DUALITY AND RATIONALITY

by

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I. Introduction.

A. Duality: theory and metatheory.

The notion of "indirect utility," dual to that of utility arises naturally in several theoretical and empirical problems in economics. Many economists, over many years, have explored this duality in special contexts, and this paper introduces a very general framework for such studies. As applications we choose several results, both new and old, involving the rationality of choice behavior.

The duality that concerns us is a very basic one, with at least four aspects. It extends to:

a) concepts, terms, and sets (e.g. "direct demand functions" and "inverse demand functions");

b) axioms (e.g. symmetry of the Slutsky matrix and symmetry of the Antonelli matrix);

c) theorems (e.g. "integrability" theorems of the Samuelson-Hurwicz and Uzawa type and theorems of the Antonelli-Allen-Hicks-Allen-Samuelson-Debreu type); and

d) proofs.

In order to study this "fourfold duality" we proceed as follows. First we present (in Part II.A,B) a general set-theoretic framework that is convenient for much of modern choice theory. Next we present (in Part II.C) a formal language corresponding to this framework. With this language, we can make precise the notion of duality for terms and sets (a). Since every sentence, as well as term or set, will have a dual, this will also make precise the notion of duality for axioms (b), theorems (c), and proofs (d). But these linguistic aspects are not of primary interest to the economist, who wants to know relationships between dual conepts:

a) concerning terms and sets he will want to know, for example, Does the existence of a utility imply the existence of a "dual utility"?

b) concerning axioms he will want to know, for example, If the Weak Axiom of Revealed Preference holds, does its dual hold?

c) concerning theorems he will want to know, Is the dual of a theorem a theorem?

d) concerning proofs he will want to know, Is the dual of a proof a proof?

Questions (a) and (b) are mathematical questions, and most of the theorems of the paper answer such questions. Questions (c) and (d) are metamathematical questions, and require a different approach. We answer them affirmatively in the Duality Metatheorem of II.C. Thus for every theorem (and proof) of our language we automatically find a new theorem (and proof); we make frequent use of this to generate new theorems and to save work in proving them.

B. Applications: rational choice.

The generality of the Duality Metathorem is very great as it applies to any theorem of our language, and our language is compatible with much of the theory of production as well as consumer choice.[1] For applications, therefore, we have focussed on several results in the theory of rational choice, this being the origin of our interest in duality.

Conditions on choice behavior that are necessary and sufficient for it to be rational (generated through preference maximization subject to budget constraints) are of interest because they completely characterize classical choice behavior; and if the conditions are testable, they provide tests for the classical theories.

Three main approaches have yielded such conditions: Integrability Theory, Revealed Preference Theory, and Differential Revealed Preference Theory. One can use with any of these approaches either "direct" demands or "inverse" demands, and our Duality Metatheorem allows us to commute between the two columns of the matrix in Figure 1, providing a symmetric synthesis of much of modern choice theory.

Figure 1 shows some of the relationships between several of the significant previous publications and the present paper.

Axioms \ Demand functions	Direct	Inverse
Integrability	Samuelson [38] Hurwicz-Uzawa [26] *	Antonelli [2] Allen [1] Hicks-Allen [20] Samuelson [38] Debreu [9] *
Revealed Preference	Samuelson [35] Houthakker [21] Uzawa [45] Richter [33] *	*
Differential Revealed Preference	*	Ville [46] *

*Discussed in the present paper.

Figure 1.

The revealed preference approaches are discussed in Part III, the integrability approaches in Part IV, and the differential revealed preference approaches in Part V.

C. References.

The survey article by Diewert [14] and the comments by Jacobsen [27], Lau [31], and Shephard [41] provide a recent overview of numerous duality results and applications in consumer, production, and general equilibrium theory, as well as historical remarks and extensive bibliography. For this reason we limit our scope and our historical references. The present work differs from these papers in being oriented primarily toward rational choice theory.[1] It also goes beyond the Euclidean spaces which have been the framework for much of the previous work, and discusses duality in more general choice models. The metamathematical aspects seem to have been only implicit in earlier duality studies.

II. Choice and Rationality Concepts.

A. Relations.

We develop choice theory in an abstract setting. Let X be a nonempty set. Intuitively, we will think of X as a fixed universe from which a decision maker chooses alternatives at various times.

Let $\mathcal{B}$ be a nonempty set. Intuitively, we will think of $\mathcal{B}$ as a family of nonempty subsets of X, interpreting a particular element of $\mathcal{B}$ as a set of alternatives from X to which the decision maker's choice is restricted at a particular time and place. Therefore we shall refer to elements of $\mathcal{B}$ as budgets. Without loss of generality we can assume that X and $\mathcal{B}$ are disjoint.[1]

Let $\mathbb{C}$ be a binary relation between elements of X and elements of $\mathcal{B}$: $\mathbb{C} \subseteq (X \times \mathcal{B}) \cup (\mathcal{B} \times X)$. We call $\mathbb{C}$ a choice relation, and if $x \in X$ & $B \in \mathcal{B}$ then we interpret $(x,B) \in \mathbb{C}$ or $(B,x) \in \mathbb{C}$ as meaning that:

a) x is one of the alternatives chosen when the decision maker is restricted to budget B, and

b) B is one of the budgets under which the decision maker chooses alternative x.

We assume that $\mathbb{C}$ is symmetric:[2] for any x and B,

(II.1) $$(x,B) \in \mathbb{C} \Leftrightarrow (B,x) \in \mathbb{C}.$$

If $(x,B) \in \mathbb{C}$, we shall also write "$x\mathbb{C}B$," if $(B,x) \in \mathbb{C}$ we shall also write "$B\mathbb{C}x$."

Under our intuitive interpretation, if $x\mathbb{C}B$, then we should require $x \in B$: any element chosen when the decision maker is restricted to budget B should belong to B. But in our abstract model we have not even required that B should be a subset of X. So we shall proceed formally, and simply assume that there is some specified "admissibility" relation A between certain elements of X and certain elements of $\mathcal{B}$.

We assume that A is symmetric:[1] for any x and B,

(II.2) $xAB \Leftrightarrow BAx.$

And we assume that:

(II.3) $\mathbb{C} \subseteq A;$

so if $(x,B)\in\mathbb{C}$, then $(x,B)\in A$, and we shall say that (x,B) is admissible. For the intuitive interpretation we should require that A be the symmetrization of the relation $\in$, but for our abstract formulation of the theory, it is advantageous to allow other relations.

We will call $(X,\mathcal{B},A,\mathbb{C})$ a choice space.[2]

Of the many ways we can "slice" the relation $\mathbb{C}$, two are particularly simple. To emphasize alternatives, we define the function[3] $h^{\mathbb{C}}: \mathcal{B} \to \mathfrak{P}X$ by:

(II.4) $\forall B_{B\in\mathcal{B}} h^{\mathbb{C}}(B) = \{x\in X: x\mathbb{C}B\}$,

and we call $h^{\mathbb{C}}$ the demand corresponding to $\mathbb{C}$. When the intended $\mathbb{C}$ is clear, we will drop the superscript and write simply "h." Clearly $h^{\mathbb{C}}$ is what is usually called a "demand correspondence" [7]. If $h^{\mathbb{C}}(B)$ is a singleton for all $B\in\mathcal{B}$, then we call $\xi^{\mathbb{C}}: \mathcal{B} \to X$, defined by:

(II.5) $\forall B_{B\in\mathcal{B}} \xi^{\mathbb{C}}(B) = x$, where $h^{\mathbb{C}}(B) = \{x\}$,

a single valued demand corresponding to $\mathbb{C}$. Clearly $\xi^{\mathbb{C}}$ is what is usually called a "demand function" [39]. We call the set of alternatives chosen for some budget the range of $h^{\mathbb{C}}$, and denote it by $\Gamma^{\mathbb{C}}$:

(II.6) $\Gamma^{\mathbb{C}} = \bigcup \{h^{\mathbb{C}}(B): B\in\mathcal{B}\}.$

Cf. Figure 2.

To emphasize <u>budgets</u>, we define the function $b^{\mathfrak{C}}: X \to \mathfrak{P}\mathcal{B}$ by:

$$\text{(II.7)} \qquad \forall x_{x \in X} b^{\mathfrak{C}}(x) = \{B \in \mathcal{B}: B \mathfrak{C} x\},$$

and we call $b^{\mathfrak{C}}$ the <u>budgeter</u> corresponding to $\mathfrak{C}$. Clearly $b^{\mathfrak{C}}$ is what would usually be called an "inverse demand correspondence," but for our duality theory we do not want $b^{\mathfrak{C}}$ to have a status secondary to $h^{\mathfrak{C}}$, so here (and elsewhere) we avoid such asymmetric terminology, even to the extent of inventing new names. If $b^{\mathfrak{C}}(x)$ is a singleton for all $x \in X$, then we call $\beta^{\mathfrak{C}}: X \to \mathcal{B}$, defined by:

$$\text{(II.8)} \qquad \forall x_{x \in X} \beta^{\mathfrak{C}}(x) = B, \text{ where } b^{\mathfrak{C}}(x) = \{B\},$$

a <u>single valued budgeter</u> corresponding to $\mathfrak{C}$. Clearly $\beta^{\mathfrak{C}}$ is what is usually called an "inverse demand function." We call the set of budgets under which some alternative is chosen, the <u>range</u> of $b^{\mathfrak{C}}$, and denote it by $\Delta^{\mathfrak{C}}$:

$$\text{(II.9)} \qquad \Delta^{\mathfrak{C}} = \bigcup \{b^{\mathfrak{C}}(x): x \in X\}.$$

Cf. Figure 2.

Figure 2.

Note that also:

$$\text{(II.10)} \quad \begin{aligned} \Gamma^{\mathbb{C}} &= \{x\in X\colon\ b^{\mathbb{C}}(x) \neq \emptyset\} \\ \Delta^{\mathbb{C}} &= \{B\in\mathcal{B}\colon\ h^{\mathbb{C}}(B) \neq \emptyset\}\ . \end{aligned}$$

Corresponding to our two slices of $\mathbb{C}$, we have two slices of the admissibility relation A. We define $A_X\colon \mathcal{B} \to \mathfrak{P}X$ so it shows for any budget $B\in\mathcal{B}$, the admissible alternatives:

$$\text{(II.11)} \qquad A_X(B) = \{x\in X\colon\ (x,B)\in A\}\ .$$

Of course, if budgets B were identified with subsets of X, then in most applications we should expect that $A_X(B) = B$. If, however a budget B were identified with a "competitive" price vector $p\in R^n$ and an expenditure level $m\in Re$, then in the classical case when X is a subset of R^n, we might expect that $A_X(p,m) = \{x\in X\colon\ p\cdot x \leqq m\}$.

And we define $A_{\mathcal{B}}\colon X \to \mathfrak{P}\mathcal{B}$ so it shows, for any $x\in X$, the admissible budgets:

$$\text{(II.12)} \qquad A_{\mathcal{B}}(x) = \{B\in\mathcal{B}\colon\ (B,x)\in A\}\ .$$

The symmetry between the roles played by X and $\mathcal{B}$ is very strongly apparent in comparing (II.4) with (II.7), or (II.6) with (II.9). We recognize this by saying that the sets X and $\mathcal{B}$ are <u>dual</u>[1] to each other, as are $h^{\mathbb{C}}$ and $b^{\mathbb{C}}$, and as are A_X and $A_{\mathcal{B}}$. We will continue our efforts to maintain this symmetry. In the next section, for example, we develop the notion of "worth" symmetrically to that of "utility," rather than through the derivative concept of "indirect utility." Our reasons for preserving symmetry in our definitions are not just aesthetic, but quite practical as well. The symmetry in our definitions will lead to symmetry in theorems: for each theorem a "dual" theorem can be stated, with a proof that is also "dual." This will be made precise in the Duality Metatheorem below in Section C, which will suggest new theorems and greatly simplify many of our proofs.

B. Rationality concepts.

We develop three concepts of rationality, corresponding to the three concepts of choice, demand, and budgeter given in Section A.

1. Rational demand.

Given a choice space $(X, \mathcal{B}, A, \mathbb{C})$, we say that $h^{\mathbb{C}}$ is a rational[1] demand if there exists a binary relation $\succcurlyeq$ on X such that:

$$\forall B_{B \in \mathcal{B}} h^{\mathbb{C}}(B) = \{x \in X: xAB \text{ \& } \forall y_{yAB} x \succcurlyeq y\} . \tag{II.13}$$

Such a relation will be called a preference for $h^{\mathbb{C}}$ or an alternative preference for $\mathbb{C}$.

If U is a real valued function defined on X such that

$$\forall B_{B \in \mathcal{B}} h^{\mathbb{C}}(B) = \{x \in X: xAB \text{ \& } \forall y_{yAB} U(x) \geqq U(y)\} , \tag{II.14}$$

then U will be called a utility for $h^{\mathbb{C}}$, or for $\mathbb{C}$. It is clear that, even when $h^{\mathbb{C}}$ admits a linear ordering preference, it may admit no utility. Under competitive budget constraints, Debreu's case of a lexicographic preference on the plane is one example [6]. In a very different direction, whenever $\mathcal{B}$ is finite but $h(B) = \emptyset$ for some nonempty $B \in \mathcal{B}$, then clearly h admits no utility.

Example 1. Let $X = \{x, y\}$, with $x \neq y$, and let $\mathcal{B} = \{B_1, B_2, B_3\}$, where

$$B_1 = \{x\}$$
$$B_2 = \{y\}$$
$$B_3 = \{x, y\}.$$

Then, if $h(B_1) = \{x\}$ and $h(b_2) = \{y\}$ and $h(B_3) = \emptyset$, clearly no utility exists.

This suggests that it will be useful to define a partial utility for $h^{\mathbb{C}}$, and a partial preference for $h^{\mathbb{C}}$, by restricting the definitions (II.13) and (II.14) to $\Delta^{\mathbb{C}}$. Thus a partial preference $\succcurlyeq$ satisfies:

$$\text{(II.15)} \qquad \forall B_{B\in\Delta^{\mathbb{C}}} h^{\mathbb{C}}(B) = \{x\in X\colon xAB \ \&\ \forall y_{yAB}\, x \succcurlyeq y\}\ ,$$

and then we say that $h^{\mathbb{C}}$ is partial rational; and a partial utility U satisfies:

$$\text{(II.16)} \qquad \forall B_{B\in\Delta^{\mathbb{C}}} h^{\mathbb{C}}(B) = \{x\in X\colon xAB\ \forall y_{yAB} U(x) \geqq u(y)\}\ .$$

If $h^{\mathbb{C}}(B)$ is a singleton for all $B\in\Delta^{\mathbb{C}}$, then we call $\xi^{\mathbb{C}}$ a single valued partial demand corresponding to $\mathbb{C}$, where $\xi^{\mathbb{C}}\colon \Delta^{\mathbb{C}} \to X$ is defined by:

$$\text{(II.17)} \qquad \forall B_{B\in\Delta^{\mathbb{C}}} \xi^{\mathbb{C}}(B) = x, \text{ where } h^{\mathbb{C}}(B) = \{x\}$$

2. Rational budgeter.

If the concepts had not been discovered long ago, the duality approach mentioned at the end of Section A would immediately suggest that we dualize the definitions (II.13), (II.14), (II.15) and (II.16), obtaining new, dual, notions of preference and utility. We postpone doing this in order to first motivate the dual concepts in a less technical way, which indeed is the manner in which they have usually been presented.

We first note that if a utility U exists for h, then it assigns "indirectly" a "utility" for some budgets: we can define a real valued function $\bar{W}^U$ on Δ by

$$\text{(II.18)} \qquad \forall B_{B\in\Delta} \bar{W}^U(B) = U(x), \text{ for any } x\in h(B).$$

Observe that if h is single valued, we can represent $\bar{W}^U$ as the composition $U \circ \xi$ on Δ. Thus $\bar{W}^U(B)$ can be considered as the highest utility of any alternative in B. $\bar{W}^U$ is usually called an "indirect utility" (cf. [28]).

We next note that, while h <u>maximizes</u> utility U for each budget B, i.e.:

$$\text{(II.19)} \qquad \forall B_{B\in\mathcal{B}} \forall \bar{x}_{\bar{x}\in h(B)} U(\bar{x}) = \max \{U(x): xAB\} ,$$

b <u>minimizes</u>[1] $\bar{W}^U$ for each alternative x that is ever chosen, i.e.:

$$\text{(II.20)} \qquad \forall \bar{B}_{\bar{B}\in b(x)} \bar{W}^U(\bar{B}) = \min \{\bar{W}^U(B): B\in\Delta \ \& \ BAx\} .$$

The asymmetry between (II.19) and (II.20) due to the requirement $B\in\Delta$ calls attention to the asymmetry between U and $\bar{W}^U$ themselves: U is defined on all of X, but $\bar{W}^U$ is only defined on Δ. We could eliminate this asymmetry by changing our definition of $\bar{W}^U$, transforming it to W^U, defined on all of $\mathcal{B}$, by, for example:

$$\text{(II.21)} \qquad \forall B_{B\in\mathcal{B}} W^U(B) = \begin{cases} \arctan \bar{W}^U(B), & \text{if } B\in\Delta \\ \frac{\pi}{2}, & \text{if } B\notin\Delta. \end{cases}$$

Then (II.20) holds for W^U, without the qualification $B\in\Delta$.

Such modifications of $\bar{W}^U$ suggest that we go all the way and treat indirect utility, not as a concept derived from utility, but as an independent concept in its own right. We perform this dualization now for demand preference as well as for utility.

Given a choice space $(X,\mathcal{B},A,\mathbb{C})$, we say that $b^{\mathbb{C}}$ is a rational[2] budgeter if there exists a binary relation $\preccurlyeq$ on $\mathcal{B}$ such that:

(II.22) $\forall x_{x\in X} b^{\mathbb{C}}(x) = \{\bar{B}\in\mathcal{B}\colon \bar{B}Ax \ \&\ \forall B_{xAB} \bar{B} \preccurlyeq B\}.$

Such a relation will be called a preference for $b^{\mathbb{C}}$ or a budget preference for $\mathbb{C}$.

If W is a real valued function defined on $\mathcal{B}$ such that:

(II.23) $\forall x_{x\in X} b^{\mathbb{C}}(x) = \{\bar{B}\in\mathcal{B}\colon \bar{B}Ax \ \&\ \forall B_{BAx} W(\bar{B}) \leq W(B)\}$,

then W is called a worth for $b^{\mathbb{C}}$, or for $\mathbb{C}$. It is clear that, even when $b^{\mathbb{C}}$ admits a preference, it may admit no worth, as seen in the next example.

Example 2. Let $X = \{x,y\}$, with $x \neq y$, and let $\mathcal{B} = \{B\}$, with $B = \{x,y\}$. If $h(B) = \{x\}$, then clearly no worth exists for b, since $b(y) = \emptyset$. This is true even though h admits a utility (define $U(x) = 1$ & $U(y) = 0$).[1]

As before, this suggests that we define partial concepts by restricting our definitions (II.22) and (II.23) to $\Gamma^{\mathbb{C}}$. Thus a partial preference $\preccurlyeq$ for $b^{\mathbb{C}}$ satisfies:

(II.24) $\forall x_{x\in \Gamma^{\mathbb{C}}} b^{\mathbb{C}}(x) = \{\bar{B}\in\mathcal{B}\colon \bar{B}Ax \ \&\ \forall B_{xAB} \bar{B} \preccurlyeq B\}$,

and then we say that $b^{\mathbb{C}}$ is partial rational; and a partial worth W for $b^{\mathbb{C}}$ satisfies:

(II.25) $\forall x_{x\in \Gamma^{\mathbb{C}}} b^{\mathbb{C}}(x) = \{\bar{B}\in\mathcal{B}\colon \bar{B}Ax \ \&\ \forall B_{xAB} W(\bar{B}) \leq W(B)\}.$

If $b^{\mathbb{C}}(x)$ is a singleton for all $x\in\Gamma^{\mathbb{C}}$, then we call $\beta^{\mathbb{C}}$ a single valued partial budgeter corresponding to $\mathbb{C}$, where $\beta^{\mathbb{C}}\colon \Gamma \to \mathcal{B}$ is defined by:

(II.26) $\forall x_{x\in \Gamma^{\mathbb{C}}} \beta^{\mathbb{C}}(x) = B$, where $b^{\mathbb{C}}(x) = B$.

3. Rational choice.[1]

From the notion of utility for demand we motivated the notion of worth, a budgeter "utility." From these notions together we can now motivate the notion of value, a "utility" for the choice relation $\mathbb{C}$ itself. Since utility is maximized by the demand's alternatives, and since worth is minimized by the budgeter's budgets, it seems reasonable that a numerical function representing $\mathbb{C}$ should have a saddle value at any $(x,B)\in\mathbb{C}$. We formalize this first for preferences, and then for numerical representations.

Given a choice space $(X,\mathcal{B},A,\mathbb{C})$, we say that $\mathbb{C}$ is a rational[2] choice if there exists a binary relation $\geqq$ on $X \times \mathcal{B}$ such that

$$(\text{II.27}) \qquad \mathbb{C} = \{(\bar{x},\bar{B})\in X \times \mathcal{B}\colon\ \bar{x}A\bar{B}\ \&\ \forall (x,B)_{\bar{x}AB\ \&\ xA\bar{B}}(\bar{x},B) \geqq (\bar{x},\bar{B}) \geqq (x,\bar{B})\}.$$

Such a relation will be called a preference for $\mathbb{C}$.

If V is a real valued function defined on $X \times \mathcal{B}$ such that

$$(\text{II.28}) \qquad \mathbb{C} = \{(\bar{x},\bar{B})\in X \times \mathcal{B}\colon\ \bar{x}A\bar{B}\ \&\ \forall (x,B)_{\bar{x}AB\ \&\ xA\bar{B}}V(\bar{x},B) \geqq V(\bar{x},\bar{B}) \geqq V(x,\bar{B})\},$$

then V is called a value for $\mathbb{C}$.

As before, we can define partial notions: a partial preference for $\mathbb{C}$

$$(\text{II.29}) \qquad \mathbb{C} = \{(\bar{x},\bar{B})\in \Gamma \times \Delta\colon\ \bar{x}A\bar{B}\ \&\ \forall (x,B)_{\bar{x}AB\ \&\ xA\bar{B}}(\bar{x},B) \geqq (\bar{x},\bar{B}) \geqq (x,\bar{B})\}$$

and we say that $\mathbb{C}$ is partial rational; and a partial value for $\mathbb{C}$ satisfies:

$$(\text{II.30}) \qquad \mathbb{C} = \{(\bar{x},\bar{B})\in \Gamma \times \Delta\colon\ \bar{x}A\bar{B}\ \&\ \forall (x,B)_{\bar{x}AB\ \&\ xA\bar{B}}V(\bar{x},B) \geqq V(\bar{x},\bar{B}) \geqq V(x,\bar{B})\}.$$

C. The Duality Metatheorem.

1. The Metatheorem.

We have already noted in Section A the symmetry between our definitions of demand and budgeter, and this has carried over to definitions of utility and worth. Such symmetry rests ultimately on the symmetry between the role played by X and the role played by $\mathcal{B}$. What we now observe is that not only every term and set, but also every sentence -- hence every axiom and theorem -- has a dual, obtained by interchanging the roles of X and $\mathcal{B}$ (and thereby interchanging the roles of demand and budgeter, and of utility and worth, etc., since those concepts were defined symmetrically in terms of X and $\mathcal{B}$).

Of course, to properly state and prove such a metatheorem about what can be proved, we have to be precise about what axiom system we are talking about, and what notion of proof we are assuming.[1] First, we take as our language a first-order predicate language $\mathcal{L}$ with equality symbol $\approx$ and, in addition to the usual binary predicate parameter ϵ of set theory, four individual constant symbols X, $\mathcal{B}$, A, and $\mathbb{C}$.[2] Second, as our set Λ of logical axioms, we take any of the standard ones; to be specific, we take that of [15], §2.4. Third, as our set $\mathcal{S}$ of nonlogical axioms we take a set of axioms for set theory, together with our conditions (II.1 & 2 & 3). For the set theory axioms we again adopt any of the standard ones; to be specific, we use the set ZFC of axioms for Zermelo-Fraenkel set theory with the axiom of choice, as given by the sentences corresponding to [3] the formulas in [5], pp. 507-508, using in the Axiom Schema of Replacement all formulas of our language $\mathcal{L}$. Fourth, we require a scheme of deduction; to be specific we again follow [15] (p. 102), and use modus ponens as our single rule of inference. Then by a deduction of a formula φ from $\mathcal{S}$ we mean a finite sequence $(\varphi_1,\ldots,\varphi_n)$ of formulas, with $\varphi_n = \varphi$, where each φ_i is either a formula of $\mathcal{S}$ or Λ or follows from some φ_j and φ_k (with $j,k < i$) by modus ponens. If there is a deduction of φ from $\mathcal{S}$

then we write $\mathcal{S} \vdash \varphi$. Finally, for any expression η of $\mathcal{L}$, we denote by η^τ the expression obtained from η by replacing the symbol X in each occurrence by the symbol $\mathcal{B}$, and replacing the symbol $\mathcal{B}$ in each occurrence by the symbol X. We call η^τ the <u>dual</u> of η, and say that η and η^τ are <u>dual</u> (expressions). And we extend this terminology to sets, saying that two sets are <u>dual</u> if they are defined by dual formulas.

<u>Duality Metatheorem</u>. If φ is a formula of $\mathcal{L}$ and $\mathcal{S} \vdash \varphi$, then $\mathcal{S} \vdash \varphi^\tau$. Indeed, if $(\varphi_1, \ldots, \varphi_n)$ is a deduction from $\mathcal{S}$, then $(\varphi_1{}^\tau, \ldots, \varphi_n{}^\tau)$ is a deduction from $\mathcal{S}$.

<u>Proof</u>. The Metatheorem is obvious: since all of our axioms treat X and $\mathcal{B}$ symmetrically, any assertions about X should be as true for $\mathcal{B}$, and <u>vice versa</u>.[1] We therefore omit details of the proof. However, any reader wanting to verify the details will have no trouble checking that the transpostion operation $(.)^\tau$, because it commutes with the logical operations (negations, implication, quantification, etc.), sends elements of Λ into elements of Λ, elements of $\mathcal{S}$ into elements of $\mathcal{S}$, and implications into implications; thus it sends deductions into deductions, and the Metatheorem follows.

2. <u>Applications</u>.

While the Metatheorem itself is obvious, we must take some care in applying it. For example, suppose we have proved (as in Theorem 1(a.i)): if $\mathbb{C}$ has a partial utility then $\mathbb{C}$ has a partial worth. Then we would like to use the Duality Metatheorem and prove: if $\mathbb{C}$ has a partial worth then $\mathbb{C}$ has a partial utility (Theorem 1(a.ii)). But to make such an application, we must show first that Theorem 1(a.i) and (a.ii) can be[2] stated in terms of our language $\mathcal{L}$; for example, we would use (II.4) to eliminate $h^{\mathbb{C}}$ from (II.16), so that partial utility would be defined in terms of $\in$, X, $\mathcal{B}$, A, and $\mathbb{C}$; and we would use (II.18) to eliminate $b^{\mathbb{C}}$ from (II.25), so that partial worth would also be defined in $\mathcal{L}$.

We would then find, however, that the concepts of partial worth and partial utility were not quite dual, and the Duality Metatheorem did not give the desired result: the inequalities in (II.16) and (II.25) are opposite; utility is defined in terms of maximization while worth is defined in terms of minimization. But that is easily taken care of, after applying the Duality Metatheorem, by a change of sign.[1]

Finally, we note what the Duality Metatheorem does not say. For example, it does not say that if there exists a utility for the demand in a particular model of the axioms (i.e., for a particular interpretation of X, $\mathcal{B}$, A, $\mathcal{C}$) then there exists a worth. The Duality Metatheorem applies only when an assertion in provable from the axioms, i.e., when it holds in all models of the axioms.

D. Relationships among the rationality concepts.

To prove that a consumer's demand maximizes a utility function, it is often easier or more natural (cf. Part IV,D, Theorem 19) to show that a worth ("indirect utility") exists, and then to derive a utility from the worth. That is one motivation for the theorems of this section which give conditions for passing between types of representation.

1. Numerical representations.

For numerical representations, the main relationships are illustrated in Figure 3 below, which summarizes Theorem 1.

Figure 3

The following definitions will be helpful. If U is a partial utility, then we define $W^U: \mathcal{B} \rightarrow Re$ as in (II.21). If W is a partial worth, then we define $U^W: X \rightarrow Re$ by

$$\text{(II.31)} \qquad U^W(x) = \begin{cases} \arctan(W(B)), \text{ for any } B \in b(x), \text{ if } x \in \Gamma^{\mathcal{C}} \\ -\frac{\pi}{2}, \text{ if } x \notin \Gamma^{\mathcal{C}}. \end{cases}$$

<u>Theorem 1</u>. Let $(X, \mathcal{B}, A, \mathcal{C})$ be a choice space. Then:

a.i) if U is a partial utility for $\mathcal{C}$, then W^U is a partial worth for $\mathcal{C}$;

a.ii) if W^U is a partial worth, then U^W is a partial utility;

b.i) if $\mathcal{C}$ has a partial utility or a partial worth, then $\mathcal{C}$ has a partial value;

b.ii) if $\mathcal{C}$ has a partial value, then $\mathcal{C}$ has a partial utility and a partial worth;

c.i) a utility for $\mathcal{C}$ is a partial utility $\mathcal{C}$;

c.ii) a partial utility for $\mathcal{C}$ is a utility for $\mathcal{C}$ if $\Delta^{\mathcal{C}} = \mathcal{B}$;

d.i) a worth for $\mathcal{C}$ is a partial worth for $\mathcal{C}$;

d.ii) a partial worth for $\mathcal{C}$ is a worth for $\mathcal{C}$ if $\Gamma^{\mathcal{C}} = X$;

e.i) a value for $\mathcal{C}$ is a partial value for $\mathcal{C}$;

e.ii) a partial value for $\mathcal{C}$ is a value for $\mathcal{C}$ if $\Delta^{\mathcal{C}} = \mathcal{B}$ & $\Gamma^{\mathcal{C}} = X$;

f.i) if $\mathcal{C}$ has a utility and a worth, then $\mathcal{C}$ has a value;

f.ii) $\mathcal{C}$ may have a value without having either a utility or a worth.

Proof. Most of the proofs either are trivial or straightforward, and we omit them. Nevertheless a few suggestions may be helpful. In (b.i) we can (by a.i & ii)) suppose U is a partial utility and W is a partial worth, and then show a partial value V can be defined by

$$V(x,B) = U(x)W(B). \tag{II.32}$$

In (b.ii) we can derive a partial utility U from a partial value V by:

$$U(x) = \min \{V(x,B) : xAB\}. \tag{II.33}$$

Finally, (f.ii) can be verified by a simple example.

2. Preferences.

The most frequently discussed preferences are reflexive, transitive, and total; we shall call such preferences regular. The situation with regard to regular-rationality -- of demand, budgeter, or choice; and partial or not -- is essentially the same as for numerical representations. The easiest way to see this is to observe that a binary relation is regular if and only if it has a nonstandard numerical representation,[1] so every regular preference can be translated nonstandard-numerically, to obtain and prove a restatement of Theorem 1 in terms of preference. We thus obtain Theorem 2, and a slight reinterpretation of Figure 3.

Theorem 2. Let $(X,\mathcal{B},A,\mathbb{C})$ be a choice space. Then:

a.i) if $\mathbb{C}$ has a regular partial demand preference, then $\mathbb{C}$ has a regular partial budgeter preference;

a.ii) if $\mathbb{C}$ has a regular partial budgeter preference, then $\mathbb{C}$ has a regular partial demand preference;

b.i) if $\mathbb{C}$ has a regular partial demand preference or a regular partial budgeter preference, then $\mathbb{C}$ has a regular partial choice preference;

b.ii) if $\mathbb{C}$ has a regular partial choice preference, then $\mathbb{C}$ has a regular partial demand preference and a regular partial budgeter preference;

c.i) a regular demand preference for $\mathbb{C}$ is a regular partial demand preference for $\mathbb{C}$;

c.ii) a regular partial demand preference is a regular demand preference if $\Delta^{\mathbb{C}} = \mathcal{B}$;

d.i) a regular budgeter preference for $\mathbb{C}$ is a regular partial budgeter preference for $\mathbb{C}$;

d.ii) a regular partial budgeter preference for $\mathbb{C}$ is a regular budgeter preference for $\mathbb{C}$ if $\Gamma^{\mathbb{C}} = X$;

e.i) a regular choice preference for $\mathbb{C}$ is a regular partial choice preference for C;

e.ii) a regular partial choice preference for $\mathbb{C}$ is a regular choice preference for $\mathbb{C}$ if $\Delta^{\mathbb{C}} = \mathcal{B}$ & $\Gamma^{\mathbb{C}} = X$;

f.i) if $\mathbb{C}$ has a regular demand preference and a regular budgeter preference, then $\mathbb{C}$ has a regular choice preference;

f.ii) $\mathbb{C}$ may have a regular choice preference without having either a regular demand preference or a regular budgeter preference.

Proof. See preceding discussion.

3. Convexity.

In the abstract framework up to this point, convexity notions did not arise. In many applications, however, both X and $\mathcal{B}$ have natural linear structures, and then convexity properties of representations are of interest.

In much of classical consumer theory, for example, the set X of alternatives is a subset of a real linear space (cf. [39], Chapter V; [7], Chapter 4), usually finite dimensional. And when the consumer is viewed as facing market prices that are independent of his purchases, then a typical budget

(II.34) $$B = \{x \in X: p \cdot x \leqq m\}$$

is determined by price vector p which is a member of the dual linear space, and the expenditure limit $m \in R^1$. Thus $\mathcal{B} \subseteq \mathfrak{P}X$, and in most applications A would be the symmetrization of $\in$.

In this context, there are two senses in which we could understand convexity notions. On the one hand, we could consider budget sets as elements of the real linear space defined from the power set $\mathfrak{P}X$ in the usual way:

(II.35) $$\forall s,t_{s,t \in R^1} \forall A,B_{A,B \in \mathfrak{P}X} sA + tB = \{sx + ty: x \in A \text{ \& } y \in B\}.$$

But this is unsatisfactory for our purposes, as a convex combination $tB_1 + (1-t)B_2$, with $t \in [0,1]$ and $B_1, B_2 \in \mathcal{B}$ may very well yield a set not in $\mathcal{B}$, as indicated in Figure 4, where $\mathcal{B}$ is the set of competitive budgets determined by positive prices and expenditure levels.

Figure 4

On the other hand we may as budgets consider, not subsets of X of type (II.34), but the price-expenditure parameters (p,m) that define them. Then B would be (p,m) itself, and in most applications we would have, for any $x \in X$ and $(p,m) \in \mathcal{B}$, $xA(p,m) \Leftrightarrow p \cdot x \leqq m$, so:

$$\text{(II.36)} \qquad A_X(p,m) = \{x \in X\colon\ p \cdot x \leqq m\}.$$

At first sight, this seems unsatisfactory because there is not a one-to-one relationship between budget sets and their parametric representations: if $(\bar{p},\bar{m})$ and $(\tilde{p},\tilde{m})$ represent two different budget sets B_1 and B_2, respectively, then the set represented by $s(\bar{p},\bar{m}) + t(\tilde{p},\tilde{m})$ depends on the particular choice of positive values for s and t, even though the sets B_1 and B_2 are represented by $s(\bar{p},\bar{m})$ and $t(\tilde{p},\tilde{m})$, respectively, for <u>all</u> positive s and t.

This difficulty can be overcome by identifying budgets B with price-expenditure vectors (p,m) in a one-to-one fashion, and for that some kind of normalization is usually performed:

Assuming that $X \subseteq R^n_{\geqq}$, we initially consider budgets in $\mathcal{B}$ as represented by elements (p,m) of the dual space $(R^{n+1})^*$ of R^{n+1}, where p, an element of $(R^n)^*$ may be represented by $(p_1,\ldots,p_n) \in R^n$. Thus through the usual identification of $(R^{n+1})^*$ with R^{n+1}, budgets in $\mathcal{B}$ may be represented as elements of R^{n+1}. Under special assumptions, one of the following normalizations may be used to represent budgets by elements of R^n, rather than by elements of R^{n+1}.

(i) If m is never zero, we may represent $(p_1,\ldots,p_n,m)$ by $(\frac{p_1}{m},\ldots,\frac{p_n}{m},1)$, or just by $(\frac{p_1}{m},\ldots,\frac{p_n}{m})$. We may correspondingly define:

$$\text{(II.37)} \qquad A_X(p_1,\ldots,p_n) = \{x \in R^n_{\geqq}\colon\ p_1x_1 + \ldots + p_nx_n \leqq 1\}.$$

(ii) If $p = (p_1,\ldots,p_n)$ is never the zero vector, we may represent $(p_1,\ldots,p_n,m)$ by

$$\left(\frac{p_1}{\sum_{i=1}^{n} p_i},\ldots,\frac{p_n}{\sum_{i=1}^{n} p_i},\frac{m}{\sum_{i=1}^{n} p_i}\right),$$

or just by

$$\left(\frac{p_1}{\sum_{i=1}^{n} p_i},\ldots,\frac{p_{n-1}}{\sum_{i=1}^{n} p_i},\frac{m}{\sum_{i=1}^{n} p_i}\right).$$

We may correspondingly define, when $p_1 + \ldots + p_{n-1} \leqq 1$:

$$\text{(II.38)} \qquad A_X(p_1,\ldots,p_{n-1},m) = \{x \in R^n_{\geqq}\colon\ p_1x_1 + \ldots + p_{n-1}x_{n-1} + (1 - \textstyle\sum_{i=1}^{n} p_i)x_n \leqq m\}.$$

(iii) If p_n is never zero, we may represent $(p_1,\ldots,p_n,m)$ by

$$\left(\frac{p_1}{p_n},\ldots,\frac{p_{n-1}}{p_n},1,\frac{m}{p_n}\right),$$

or just by

$$\left(\frac{p_1}{p_n},\ldots,\frac{p_{n-1}}{p_n},\frac{m}{p_n}\right).$$

We may correspondingly define:

(II.39) $$A_X(p_1,\ldots,p_{n-1},m) = \{x\in R^n_{\geqq}:\ p_1x_1+\ldots+p_{n-1}x_{n-1}+x_n \leqq m\}.$$

We now define for each integer $n \geqq 1$, an <u>n-dimensional classical choice space of type (i), (ii), or (iii)</u>, respectively, to be a choice space $(X,\mathcal{B},A,\mathcal{C})$ in which $X \subseteq R^n_{\geqq}$ and $\mathcal{B} \subseteq (R^n)^*_{\geqq}$, and where, identifying $(R^n)^*$ with R^n:

(II.40) i) for all $(p_1,\ldots,p_n)\in\mathcal{B}$,

$$A_X(p_1,\ldots,p_n) = \{x\in X:\ p_1x_1+\ldots+p_nx_n \leqq 1\}, \quad \text{or}$$

ii) for all $(p_1,\ldots,p_{n-1},m)\in\mathcal{B} \subseteq \{(\bar{p}_1,\ldots,\bar{p}_{n-1},\bar{m})\in R^n_{\geqq}:\ \bar{p}_1+\ldots+\bar{p}_{n-1} \leqq 1\}$,

$$A_X(p_1,\ldots,p_{n-1},m) = \{x\in X:\ p_1x_1+\ldots+p_{n-1}x_{n-1}+(1-\sum_{i=1}^{n-1}p_i)x_n \leqq m\}, \quad \text{or}$$

iii) for all $(p_1,\ldots,p_{n-1},m)\in\mathcal{B}$,

$$A_X(p_1,\ldots,p_{n-1},m) = \{x\in X:\ p_1x_1+\ldots+p_{n-1}x_{n-1} + x_n \leqq m\},$$

respectively. By a <u>classical choice space</u> we mean one of any of these three types.

<u>Remark</u>. Note that in classical choice spaces we have $X \subseteq R^n_{\geqq}$ and $\mathcal{B} \subseteq (R^n)^*_{\geqq}$, so X and $\mathcal{B}$ are disjoint as desired (cf. page 7, footnote 1) for applications of the Duality Metatheorem. In our applications, however, it is easier to identify $(R^n)^*_{\geqq}$ with $R^n_{\geqq}$, as in (II.40), and we shall do this without further mention.

More generally, we define a <u>linear</u> (respectively, <u>topological linear</u>) <u>choice space</u> as a choice space $(X,\mathcal{B},A,\mathcal{C},L_X,L_{\mathcal{B}})$ in which X and $\mathcal{B}$ are subsets of real linear (respectively, topological linear) vector spaces L_X and $L_{\mathcal{B}}$; if these linear spaces are each of dimension n, then we will say that $(X,\mathcal{B},A,\mathcal{C},L_X,L_{\mathcal{B}})$ is of <u>dimension n</u>.

Simple calculations show that for the classical choice spaces we have the dual conditions:

(II.41) $\qquad \forall B_1, B_{2\,B_1, B_2 \in \mathcal{B}} \forall t_{t\in[0,1]} A_X(tB_1 + (1-t)B_2) \subseteq A_X(B_1) \cup A_X(B_2)$,

(II.42) $\qquad \forall x, y_{x,y\in X} \forall t_{t\in[0,1]} A_{\mathcal{B}}(tx + (1-t)y) \subseteq A_{\mathcal{B}}(x) \cup A_{\mathcal{B}}(y)$;

thus A_X and $A_{\mathcal{B}}$ are each quasi-convex correspondences ([40], p. 182). By a <u>quasi-convex (topological) linear choice space</u> we shall mean any (topological) linear choice space satisfying (II.41) and (II.42).

We will soon observe that, on convex linear choice spaces, utilities, worths, and values can be taken to have convexity properties. But we require a few preliminary definitions: we define convexity notions relative to a given set, so that the usual properties are required of a convex combination only when that convex combination is in the given set. Let Y, Z, Y_1, and Z_1 be subsets of real linear vector spaces. Then:

a. If $Y \subseteq Z$ and $f: Z \to R^1$, then f is quasi-concave [resp. quasi-convex] <u>relative to Y</u> if, for all real α and all $x, y \in Z$ with $f(x) \geqq \alpha$ & $f(y) \geqq \alpha$ [$f(x) \leqq \alpha$ & $f(y) \leqq \alpha$], we have $f(tx + (1-t)y) \geqq \alpha$ [$f(tx + (1-t)y] \leqq \alpha$) for all $t\in[0,1]$ such that $tx + (1-t)y \in Y$.

b. If $Y \subseteq Z$ & regular $\succcurlyeq \subseteq Z \times Z$ and, for all $x, y \in Z$ with $y \succcurlyeq x$, and for all $t\in[0,1]$ with $tx + (1-t)y \in Y$, we have $tx + (1-t)y \succcurlyeq x$ [resp. $\preccurlyeq y$], then we say that $\succcurlyeq$ is quasi-concave [quasi-convex] <u>relative to Y</u>.

c. If $Y_1 \subseteq Z_1$ & $Y_2 \subseteq Z_2$ & $f: Z_1 \times Z_2 \to R^1$, then we say that f is quasi-concave-convex <u>relative to $Y_1 \times Y_2$</u> if, for all $z_2 \in Z_2$, $f(\cdot, z_2): Z_1 \to R^1$ is quasi-concave relative to Y_1 and, for all $z_1 \in Z_1$, $f(z_1, \cdot): Z_2 \to R^1$ is quasi-convex relative to Y_2.

d. If $Y_i \subseteq Z_i$ $(i = 1,2,3,4)$ & $\geqq \subseteq (Z_1 \times Z_2) \times (Z_3 \times Z_4)$, then we say that $\geqq$ is quasi-concave-convex <u>relative to $(Y_1 \times Y_2) \times (Y_3 \times Y_4)$</u> if, for all $(z_3, z_4) \in Z_3 \times Z_4$, the restriction of $\geqq$ to $Z_1 \times Z_2 \times \{(z_3, z_4)\}$ is regular and quasi-concave relative to $Y_1 \times Y_2$, and for all $(z_1, z_2) \in Z_1 \times Z_2$ the restriction of $\geqq$ to $\{(z_1, z_2)\} \times Z_3 \times Z_4$ is regular and quasi-convex relative to $Y_3 \times Y_4$.

Theorem 3. Let $(X,\mathcal{B},A,\mathcal{C})$ be a quasi-convex linear choice space. If $\mathcal{C}$ has:

a partial utility, or
a partial worth, or
a partial value,

then $\mathcal{C}$ has:

a partial utility that is quasi-concave relative to $\Gamma^{\mathcal{C}}$, and
a partial worth that is quasi-convex relative to $\Delta^{\mathcal{C}}$, and
a partial value that is quasi-concave-convex relative to $\Gamma^{\mathcal{C}} \times \Delta^{\mathcal{C}}$.

Proof. By Theorem 1, the existence of any of the first three functions implies the existence of the other two, so it only remains to show that they can be given the indicated convexity properties. Let U be a partial utility; we will obtain a partial utility $\tilde{U}$ that is quasi-concave relative to Γ, by defining, for all $x \in X$:

$$\text{(II.43)} \qquad \tilde{U}(x) = \begin{cases} \arctan U(x), & \text{if } x \in \Gamma \\ -\frac{\pi}{2}, & \text{if } x \notin \Gamma. \end{cases}$$

Clearly $\tilde{U}$ is still a partial utility. Let $\tilde{U}(x) \geqq \alpha$ & $\tilde{U}(y) \geqq \alpha$; we will show that, for any $t \in [0,1]$, we have $\tilde{U}(tx+(1-t)y) \geqq \alpha$. Let $\bar{x} = tx+(1-t)y \in \Gamma$. If $x \notin \Gamma$ or $y \notin \Gamma$, then $\alpha \leqq -\frac{\pi}{2}$, hence $\tilde{U}(\bar{x}) \geqq \alpha$. So we may assume $x,y \in \Gamma$, say $x \in A_X(B_1)$ & $y \in A_X(B_2)$. Since $x \in \Gamma$, say $\bar{x} \in h(\bar{B})$, so $\bar{B} \in A_{\mathcal{B}}(\bar{x})$. Then by (II.41), either $xA\bar{B}$ or $yA\bar{B}$. In the former case, $U(\bar{x}) \geqq U(x)$, and in the latter case, $U(\bar{x}) \geqq U(y)$, since U is a partial utility. Thus $\tilde{U}(\bar{x}) \geqq \tilde{U}(x) \geqq \alpha$ in the former case, and $\tilde{U}(\bar{x}) \geqq \tilde{U}(y) \geqq \alpha$ in the latter.

The proof that a partial worth can be made quasi-convex relative to Δ is dual.

As for partial values, note that the existence of a partial value implies, by Theorem 1(b.ii) and (a) and (b), the existence of a partial utility U that is quasi-concave relative to Γ, and a partial worth W that is quasi-convex relative to Δ. It is easily verified that if we define $V: X \times \mathcal{B} \to R^1$ by: for all $(x,B) \in X \times \mathcal{B}$,

$$V(x,B) = U(x)W(B), \tag{II.44}$$

then V is a partial value that is quasi-concave-convex relative to $\Gamma^{\mathbb{C}} \times \Delta^{\mathbb{C}}$.

As before, there is a preference version.

<u>Theorem 4</u>. Let $(X,\mathcal{B},A,\mathbb{C})$ be a quasi-convex linear choice space. If $\mathbb{C}$ has:

a regular partial demand preference, or
a regular partial budgeter preference, or
a regular partial choice preference,

then $\mathbb{C}$ has:

a regular partial demand preference that is quasi-concave on $\Gamma^{\mathbb{C}}$, and
a regular partial budgeter preference that is quasi-convex on $\Delta^{\mathbb{C}}$, and
a regular partial choice preference that is quasi-concave-convex on $\Gamma^{\mathbb{C}} \times \Delta^{\mathbb{C}}$.

The proof of Theorem 4 can again be derived by nonstandard analysis.

4. Smoothness.

In many applications in economics, X, $\mathcal{B}$, Δ, and Γ have natural topological or manifold structures. For example, Γ is often the strictly positive orthant of n-space (as would follow, for example, from Cobb-Douglas utility functions and positive competitive prices and expenditures), and Δ is often the set of budgets determined, as earlier, by strictly positive price-expenditure vectors $(p_1, \ldots, p_n, m)$.

By a topological choice space $(X, \mathcal{B}, A, \mathcal{C}, T_X, T_{\mathcal{B}})$ we mean a choice space together with a topological space T_X including X and a topological space $T_{\mathcal{B}}$ including $\mathcal{B}$.

a) We first show that, when appropriate demands or budgeters are single-valued, then continuity and differentiability carry over between partial utility and partial worth.

It should be noted that one reason we have to work so much harder to get the continuity results below (especially Theorems 6, 7, 8, and 9) than one would expect from the theorems of [14], p. 121 and p. 123, is a significant difference in the concept of duality: our dual functions are required (indeed defined) to preserve rationality conditions (cf. (II.14) and (II.23)).[1]

Theorem 5. Let $(X, \mathcal{B}, A, \mathcal{C})$ be a choice space and suppose that $\Gamma^{\mathcal{C}}$ and $\Delta^{\mathcal{C}}$ are topological spaces or C^k manifolds $(k \geqq 0)$, respectively. Then:

i) If $\mathcal{C}$ has a single-valued partial demand $\xi^{\mathcal{C}}: \Delta^{\mathcal{C}} \rightarrow \Gamma^{\mathcal{C}}$ that is continuous or C^k, respectively, and if $\xi^{\mathcal{C}}$ has a partial utility U that is continuous or C^k, respectively, on $\Gamma^{\mathcal{C}}$, then W^U is a partial worth whose restriction to $\Delta^{\mathcal{C}}$ is continuous or C^k, respectively;

ii) If $\mathbb{C}$ has a single-valued partial budgeter $\beta^{\mathbb{C}}: \Gamma^{\mathbb{C}} \to \Delta^{\mathbb{C}}$ that is continuous or C^k, respectively, and if $\beta^{\mathbb{C}}$ has a partial worth W that is continuous or C^k, respectively, on $\Delta^{\mathbb{C}}$ then U^W is a partial utility whose restriction to $\Gamma^{\mathbb{C}}$ is continuous or C^k, respectively;

iii) If $(X, \mathcal{B}, A, \mathbb{C})$ is also a quasi-convex linear choice space, then W^U in (i) is quasi-convex relative to $\Delta^{\mathbb{C}}$, and U^W in (ii) is quasi-concave relative to $\Gamma^{\mathbb{C}}$.

The proof is straightforward. In (ii), for example, the restriction of U^W to $\Gamma^{\mathbb{C}}$ equals (cf. (II.40)) the restriction of $\arctan \circ W \circ \beta$, which is clearly continuous or C^k, respectively; the quasi-concavity is shown as in the proof of Theorem 3.

b) Preserving continuity in passing between utility and worth is more difficult without the single-valuedness hypotheses of Theorem 5. Our next theorems show how, when h and b are correspondences, selection theorems allow us to use the methods of Theorem 5.

Theorem 6. Let $(X, \mathcal{B}, A, \mathbb{C}, L_X, L_{\mathcal{B}})$ be a finite dimensional topological linear choice space, and suppose that:

i.1) $\Gamma^{\mathbb{C}}$ is given the relative topology induced by L_X; and

i.2) $\Delta^{\mathbb{C}}$ is given the relative topology induced by $L_{\mathcal{B}}$; and

ii) either:

ii.1) $b^{\mathbb{C}}(y)$ is convex, for all $y \in \Gamma^{\mathbb{C}}$, or

ii.2) $b^{\mathbb{C}}(y)$ is closed, for all $y \in \Gamma^{\mathbb{C}}$; and

iii) $b^{\mathbb{C}}: \Gamma^{\mathbb{C}} \to \Delta^{\mathbb{C}}$ is a lower hemi-continuous[1] correspondence.

If $\mathbb{C}$ admits a partial worth W that is continuous relative to $\Delta^{\mathbb{C}}$, then U^W is a partial utility that is continuous relative to $\Gamma^{\mathbb{C}}$.

Proof. By selection theorems of Michael [32] (p.368 Theorem 3.1''', for case (ii.1); p.367, Theorem 3.2'', for case (ii.2)), there exists a continuous function $\tilde{\beta}$: $\Gamma^{\mathbb{C}} \rightarrow \Delta^{\mathbb{C}}$, such that $\tilde{\beta}(x) \in b^{\mathbb{C}}(x)$, for all $x \in \Gamma^{\mathbb{C}}$. Then the method of proof used for Theorem 5 completes the present proof.

As usual, there is a dual version.

Theorem 7. Let $(X, \mathcal{B}, A, \mathbb{C}, L_X, L_{\mathcal{B}})$ be a finite dimensional topological linear choice space, and suppose that:

i.1) $\Delta^{\mathbb{C}}$ is given the relative topology induced by $L_{\mathcal{B}}$, and

i.2) $\Gamma^{\mathbb{C}}$ is given the relative topology induced by L_X; and

ii) either:

ii.1) $h^{\mathbb{C}}(b)$ is convex for all $B \in \Delta^{\mathbb{C}}$, or

ii.2) $h^{\mathbb{C}}(B)$ is closed, for all $B \in \Delta^{\mathbb{C}}$; and

iii) $h^{\mathbb{C}}$: $\Delta^{\mathbb{C}} \rightarrow \Gamma^{\mathbb{C}}$ is a lower hemi-continuous correspondence.

If $\mathbb{C}$ admits a partial utility U that is continuous relative to $\Gamma^{\mathbb{C}}$, then W^U is a partial worth that is continuous relative to $\Delta^{\mathbb{C}}$.

Remark 1. In view of Theorems 6 and 7 it is of interest to know when $b^{\mathbb{C}}$ and $h^{\mathbb{C}}$ are lower hemi-continous. As a first case, we note that, when $h^{\mathbb{C}}$ has a single-valued demand $\xi^{\mathbb{C}}$: $\Delta^{\mathbb{C}} \rightarrow \Gamma^{\mathbb{C}}$, then the budgeter $b^{\mathbb{C}}$ is lower hemi-continuous if and only if $\xi^{\mathbb{C}}$ is an open mapping (i.e. carries $\Delta^{\mathbb{C}}$ - open sets onto $\Gamma^{\mathbb{C}}$ - open sets) (cf.[32], p.362, Example 1.1*; [30], p.174, theorem 4). A situation in which a continuous worth W does not give a continuous U^W is Example 3 of [23], pp. 64-66. There U^W fails to be continuous relative to $\Gamma^{\mathbb{C}}$; the single valued continuous demand ξ is clearly not open (yet Theorem 17 below guarantees the existence of a worth W that is continuous relative to $\Delta^{\mathbb{C}}$).

That lower hemi-continuity of $b^{\mathbb{C}}$ is not necessary for the existence of a continuous, strictly monotone utility is shown by the example of Figure 5. In a classical choice framework, with $\Gamma = R^n_{\geqq}$, such a utility may exist even though, for an open neighborhood N of B, $\xi(N)$ is the non-open set T; thus $\xi^{\mathbb{C}}$ is not an open mapping, so $b^{\mathbb{C}}$ fails to be lower hemi-continuous.

Figure 5

As a second case, we note that if $b^{\mathbb{C}}$ is singleton-valued, with single-valued budgeter $\beta^{\mathbb{C}}: \Gamma^{\mathbb{C}} \to \Delta^{\mathbb{C}}$, then it is lower hemi-continuous if and only if $\beta^{\mathbb{C}}$ is continuous relative to $\Gamma^{\mathbb{C}}$. Then we are in the framework of Theorem 5.

As usual, these observations have dual versions.

Remark 2. The hypotheses of Theorems 6 and 7 can be considerably weakened, to allow, for example, infinite dimensional spaces. For Michael's selection theorems, on which Theorems 6 and 7 rest, are useful in a much more general framework than we have used.

c) Our next theorems, which also show how continuity may be preserved in passing between utility and worth, are based on the Maximum Theorem, rather than the selection theorems used in (b). First we give some definitions.

Let $f\colon Y \to Z$ be a correspondence from topological space Y to topological space Z.

f is pre-continuous means that, for every compact $K \subseteq Z$, the correspondence $g\colon Y \to Z$, defined by:

$$\forall y_{y \in Y}\, g(y) = f(y) \cap K$$

is continuous.[1]

f is inner-proper means that, for every $z \in Z$, there exists a neighborhood C of z and a compact set $K \subseteq Y$ such that $C \subseteq \bigcup \{f(y)\colon y \in K\}$.

Theorem 8. Let $(X, \mathcal{B}, A, \mathbb{C}, T_X, T_{\mathcal{B}})$ be a topological choice space, and suppose that $\Delta^{\mathbb{C}}$ and $\Gamma^{\mathbb{C}}$ are given the relative topologies. Suppose that the correspondence $A_{\mathcal{B}}\colon X \to \mathcal{B}$ is precontinuous and that the correspondence $h^{\mathbb{C}}|\Delta^{\mathbb{C}}\colon \Delta^{\mathbb{C}} \to \Gamma^{\mathbb{C}}$ is inner-proper. If there exists a partial worth W that is continuous relative to $\Delta^{\mathbb{C}}$, then U^W is a partial utility that is continuous relative to $\Gamma^{\mathbb{C}}$.

Proof. Since U^W is a partial utility by Theorem 1(a.i), it suffices to show that for each $x \in \Gamma$, there is a Γ-neighborhood C of X such that U^W is continuous relative to C.

So let $\bar{x} \in \Gamma$. The hypothesis on h implies there is a neighborhood C of $\bar{x}$ and a Δ-compact (hence X-compact) set $K \subseteq \Delta$ such that $C \subseteq \bigcup\{h(B)\colon B \in K\}$. We will show that $U^W|C\colon C \to R^1$ is continuous.

Define the correspondence $\eta\colon X \to \mathcal{B}$ by:

$$\forall x_{x \in X}\, \eta(x) = A_{\mathcal{B}}(x) \cap K.$$

Since K is X-compact the hypothesis on $A_{\mathcal{B}}$ implies that η is a continuous correspondence. In particular, for each $x \in X$, $\eta(x)$ is a $\mathcal{B}$-compact (hence Δ-compact) subset of Δ (cf.[4],p.110, Theorem 2). It is also easy to verify that $\eta(x) \neq \emptyset$ for all $x \in C$, using the inner-properness of $h|\Delta$.

Then it easily follows that the restriction $\eta \mid C\colon C \to \Delta$ is a continuous correspondence with respect to the relative topologies on C and Δ, such that $\eta(x)$ is nonempty for all $x \in C$. Since the continuity hypothesis on W implies that $W|C$ is continuous relative to C, it then follows from the Maximum Theorem ([4], p. 116) that the function $\tilde{u}\colon C \to R^1$ given by

$$\forall x_{x \in C}\, \tilde{u}(x) = \min \{W(B)\colon B \in \eta(x)\}$$

is well defined and continuous relative to C.

Our proof will be completed by showing that $U^W|C = \arcsin \tilde{u}$. For this it clearly suffices to show that, for all $x \in C$, $\min\{W(B)\colon B \in \eta(x)\} = \min\{W(B)\colon B \in A_{\mathcal{B}}(x)\}$. Since $\eta(x) \subseteq A_{\mathcal{B}}(x)$, it thus suffices to show that, for all $x \in C$, $\min \{W(B)\colon B \in \eta(x)\} \leqq \min \{W(B)\colon B \in A_{\mathcal{B}}(x)\}$. If $x \in C \subseteq \Gamma$, then there exists $\bar{B} \in \mathcal{B}$ such that $x \in h(\bar{B}) \cap C$ & $\bar{B} \in A_{\mathcal{B}}(x)$, hence $\bar{B} \in K \cap A_{\mathcal{B}}(x)$. Thus $\bar{B} \in \eta(x)$, so $\min \{W(B)\colon B \in \eta(x)\} \leqq W(\bar{B})$. However, since W is a partial worth and $\bar{B} \in b^{\mathbb{C}}(x)$, (II.25) implies that $W(\bar{B}) \leqq \min \{W(B)\colon B \in A_{\mathcal{B}}(x)\}$, and the proof is complete.

Remark 1. If $\mathbb{C}$ admits a single-valued partial demand $\xi\colon \Lambda \to \Gamma$, then our inner-proper condition on h is weaker requiring that ξ be a proper map (i.e., that $\xi^{1}(C)$ be compact for every compact C (cf. [18], p. 17)). For example, in the continuous utility-maximization situation pictured in Figure 6, using the classical choice space representation of type (ii), we see that for the monotone preferences shown, $\xi^{-1}(\bar{x}) = \{r, 1-r, r)\colon r \in (0,1]\}$, which is not a compact subset of $\Delta = \{(r, 1-r, m)\colon r > 0 \;\&\; m > 0\}$. Since $\{\bar{x}\}$ is a compact subset of Γ, ξ is not proper. Yet ξ satisfies our inner proper condition, $\{(1,0,1)\}$ is a compact subset of Δ whose ξ-image is $\{\bar{x}\}$.

Figure 6

Remark 2. An example in which the inner proper condition fails for a single-valued demand is given in [23], p.64, Example 3. And there it is shown that no utility exists that is continuous relative to Γ.

Remark 3. But the inner-properness condition is not necessary, even in the classical framework of continuous utility maximization. Cf. Figure 4, where $\Gamma = R^n_> \cup \{\bar{x}\}$. Clearly there is no compact Δ-neighborhood of $\bar{B}$ whose ξ-image covers a Γ-neighborhood of $\bar{x}$.

Remark 4. We can motivate the inner-properness condition with several observations. First in classical choice spaces it appears to be closely related to "desirability conditions" in common use (cf. [12], P.9(A)), which follow from maximization of continuous, strictly monotone utility. Second, it follows from Debreu's desirability condition ([8], p.388(A); [12], p.9(D)) in the classical framework with fixed endowment vector and single-valued demand ξ. Finally, if Δ is compact (as with S_ε in [11], for example), then the inner-properness condition is necessary for a single-valued demand ξ to be continuous: If A is a closed Γ-neighborhood of x, then $\xi^{-1}(A)$ is closed, hence compact.

Again we have a dual theorem.

Theorem 9. Let $(X,\mathcal{B},A,\mathbb{C},T_X,T_{\mathcal{B}})$ topological choice space, and suppose that $\Gamma^{\mathbb{C}}$ and $\Delta^{\mathbb{C}}$ are given the relative topologies. Suppose that the correspondence $A_X\colon \mathcal{B} \to X$ is precontinuous and that the correspondence $b^{\mathbb{C}}|\Gamma^{\mathbb{C}}\colon \Gamma^{\mathbb{C}} \to \Delta^{\mathbb{C}}$ is inner-proper. If there exists a partial utility U that is continuous relative to $\Gamma^{\mathbb{C}}$, then W^U is a partial worth that is continuous relative to $\Delta^{\mathbb{C}}$.

d) Sometines a weaker notion of smoothness, namely semi-continuity, is of interest, so our next theorem will give sufficient conditions under which that is preserved in passing between worth and utility.

Theorem 10. Let $(X,\mathcal{B},A,\mathbb{C})$ be a classical choice space. Then:

a) If $\mathbb{C}$ has an upper semi-continuous partial worth W, and if $\Delta^{\mathbb{C}} = \mathcal{B}$, then U^W is a utility that is upper semi-continuous relative to $\Gamma^{\mathbb{C}}$ and quasi-concave relative to $\Gamma^{\mathbb{C}}$.

b) If $\mathbb{C}$ has a lower semi-continuous partial utility U, and if $\Gamma^{\mathbb{C}} = X$, then W^U is a worth that is lower semi-continuous relative to $\Delta^{\mathbb{C}}$.

Proof. We sketch the proof of (a) for classical choice spaces of type (ii) or (iii). Let W be an u.s.c. worth. As in the proof of Theorem 1(a.ii), $U = U^W$ is a partial utility; U is in fact a utility, by Theorem 1(c.i), since $\Delta = \mathcal{B}$.

To see that U is also u.s.c. on Γ, let $\alpha \in Re$. We must show that $\{x \in \Gamma: U(x) < \alpha\}$ is open (in the relative topology on X; cf. footnote 1, p.29). Let $\bar{x} \in \Gamma$ & $U(\bar{x}) < \alpha$; we will show there is a Γ-open neighborhood N of $\bar{x}$ with $U(x) < \alpha$ for all $x \in N$. Since $\bar{x} \in \Gamma$, we can say $\bar{x} \in h(\bar{p},\bar{m})$ & $U(\bar{x}) = \arctan (W(\bar{p},\bar{m})) < \alpha$. Because W is u.s.c., $\arctan (W(\bar{p},\bar{m}+\varepsilon)) < \alpha$ for some small $\varepsilon > 0$; and since $\Delta = \mathcal{B}$ we can let $y \in h(\bar{p},\bar{m}+\varepsilon)$, so $U(y) = \arctan (W(\bar{p},\bar{m}+\varepsilon))$. Because we are supposing a classical choice space of type (ii) or (iii), there is a Γ-open neighborhood N of $\bar{x}$ with $N \subseteq A_X(\bar{p},\bar{m}+\varepsilon)$; thus, since U is a utility, $U(y) \geqq U(x)$ for all $x \in N$. We have, therefore,

$$(U(x) \leqq U(y) = \arctan (W(\bar{p},\bar{m}+\varepsilon)) < \alpha$$

for all $x \in N$, and the proof of (a) is complete. The proof of (b) is dual.

<u>Remark 1</u>. It is clear that Theorem 10 holds for a much larger family than the classical choice spaces. By postulating appropriate "monotonicity" conditions on A, it is apparent that the result extends to a wide class of topological choice spaces.

III. Revealed Preference and Rationality.

We present some of the main definitions, axioms, and theorems of revealed preference theory. For much of this material, more complete discussions can be found in [34] and [23].

A. Revealed demand preference definitions.

1. Direct revelation.

Samuelson's original notion [35] of revealed preference applied to R^n. Let X be a subset of $R^n_{\geqq}$, let $\mathcal{B} = \{(p,m) \in R^{n+1}: 0 < p \in R^n \ \& \ 0 \leqq m \in R^1\}$, and, for $x \in X$ & $(p,m) \in \mathcal{B}$, let $xA(p,m)$ mean $p \cdot x \leqq m$. Let $\mathbb{C}$ be a choice with respect to X and $\mathcal{B}$.

If $y \in X$ & $x \in h(p,m)$ & $x \neq y$, $p \cdot y \leqq p \cdot x$ for some $(p,m) \in \mathcal{B}$, then y could have been chosen when x was chosen instead, so we say that x is directly revealed preferred to y,[1] and we write $x\bar{S}y$.[2] We can justify the preference terminology by noting that, if single valued demand ξ has any demand preference $\succcurlyeq$, with asymmetric part[3] $\succ$, then $x\bar{S}y \Rightarrow x \succ y$.

A stronger notion of direct revelation is also of interest, as we shall see in Part V. This is a notion implicit in [39](p.110) and [29]. The weak inequality defining direct revelation is replaced by a strict inequality: If $y \in X$ & $x \in h(p,m)$ $p \cdot y < p \cdot x$ for some $(p,m) \in \mathcal{B}$, then we say that x is <u>directly strictly revealed preferred to y</u>, and we write xTy.

We can capture the intuition of $\bar{S}$ even in an abstract choice space $(X,\mathcal{B},A,\mathbb{C})$ without the linear vector space structure used to define $\bar{S}$ (cf. [34]). If, for some $B \in \mathcal{B}$ and $x,y \in X$ we have $x \neq y$ & $x \in h(B)$ & yAB, then we again say that <u>x is directly revealed preferred to y</u>, and we write xSy.

A useful weaker notion is the following. If $x \in h(B)$ & yAB, for some $B \in \mathcal{B}$ and some $x, y \in X$, then we say that x is directly revealed as good as y, and we write xVy. Again, the intuition is that x was chosen while y could have been chosen, but now there is no requirement that y be different from x.

2. Revelation.

We now extend $\bar{S}$, T, S, and V to their transitive closures.[1] If either $x\bar{S}y$ or $x\bar{S}u\bar{S}y$ or $x\bar{S}u_1\bar{S}\ldots\bar{S}u_k\bar{S}y$ for some $u, u_1, \ldots, u_k \in X$, then we drop the word "direct," and say that x is revealed preferred to y; we then write $x\bar{H}y$. Clearly $\bar{H}$ is the transitive closure of $\bar{S}$. We denote the transitive closure of T by Q, the transitive closure of S by H, and the transitive closure of V by W.

B. Revealed budgeter preference definitions.

1. Direct revelation.

Each of the revealed demand preference definitions in (A) is the dual of a revealed budgeter preference definition.

For example, $(X, \mathcal{B}, A, \mathcal{C})$ is a choice space, then, interchanging the roles of budgets and alternatives in the definition of S: if $\bar{B}, \tilde{B} \in \mathcal{B}$ & $\bar{B} \neq \tilde{B}$ & $\exists x_{x \in X} \bar{B} \in b(x)$ & $xA\tilde{B}$, then we say that $\bar{B}$ is directly revealed worse than $\tilde{B}$, and we write $\bar{B}s\tilde{B}$. The intuition is analogous to that for S: if single valued budgeter β admits any budgeter preference $\leqslant$, with asymmetric part $<$, then $\bar{B}s\tilde{B} \Rightarrow \bar{B} < \tilde{B}$.

If we drop the requirement $\bar{B} \neq \tilde{B}$ in the above definition, then we say that $\bar{B}$ is directly revealed as bad as $\tilde{B}$, and we write $\bar{B}v\tilde{B}$.

And if X is a subset of R^n and $\mathcal{B}$ is identified with the set of vectors (p, m) for $0 < p \in R^n$ & $0 < m \in R^1$, then this notion of revelation translates as follows.

If $(\bar{p},\bar{m}),(\tilde{p},\tilde{m}) \in \mathcal{B}$, and if the budget sets corresponding to $(\bar{p},\bar{m})$ and to $(\tilde{p},\tilde{m})$ are different, and if $\exists x_{x\in X}(\bar{p},\bar{m}) \in b(x)$ & $\tilde{p}\cdot x \leqq \tilde{m}$, then this is a special case of $(\bar{p},\bar{m})s(\tilde{p},\tilde{m})$; and we write $(\bar{p},\bar{m})\bar{s}(\tilde{p},\tilde{m})$.

A stronger notion of direct revelation is again obtained if we replace the weak inequality by a strict inequality: If $(\tilde{p},\tilde{m})\in\mathcal{B}$ and $(\bar{p},\bar{m})\in b(x)$ & $\tilde{p}\cdot x < \tilde{m}$ for some $x\in X$, then we say that $(\bar{p},\bar{m})$ is <u>directly strictly revealed worse than $(\tilde{p},\tilde{m})$</u>, and we write $(\bar{p},\bar{m})t(p,m)$.

2. <u>Revelation</u>.

As before, we can extend s, v, $\bar{s}$, and t to their transitive closures, which we denote by h, w, $\bar{h}$, and q, respectively.

C. <u>Revealed preference axioms</u>.

Samuelson's original idea [35] was that, if alternative x was directly revealed preferred to alternative y by a consumer's choice, then it seemed plausible that the consumer's choices should not also directly reveal y preferred to x. He postulated this condition, the asymmetry of $\bar{S}$, and was able to deduce several consequences for demand behavior, including negative semi-definiteness of the Slutsky substitution matrix (cf. [39], pp. 107-116, also).

Samuelson sought to go further (cf. [38], p. 370) and prove the existence of a utility by postulating asymmetry of $\bar{H}$. But it was not until Houthakker's paper [21] that this was done. Extensions and generalizations, together with some new axioms, are to be found in [45], [34], and [23]. In this section we shall list many of the axioms which are useful in rationalization theory, making sure we include dual versions for all of them. In Section D and in Part V we shall see applications to rationality theorems.

1. Revealed demand preference axioms.

a) The Weak Axiom of Revealed Demand Preference (WARDP).

This asserts that S is asymmetric: $xSy \Rightarrow$ not ySx. In the context of the linear space definition of $\bar{S}$ (Section A.1), when there is also a demand ξ as in (II.5) this becomes:

$$\text{(III.1)}\qquad \begin{array}{l} \text{for all } (\bar{p},\bar{m}),(\tilde{p},\tilde{m}) \in \mathcal{B} \text{ for which } \xi(\bar{p},\bar{m}) \neq \xi(\tilde{p},\tilde{m}), \\ \bar{p}\cdot\xi(\tilde{p},\tilde{m}) \leqq \bar{p}\cdot\xi(\bar{p},\bar{m}) \Rightarrow \tilde{p}\cdot\xi(\bar{p},\bar{m}) > \tilde{p}\cdot\xi(\tilde{p},\tilde{m}); \end{array}$$

i.e.,

$$\text{(III.2)}\qquad \begin{array}{l} \text{where } \Delta\xi \neq 0: \\ \bar{p}\cdot\Delta\xi \leqq 0 \Rightarrow (\bar{p}+\Delta p)\cdot\Delta\xi < 0. \end{array}$$

See Figure 7 for a violation.

Figure 7

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b) The Weak Weak Axiom of Revealed Demand Preference (W^2ARDP).

This axiom, due to [29], asserts, in the context of the linear space definitions of $\bar{S}$ and T (Section A.1), that, for all $x,y\in X$, $xTy \Rightarrow$ not ySx. Again, for the special case of a demand ξ as in (II.5) this becomes:[1]

$$\text{(III.3)}\qquad \bar{p}\cdot\xi(\tilde{p},\tilde{m}) < \bar{p}\cdot\xi(\bar{p},\bar{m}) \Rightarrow \tilde{p}\cdot\xi(\bar{p},\bar{m}) > \tilde{p}\cdot\xi(\tilde{p},\tilde{m}).$$

Clearly, since the current hypothesis is stronger than that of the WARDP, the current axiom is a weaker axiom. Cf. Figures 7 and 8a.

Figure 8

c) <u>The Weak Weak Weak Axiom of Revealed Demand Preference (W^3ARDP)</u>.

The only difference between (III.1) and (III.3) is that the former has a weak inequality in its hypothesis. It is natural to ask, therefore, what happens if (III.3) is modified so that its conclusion, rather than its hypothesis, has a weak inequality. Thus in the context of the linear space definition of T (Section A.1), it is natural to consider the axiom, W^3ARDP, that asserts the asymmetry of T: $xTy \Rightarrow$ not yTx. Again, for the special case of a demand ξ as in (II.5) this becomes:

$$\text{(III.4)} \qquad \bar{p}\cdot\xi(\tilde{p},\tilde{m}) < \bar{p}\cdot\xi(\bar{p},\bar{m}) \Rightarrow \tilde{p}\cdot\xi(\bar{p},\bar{m}) \geqq \tilde{p}\cdot\xi(\tilde{p},\tilde{m}).$$

Since the current conclusion is weaker than that of the W^2ARDP, the current axiom is clearly a weaker axiom. Cf. Figure 9.

Figure 9

d) The V-Axiom.

This axiom asserts that, if $B \in \mathcal{B}$ & xAB & $\forall y_{yAB} xVy$, then $x \in h(B)$. It was introduced in [34].

When we consider the transitive closures of S, T, and V, we are led to the following "strong" versions of these axioms.

e) The Strong Axiom of Revealed Demand Preference (SARDP).

This axiom asserts that H is asymmetric: $xHy \Rightarrow$ not yHx. Stated in terms of the acyclicity of $\bar{S}$, this formed the basis of Houthakker's approach [21].

f) The Weak Weak Strong Axiom of Revealed Demand Preference (W^2SARDP).

This axiom asserts, in the context of the linear space definitions of $\bar{S}$ and T (Section A.1) that Q is asymmetric: $xQy \Rightarrow$ not yQx.

g) The W-Axiom.

This axiom asserts that, if $B \in \mathcal{B}$ & $\forall y_{yAB} xWy$, then $x \in h(B)$. It was introduced in [34]. If $h(B) \neq \emptyset$ for all $B \in \mathcal{B}$, it is equivalent to the next axiom.

h) The Congruence Axiom of Revealed Demand Preference (CARDP).

This axiom asserts that, if $B \in \mathcal{B}$ & $x \in h(B)$ & yAB & yWx, then $y \in h(B)$. This axiom, which was introduced in [33], is equivalent to the W-axiom when $h(B) \neq \emptyset$ for all $B \in \mathcal{B}$, i.e., when $\Delta = \mathcal{B}$. The SARDP implies the CARDP, and when the demand is single valued they are equivalent.

Remark. Clearly several other interesting axioms could be formulated. For example, in a classical choice space we may wish to prohibit revelations like $xTu_1 S \ldots Su_k Sy$ & yTx. For simplicity we restrict our attention to the listed axioms.

2. Revealed budgeter preference axioms.

Each of the demand axioms in Section (C.1) is dual to a budgeter axiom, expressible in terms of the notions s, $\bar{s}$, h, $\bar{h}$, t, q, v, and w.

a) The Weak Axiom of Revealed Budgeter Preference (WARBP).

This asserts the asymmetry of s: $\bar{B}s\tilde{B} \Rightarrow$ not $\tilde{B}s\bar{B}$. See Figures 10 and 11a for violations.

b) The Weak Weak Axiom of Revealed Budgeter Preference (W^2ARBP).

This asserts that $\bar{B}t\tilde{B} \Rightarrow$ not $\tilde{B}s\bar{B}$. See Figure 11.

c) The Weak Weak Weak Axiom of Revealed Budger Preference (W^3ARBP).

This asserts the asymmetry of t: $\bar{B}t\tilde{B} \Rightarrow$ not $\tilde{B}t\bar{B}$. Cf. Figure 12.

d) The v-Axiom.

This asserts that, if $x \in X$ & $xA\bar{B}$ & $\forall \tilde{B}_{xA\tilde{B}} \bar{B}v\tilde{B}$, then $\bar{B} \in b(x)$.

e) The Strong Axiom of Revealed Budgeter Preference (SARBP).

This asserts the asymmetry of h: $\bar{B}h\tilde{B} \Rightarrow$ not $\tilde{B}h\bar{B}$.

f) The Weak Weak Strong Axiom of Revealed Budgeter Preference W^2SARBP).

This asserts that q is asymmetric: $\bar{B}q\tilde{B} \Rightarrow$ not $\tilde{B}q\bar{B}$.

g) The w-Axiom.

This asserts that, if $x \in X$ & $xA\bar{B}$ & $\forall \tilde{B}_{xA\tilde{B}} \bar{B}w\tilde{B}$, then $\bar{B} \in b(x)$.

h) The Congruence Axiom of Revealed Budgeter Preference (CARBP).

This asserts that, if $x \in X$ & $b = b(x)$ & $xA\tilde{B}$ & $\tilde{B}w\bar{B}$, then $\tilde{B} \in b(x)$. If $b(x) \neq \emptyset$ for all $x \in X$, i.e., if $\Gamma = X$, then CARBP is equivalent to the w-Axiom.

3. Relations between demand and budgeter axioms.

It is clear from Figure 13a that the SARDP does not even imply, on the dual side, the WARBP.

Figure 13

And dually, it is clear from Figure 13b that the SARBP does not imply the WARDP. So it is reassuring that we can prove the equivalence of the more general congruence axioms.

Theorem 11. The CARDP and the CARBP are equivalent.

Proof. By Theorem 2(a) and Theorem 13 in (D) below.

At a less general level, the following duality equivalences are useful.

Theorem 12. Let $(X, \mathcal{B}, A, \mathbb{C})$ be a classical choice space, and suppose that the budget identity holds:

$$\text{(III.5)} \qquad \forall x_{x \in X} \forall (p,m)_{(p,m) \in \mathcal{B}} [(x,(p,m)) \in \mathbb{C} \rightarrow p_1 x_1 + \ldots + p_n x_n = m],$$

where $m = 1$, $\sum_{i=1}^{n} P_i = 1$, or $P_n = 1$ according as whether the choice space is of type (i), (ii), or (iii). Then:

a) the W^3ARBP and the W^3ARDP are equivalent;

b) the W^2ARBP and the W^2ARDP are equivalent;

c) the W^2SARBP and the W^2SARDP are equivalent.

Proof. Because the proofs are straightforward, we just show that a violation of W^3ARBP implies a violation of W^3ARDP. Say $(\bar{p},\bar{m}) t (\tilde{p},\tilde{m})$ & $(\tilde{p},\tilde{m}) t (\bar{p},\bar{m})$; thus $(\bar{p},\bar{m}) \neq (\tilde{p},\tilde{m})$ and the first revelation implies

$$\exists \bar{x}_{\bar{x} \in X} (p,m) \in b(\bar{x}) \ \& \ \tilde{p} \cdot \bar{x} < \tilde{m}$$

while the second implies

$$\exists \tilde{x}_{\tilde{x} \in X} (\tilde{p},\tilde{m}) \in b(\tilde{x}) \ \& \ \bar{p} \cdot \tilde{x} < \bar{m}.$$

Because of the budget identity, the first condition implies $\tilde{x} T \bar{x}$, while the second implies $\bar{x} T \tilde{x}$, violating the W^3ARDP. Q.E.D.

D. Revealed preference rationality theorems.

We list without proof some of the main rationality theorems that have been and can be proved with revealed preference axioms.

Theorem 13. Let $(X,\mathcal{B},A,\mathbb{C})$ be a choice space. Then:

a.i) $\mathbb{C}$ has a demand preference if and only if the V-Axiom holds.

a.ii) $\mathbb{C}$ has a budgeter preference if and only if the v-Axiom holds.

b.i) $\mathbb{C}$ has a regular partial demand preference if and only if the CARDP holds.

b.ii) $\mathbb{C}$ has a regular partial budgeter preference if and only if the CARBP holds.

c.i) If $\mathbb{C}$ has a single valued demand, then $\mathbb{C}$ has a regular partial demand preference if and only is the SARDP holds.

c.ii) If $\mathbb{C}$ has a single valued budgeter, then $\mathbb{C}$ has a regular partial budgeter preference if and only if the SARBP holds.

Proof. (a.i) is proved in [34] (B,2); and (a.ii) is dual. (b.i) is essentially what the proof of [33] (Theorem 1) shows; and (b.ii) is dual. (c.i) follows from (b.i) and the equivalence (C.1.h) of the CARDP and the SARDP when demand is single valued.

Theorem 14. Let $(X,\mathcal{B},A,\mathbb{C})$ be a classical choice space. Then:

a) If $\Gamma^{\mathbb{C}}$ is convex, then $\mathbb{C}$ has a partial utility that is upper semi-continuous on $\Gamma^{\mathbb{C}}$ and strictly quasi-concave on $\Gamma^{\mathbb{C}}$, if and only if SARDP holds.

b) If $\Delta^{\mathbb{C}}$ is convex, then $\mathbb{C}$ has a partial worth that is lower semi-continuous on $\Delta^{\mathbb{C}}$ and strictly quasi-convex on $\Delta^{\mathbb{C}}$, if and only if SARBP holds.

Proof. A slightly stronger version of (a) is proved in [23] (Theorem 1); (b) is dual.

IV. Integrability and Rationality.

For motivation and the history of the integrability approach to the rationality problem, we refer to Hurwicz's survey [22]. Note that Hurwicz clearly indicates a duality between the "direct" and "indirect" approaches (cf. [22], p. 205, footnote 62).

A. Standing hypotheses for Part IV.

Throughout Part IV we work in the context of classical consumer theory. For definiteness we work in a classical choice space of type (iii), although our results could easily be reformulated for the other types. In particular, we will work with the budget identity:

$$\text{(IV.1)} \qquad \forall x_{x \in X} \forall (p,m)_{(p,m) \in \mathcal{B}} [(x,(p,m)) \in \mathcal{C} \Rightarrow p_1 x_1 + \ldots + p_{n-1} x_{n-1} + x_n = m] .$$

Because the main approach (with respect to definitions, theorems, and methods of proof) described here has been through classical differential equations, we shall always assume in this section that the choice $\mathcal{C}$ has either a single valued demand ξ as in (II.5), or that $\mathcal{C}$ has a single valued budgeter β as in (II.8).[1]

B. Integrability axioms.

1. Budgeter axioms.

Following Antonelli ([2], [3]), we assume given a differentiable single valued budgeter $\beta = (\beta^1,\ldots,\beta^n)$. We may picture the situation as in Figure 14, where to each $x \epsilon R^2_>$ a unique slope, $-\beta^1(x) = -p_1$, is determined, and a unique expenditure, $\beta^2(x) = m$, is determined by $\beta(x) = (\beta^1(x), \beta^2(x))$.

Figure 14

With a vector field thus defined on $R^2_>$, it is natural to seek integral curves -- i.e., solutions to the ordinary differential equation

$$\text{(IV.2)} \qquad \frac{dx_2}{dx_1} = -\beta^1(x_1, x_2)$$

-- and use them as level curves of a function that might serve as a utility for $\mathbf{C}$. (This is in the spirit of Antonelli [2], Hicks and Allen [20], Samuelson [38], and Debreu [9].) Of course, when $n > 2$, (IV.2) is replaced by system of partial differential equations. General conditions under which this program can be carried out for $n \geqq 2$ will be given in terms of a matrix whose components we now define.

For each $i,j = 1,\ldots,n-1$, and for each $x \in X$, write

$$\text{(IV.3)} \qquad A^{ij}(x) = \frac{\partial\beta^i}{\partial x_j}(x) - \beta^j(x)\frac{\partial\beta^i}{\partial x_n},$$

or, writing $\beta^i_j(x) = \dfrac{\partial\beta^i(x)}{\partial x_j}$,

$$\text{(IV.4)} \qquad A^{ij}(x) = \beta^i_j(x) - \beta^j(x)\beta^i_n(x).$$

And then, for each $x \in X$, we define the matrix

$$\text{(IV.5)} \qquad A(x) = \begin{bmatrix} A^{11}(x) & \cdots & A^{1,n-1}(x) \\ \vdots & & \vdots \\ A^{n-1,1}(x) & \cdots & A^{n-1,n-1}(x) \end{bmatrix}.$$

The Negative Semi-Definiteness Budgeter Axiom (NSBDA) asserts that, for each $x \in X$, the matrix $A(x)$ is negative semi-definite.[1]

The Symmetry Budgeter Axiom (SBA) asserts that, for each $x \in X$, the matrix $A(x)$ is symmetric.[2]

For the present, these axioms appear as technical, economically unmotivated conditions on β. Nevertheless, we will use the duality theory of Part II and the axioms of Parts III and V to motivate these axioms.

2. Demand axioms.

In the spirit of Slutsky ([42], [43]), we start with a differentiable $\xi = (\xi^1, \ldots, \xi^n)$. We may picture the situation as in Figure 15, where to each $(p_1, m) \in R^2_>$ a unique slope, $\xi^1(p_1, m) = x_1$, is determined,

Figure 15

and a unique commodity quantity, $\xi^2(p_1, m) = x_2$, is determined by $\xi(p_1, m) = (\xi^1(p_1, m), \xi^2(p_1, m))$.

With a vector field thus defined on $R^2_>$, it is natural to seek integral curves -- i.e., solutions to the ordinary differential equation

$$\text{(IV.6)} \qquad \frac{dm}{dp_1} = \xi^1(p_1, m)$$

-- and use them as level curves of a function that might serve as a worth for $\mathcal{C}$. Of course, when $n > 2$, (IV.6) is replaced by a system of partial differential equations (cf. (IV.9) below). General conditions under which this program can be carried out for $n \geqq 2$ will be given in terms of a matrix whose components we now define.

For each $i,j = 1,\dots,n-1$, and for each $x \in X$, let

$$\text{(IV.7)} \qquad S^{ij}(p,m) = \frac{\partial \xi^i}{\partial p_j}(p,m) + \xi^j(p,m)\frac{\partial \xi^i}{\partial m}(p,m);$$

or, writing $\xi^i_j(p_1,\dots,p_{n-1},m) = \dfrac{\partial \xi^i(p_1,\dots p_{n-1},m)}{\partial p_j}$ for $j = 1,\dots,n=1$,

and $\xi^i_n(p_1,\dots,p_{n-1},m) = \dfrac{\partial \xi^i(p_1,\dots,p_{n-1},m)}{\partial m}$,

$$\text{(IV.8)} \qquad S^{ij}(p\ ,\dots,p_{n-1},m) = \xi^i_j(p_1,\dots,p_{n-1},m) + \xi^j(p_1,\dots,p_{n-1},m)\,\xi^j_n(p_1,\dots,p_{n-1},m).$$

And then, for each $(p,m) \in \mathcal{B}$, we define the matrix

$$S(p,m) = \begin{bmatrix} S^{11}(p,m) & \dots & S^{1,n-1}(p,m) \\ \vdots & & \vdots \\ S^{n-1,1}(p,m) & \dots & S^{n-1,n-1}(p,m) \end{bmatrix} .$$

The <u>Negative Semi-Definiteness Demand Axiom (NSDDA)</u> asserts that, for each $(p,m) \in \mathcal{B}$, the matrix $S(p,m)$ is negative semi-definite.

The <u>Symmetry Demand Axiom (SDA)</u> asserts that, for each $(p,m) \in \mathcal{B}$, the matrix $S(p,m)$ is symmetric.

Again, it is difficult to see an economic or behavioral motivation for these axioms, although they are natural in the mathematical context of this approach. The economic motivation will, however, emerge from the duality theory of Part II and the axioms of Parts III and V. We shall also present simplified proofs that together they are necessary and sufficient conditions for the existence of a worth for $\mathcal{C}$, and thereby a utility if $\Delta = \mathcal{B}$.

C. Necessary conditions for regular-rationality.

Our first theorem combines our Theorem 2 with Theorem 1 of Hurwicz and Uzawa [26], extending the work of Slutsky [42],

Theorem 15. Let $(X, \mathcal{B}, A, \mathbb{C})$ be a classical choice space of type (iii), where $\mathcal{B} = R^n_>$. Let the single valued demand ξ be differentiable and satisfy the budget equality (IV.1). If ξ has a regular demand preference, or if $\mathbb{C}$ has a regular partial budgeter preference, then the NSDDA and the SDA hold.

Proof. The only significant difference between Theorem 15 and Theorem 1 of [26] is the insertion in Theorem 15 of the alternate hypothesis of a regular partial budgeter preference. But, by our Theorem 2(a.ii), if $\mathbb{C}$ has a regular partial budgeter preference, then ξ has a regular partial demand preference which (under our assumption that $\xi: \mathcal{B} \to X$, so $\Delta = \mathcal{B}$) implies that ξ has a regular demand preference. Thus Theorem 15 follows immediately from Theorem 1 of [26].

To some extent, Theorem 15 provides motivation for the pair of axioms, the NSDDA and the SDA: they are essentially implied by regular-rationality of demand. Dually we obtain the following joint motivation for the budgeter axioms, the NSDBA and the SBA.

Theorem 16. Let $(X, \mathcal{B}, A, \mathbb{C})$ be a classical choice space of type (iii), where $X = R^n_>$. Let the single valued budgeter β be differentiable and satisfy the budget equality (IV.1). If β has a regular budgeter preference, or if $\mathbb{C}$ has a regular partial demand preference, then the NSDBA and the SBA hold.

It would also be nice to motivate each of the axioms -- the negativity and symmetry -- individually. This has already been done by Kihlstrom, Mas-Colell, and Sonnenschein [29] for the NSDDA (and, by duality, for NSDBA), as we show in Part V. For the symmetry axioms it will also be done in Part V, following Ville and applying duality.

D. Sufficient conditions for utility and worth.

We present some of the main rationality results provable from integrability axioms. For history, further motivation, and many details, the reader should consult [22].

Our first theorem generalizes Hurwicz and Uzawa's Theorem 2 of [26]. We sketch a proof to show how the duality theorems of Part II apply.

Theorem 17. Let $(X, \mathcal{B}, A, \mathcal{C})$ be a classical choice space of type (iii), where $\mathcal{B} = R^n_>$, and let the single valued demand ξ satisfy the budget identity (IV.1). Suppose that ξ also satisfied one of the following smoothness conditions (a) or (b):

a.i) ξ is differentiable, and

a.ii) for each $(\bar{p}, \bar{m}) \in \mathcal{B}$, there exists a neighborhood N of $(\bar{p}, \bar{m})$ and a positive real K_N such that, for all $(p,m) \in N$ and all $i = 1,\ldots,n-1$:

$$|\xi^i_n(p,m)| \leqq K_N;$$

or

b) ξ is C^k, for some $k \geqq 1$.

If the NSDDA and the SDA hold, then:

i) in case (a) there exists a quasi-convex continuous partial worth, and a utility that is quasi-concave relative to Γ and upper semi-continuous relative to Γ;

ii) in case (b) there exists a quasi-convex C^k partial worth, and a utility that is quasi-concave relative to Γ and upper semi-continuous relative to Γ.

Proof sketch.

In case (b) Frobenius' Theorem (cf. [13], p. 310 (10.9.5)) - - and in case (a) Tsuji's extension ([44]; cf. [26], p. 142)) - - gives us continuous and unique local solutions of the following system of partial differential equations:

$$\text{(IV.9)} \qquad \begin{aligned} \frac{\partial m}{\partial p_1} &= \xi^1(p,m) \\ &\vdots \\ \frac{\partial m}{\partial p_{n-1}} &= \xi^{n-1}(p,m). \end{aligned}$$

We thus obtain, for each $(\bar{p},\bar{m})\in\mathcal{B}$, a neighborhood $N(\bar{p},\bar{m})$ of $(\bar{p},\bar{p},\bar{m})$ and a unique continuous real valued function $\mu(\cdot,\bar{p},\bar{m})$ (which is C^k if ξ is C^k) such that, for any $(\tilde{p},\tilde{p},\tilde{m}),(p,\tilde{p},\tilde{m}) \in N(\bar{p},\bar{m})$:

$$\text{(IV.10)} \qquad \begin{aligned} &\mu(\tilde{p},\tilde{p},\tilde{m}) = \tilde{m} \\ &\mu_i(p,\tilde{p},\tilde{m}) = \xi^i(p,\mu(p,\tilde{p},\tilde{m})) \text{ for each } i = 1,\dots,n-1. \end{aligned}$$

Now pick any $p^*\in R^{n-1}_{>}$. Following [26] we can define a function W (which we will verify as a worth) by:

$$\text{(IV.11)} \qquad \forall(p,m)_{(p,m)\in\mathcal{B}} W(p,m) = \mu(p^*,p,m).$$

To justify this definition, we must show that μ, whose existence we have only claimed locally, is actually defined at (p^*,p,m) for all $(p,m)\in\mathcal{B}$. This would follow immediately from the Hurwicz-Uzawa extension ([26], Appendix, Existence Theorem III) of the classical literature in case (b). However, case (a) gives weaker conditions than stated in that extension, and so we note that an argument essentially dual to that used by Debreu in [9] (pp. 608-610; cf. [10]) will show (even without[1] the C^1 character of Debreu's context) that μ is defined globally.

Having defined W by (IV.11), we will show that W is a partial worth, i.e., that (II.25) holds. Toward this end, we first note that W is quasi-convex.[1]

We next verify (II.25) in two steps. First, let $(\bar{p},\bar{m}) \in b(x)$; we will show that

$$(\bar{p},\bar{m}) \text{ minimizes } W \text{ over } A_{\mathcal{B}}(x) = \{(p,m) \in \mathcal{B}\colon p_1 x_1 + \ldots + p_{n-1} x_{n-1} + x_n - m \leqq 0\}.$$

Note that (by (IV.11) and (IV.10)) the level hypersurface of W through $(\bar{p},\bar{m})$ has tangent hyperplane with normal r given by

$$\text{(IV.12)} \qquad r = (\xi^1(\bar{p},\bar{m}), \ldots, \xi^{n-1}(\bar{p},\bar{m}), -1).$$

And r is also clearly normal to the boundary of the constraint set

$$K = A_{\mathcal{B}}(\xi(\bar{p},\bar{m})) = \{(p,m) \in \mathcal{B}\colon p_1 \xi^1(\bar{p},\bar{m}) + \ldots + p_{n-1} \xi^{n-1}(\bar{p},\bar{m}) + \xi^n(\bar{p},\bar{m}) - m \leqq 0\}.$$

Thus the level hypersurface is tangent at $(\bar{p},\bar{m})$ to the boundary of K, and so it is easily seen that quasi-convex W reaches its minimum on convex K at $(\bar{p},\bar{m})$.[2]

Conversely, let $x \in \Gamma$ -- say $x = \xi(\bar{p},\bar{m})$ -- and suppose $(\tilde{p},\tilde{m})$ minimizes W over K; we will show that $x = \xi(\tilde{p},\tilde{m})$. Since $(\tilde{p},\tilde{m})$ minimizes W over K,

$$\text{(IV.13)} \qquad (\tilde{p},1) \cdot \xi(\bar{p},\bar{m}) = \tilde{m},$$

and the level hypersurface of W through $(\tilde{p},\tilde{m})$ is tangent at $(\tilde{p},\tilde{m})$ to to the boundary of K (this is really a theorem of the Kuhn-Tucker necessity type). Since a normal at $(\tilde{p},\tilde{m})$ of the latter is given by (IV.12), while a normal at $(\tilde{p},\tilde{m})$ of the hypersurface is similarly given by $(\xi^1(\tilde{p},\tilde{m}), \ldots, \xi^{n-1}(\tilde{p},\tilde{m}), -1)$, we see that

(IV.14) $\xi^i(\tilde{p},\tilde{m}) = \xi^i(\bar{p},\bar{m}) = x_i$ for $i = 1,\ldots,n$.

It only remains to show $\xi^n(\tilde{p},\tilde{m}) = \xi^n(\bar{p},\bar{m})$; but that is immediate from (IV.1), (IV.13), and (IV.14).

The proof of Theorem 17 is now completed by passing from the continuous partial worth W to a quasi-concave utility that is upper semi-continuous relative to Γ, by Theorem 10(a).

As usual, we have a dual version.[1]

Theorem 18. Let $(X,\mathcal{B},A,\mathbb{C})$ be a classical choice space of type (iii), where $X = R^n_>$, and let the single-valued budgeter β satisfy the budget identity (IV.1). Suppose that β also satisfies one of the following smoothness conditions (a) or (b):

a.i) β is differentiable, and

a.ii) for each $\bar{x} \in X$ there exists a neighborhood N of $\bar{x}$ and a positive real K_N such that, for all $x \in N$ and all $i = 1,\ldots,n-1$:

$$|\beta^i_n(x)| \leqq K_N;$$

or

b) β is C^k, for some $k \geqq 1$.

If the NSDBA and the SBA hold, then:

i) in case (a) there exists a quasi-concave continuous partial utility, and a worth that is quasi-convex relative to Δ and lower semi-continuous relative to Δ;

ii) in case (b) there exists a quasi-concave C^k partial utility, and a worth that is quasi-convex relative to Δ and lower semi-continuous relative to Δ.

V. Connections Between the Revealed Preference and Integrability Approaches.

Revealed preference axioms for rationality are often considered to have more intuitive appeal than the negativity and symmetry axioms of the integrability approach to rationality. In this Part we display relationships between the two types of axioms. First in Section A we note with Samuelson ([39], pp. 111-113; [35]; [36]) that the negativity axioms follow from the Weak Axioms of Revealed Preference; indeed, with Kihlstrom, Mas-Colell, and Sonnenschein [29], we observe that the negativity axioms are equivalent to the Weak Weak Axioms of Revealed Preference. Then in Section B we note, following Ville ([46], [47]), that the symmetry axioms are equivalent to differential versions of revealed preference axioms.

A. The Weak Axioms and Negative Semi-Definiteness.

Theorem 19. In a classical choice space of type (iii), suppose $\mathcal{B} = R^n_>$, and let the simgle-valued demand ξ be differentiable and satisfy the budget identity (IV.1). Then ξ satisfies NSDDA if and only if ξ satisfies W^2ARDP (equivalently W^2ARBP).

Proof. By Kihlstrom, Mas-Colell, and Sonnenschein [29], Theorems 1 and 2.[1] (The equivalence of W^2ARBP and W^2ARDP is from our Theorem 12(b).)

By the Duality Metatheorem we have a dual:

Theorem 20. In a classical choice space of type (iii), suppose $X = R^n_>$, and let the single-valued budgeter β be differentiable and satisfy the budget identity (IV.9). Then β satisfies NSDBA if and only if β satisfies W^2ARBP (equivalently W^2ARDP).

Thus the Weak Weak Axiom of Revealed Demand Preference is equivalent to negative semi-definiteness of the (normalized) Slutsky or Antonelli matrix, when the demand or budgeter is differentiable, respectively.

B. Differential Revealed Preference and Symmetry.

1. Paths.

For the rest of this paper we work with an n-dimensional classical choice space $(X, \mathcal{B}, A, \mathbb{C})$ of type (iii), and we consider "time paths" of commodities and budgets. By a commodity path we mean a function $x(\cdot) = (x^1(\cdot),\ldots,x^n(\cdot))\colon [0,r] \to X$, for some real $r > 0$; by a budget path we mean a function $c(\cdot) = (c^1(\cdot),\ldots,c^n(\cdot))\colon [0,r] \to \mathcal{B}$, for some real $r > 0$. By a path we mean a function $(x(\cdot),c(\cdot))\colon [0,r] \to X \times \mathcal{B}$, for some real $[0,r]$.

These path notions acquire interest when they agree with choice. By a choice path we mean a path $(x(\cdot),c(\cdot)\colon [0,r] \to X \times \mathcal{B}$ such that

(V.1) $$\forall\tau_{\tau\in[0,r]}\ (x(\tau),c(\tau))\in\mathbb{C}.$$

Thus $c^i(\tau)$ represents the price $P^i(\tau)$ of $x^i(\tau)$ for $i = 1,\ldots,n-1$, and $c^n(\tau)$ represents income $m(\tau)$.

In view of the intended interpretation (cf. (II.40(iii)) of classical choice spaces of type (iii), we say that such a choice path $(x(\cdot),c(\cdot))$ satisfies the budget identity if:

(V.2) $$\forall\tau_{\tau\in[0,r]}\, c^1(\tau)x^1(\tau) +\ldots+ c^{n-1}(\tau)x^{n-1}(\tau) + x^n(\tau) = c^n(\tau).$$

2. <u>Commodity cycles</u>.

Suppose $(x(\cdot), c(\cdot)) \colon [0,r] \to X \times \mathcal{B}$ is a choice path satisfying the budget identity, and suppose, for two times $\tau_1 < \tau_2$ we observe that

$$\text{(V.3)} \qquad c^1(\tau_2)x^1(\tau_2) + \ldots + c^{n-1}(\tau_2)x^{n-1}(\tau_2) + x^n(\tau_2) > c^1(\tau_2)x^1(\tau_1) + \ldots + c^{n-1}(\tau_2)x^{n-1}(\tau_1) + x^n(\tau_1),$$

or, writing $P(\cdot) = (c^1(\cdot), \ldots, c^{n-1}(\cdot), 1)$,

$$\text{(V.4)} \qquad P(\tau_2) \cdot x(\tau_2) > P(\tau_2) \cdot x(\tau_1).$$

Then in view of the budget identity, $x(\tau_2)$ is directly strictly revealed preferred to $x(\tau_1)$ i.e., $x(\tau_2)Tx(\tau_2)$ (cf. Part III.A.1); and $c(\tau_1)$ is directly strictly revealed worse than $c(\tau_1)$, i.e., $c(\tau_1)tc(\tau_2)$ (cf. Part III.B.1). And thus according to even the Weak Weak Strong Axiom of Revealed Demand Preference (the W^2SARDP; cf. Part III.C.1(f)), we should never find points in time $\tau_1' < \tau_2' < \ldots \tau_k'$ such that $x(\tau_1)Tx(\tau_1')T\ldots Tx(\tau_k')Tx(\tau_2)$, i.e., that yield $x(\tau_1)Qx(\tau_2)$, where Q is the transitive closure of T (cf. Part III.A.2). And dually, according to even the Weak Weak Strong Axiom of Revealed Budgeter Preference (the W^2SARBP; cf. Part III.C.2(f)), we should never find points in time that yield $c(\tau_2)qc(\tau_1)$, where q is the transitive closure of t (cf. Parts III.B.2).

We can rewrite (V.4) as

$$\text{(V.5)} \qquad P(\tau_2) \cdot (x(\tau_2) - x(\tau_1)) > 0,$$

or as

$$\text{(V.6)} \qquad P(\tau_2) \cdot \Delta x > 0,$$

which suggests, as a natural differential analogue,

$$\text{(V.7)} \qquad P(\tau)\cdot\dot{x}(\tau) > 0$$

if $x(\cdot)$ is differentiable. Then according to the preceding paragraph, a natural differential analogue of the W^2SARDP and of the W^2SARBP would be the condition that there exist no choice path satisfying the budget identity, making a "cycle" starting and ending at the same $x(0) = x(r) \in X$, and satisfying (V.7) throughout. This would say that we cannot continually be moving toward revealed preferred commodity bundles (dually, budgets), eventually returning to the starting bundle (budget).

Another way to motivate such a condition is to note that, if $\xi: \mathcal{B} \to R^n_>$ has a C^1 utility U for $\mathcal{C}$, then, writing U_i for the i-th partial derivative of U,

$$\text{(V.8)} \qquad \frac{dU(x(\tau))}{d\tau} = \sum_{i=1}^{n} U_i(x(\tau))\dot{x}^i(\tau) = \lambda(\tau)P(\tau)\cdot\dot{x}(\tau)$$

where $\lambda = U_n$ and we have assumed $x(\cdot)$ is differentiable. Assuming $U_n > 0$, then (V.8) implies that $\frac{dU}{dt}(x(\tau)) > 0$ whenever (V.7) holds; so it is natural to follow Allen [1] and Georgescu-Roegen [17] and call $\dot{x}(\tau)$ a <u>direction of increasing revealed demand preference at $x(\tau)$</u> when (V.7) holds. So the condition can now be stated as prohibiting cycles that move always in a direction of increasing revealed preference, yet return to their starting point.

3. <u>Budget cycles</u>.

A motivation dual to that of Subsection 2 can be given for ruling out cycles dual to those of Subsection 2. Thus we now discuss paths on which $\sum_{i=1}^{n-1} x^i(\tau)\dot{P}^i(\tau) - \dot{m}(\tau) < 0$, instead of $\sum_{i=1}^{n} P^i(\tau)\dot{x}^i(\tau) > 0$.

Actually, because of the budget identity (V.2),

$$\sum_{i=1}^{n} P^i(\tau)\dot{x}^i(\tau) + \sum_{i=1}^{n-1} x(\tau)\dot{P}^i(\tau) - \dot{m}(\tau) = 0,$$

for all τ, so a cycle on which one of these inequalities holds is also a cycle on which the other inequality holds; thus the motivation of Subsection 1 would also be sufficient to rule out the "budget cycles" of this Subsection. Nevertheless, it is important for our duality principle that a dual motivation can be given as follows.

Suppose $(x(\cdot),c(\cdot))\colon [0,r] \to X \times \mathcal{B}$ is a choice path satisfying the budget identity, and suppose that, for two times $\tau_1 < \tau_2$ we observe that

(V.9) $$c^1(\tau_2)x^1(\tau_1) +\ldots+ c^{n-1}(\tau_2)x^{n-1}(\tau_1) + x^n(\tau_1) > c^n(\tau_2),$$

or, writing $m(\cdot) = c^n(\cdot)$,

(V.10) $$P(\tau_2)\cdot x(\tau_1) < m(\tau_2).$$

Then, in view of the budget identity, $(P(\tau_1),m(\tau_1))$ is directly strictly revealed worse than $(P(\tau_2),m(\tau_2))$, i.e., $(P(\tau_1),m(\tau_1))t(P(\tau_2),m(\tau_2))$ (cf. Part III.B.1); and $x(\tau_2)$ is directly strictly revealed preferred to $x(\tau_1)$, i.e., $x(\tau_2)Tx(\tau_1)$ (cf. Part III.A.1). And thus according to even the Weak Weak Strong Axiom of Revealed Budgeter Preference (the W^2SARBP, cf. Part II.C.2(f)), we should not have $(P(\tau_2),m(\tau_2))q(P(\tau_1),m(\tau_1))$, where q is the transitive closure of t (cf. Part III.B.2). And dually, according to even the Weak Weak Strong Axiom of Revealed Demand Preference (the W^2SARDP; cf. Part III.C.2(f)), we should not have $x(\tau_1)Qx(\tau_2)$, where Q is the transitive closure of T (cf. Part III.A.2).

In view of the budget identity, we can rewrite (V.10) as:

(V.11) $$\sum_{i=1}^{n-1} x^i(\tau_1)(P^i(\tau_2)-P^i(\tau_1)) - (m(\tau_2)-m(\tau_1)) < 0,$$

or as:

(V.12) $$x(\tau_1)\cdot\Delta P - \Delta m < 0,$$

which suggests, as a natural differential analogue,

$$\text{(V.13)} \qquad x(\tau)\cdot\dot{P}(\tau) - \dot{m}(\tau) < 0$$

if $P(\cdot)$ is differentiable. Then according to the preceding paragraph, a natural differential analogue of the W^2SARBP and of the W^2SARDP would be the condition that there exist no choice path satisfying the budget identity, making a "cycle" starting and ending at the same $(P(0),m(0)) = (P(r),m(r))$, and satisfying (V.13) throughout. This would say that we cannot continually be moving toward revealed worse budgets (dually, commodity bundles), eventually returning to the starting budget (bundle).

Dual to Subsection 1, we could also call $(\dot{P}(\tau),\dot{m}(\tau))$ a <u>direction of decreasing (budget) preference at $(P(\tau),m(\tau))$</u> if (V.13) held; and we could motivate it by a dual argument, assuming a worth existed. And then we would require that there be no cycles moving always in a direction of decreasing (budget) preference, yet returning to their starting point.

4. <u>Ville Cycles and Axioms</u>.

Let $(x(\cdot),c(\cdot))$: $[0,r] \to X \times \mathcal{B}$ be a C^k path for some $k \geq 1$ and some real $r > 0$. Then the path is called a <u>C^k positive Ville commodity cycle</u> if $x(0) = x(r)$ & (V.7) holds for all $\tau\in[0,r]$. And the path is called a C^k <u>positive Ville budget cycle</u> if

$$\text{(V.14)} \qquad \begin{aligned} & c(0) = c(r) \text{ and} \\ & \forall\tau_{\tau\in[0,r]}\, x^1(\tau)\dot{c}^1(\tau) + \ldots + x^{n-1}\dot{c}^{n-1}(\tau) - \dot{c}^n(\tau) > 0, \end{aligned}$$

or, with $P(\cdot)$ defined as before and $m(\cdot)$ defined by $m(\tau) = c^n(\tau)$:

$$\text{(V.15)} \qquad \forall\tau_{\tau\in[0,r]}\, x(\tau)\cdot\dot{P}(\tau) > \dot{m}(\tau).$$

Of course <u>negative</u> Ville cycles are defined by reversing the inequalities in (V.7) or (V.15). Note that, by traversing a positive or negative Ville cycle in the opposite direction, we change its sign; for this reason we do not need to distinguish between the positive and negative ones, and are justified in referring to both as Ville cycles.

If ξ is a single valued demand, then a budget path $c(\cdot)$ determines a choice path $(\xi(c(\cdot)),c(\cdot))$; indeed, every choice path -- hence every Ville cycle -- is of the form $(\xi(c(\cdot)),c(\cdot))$ for some budget path $c(\cdot)$. Dually, if β is a single valued budgeter, then the choice paths are those paths of the form $(x(\cdot),\beta(x(\cdot)))$ for some commodity path $x(\cdot)$.

If E is a subset of X, then we say that the C^k Differential Demand Axiom of Revealed Preference (C^kDDARP) holds on E if E contains no C^k Ville commodity cycle. If E is a subset of $\mathcal{B}$, then we say that the C^k Differential Budget Axiom of Revealed Preference (C^kDBARP) holds on E if E contains no C^k Ville budget cycles. Note that if the budget identity holds, then C^kDDARP and C^kDBARP are equivalent, so we may then refer simply to the C^k Differential Axiom of Revealed Preference (C^kDARP).

5. Revealed preference and differential revealed preference.

By taking discrete approximations to (V.7) or (V.13) over a compact time interval, it is easy to use the arguments (B.2 & 3) above and show that W^2SARDP and W^2SARBP imply C^kDARP.

Theorem 21. Let $(X,\mathcal{B},A,\mathbb{C})$ be an n-dimensional classical choice space of type (iii), satisfying the budget identity (IV.1). If $\mathbb{C}$ satisfies the equivalent[1] conditions, the W^2SARBP and the W^2SARDP, then $\mathbb{C}$ satisfies the C^kDARP.

Remark. Very simple examples show that the converse of Theorem 21 is false. Indeed, the C^kDARP does not even imply the W^3ARDP (equivalently, the W^3ARBP)(cf. [25], Section V).

We will now show that the C^kDARP characterizes the classical Slutsky and Antonelli symmetry conditions.

6. The Ville connection: differential revealed preference and symmetry.

First we relate symmetry of the Antonelli matrix to the differential revealed preference axiom.

Theorem 22. Let $(X,\mathcal{B},A,\mathbb{C})$ be an n-dimensional classical choice space of type (iii), with $X = R^n_>$, and with single valued C^k $(k \geqq 1)$ budgeter β which never vanishes. Let E be a nonempty open subset of X. If the SBA holds on E, then E contains no C^1 Ville budget cycles. If E contains no C^∞ Ville budget cycle, then the SBA holds on E. Thus SBA holds on E if and only if C^k DDARP holds on E.

Proof. By [25], Theorem 1 (which rests on the integrability theorem of [24].

Theorem 22 is an extension of a result of Ville ([46], [47]); for a discussion of the relationship, see [25], Section VII.

Next, by the Duality Metatheorem we relate symmetry of the Slutsky matrix to the differential revealed preference axiom.

Theorem 23. Let $(X,\mathcal{B},A,\mathbb{C})$ be an n-dimensional classical choice space of type (iii), with $\mathcal{B} = R^n_>$, and with a single valued C^k $(k \geqq 1)$ demand ξ which never vanishes. Let E be a nonempty open subset of $\mathcal{B}$. If the SBA holds on E, then E contains no C^1 Ville commodity cycles. If E contains no C^∞ Ville commodity cycles then the SBA holds on E. Thus SDA holds on E if and only if C^k DBARP holds on E.

7. Revealed Preference and Differential Revealed Preference.

In Subsections 2 and 3 we motivated the differential revealed preference axioms by appealing to weak versions of the Strong Axioms of Revealed Preference. Indeed, although he did not use revealed preference terminology, Ville's axiom is very close in spirit to Houthakker's Strong Axiom of Revealed Preference. Nevertheless, it is important to note that these axioms are not equivalent: while the Strong Axioms imply both the negativity and symmetry axioms for differentiable demand or budgeter (Theorems 15 and 16, in conjunction with Theorems 13(c) and 2(c.ii & d.ii)), the differential axioms imply only symmetry -- but not negativity, hence not the convexity properties of even the Weak Weak Axioms. For details, see [25], Section V.

FOOTNOTES

* This research was aided by National Science Foundation Grant GS35682X. It represents one facet of studies over several years, extending the research reported in [34]. Some independent work on duality in revealed preference theory by Y. Sakai and by J. Little overlaps some of our results in Part III. We are indebted to Kim Border for helpful comments.

Page 3, n. 1. Indeed, production theory has been one of the important generators of recent duality research (cf. [40]).

Page 5, n. 1. For example, when Diewert ([14], p. 123) passes from an "indirect utility function" to the corresponding "direct utility function," he is not concerned with preserving rationality. Indeed, rationality may not be preserved. Cf. Part II.D.4(a), below.

Page 6, n. 1. For example, we can replace X by $X \times \{0\}$, and $\mathcal{B}$ by $\mathcal{B} \times \{1\}$.

Page 6, n. 2. This is not essential. We could instead assume, for example, that $\mathbb{C}$ is asymmetric (and then we might as well assume, for example, that $\mathbb{C} \subseteq X \times \mathcal{B}$). But assumption (II.1) will simplify our notation later.

Page 7, n. 1. Because $X \cap \mathcal{B} \neq 0$, symmetrizing an admissibility relation does not change its meaning through the introduction of unintended relationships.

Page 7, n. 2. This terminology conflicts with that in [34], but the greater scope of the present investigation warrants new terminology. Nevertheless, many of the concepts are very similar, so [34] may be consulted for more extensive discussion of many aspects.

Page 7, n. 3. $\mathfrak{P}X$ is the power set of X, the set of all subsets of X.

Page 9, n. 1. In many important applications (cf. Part IV) in classical competitive consumer theory, $\mathcal{B}$ can be identified with a subset of linear space (of "price vectors") that is the dual vector space (in the linear algebra sense) of the linear space X (of "commodity vectors"). Thus our duality terminology plans a dual role.

Page 10, n. 1. If $\geqslant$ has some additional property denoted by adjective P (e.g., transitivity) then we shall say that h is P-rational. Cf. [34].

Page 12, n. 1. Here, and elsewhere we shall omit proofs that are entirely elementary and straightforward.

Page 12, n. 2. As in footnote 1, page 10, if P is an adjective applicable to $\leqslant$, then b will be called P-rational.

Page 13, n. 1. We are now able to abserve that, analogously, while no utility exists in Example 1, a worth does exist (define $T(B_1) = 1$ & $T(B_2) = 1$ & $T(B_3) = 2$).

Page 14, n. 1. Rational choice is here defined differently than in [34]; what is called rational choice in [34] is what we have here called rational demand.

Page 14, n. 2. We modify as in footnote 1, page 10, where applicable.

Page 15, n. 1. Although the Metatheorem has important applications, it is very elementary (as well as "obvious"). Hence we shall be very brief in our exposition of it. Readers wishing more details will find [15] a useful source for much background material.

Page 15, n. 2. Cf. [15], pp. 68-69.

Page 15, n. 3. That is, we take the universal closures of the formulas in [5], pp. 507-508.

Page 16, n. 1. Indeed, since the Metatheorem simply says that "names do not matter," it is valid more generally than in our statement for just the two constants X and $\mathcal{B}$. So it is just a special case of a general "Symmetry Metatheorem."

Page 16, n. 2. For our more difficult theorems, we make use of the fact that the real numbers, R^n, and notions of continuity and differentiability, are also definable in $\mathfrak{L}$.

Page 17, n. 1. Of course that will change quasi-concavity into quasi-convexity, upper semi-continuity into lower semi-continuity, etc. (Cf. Theorems 3, 4, 5, 6, 8, 11, and 12).

Page 19, n. 1. This is just another, more complicated, way of saying that the statements and proofs of Theorem 1 could be translated into purely preference statements.

Page 28, n. 1. Another reason is that, for our purposes, to simply assume that our utility or worth was "continuous enough" would be to assume away the essential difficulty.

Page 32, n. 1. Continuity for correspondences is defined in [4].

Page 37, n. 1. In Samuelson's original terminology, we would say that $\xi(p,m)$ is selected over y ([35], p. 65) or chosen over y ([37], p. 246). This nonpreference terminology has the definite advantage of not sounding as if the existence of a preference is already being assumed, but the revealed preference terminology of [37] and [38] (to which we have added the word "direct") is very suggestive.

Page 37, n. 2. Since we have not required that $p \cdot \xi(p,m) = m$, but only $p \cdot \xi(p,m) \leqq m$, we could define a different, weaker, notion of revelation by requiring $p \cdot y \leqq m$ rather than $p \cdot y \leqq p \cdot \xi(p,m)$. This same observation will apply to many of the revelation notions that follow.

Page 37, n. 3. By the asymmetric part of a binary relation R we mean the relation P defined by: $xPy \Leftrightarrow xRy$ & not yRx.

Page 38, n. 1. By the transitive closure of a binary relation R on X we mean the smallest transitive relation on X that includes R.

Page 40, n. 1. We could change the axiom, so that (III.3) would read:

$$\bar{p} \cdot \xi(\tilde{p},\tilde{m}) < p \cdot \xi(\bar{p},\bar{m}) \Rightarrow \tilde{p} \cdot \xi(\bar{p},\bar{m}) > \tilde{m} .$$

This is just one of many points at which the theory bifurcates, leading in the end to a very large catalogue of definitions, axioms and theorems. Part of the problem is to pick out the most useful ones. In any case, it is usually assumed that, for all $(p,m) \in \mathcal{B}$, $p \cdot \xi(p,m) = m$; and then these differences, at least, vanish.

Page 47, n. 1. Recent developments in the theory of generalized differential equations would perhaps permit this assumption to be weakened.

Page 49, n. 1. I.e., for all $(y_1, \ldots, y_n) \in R^n$, $\sum_{i=1}^{n} \sum_{j=1}^{n} A^{ij}(x) y_i y_j \leqq 0$.

Page 49, n. 2. I.e., for all $i,j = 1,\ldots,n-1$, $A^{ij}(x) = A^{ji}(x)$.

Page 54, n. 1. Debreu's extension rests on the basic extension theorem of ordinary differential equations (cf. [19], pp. 12-13, Theorem 3.1), which requires only continuity of f.

Page 55, n. 1. Each level surface of W, through any point $(\bar{p},\bar{m})$, is described by the function $\mu(\cdot,p,m)$ which is concave, since, by (IV.12):

$$\mu_i(p,\bar{p},\bar{m}) = \xi^i(p,\mu(p,\bar{p},\bar{m})), \qquad (i = 1,\ldots,n-1)$$

so

$$\mu_{ij}(p,\bar{p},\bar{m}) = \xi^i_j(p,\mu(p,\bar{p},\bar{m})) + \xi^i_n(p,\mu(p,\bar{p},\bar{m}))\mu_j(p,\bar{p},\bar{m}) \qquad (i = 1,\ldots,n-1)$$

$$= \xi^i_j(p,\mu(p,\bar{p},\bar{m})) + \xi^i_n(p,\mu(p,\bar{p},\bar{m}))\xi^j(p,\bar{p},\bar{m})$$

$$= S^{ij}(p,\mu(p,\bar{p},\bar{m})).$$

Since $S(p,\mu(p,\bar{p},\bar{m}))$ is negative semi-definite by NSDDA, $\mu(\cdot,\bar{p},\bar{m})$ is (strictly) concave on $R^{n-1}_{>}$ (cf. [16], pp. 87-88, Theorem 35; [4], pp. 199-200, Corollary 2). From this it follows easily that W is quasi-convex.

Page 55, n. 2. We indicate the details briefly. Since $L = \{(p,m): W(p,m) \leqq W(\bar{p},\bar{m})\}$ is convex, it has a supporting hyperplane at $(\bar{p},\bar{m})$, say with normal $\tilde{r}$, so that, for some real α and all $(p,m)\epsilon A$, $r\cdot(p,m) \geqq \alpha$. The boundary of L is given by the level surface of $W(\bar{p},\bar{m})$, which has at $(\bar{p},\bar{m})$ the normal r of (IV.14), which can therefore be taken equal to $\tilde{r}$, so $\alpha = r\cdot(\bar{p},\bar{m}) = p_1\xi^1(\bar{p},\bar{m}) +\ldots+ p_{n-1}\xi^{n-1}(\bar{p},\bar{m}) - \bar{m} = -\xi^n(\bar{p},\bar{m})$ (by IV.8). Suppose that $(\bar{p},\bar{m})$ did not minimize W on K; say $W(p,m) < W(\bar{p},\bar{m})$ for some $(p,m)\epsilon K$. Since W is continuous, $W(p,m+\delta) < W(\bar{p},\bar{m})$ for some real $\delta > 0$, so $(p,m+\delta)\epsilon A$, and consequently $r\cdot(p,m+\delta) \geqq \alpha = -\xi^n(\bar{p},\bar{m})$; that is, $p_1\xi^1(\bar{p},\bar{m}) +\ldots+ p_{n-1}\xi^{n-1}(\bar{p},\bar{m}) - m - \delta \geqq -\xi^n(\bar{p},\bar{m})$, so $p_1\xi^1(\bar{p},\bar{m}) +\ldots+ p_{n-1}\xi^{n-1}(\bar{p},\bar{m}) + \xi^n(\bar{p},\bar{m}) > m$, contradicting the fact that $(p,m)\epsilon K$.

Page 56, n. 1. For $k = 1$, part (b) is essentially Debreu's formulation ([9], pp. 606-610) of the Antonelli approach ([2], [3]) (since [9] assumes that $\Gamma = X$, so a partial utility is a utility). (The assertion that the utility is C^2 ([9], p. 610) does not follow when β is only C^1; cf. [10].)

Page 57, n. 1. Although [29] postulates a C^1 demand, only differentiability is actually needed for these theorems.

Page 60, n. 1. Here and elsewhere we interpret derivatives at 0 and r as being from the right and from the left, respectively.

Page 63, n. 1. By Theorem 12.

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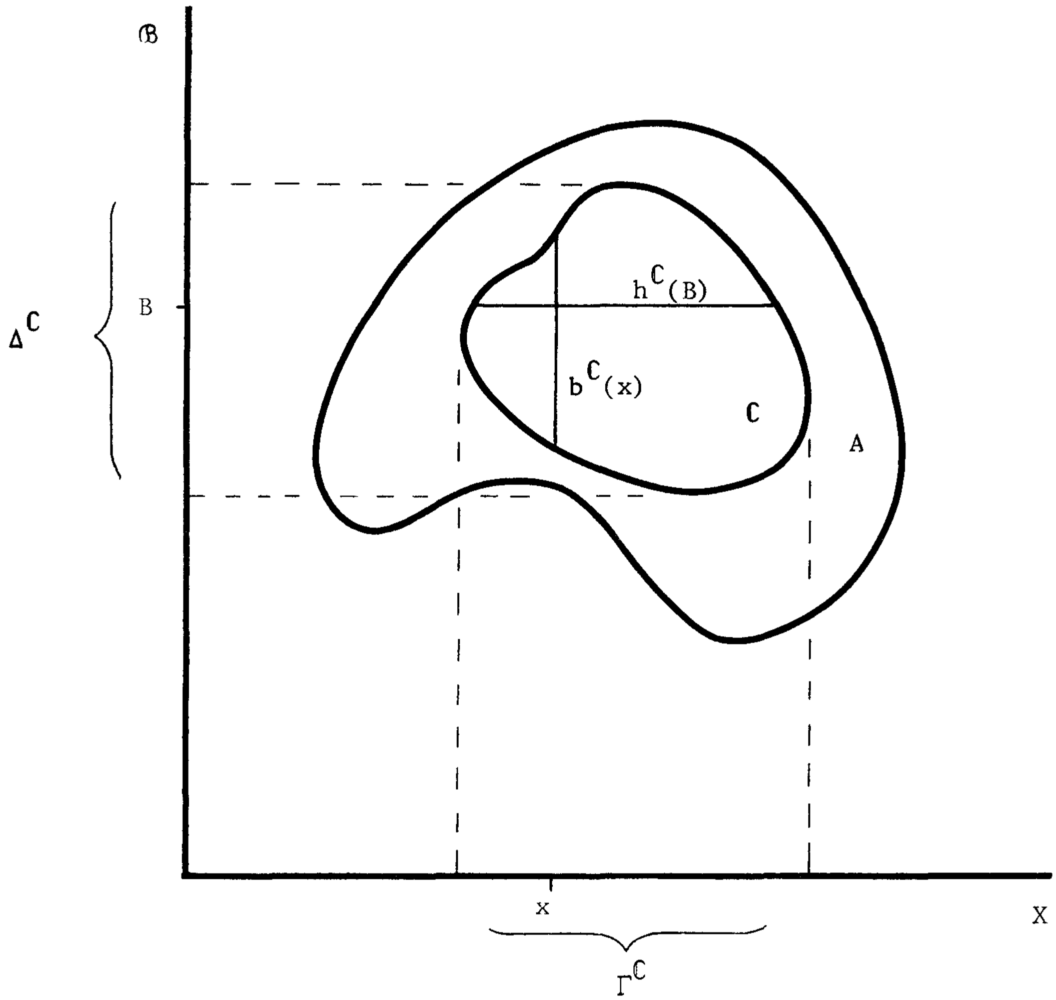
Restrictions to $X \times \mathcal{B}$



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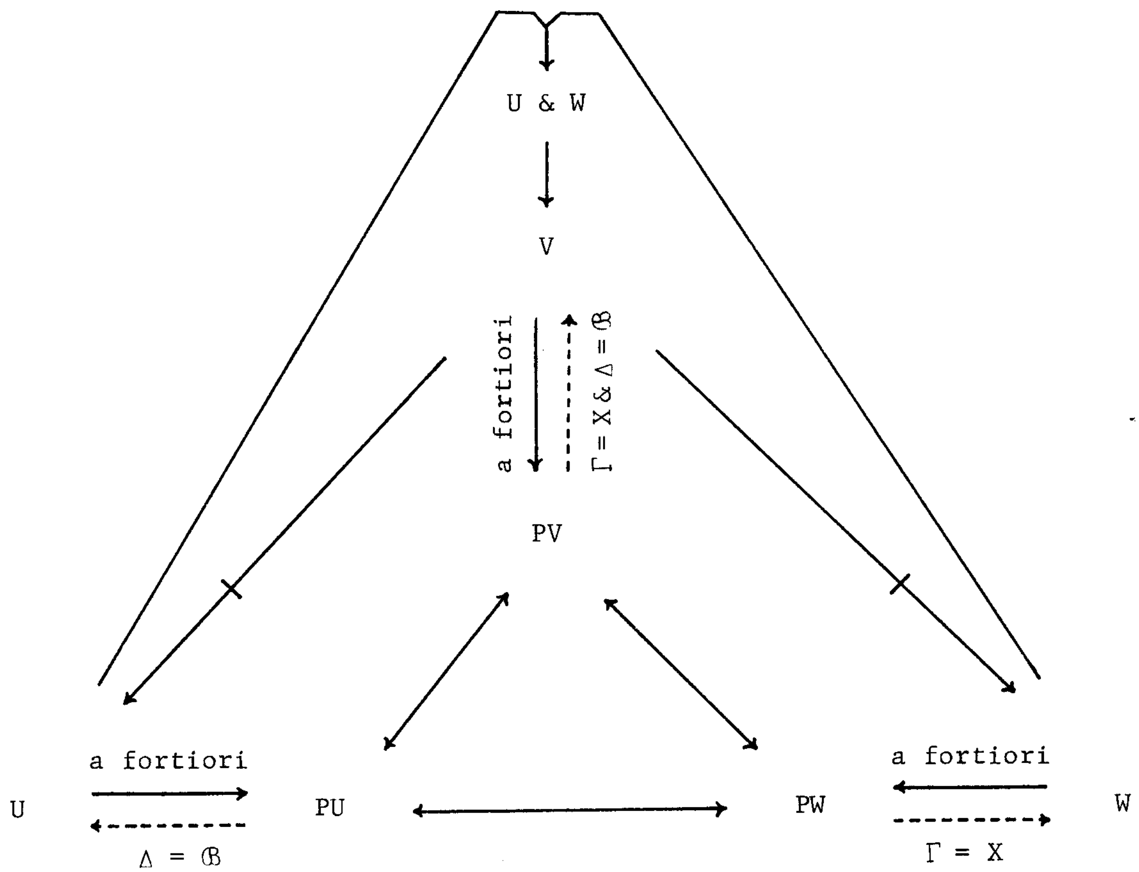

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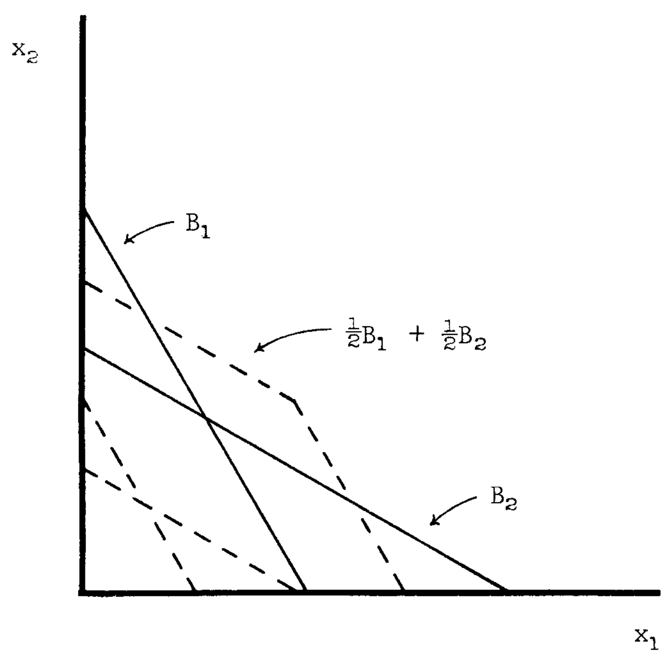

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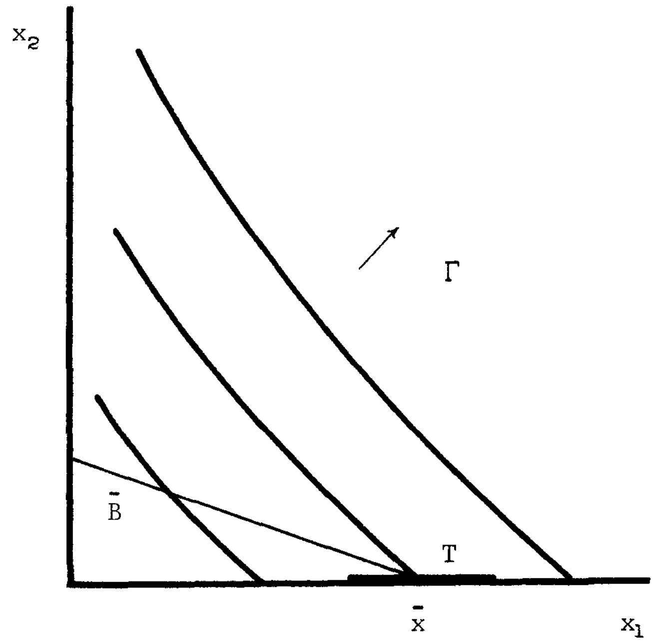

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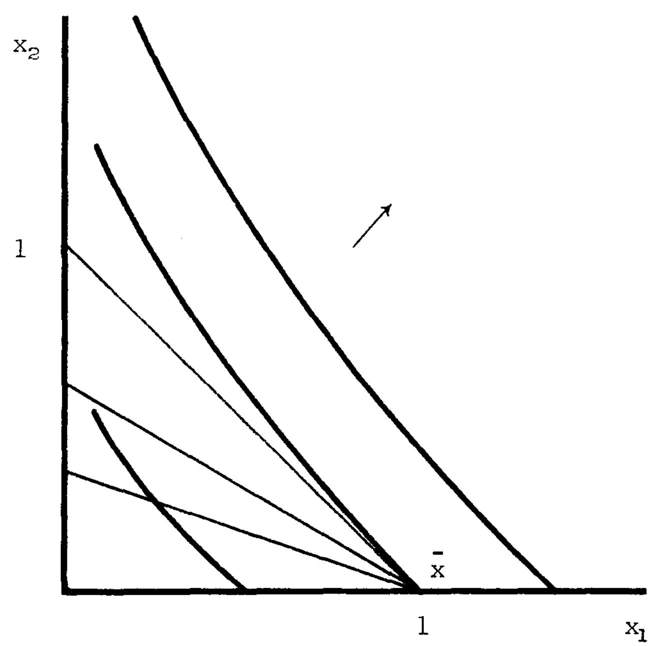

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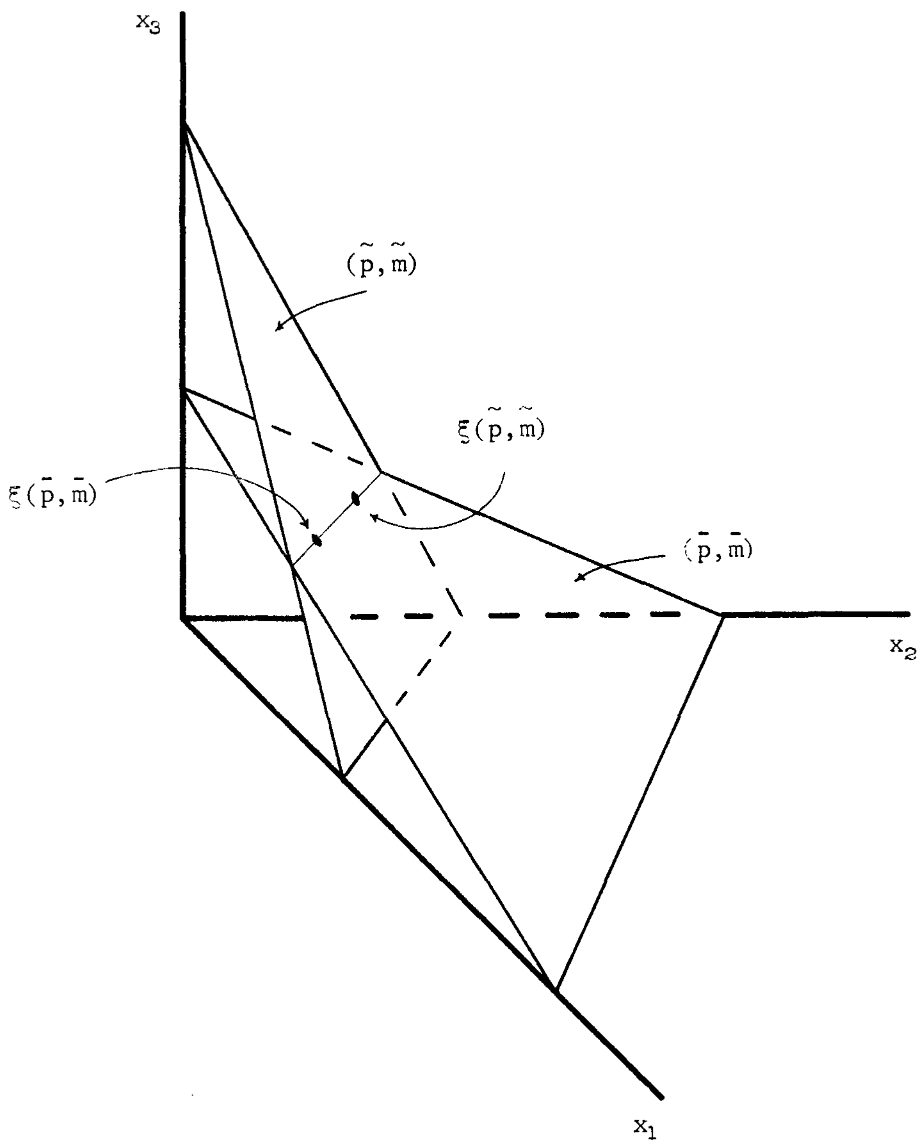


x_3
$(\tilde{p}, \tilde{m})$
$\xi(\tilde{p}, \tilde{m})$
$\xi(\bar{p}, \bar{m})$
$(\bar{p}, \bar{m})$
x_2
x_1

Figure 7

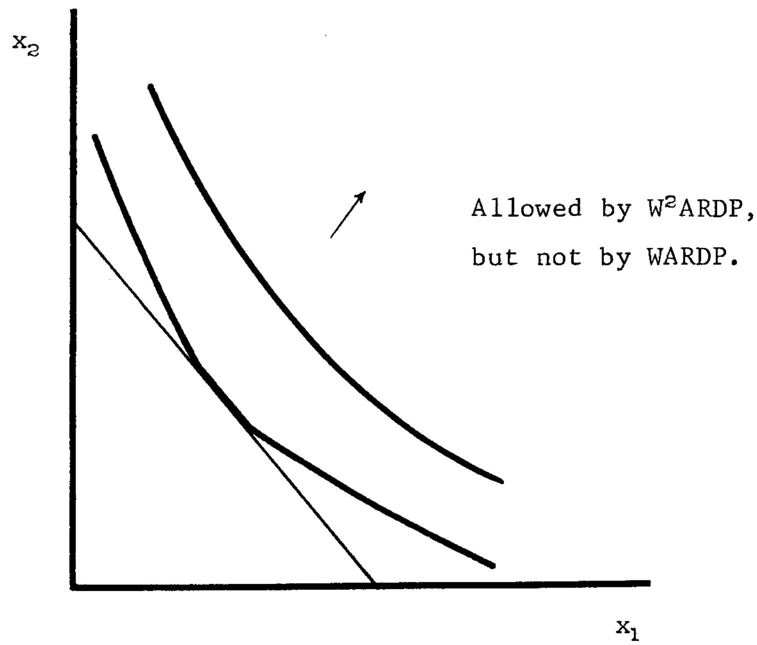


Figure 8a

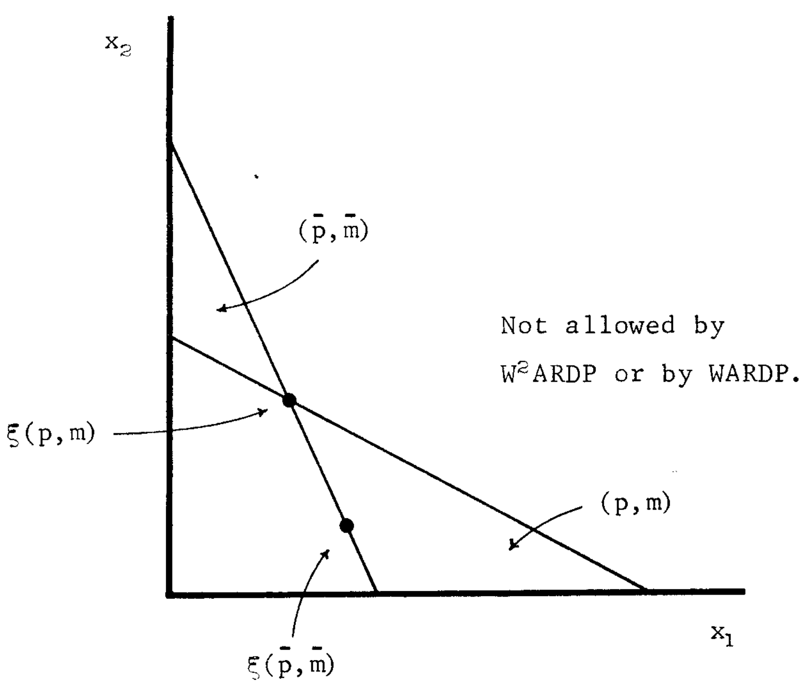
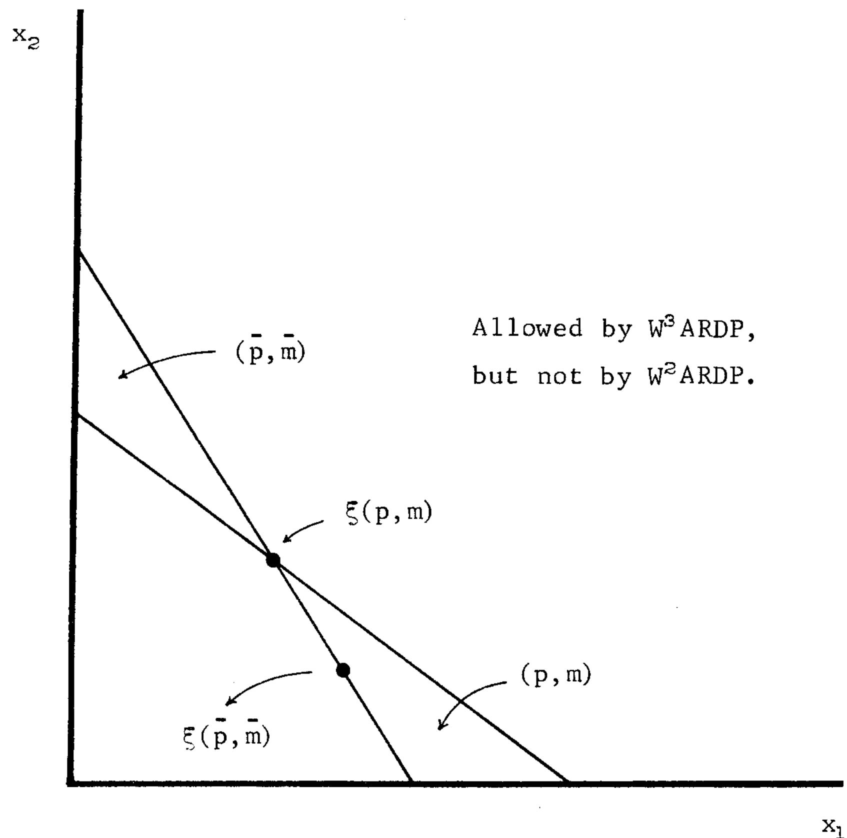
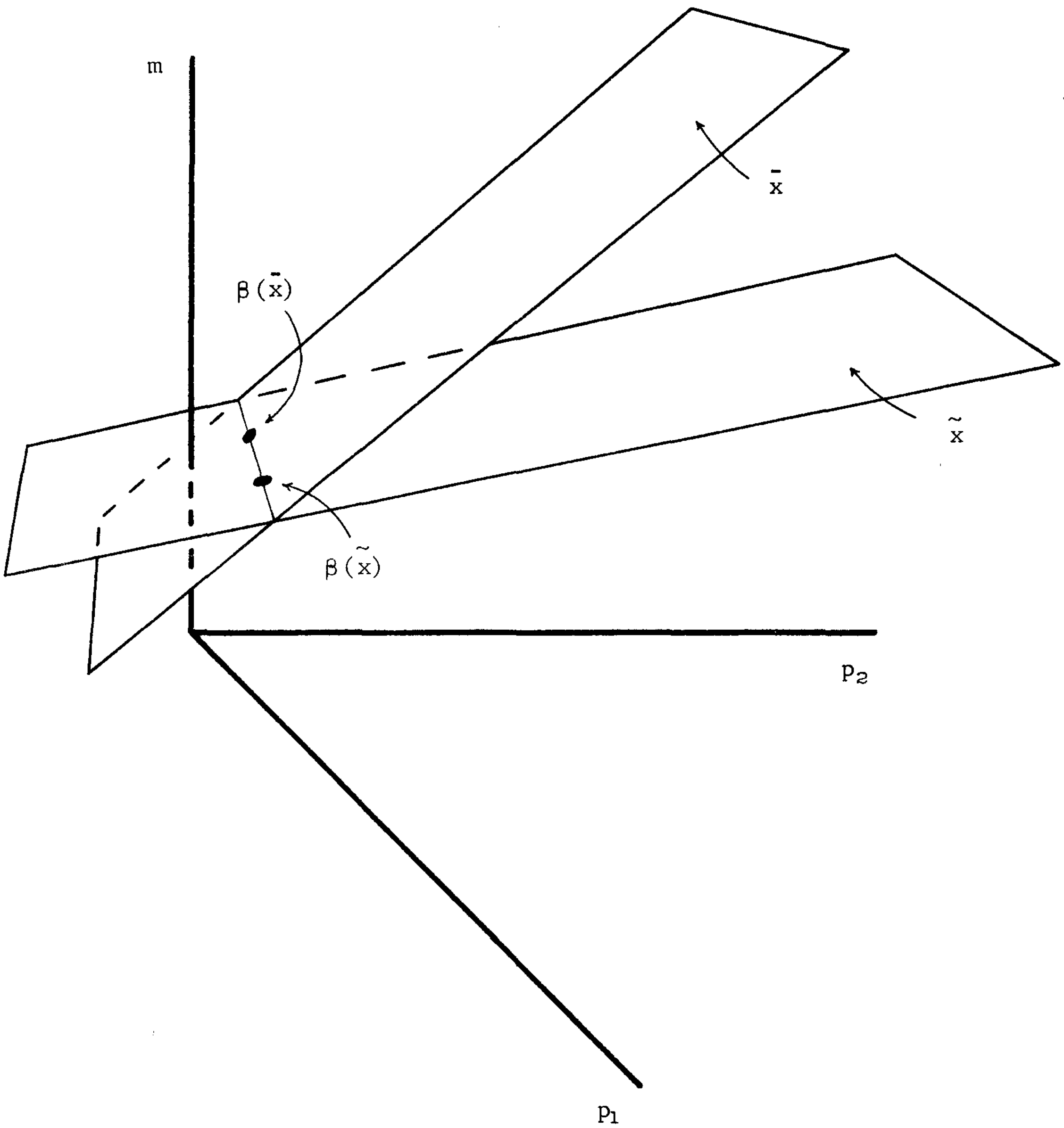


Figure 8b



x_2
Allowed by W^3ARDP,
but not by W^2ARDP.
$(\bar{p}, \bar{m})$
$\xi(p, m)$
(p, m)
$\xi(\bar{p}, \bar{m})$
x_1

Figure 9



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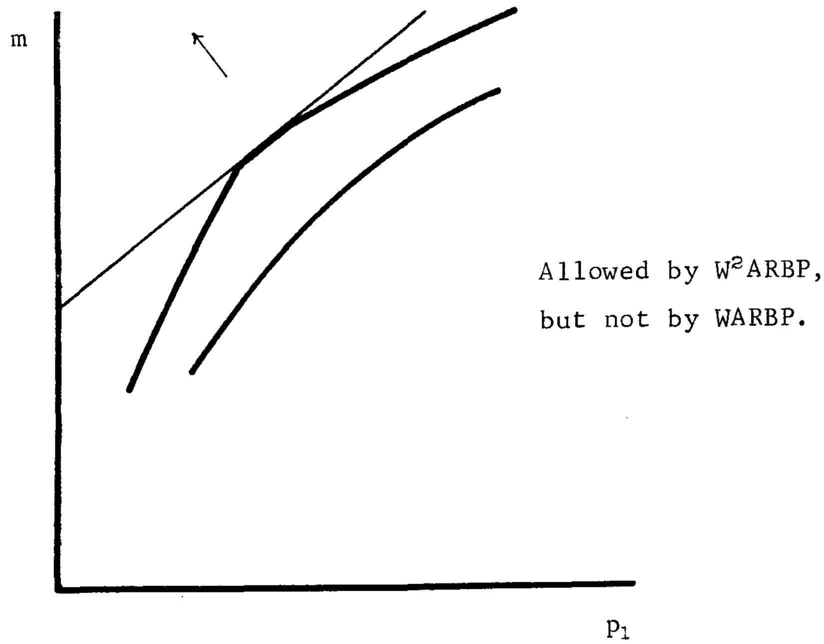

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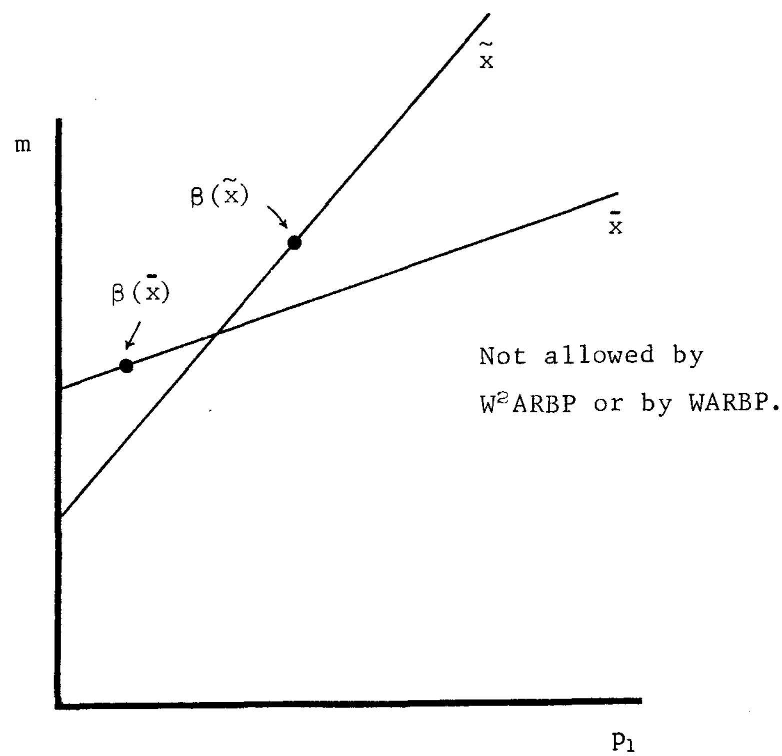
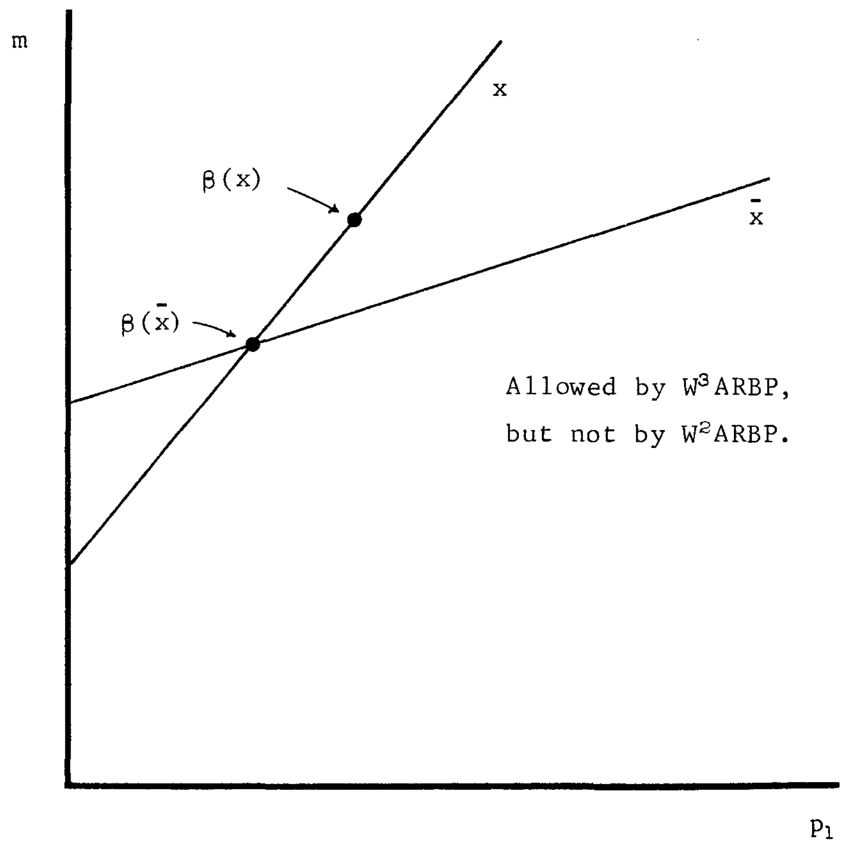


Figure 11b



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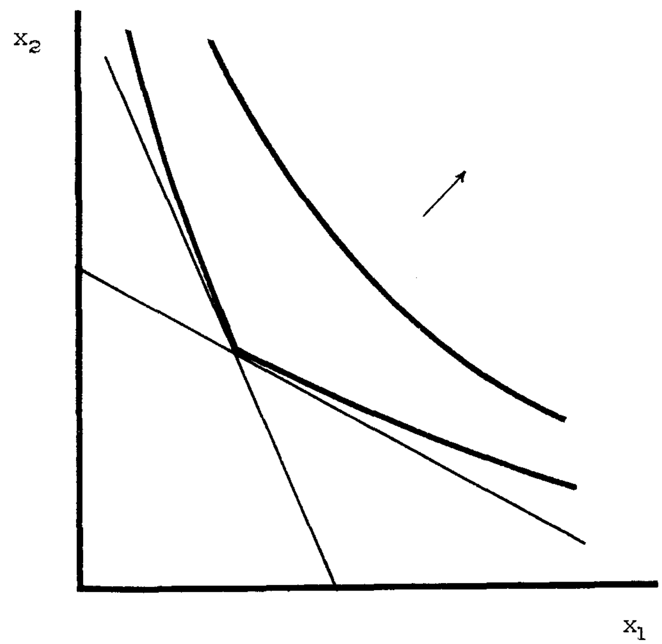

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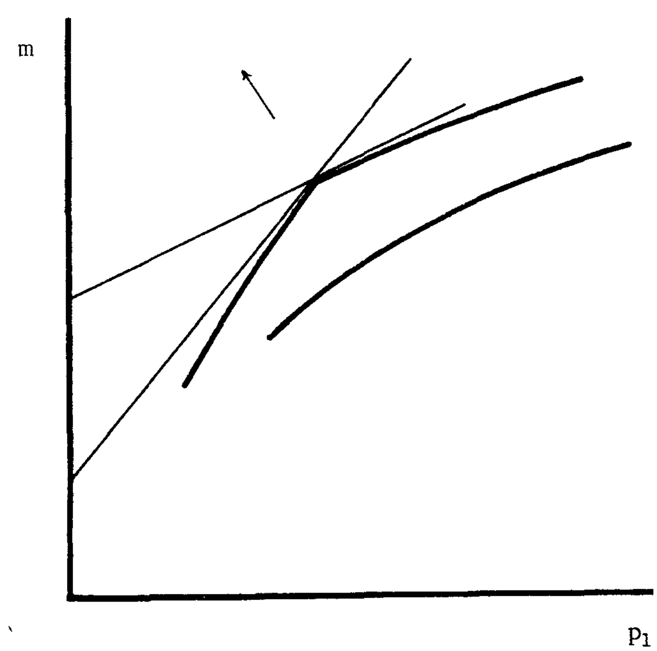

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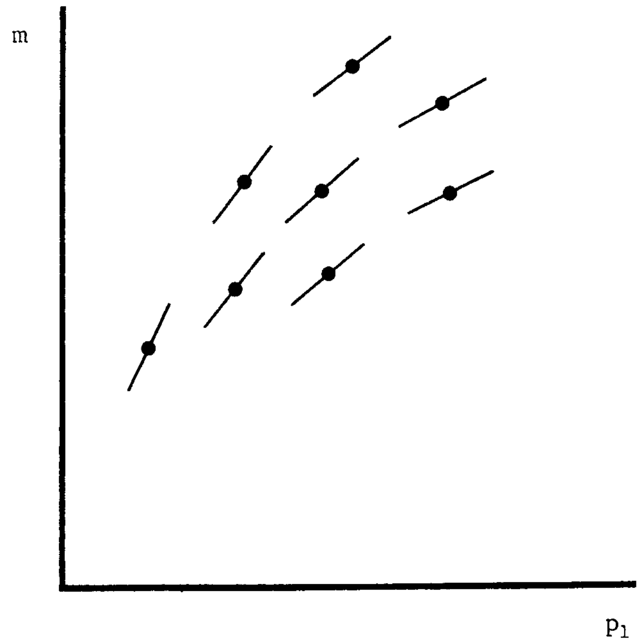

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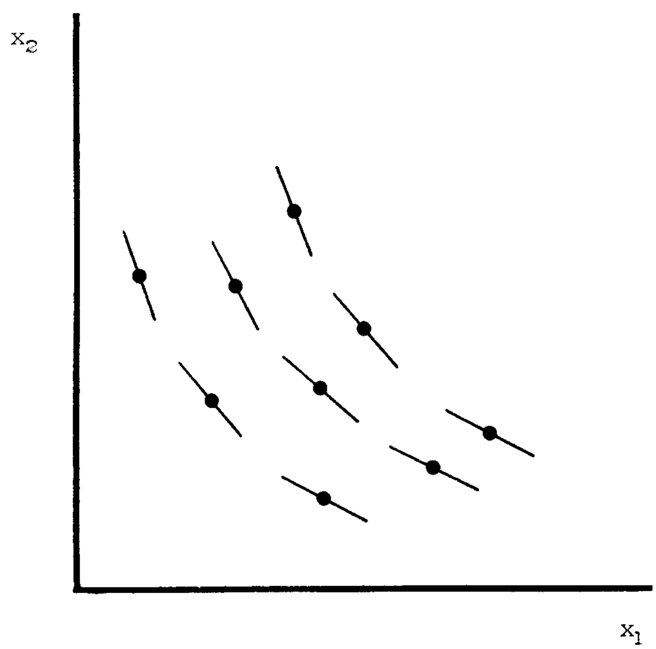

Figure 15