5, n. 1. Subscripts on U denote partial differentiation.

Page 5, n. 2. The superscript dot denotes differentiation with respect to the argument t.

Page 7, n. 1. It might be less confusing to use, instead of "revealed preferred to," an expression such as "selected over" ([29], p. 65) or "chosen over" ([30], p. 246) in order to emphasize that only choice acts are being described, rather than preferences. Cf. [28], p. 65.

Page 8, n. 1. The superscript dot denotes differentiation with respect to the "time" variable τ. The derivatives in (III.9) are to be interpreted as right or left hand derivatives as appropriate at the boundaries of $[0,r]$.

Page 8, n. 2. Cf. Georgescu-Roegen's discussion of "'illusions'," [10], pp. 566-568. See also Samuelson's later discussion [32], pp. 367-372.

Page 9, n. 1. We use revealed preference terminology to emphasize that we are referring only to characteristics of choice acts, rather than preferences. Cf. footnote 1, page 7. A better terminology might be "direction of choice" or "selection direction" rather than "revealed preference direction."

Page 13, n. 1. The SARDP implies the Weak Weak Axiom of Revealed Demand Preference, which is equivalent (by [27], Theorem 12) to the Weak Weak Axiom of Revealed Budgeter Preference ([27], Part III.C), which in turn is equivalent (by [27], Theorem 20) to the negative semi-definiteness of the Antonelli matrix when β is differentiable.

Page 19, n. 1. In a slightly weaker sense, allowing a finite number of zero values.

Page 19, n. 2. This is essentially Darboux's theorem as stated in [5], p. 26, section V. Cf. Goursat [12], p. 14; Sternberg [33], p. 141, Theorem 6.2; Dieudonné [9], p. 105, Problem 3(c).

Although these authors either do not mention differentiability hypotheses or assume the functions X^i and x^i are C^∞, we have taken pains to indicate precisely the differentiability hypothesis assumed for the method of proof given by Darboux and Goursat.

Page 19, n. 3. For our present purposes, nothing is lost by thinking of M as an open subset of R^n.

Page 19, n. 4. Cf. [22], pp. 134-135 and [13], p. 163 (Proposition), which show that we can let r equal m.

Page 19, n. 5. That is, for any $a \in S$ and any coordinate chart $(z^1, \ldots, z^m)$ in a neighborhood of a, the matrix

$$\begin{bmatrix} \left.\frac{\partial x^1}{\partial z^1}\right|_a & \cdots & \left.\frac{\partial x^1}{\partial z^m}\right|_a \\ \vdots & & \vdots \\ \left.\frac{\partial x^r}{\partial z^1}\right|_a & \cdots & \left.\frac{\partial x^r}{\partial z^m}\right|_a \end{bmatrix}$$

has rank r.

Page 21, n. 1. Note that since we are assuming that integrability fails on open subsets of N, we know that $n \geqq 3$, so if $k < \infty$, then $k + 2 - 2n < k + 3 - 2n < k$.

Page 23, n. 1. This is essentially Ville's construction, although he seems to implicitly assume that the z_i can all be chosen equal to zero.

Page 23, n. 2. In the weaker sense of footnote 1, page 19.

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